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An Identification Problem for an Abstract System of Linear Evolution Equations in a Banach Space

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Synopsis

- The Identification Problem
- Historical comments
- The main result
- Auxiliary results
- Proof of Theorem 1
- Application to abstract parabolic problems
- A second-order linear evolution equation

The Identification Problem.

- X is a Banach space with the norm $\|\cdot\|$
- $\mathcal{L}(X)$ is the space of all linear continuous operators $B : X \to X$, endowed with the norm $\|B\|_{\mathcal{L}(X)} = \sup\{\|Bx\|;\ \|x\| = 1\}$
- $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup of contractions
- $F : [0,1] \to \mathcal{L}(X)$ is a given function
- Σ is the σ-field of Lebesgue measurable subsets in $[0,1]$ and $\mu : \Sigma \to \mathcal{L}(X)$ is a *countably additive vector measure*. See Diestel and Uhl [2], Definition 1, p. 1–2.
- if $x \in X$, $\mu(\cdot)x : \Sigma \to X$ is the countably additive vector measure defined by $\mu(E)x = \mu(E)(x)$ for each $E \in \Sigma$
- the *variation* of μ, denoted by $|\mu|$, is defined by $|\mu|(E) = \sup_{\pi} \sum_{G \in \pi} \|\mu(G)\|_{\mathcal{L}(X)}$ where the supremum is taken over all finite partitions π of E into measurable subsets.

Now, we can state the identification problem.

Let $u_0, u_1 \in X$, let $F \in C^1([0,1]; \mathcal{L}(X))$ and let us assume that $\mu([0,1])$ invertible. The identification problem we are considering here consists in finding a function $u : [0,1] \to X$ and an element $z \in X$ satisfying

$$\begin{cases} u'(t) = Au(t) + F(t)z, & t \in [0,1] \\ u(0) = u_0, \\ \mu([0,1])^{-1} \displaystyle\int_0^1 d\mu(t)u(t) = u_1. \end{cases} \tag{1}$$

We notice that, when $\mu = \delta(1) \cdot I$, i.e. the Dirac delta measure concentrated at $t = 1$ multiplied by the identity on X, the integral condition simplifies to $u(1) = u_1$, the so-called *final condition*.

Historical comments

• Prilepko and Kostin [5](1993) consider the identification problem (1) in an ordered Banach space, with A the infinitesimal operator of a positive, compact C_0-semigroup and of negative exponential type, $F \in C^1([0,1];\mathcal{L}(X))$ and $\mu = d\varphi I$, with φ either absolutely continuous or a Heaviside functions and they prove the existence and uniqueness of the solution for each $u_0, u_1 \in D(A)$;

• Prilepko and Tikhonov [6] (1994) consider the identification problem (1) with $F \in C^1([0,1];\mathcal{L}(X))$ and $\mu = d\varphi I$ with φ of bounded variation and prove the well-posedness, for $u_0, u_1 \in D(A)$, and stability with respect to the overdetermination φ;

• Tikhonov and Eidel'man [8](1994) consider the identification problem (1) with $F(t) = g(t)I$, g continuous and with bounded variation and $\mu = d\varphi I$ with φ of bounded variation. In the following four cases: (a) A norm continuous (b) A generates a c_0-semigroup which is equicontinuous at some $t > 0$ (c) φ is absolutely continuous and $\varphi(0) = 0$ (e) g is absolutely continuous, they prove a necessary and sufficient condition for the well-posedness of (1) for each $u_0, u_1 \in D(A)$;

• Prilepko, Piskarev and Shaw [7] (2007) use an iteration-approximation method to investigate inverse problems of the form (1) for parabolic equations subjected to a final condition;

• Anikonov and Lorenzi [1] (2007) assume that A generates an analytic C_0-semigroup of contractions, $F = f \cdot I$ where $f \in C^\alpha([0,1];\mathbb{R})$, with $\alpha \in (0,1)$, $\mu = \lambda \cdot I$, λ being a Borel *positive finite measure* and $u_0, u_1 \in D(A)$ and prove that the identification problem above has exactly one solution which admits an explicit representation in terms of A, the C_0-semigroup generated by A, F, μ, $u_0 \in D(A)$ and of $u_1 \in D(A)$.

