Hardy inequalities and singular inverse-square potentials in controllability theory

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Controllability of standard PDE's with an inverse-square potential

$\Omega \subset \mathbb{R}^N$ bounded

Question : replace $-\Delta$ by $-\Delta - V(x)I$ in standard PDE's (heat equation, wave equation, Schrödinger equation) ?

- $V \in L^\infty(\Omega)$: standard results holds (well-posedness + controllability)
- what about singular inverse-square potentials ?

$$0 \in \Omega \quad \text{and} \quad V(x) = \frac{\lambda}{|x|^2}, \; x \in \Omega$$

Models with inverse-square potentials :

- quantum mechanics : Baras-Goldstein, 1994
- linearized combustion models around singular stationary solutions : Mignot-Puel (1988), Brézis-Vázquez (1997)

Role of Hardy inequalities I

and interesting phenomena generated by such singular potentials

Essential tool : Hardy inequality

$$\forall z \in H_0^1(\Omega), \qquad \underbrace{\frac{(N-2)^2}{4}}_{=:\lambda_\star(N)} \int_\Omega \frac{z^2}{|x|^2}, dx \leq \int_\Omega |\nabla z|^2 \, dx$$

Consequences : for $N \neq 2$, $z \in H_0^1(\Omega) \Rightarrow z/|x| \in L^2(\Omega)$;
$-\Delta - \lambda|x|^{-2}I$ coercive in $H_0^1(\Omega)$ when $\lambda < \lambda_\star(N)$ and at least nonnegative when $\lambda \leq \lambda_\star(N)$

Example : well-posedness of the heat equation with an inverse square-potential

- **super-critical case : $\boldsymbol{\lambda > \lambda_\star(N)}$**
  Baras-Goldstein (Trans. AMS, 1984) : severely ill-posed, instantaneous and complete blow-up of positive solutions

Role of Hardy inequalities II

and interesting phenomena generated by such singular potentials

- **sub-critical case : $\boldsymbol{\lambda < \lambda_\star(N)}$**
  global existence in standard functional setting
  $u_0 \in L^2(\Omega) \Rightarrow u \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$
- **critical case : $\boldsymbol{\lambda = \lambda_\star(N)}$** , *Vázquez-Zuazua, 2000*
  same result but replace $H_0^1(\Omega)$ by $H := \overline{H_0^1(\Omega)}^{\|\cdot\|_H}$

$$\|z\|_H^2 := \int_\Omega \left[|\nabla z|^2 - \lambda_\star(N) \frac{z^2}{|x|^2} \right] dx$$

$$u_0 \in L^2(\Omega) \Rightarrow u \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H)$$

Remark : $H_0^1(\Omega) \underset{\neq}{\subset} H \underset{\neq}{\subset} \cap_{q<2} W^{1,q}(\Omega)$

$$1 \le q < 2, \qquad \int_\Omega \left[|\nabla z|^2 - \lambda_\star(N) \frac{z^2}{|x|^2} \right] dx \ge C_q \|z\|^2_{W^{1,q}(\Omega)}$$

The heat equation I

The control problem : $T > 0$, $0 \in \Omega \subset \mathbb{R}^N$ bdd, $\emptyset \neq \omega \subset \Omega$, $0 \notin \omega$

$$\begin{cases} u_t - \Delta u - \dfrac{\lambda}{|x|^2} u = h\chi_\omega & (0,T) \times \Omega \\ u = 0 & (0,T) \times \partial\Omega \\ u(0,x) = u_0(x) & \Omega \end{cases}$$

- *Fursikov-Imanuvilov, 1996* :
  $V \in L^\infty(\Omega) \rightsquigarrow$ null controllability (in any time $T > 0$).
  Proof : Carleman estimates $\Rightarrow$ observability $\Rightarrow$ N.C.
- *Imanuvilov-Yamamoto, 2003* : $V \in L^p(\Omega), p < 2N/3$.
- *V.-Zuazua (J. Funct. Anal., 2008)* : $V(x) = \lambda/|x|^2$
  - $\mathbf{N = 1}$ **and** $\boldsymbol{\lambda \leq \lambda_\star(1)}$: null controllability still holds.
    Proof : Hardy inequalities + *new specific Carleman estimates* (weights inspired by *Cannarsa-Martinez-V., SICON 2008*)

The heat equation II

- $\mathbf{N \geq 3,\ \lambda \leq \lambda_\star(N)}$: same result but under specific geometric conditions (ω circles the singularity).
  Tools : cut-off arguments + spherical harmonics + previous $1-d$ Carleman estimates for singular problems.

- *Ervedoza (preprint, 2008)* : $V(x) = \lambda/|x|^2$
  - $\mathbf{N \geq 3,\ \lambda \leq \lambda_\star(N)}$: removed the geometric condition.
    Tools : extension of previous $1-d$ Carleman estimates.
  - $\boldsymbol{\lambda > \lambda_\star(N)}$: lack of null controllability.
    Proof : regularized potential $V_\varepsilon(x)$ + sequence of eigenfunctions whose energies concentrate around the singularity $\rightsquigarrow$ lack of uniform observability$/\varepsilon$ (when $0 \notin \omega$).

Conclusion : standard controllability results hold for the heat equation when $\lambda \leq \lambda_\star(N)$.
Question : what about the wave equation ?

The wave equation I

The control problem : $0 \in \Omega \subset \mathbb{R}^N$ bdd, $\Gamma = \partial\Omega$ $\mathcal{C}^2$, $\emptyset \neq \Gamma_0 \subset \Gamma$

$$\begin{cases} u_{tt} - \Delta u - \dfrac{\lambda}{|x|^2} u = 0 & (0,T)\times\Omega \\ u(t,x) = h(t,x) & (0,T)\times\Gamma_0 \\ u(t,x) = 0 & (0,T)\times\Gamma\setminus\Gamma_0 \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \Omega \end{cases}$$

Question : exact controllability ?

References in the case $\lambda = 0$:
Dirichlet boundary control : *Lions (88), Triggiani (88)*
Neumann boundary control : *Triggiani (86), Lasiecka-Triggiani (89)*

The wave equation II

Exact controllability result in the case $\boldsymbol{\lambda = 0}$: *assume* $\Gamma_0 := \{x \in \Gamma \mid (x - x_0) \cdot \nu \geq 0\}$ *and* $T > 2\max\{|x - x_0|, x \in \Omega\}$. *Then, for all* (u_0, u_1) *and* $(\bar{u}_0, \bar{u}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *there exists* $h \in L^2((0, T) \times \Gamma_0)$ *such that* $(u(T), u_t(T)) = (\bar{u}_0, \bar{u}_1)$

Question : extends this result to the case $\lambda \leq \lambda_\star(N)$? provided of course that $H_0^1(\Omega)$ is replaced by H_λ defined by

$$H_\lambda := \overline{H_0^1(\Omega)}^{\|\cdot\|_{H_\lambda}} \text{ with } \|z\|_{H_\lambda}^2 := \int_\Omega \left(|\nabla z|^2 - \lambda \frac{z^2}{|x|^2} \right) dx$$

- $\mathbf{N \neq 2,\ \boldsymbol{\lambda} < \boldsymbol{\lambda}_\star(N)}$: Hardy inequality $\Rightarrow$ $H_\lambda = H_0^1(\Omega)$ since

$$\left(1 - \frac{\max(0, \lambda)}{\lambda_\star}\right) \|z\|_{H_0^1(\Omega)} \leq \|z\|_{H_\lambda} \leq \left(1 - \frac{\min(0, \lambda)}{\lambda_\star}\right) \|z\|_{H_0^1(\Omega)}$$

- $\mathbf{N \neq 2,\ \boldsymbol{\lambda} = \boldsymbol{\lambda}_\star(N)}$: $H_\lambda = H$ described by Vázquez-Zuazua

The wave equation III

Main results : *V.-Zuazua (preprint, 2008)*

1. $\boldsymbol{\lambda \leq \lambda_\star(N)}$: Exact controllability with an observation on $\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}$.