Here we extend the result in Anikonov and Lorenzi [1] to the general case of infinitesimal generators of C_0-semigroups of contractions (possibly non-analytic) by assuming that $F \in C^1([0,1]; \mathcal{L}(X))$, and we relax the conditions on both μ and u_0, u_1 by assuming that μ is an operator-valued vector measure and

$$u_1 - \int_0^1 d\mu(\theta) S(\theta) u_0 \in D(A).$$

The main result

For the sake of simplicity, we will assume that $\mu([0,1]) = I$ and so the last condition in (1) takes the simpler form

$$\int_0^1 d\mu(t) u(t) = u_1. \tag{2}$$

More precisely, we have

Theorem 1 *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions, let $F \in C^1([0,1]; \mathcal{L}(X))$, let $\mu : \Sigma \to \mathcal{L}(X)$ be a countably additive vector measure on $[0,1]$, with $\mu([0,1]) = I$, and let $u_0, u_1 \in X$. If $Q = \int_0^1 d\mu(t)F(t)$ is invertible with continuous inverse, Q^{-1}, and*

$$\left\| Q^{-1} \left[\int_0^1 d\mu(t)S(t)F(0) + \int_0^1 d\mu(t) \int_0^t S(t-s)F'(s)\, ds \right] \right\|_{\mathcal{L}(X)} < 1. \tag{3}$$

Then a necessary and sufficient condition in order that the problem (1) have a unique solution $(u, z) \in C([0,1]; X) \times X$ is that

$$u_1 - \int_0^1 d\mu(\theta)S(\theta)u_0 \in D(A), \tag{4}$$

case in which

$$z = \left[\int_0^1 d\mu(t) A \left(\int_0^t S(t-s)F(s)\, ds \right) \right]^{-1} \left[A \left(u_1 - \int_0^1 d\mu(\theta)S(\theta)u_0 \right) \right] \tag{5}$$

and

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s)z\, ds \tag{6}$$

with z given by (5).

If, in addition, $u_0 \in D(A)$ then u, given by (6), is a classical solution of the Cauchy Problem in (1).

A sufficient condition, for (3) to hold, following from Theorem 7 in [8], is stated below.

Proposition 2 *Let* $\mu = \theta \cdot I$*,* θ *being a positive finite Borel measure on* $[0,1]$*, let* $F = f \cdot I$*, with* $f : [0,1] \to \mathbb{R}$*, and let us assume that:*

(i) *there exists* $\omega > 0$ *such that* $\|S(t)\| \leq e^{-\omega t}$ *for each* $t \geq 0$;

(ii) $f(t) \geq 0$ *for each* $t \in [0,1]$;

(iii) $f'(t) \geq 0$ *for each* $t \in [0,1]$;

(iv) $\displaystyle\int_0^1 \left(e^{-\omega t} \int_0^t f(s) e^{\omega s} \, ds \right) d\theta(t) > 0.$

Then (3) *holds true.*

Remark 3 If θ is the Lebesgue measure on $[0,1]$ and f satisfies (ii) and (iii) in Proposition 2 as well as $\int_0^1 f(t)\,dt > 0$, then the condition (iv) is also satisfied. Moreover, if $\theta = \delta(1)$, then (iv) again simplifies to $\int_0^1 f(s)\,ds > 0$.

Auxiliary results

Proposition 4 *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup,* $\{S(t);\ t \geq 0\}$*, and let* $F \in C^1([0,1]; \mathcal{L}(X))$*. Then, for each* $x \in X$*, we have*

$$\int_0^t S(t-s)F(s)x\,ds \in D(A), \tag{7}$$

and

$$A\left(\int_0^t S(t-s)F(s)x\,ds\right) = S(t)F(0)x + \int_0^t S(t-s)F'(s)x\,ds - F(t)x. \tag{8}$$

Corollary 5 *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup,* $\{S(t);\ t \geq 0\}$*, and let* $F \in C^1([0,1]; \mathcal{L}(X))$*. Then, for each* $x \in X$*, the function* $t \mapsto A\left(\int_0^t S(t-s)F(s)x\,ds\right)$ *is well-defined and continuous from* $[0,1]$ *to* X*.*

Proposition 6 *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup,* $\{S(t);\ t \geq 0\}$*, let* $F \in C^1([0,1]; \mathcal{L}(X))$ *and* $\mu : \Sigma \to \mathcal{L}(X)$ *be a countably additive vector measure on* $[0,1]$*. Then, for each* $x \in X$*, we have*

$$\int_0^1 d\mu(t)\left(\int_0^t S(t-s)F(s)\,ds\right)x \in D(A), \tag{9}$$

and

$$A\left[\int_0^1 d\mu(t)\left(\int_0^t S(t-s)F(s)\,ds\right)x\right] = \int_0^1 d\mu(t)A\left(\int_0^t S(t-s)F(s)\,ds\right)x. \tag{10}$$