Theorem (Exact controllability)

Assume $\lambda \leq \lambda_\star(N)$ and $T > T_0 = 2R_\Omega := 2max\{|x|, x \in \Omega\}$. Then for all (u_0, u_1) and $(\bar{u}_0, \bar{u}_1) \in L^2(\Omega) \times H_\lambda{}'$, there exists $h \in L^2((0,T) \times \Gamma_0)$ such that $(u(T), u_t(T)) = (\bar{u}_0, \bar{u}_1)$.

2. $\boldsymbol{\lambda > \lambda_\star(N)}$: Lack of controllability (when $0 \notin \omega$).

$\lambda \leq \lambda_\star(N)$: exact controllability I

STEP 1 : consider the adjoint problem

$$\begin{cases} v_{tt} - \Delta v - \dfrac{\lambda}{|x|^2} v = 0 & (0,T)\times\Omega, \\ v(t,x) = 0 & (0,T)\times\Gamma, \\ v(0,x) = v_0(x),\ \ v_t(0,x) = v_1(x) & \Omega. \end{cases}$$

- Well-posed in the energy space $H_\lambda \times L^2(\Omega)$ ($=$ standard energy space $H_0^1(\Omega)\times L^2(\Omega)$ when $\lambda < \lambda_\star(N)$).
- Generalized energy for all $\lambda \leq \lambda_\star(N)$:

$$E_v^\lambda(t) := \frac{1}{2}\int_\Omega \left(|\nabla v|^2 - \lambda \frac{v^2}{|x|^2} + v_t^2 \right) dx = \frac{1}{2}\left(\|v\|_{H_\lambda}^2 + \|v_t\|_{L^2(\Omega)}^2 \right).$$

Lemma. For all $\lambda \leq \lambda_\star(N)$, $t \mapsto E_v^\lambda(t)$ is constant.

$\lambda \leq \lambda_\star(N)$: exact controllability II

- Classical energy for $\lambda < \lambda_\star(N)$ (but $\lambda \neq \lambda_\star(N)$) :

$$E_v^0(t) := \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + v_t^2\right) dx = \frac{1}{2}\left(\|v\|^2_{H_0^1(\Omega)} + \|v_t\|^2_{L^2(\Omega)}\right),$$

equivalent to $E_v^\lambda(t)$ but not constant in time.

STEP 2 : extends "hidden regularity" results of the normal derivative known for the standard wave equation (i.e. when $\lambda = 0$).

Theorem (Hidden regularity or direct inequality)

Assume $\lambda \leq \lambda_\star(N)$ and $T > 0$.

$$\int_0^T \int_\Gamma \left|\frac{\partial v}{\partial \nu}\right|^2 \leq C_{T,\lambda} E_v^\lambda(0).$$

$\lambda \leq \lambda_\star(N)$: exact controllability III

Proof : recall the classical multipliers Lemma

Lemma

For all $q = (q_k)_k \in \mathcal{C}^1(\overline{\Omega})^N$,

$$\frac{1}{2}\int_0^T \int_\Gamma q\cdot\nu \left|\frac{\partial z}{\partial \nu}\right|^2 = \Big[\int_\Omega z_t q\cdot\nabla z\Big]_0^T$$
$$+\frac{1}{2}\int_0^T\int_\Omega (z_t^2 - |\nabla z|^2)\, div\, q + \sum_{j,k}\int_0^T\int_\Omega \frac{\partial q_k}{\partial x_j}\frac{\partial z}{\partial x_j}\frac{\partial z}{\partial x_k} - \int_0^T\int_\Omega Fq\cdot\nabla z$$

for all z *solution of*

$$\begin{cases} z_{tt} - \Delta z = F & (0,T)\times\Omega, \\ z(t,x) = 0 & (0,T)\times\Gamma. \end{cases}$$

$\lambda \leq \lambda_\star(N)$: exact controllability IV

Apply multipliers Lemma with $z = v$, $F = \lambda v/|x|^2$ and $q \in \mathcal{C}^1(\overline{\Omega})^N$ such that $q = \nu$ on Γ and $q = 0$ near $x = 0$.

Construction of q :

- Γ of class $\mathcal{C}^2 \Rightarrow \exists q_0 \in \mathcal{C}^1(\overline{\Omega})^N$ s.t. $q_0 = \nu$ on Γ.
- ϕ $\mathcal{C}^\infty$ cut-off function s.t. $\phi \equiv 0$ near $x = 0$ and $\phi \equiv 1$ near Γ.
- $q := q_0\phi$. $\square$

STEP 3 : existence of very weak solutions to the problem.