From Propositions 4 and 6, we get

Corollary 7 *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup,* $\{S(t);\ t \geq 0\}$*, let* $F \in C^1([0,1]; \mathcal{L}(X))$ *and let* $\mu : \Sigma \to \mathcal{L}(X)$ *be a countably additive vector measure on* $[0,1]$*. Then, for each* $x \in X$*, we have*

$$\int_0^1 d\mu(t) A \left(\int_0^t S(t-s) F(s)\, ds \right) x \tag{11}$$

$$= \int_0^1 d\mu(t) S(t) F(0) x + \int_0^1 d\mu(t) \left(\int_0^t S(t-s) F'(s)\, ds \right) x - \int_0^1 d\mu(t) F(t) x.$$

We will show that, under the assumption (3), $x \mapsto \int_0^1 d\mu(t) A \left(\int_0^t S(t-s) F(s)\, ds \right) x$ is invertible. To this end, let us observe that, in view of Corollary 7, we have to show that the operator $\mathcal{T} - I$ is invertible, where $\mathcal{T} : X \to X$ is defined by

$$\mathcal{T} x = Q^{-1} \left[\int_0^1 d\mu(t) S(t) F(0) x + \int_0^1 d\mu(t) \left(\int_0^t S(t-s) F'(s)\, ds \right) x \right], \tag{12}$$

with

$$Q = \int_0^1 d\mu(t) F(t).$$

In this respect, we have the following simple but useful

Lemma 8 *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup, $\{S(t);\ t \geq 0\}$, let $F \in C^1([0,1];\mathcal{L}(X))$ and let $\mu : \Sigma \to \mathcal{L}(X)$ be a countably additive vector measure on $[0,1]$. If (3) holds, then the operator $\mathfrak{T} - I$, where $\mathfrak{T}$ is given by (12), is invertible with continuous inverse.*

Proof of Theorem 1

Proof. *Necessity* Let $(u,z) \in C([0,1];X) \times X$ be a solution of (1). Then u is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s)z\,ds. \tag{13}$$

By virtue of (13), the condition (2) takes the form

$$u_1 - \int_0^1 d\mu(t)S(t)u_0 = \int_0^1 d\mu(t)\left(\int_0^t S(t-s)F(s)\,ds\right)z.$$

Thanks to Proposition 6, the right-hand side of this equality belongs to $D(A)$ and thus the left-hand side enjoys the very same property and this completes the proof of the necessity.

Sufficiency. If (4) holds, we can apply A both sides of the equality in above, and using (10) in Proposition 4, we get

$$A\left(u_1 - \int_0^1 d\mu(t)S(t)u_0\right) = \int_0^1 d\mu(t)A\left(\int_0^t S(t-s)F(s)\,ds\right)z.$$

By Lemma 8, the operator on the right-hand side is invertible with continuous inverse. Applying the inverse to both sides of the equality above, we get (5). Plugging z, given by (5), into (13), we get (6).

We conclude by observing that, if $u_0 \in D(A)$, then u is a *classical, i.e.* a C^1*-solution* to the Cauchy Problem in (1), so that (u, z) is a classical solution to our identification problem. This completes the proof. □

Application to abstract parabolic problems

If μ is the Lebesgue measure on $[0, 1]$, we can obtain an existence and uniqueness result for the identification problem without assuming that the semigroup has an exponential decay. Instead, we have to assume that X is reflexive and A generates a compact semigroup.

Namely, let us consider the identification problem:

$(\mathcal{IP}_1)$ *given* $u_0, u_1 \in X$ *and* $f : [0, 1] \to \mathbb{R}$, $f \not\equiv 0$, *find* $z \in X$ *and* $u : [0, 1] \to X$ *satisfying the Cauchy Problem*

$$\begin{cases} u'(t) = Au(t) + f(t)z \\ u(0) = u_0 \end{cases} \tag{14}$$

and the additional condition

$$\int_0^1 u(t)\, dt = u_1. \tag{15}$$

Theorem 9 *Let X be reflexive and let A generate a compact C_0-semigroup of contractions, $\{S(t);\ t \geq 0\}$, let $u_0, u_1 \in D(A)$, $f \in C^1([0,1];\mathbb{R})$, $f(t) \geq 0$, $f'(t) \geq 0$ and let the operator*

$$z \mapsto \left\{ \int_0^1 f(s)[I - S(1-s)]\, ds \right\} z =: T_0 z \tag{16}$$

be invertible. Then there exists a unique solution (u, z) to the problem $(\mathcal{IP}_1)$ admitting the representation

$$u(t) = S(t)u_0 + \int_0^t f(s)S(t-s)T_0^{-1}\big[S(1)u_0 - u_0 - Au_1\big]\, ds$$