Theorem

For every $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ *and* $h \in L^2((0,T) \times \Gamma_0)$, *there exists a unique solution* $u \in \mathcal{C}([0,T]; L^2(\Omega)) \cap \mathcal{C}^1([0,T]; H'_\lambda)$.

Proof : hidden regularity results + method of transposition.

$\lambda \leq \lambda_\star(N)$: exact controllability V

STEP 4 : main result = observability under the condition that the observation holds on $\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}$.

Theorem (Observability or inverse inequality)

Assume $\lambda \leq \lambda_\star(N)$ and $T > T_0 = 2R_\Omega := 2 \max_{x \in \Omega} |x|$.

$$\int_\Omega \left(|\nabla v(0,x)|^2 - \lambda \frac{|v(0,x)|^2}{|x|^2} + |v_t(0,x)|^2 \right) \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial v}{\partial \nu} \right|^2$$

$$\text{i.e.} \qquad E_v^\lambda(0) \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial v}{\partial \nu} \right|^2 .$$

Consequence : direct + inverse inequalities $\Rightarrow$ exact controllability.

$\lambda \le \lambda_\star(N)$: exact controllability VI

Proof. Idea = apply multipliers Lemma with $z = v$, $F = \lambda v/|x|^2$, $q(x) = x, \ \forall x \in \overline{\Omega}$, (classical multiplier $x - x_0$ centered at $x_0 = 0$).

$$\Big[(v_t, x\cdot\nabla v)_{L^2(\Omega)} \Big]_0^T + \frac{N}{2}\int_0^T\int_\Omega (v_t^2 - |\nabla v|^2) + \int_0^T\int_\Omega |\nabla v|^2$$
$$= \underbrace{\frac{1}{2}\int_0^T\int_\Gamma x\cdot\nu\left(\frac{\partial v}{\partial \nu}\right)^2}_{\le R_\Omega \int_0^T\int_{\Gamma_0}\left|\frac{\partial v}{\partial \nu}\right|^2} + \lambda \underbrace{\int_0^T\int_\Omega \frac{v}{|x|^2} x\cdot\nabla v}_{= -\frac{N-2}{2}\int_0^T\int_\Omega \frac{v^2}{|x|^2}}$$

using the condition on Γ_0 and the definition of R_Ω.

$\lambda \leq \lambda_\star(N)$: exact controllability VII

Using the definition of E_v^λ :

$$\Big[(v_t, x \cdot \nabla v)_{L^2(\Omega)} \Big]_0^T + \int_0^T E_v^\lambda(t) + \frac{N-1}{2} \underbrace{\int_0^T \int_\Omega \left(v_t^2 - |\nabla v|^2 + \lambda \frac{v^2}{|x|^2} \right)}_{= \left[\int_\Omega v_t v \right]_0^T} \leq \frac{R_\Omega}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial v}{\partial \nu} \right|^2 .$$

Using $E_v^\lambda(t) = E_v^\lambda(0)$ for all $t \geq 0$:

$$\Big[(v_t, x \cdot \nabla v + \frac{N-1}{2} v)_{L^2(\Omega)} \Big]_0^T + T E_v^\lambda(0) \leq \frac{R_\Omega}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial v}{\partial \nu} \right|^2 .$$

$\lambda \leq \lambda_\star(N)$: exact controllability VIII

It remains to estimate the following quantity at $t = 0$ and $t = T$:

$$
\begin{aligned}
&|(v_t, x \cdot \nabla v + \frac{N-1}{2} v)_{L^2(\Omega)}(t)| \\
&\leq \frac{C}{2}\|v_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2C}\|x \cdot \nabla v(t)\|^2_{L^2(\Omega)} - \frac{1}{2C}\frac{N^2-1}{4}\|v(t)\|^2_{L^2(\Omega)} \\
&\qquad \underbrace{\leq}_{??} \frac{T_0}{2} E_v^\lambda(0) \quad \text{for some suitable } T_0 > 0\,?
\end{aligned}
$$

Remark in the sub-critical case $\lambda < \lambda_\star(N)$:

- $\|x \cdot \nabla v(t)\|_{L^2(\Omega)} \leq R_\Omega \|\nabla v(t)\|_{L^2(\Omega)}$
- $\nabla v(t)$ bounded in $L^2(\Omega)$ in terms of $E_v^0(t)$
- $E_v^0(t)$ and $E_v^\lambda(t)$ equivalent and $t \mapsto E_v^\lambda(t)$ constant

$\Rightarrow$ OK but $T_0 = T_0(\lambda) \to +\infty$ if $\lambda \to \lambda_\star(N)$

$\Rightarrow$ $T_0(\lambda)$ not uniform/λ and no result for $\lambda = \lambda_\star(N)$.

$\lambda \leq \lambda_\star(N)$: exact controllability IX

Question : uniform time of controllability T_0 and critical case ?
$\lambda = \lambda_\star(N) \Rightarrow \nabla v(t) \notin L^2(\Omega) \rightsquigarrow$ need a (uniform) bound of $\|x \cdot \nabla v(t)\|^2_{L^2(\Omega)} \rightsquigarrow$ derive suitable Hardy-type inequalities :

$$\|x \cdot \nabla z\|^2_{L^2(\Omega)} \leq C_\Omega^2 \|z\|^2_{H_{\lambda_\star}} \text{ for some } C_\Omega > 0,$$

i.e

$$\int_\Omega |x \cdot \nabla z|^2 \, dx \leq C_\Omega^2 \int_\Omega \left[|\nabla z|^2 - \lambda_\star \frac{z^2}{|x|^2} \right] dx.$$

$$\Rightarrow |(v_t, x \cdot \nabla v + \frac{N-1}{2} v)_{L^2(\Omega)}(t)| \leq C_\Omega E_v^\lambda(t) = C_\Omega E_v^\lambda(0)$$

$\rightsquigarrow$ required inequality with $T_0 = 2C_\Omega$ but not $T_0 = 2R_\Omega$.

$\lambda \leq \lambda_\star(N)$: exact controllability X

Question : retrieve the expected minimal time $T_0 = 2R_\Omega$?
$R_\Omega := \max\{|x|, x \in \Omega\}$
Idea = sharper Hardy inequality + use the non-positive term

Lemma (V.-Zuazua, preprint 2008)

$$\|x \cdot \nabla z\|^2_{L^2(\Omega)} \leq R_\Omega^2 \|z\|^2_{H_{\lambda_\star}} + \frac{N^2-4}{4}\|z\|^2_{L^2(\Omega)}, \qquad i.e.$$

$$\int_\Omega |x \cdot \nabla z|^2\, dx \leq R_\Omega^2 \int_\Omega \left[|\nabla z|^2 - \lambda_\star \frac{z^2}{|x|^2}\right] dx + \frac{N^2-4}{4}\int_\Omega z^2\, dx.$$

$$\Rightarrow |(v_t, x \cdot \nabla v + \frac{N-1}{2} v)_{L^2(\Omega)}(t)| \leq R_\Omega E_v^\lambda(t) = \frac{T_0}{2} E_v^\lambda(0)$$

$\rightsquigarrow$ required inequality with $T_0 = 2R_\Omega$. □

$\lambda > \lambda_\star(N)$: Lack of controllability I

To simplify : $N \geq 3$, $\Omega = B_2 = B(0,2) \rightsquigarrow \Gamma_0 = \partial B_2$, $T_0 = 4$.

$$\begin{cases} v_{tt} - \Delta v - \dfrac{\lambda}{|x|^2 + \varepsilon^2} v = 0 & (0,T) \times \Omega \\ v(t,x) = 0 & (0,T) \times \Gamma \end{cases}$$

$\varepsilon > 0 \Rightarrow \forall \lambda \in \mathbb{R}$, well-posed in $H_0^1(\Omega) \times L^2(\Omega)$ + observability inequality holds : $\forall \lambda \in \mathbb{R}$, $\forall \varepsilon > 0$, $\forall T > 4$, $\exists C_\lambda(\varepsilon) > 0$ such that

$$\int_{B_2} \left(|\nabla v(0,x)|^2 + |v_t(0,x)|^2 \right) \leq C_\lambda(\epsilon) \int_0^T \int_{\partial B_2} \left| \frac{\partial v}{\partial \nu} \right|^2 .$$