$$z = T_0^{-1}\big[S(1)u_0 - u_0 - Au_1\big].$$

Remark 10 Let us assume that $\|S(t)\|_{\mathcal{L}(X)} \leq q < 1$ for all $t \in [\alpha, 1]$ and some $\alpha \in (0,1)$, and $\int_0^{1-\alpha} f(s)\, ds > 0$. Then, from the obvious inequality

$$\int_0^1 f(s)\|S(1-s)\|_{\mathcal{L}(X)}\, ds = \int_0^1 f(1-s)\|S(s)\|_{\mathcal{L}(X)}\, ds$$

$$\leq \int_0^\alpha f(1-s)\, ds + q\int_\alpha^1 f(1-s)\, ds < \int_0^1 f(s)\, ds,$$

we deduce that the linear operator in (16) is invertible in $\mathcal{L}(X)$.

We may now pass to the proof of Theorem 15.

Proof. Let $\omega \in (0,1]$ and let us consider the following identification problem:

($\mathcal{IP}1_\omega$) *find $z_\omega \in X$ and $u_\omega : [0,1] \to X$ satisfying the Cauchy problem*

$$\begin{cases} u_\omega'(t) = A_\omega u_\omega(t) + f(t) z_\omega \\ u_\omega(0) = u_0, \end{cases} \tag{17}$$

where $A_\omega = A - \omega I$, and the additional condition

$$\int_0^1 u_\omega(t)\, dt = u_1. \tag{18}$$

Clearly A_ω generates the C_0-semigroup of contractions $\{S_\omega(t);\ t \geq 0\}$ given by

$$S_\omega(t) = e^{-\omega t} S(t)$$

for each $t \geq 0$. Let us observe that, in view of Proposition 2, the hypotheses of Theorem 1 are satisfied for each $\omega \in (0,1]$. So, for each such ω, the identification problem ($\mathcal{IP}1_\omega$) has a unique solution (z_ω, u_ω). As $u_0 \in D(A)$ and $f \in C^1([0,1];\mathbb{R})$, u_ω is differentiable on $(0,1)$ and Au_ω is continuous in $(0,1)$. Integrating both sides of (17) over $[0,1]$ with respect to μ, and making use of the representation formula

$$u_\omega(t) = e^{-\omega t} S(t) u_0 + \int_0^t f(s) e^{-\omega(t-s)} S(t-s) z_\omega\, ds,$$

we get

$$e^{-\omega} S(1) u_0 - u_0 - A_\omega u_1 = \left\{ \int_0^1 f(t) [I - e^{-\omega(1-t)} S(1-t)]\, dt \right\} z_\omega.$$

Let us define the linear operator $T_\omega : X \to X$ by

$$T_\omega z = \left\{ \int_0^1 f(t)[I - e^{-\omega(1-t)} S(1-t)]\, dt \right\} z$$

for each $z \in X$. Since the map $\omega \mapsto T_\omega$ is continuous from $[0,1]$ to $\mathcal{L}(X)$ in the uniform operator topology and, by (16), T_0 is invertible, it follows that there exists $\gamma \in (0,1]$ such that, for each $\omega \in (0,\gamma]$, T_ω is invertible. In addition, there exists $a > 0$, independent of $\omega \in (0,\gamma]$, such that

$$\|T_\omega^{-1}\|_{\mathcal{L}(X)} \leq a$$

fore each $\omega \in (0,\gamma]$. We deduce

$$\|z_\omega\| \leq a\|e^{-\omega} S(1)u_0 - u_0 - Au_1 - \omega u_1\| \leq a\,(2\|u_0\| + \|Au_1\| + \|u_1\|)$$

for each $\omega \in (0,\gamma]$. Hence $\{u_\omega;\ \omega \in (0,\gamma]\}$ is bounded too, and therefore

$$\lim_{\omega \downarrow 0} \omega u_\omega(t) = 0$$

uniformly for $t \in [0,1]$. Let $\omega_n \downarrow 0$ be a sequence in $(0,1]$ and let us define $z_n = z_{\omega_n}$ and by $u_n = u_{\omega_n}$. As X is reflexive, we conclude that there exists $z \in X$ such that, for at least a subsequence, $\lim_n z_n = z$ weakly in X. Further, since the semigroup generated by A is compact, in view of Theorem 8.4.2, p. 196 in Vrabie [11], there exists $u \in C([0,1];X)$ such that, for at least a subsequence, $\lim_n u_n = u$ strongly in $C([0,1];X)$. Next, since $\lim_n f(t)z_n = f(t)z$ weakly in $L^1(0,1;X)$, from Remark 3.3.4,

p. 105 in Vrabie [10], we deduce that we can pass to the uniform limit in both sides in

$$u_n(t) = e^{-\omega_n} S(t)u_0 + \int_0^t e^{-\omega_n(t-s)} S(t-s)f(s)z_n\,ds$$

as $n \to +\infty$. Thus u satisfies (14). Finally, passing to the limit as $n \to +\infty$ in both sides in

$$\int_0^1 u_n(t)\,dt = u_1,$$

we conclude that u satisfies (15), and thus (u, z) is a solution of the problem $(\mathcal{IP}1_0)$.