Theorem (V.-Zuazua, preprint 2008)

$\lambda > \lambda_\star(N) \Rightarrow$ *no uniform observability/ε (for any time $T > 0$) i.e. $C_\lambda(\varepsilon)$ blows up as $\varepsilon \to 0^+$.*

$\lambda > \lambda_\star(N)$: Lack of controllability II

Proof by contradiction : assume $\exists C_\lambda > 0$ (indep. of ε) s.t.

$$\int_{B_2} \left(|\nabla v(0,x)|^2 + |v_t(0,x)|^2\right) \leq C_\lambda \int_0^T \int_{\partial B_2} \left|\frac{\partial v}{\partial \nu}\right|^2$$

STEP 1. $\exists C_\lambda > 0$ (indep. of ε) s.t.

$$\int_0^T \int_{\partial B_2} \left|\frac{\partial v}{\partial \nu}\right|^2 \leq C_\lambda \int_0^T \int_{B_2 \setminus B_1} \left(|\nabla v|^2 + v_t^2\right).$$

Proof : multiplier $q(t,x) := t(T-t)q_0(x)\phi(x)$.

STEP 2. Step 1 $\Rightarrow$ sufficient to contradict :

$$\int_{B_2} \left(|\nabla v(0,x)|^2 + |v_t(0,x)|^2\right) \leq C_\lambda \int_0^T \int_{B_2 \setminus B_1} \left(|\nabla v|^2 + v_t^2\right).$$

$\lambda > \lambda_\star(N)$: Lack of controllability III

Idea = radial solutions + $\bar{v}(t,r) = r^{(N-1)/2} v(t,r)$
$\Rightarrow$ one needs to contradict

$$\int_0^2 \left(|\bar{v}_r(0,r)|^2 + \frac{(N-1)(N-3)}{4} \frac{|\bar{v}(0,r)|^2}{r^2+\varepsilon^2} + |\bar{v}_t(0,r)|^2 \right) dr$$
$$\leq C_\lambda \int_0^T \int_1^2 \left(\bar{v}_r^2 + \frac{(N-1)(N-3)}{4} \frac{\bar{v}^2}{r^2+\varepsilon^2} + \bar{v}_t^2 \right) drdt$$

$$\text{where } \begin{cases} \bar{v}_{tt} - \bar{v}_{rr} - \dfrac{K}{r^2+\varepsilon^2}\bar{v} = 0 & (t,r) \in (0,T)\times(0,2), \\ \bar{v}(t,0) = 0 = v(t,2) & t \in (0,T) \end{cases}$$

Remark : $\lambda > \lambda_\star(N) \Rightarrow K = \lambda - \dfrac{(N-1)(N-3)}{4} > \dfrac{1}{4}.$

$\lambda > \lambda_\star(N)$: Lack of controllability IV

STEP 3. *Ervedoza, 2008* :

$K > 1/4 \Rightarrow$ the operator $L^\varepsilon \Phi := -\Phi_{rr} - \dfrac{K}{r^2+\varepsilon^2}\Phi$ with Dirichlet conditions, admits a first eigenfunction Φ_0^ε such that

$$\begin{cases} L^\varepsilon \Phi_0^\varepsilon = \lambda_0^\varepsilon \Phi_0^\varepsilon, & \lambda_0^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} -\infty, \\ \|\Phi_0^\varepsilon\|_{L^2(0,2)} = 1, & \|\Phi_0^\varepsilon\|_{H^1(1,2)} \underset{\varepsilon\to 0}{\longrightarrow} 0. \end{cases}$$

$\varepsilon > 0$ small $\Rightarrow \lambda_0^\varepsilon < 0 \rightsquigarrow$ denote $\omega_0^\varepsilon = \sqrt{-\lambda_0^\varepsilon}$

Computations with $\bar{v}(t,r) = e^{-\omega_0^\varepsilon t}\Phi_0^\varepsilon$ (solution of our problem)
$\Rightarrow \ (\omega_0^\varepsilon)^2 \le C_\lambda \left(\frac{1}{\omega_0^\varepsilon} + \omega_0^\varepsilon\right)$
$\Rightarrow$ contradiction since $\omega_0^\varepsilon \to +\infty$ as $\varepsilon \to 0$. □

The Schrödinger equation I

The control problem : $T > 0$, $0 \in \Omega \subset \mathbb{R}^N$ bdd $\mathcal{C}^3$, $\emptyset \neq \omega \subset \Omega$,
$\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}$

$$
\begin{cases}
iu_t + \Delta u + \dfrac{\lambda}{|x|^2} u = 0 & (0, T) \times \Omega, \\
u(t,x) = h(t,x) & (0, T) \times \Gamma_0, \\
u(t,x) = 0 & (0, T) \times \Gamma \setminus \Gamma_0, \\
u(0,x) = u_0(x) & \Omega.
\end{cases}
$$

Theorem (V.-Zuazua, preprint 2008)

Assume $\lambda \leq \lambda_\star(N)$ *and* $T > 0$. *Then* $\forall u_0 \in H_\lambda{}'$,
$\exists h \in L^2((0, T) \times \Gamma_0)$ *s.t.* $u(T) \equiv 0$.

- extends the result of *Machtyngier, 1994* (in the case $\lambda = 0$) ;
- lack of controllability when $\lambda > \lambda_\star(N)$.

The Schrödinger equation II

Main step : consider the adjoint problem

$$\begin{cases} iv_t + \Delta v + \dfrac{\lambda}{|x|^2} v = 0 & (t,x) \in (0,T) \times \Omega, \\ v(t,x) = 0 & (t,x) \in (0,T) \times \Gamma. \end{cases}$$

Theorem (direct and inverse inequalities)

Assume $T > 0$ and $\lambda \leq \lambda_\star(N)$.

$$\int_0^T \int_\Gamma \left(\frac{\partial v}{\partial \nu}\right)^2 \leq C_1 \|v(0)\|^2_{H_\lambda} \ \textit{and} \ \|v(0)\|^2_{H_\lambda} \leq C_2 \int_0^T \int_{\Gamma_0} \left|\frac{\partial v}{\partial \nu}\right|^2.$$

Proof : multiplier Lemma for Schrödinger equation + the suitable Hardy inequality to bound $x \cdot \nabla v$ in L^2 + compactness-uniqueness argument as in Machtyngier.

Comments and open problems I

Limits of multipliers method

- Limit of the method : multiplier $q(x) = x$ centered at *the* singularity $\rightsquigarrow$ limited to a single singularity + restriction on the region of control $\Gamma_0 = \{x \in \Gamma \mid x \cdot \nu \geq 0\}$
(e.g., Ω convex $\rightsquigarrow$ control on the *whole* boundary).
- More general geometries like $\Gamma_0 = \{x \in \Gamma \mid (x - x_0) \cdot \nu \geq 0\}$?
$q(x) = x - x_0 \rightsquigarrow$ two extra terms :

$$\left[\int_\Omega v_t x_0 \cdot \nabla v \, dx \right]_0^T \text{ and } \int_{Q_T} \frac{v}{|x|^2} x_0 \cdot \nabla v \, dx.$$

First term : estimated by $E_v^\lambda(0)$ when $\lambda < \lambda_\star(N)$ using $v(t) \in H_0^1(\Omega)$ + equivalence between $E_v^0(0)$ and $E_v^\lambda(0)$.
But not uniform/λ and no result for $\lambda = \lambda_\star(N)$.

Comments and open problems II

Limits of multipliers method

Second term : worse since

$$\int_{Q_T} \frac{v}{|x|^2} x_0 \cdot \nabla v \, dxdt = \int_{Q_T} \frac{x \cdot x_0}{|x|^4} v^2 \, dxdt,$$

not estimated by $E_v^\lambda(0)$. (And no definite sign when $0 \in \Omega$).

- Current work with L. Baudouin : weaken the geometric conditions ? multi-polar singularities ? (hyperbolic Carleman estimates for problems with singular potentials ?)