To show that the solution (u, z) is unique it suffices to show that the problem

$$\begin{cases} u'(t) = Au(t) + f(t)z \\ u(0) = 0 \\ \displaystyle\int_0^1 u(t)\,dt = 0 \end{cases} \tag{19}$$

has only the solution $z = 0$ and $u \equiv 0$. Since u is represented by

$$u(t) = \int_0^t f(s)S(t-s)z\,ds, \quad t \in [0, 1],$$

by integrating the first equality in (19) over $[0, 1]$ we get the following operator equation for z:

$$0 = \left\{\int_0^1 f(s)[I - S(1-s)]\,ds\right\} z = T_0 z.$$

Since T_0 is invertible, we deduce $z = 0$, implying, in turn, $u = 0$. We have thus proved the uniqueness of the solution to problem $(\mathcal{IP}_1)$.

Finally, to get the representation for (u, z) in the statement of the theorem, first we solve the Cauchy problem in $(\mathcal{IP}_1)$ and find the representation for u in terms of z, i.e.

$$u(t) = S(t)u_0 + \int_0^t f(s)S(t-s)z\,ds.$$

Integrating over $[0, 1]$ the equality in $(\mathcal{IP}_1)$, we obtain

$$S(1)u_0 - u_0 - Au_1 = \left\{ \int_0^1 f(s)[I - S(1-s)]\,ds \right\} z = T_0 z.$$

Since T_0 is invertible, we deduce the representation for z, and, consequently, the one for u. The proof is now complete. □

Example 11 Let $\Omega \subset \mathbb{R}^n$ be an open bounded set lying on one side with respect to its boundary $\partial\Omega$ of class $C^{1,1}$. Let $A : D(A) \subseteq L^2(\Omega) \to L^2(\Omega)$ be the operator defined by $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and

$$Au = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) - a_0(x)u, \quad u \in D(A),$$

where $a_{ij} \in C^{0,1}(\overline{\Omega})$, $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n$, and $a_0 \in C(\overline{\Omega})$ satisfy

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad a_0(x) > 0,$$

for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$ and some constant $\nu > 0$.

We recall that there exist two sequences $\{\lambda_k\}_{k=1}^{+\infty} \subset (0,+\infty)$ – increasing to $+\infty$ – and $\{\varphi_k\}_{k=1}^{+\infty} \subset H^2(\Omega) \cap H_0^1(\Omega)$ consisting, respectively, of eigenvalues of $-A$ and of eigenfunctions of A, which constitute an orthonormal basis in $L^2(\Omega)$. So, each $v \in L^2(\Omega)$ admits the representation

$$v = \sum_{k=1}^{+\infty} \langle v, \varphi_k \rangle \varphi_k \quad \text{(convergence in } L^2(\Omega)),$$

where $\langle v, \varphi_k \rangle = \int_\Omega v(x)\varphi_k(x)\,dx$. Moreover, v satisfies the Parseval equality

$$\|v\|_{L^2(\Omega)}^2 = \sum_{k=1}^{+\infty} |\langle v, \varphi_k \rangle|^2.$$

Further, A generates a compact C_0-semigroup of contractions $\{S(t);\ t \geq 0\}$,

$$[S(t)v](x) = \sum_{k=1}^{\infty} \langle v, \varphi_k \rangle e^{-\lambda_k t} \varphi_k,$$

for each $t \geq 0$ and each $v \in L^2(\Omega)$.

We now consider the following identification problem:

$(\mathcal{IP}_2)$ *given* $u_0, u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $f \in C^1([0,1];\mathbb{R})$, $f \not\equiv 0$, $f(t) \geq 0$, $f'(t) \geq 0$, $t \in [0,1]$, $\int_0^1 f(s)\,ds \leq 1$, *find* $z \in L^2(\Omega)$ *and a function* $u \in C^1([0,1];L^2(\Omega)) \cap C([0,1];H^2(\Omega) \cap H_0^1(\Omega))$ *satisfying*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + f(t)z, & t \in [0,1],\ x \in \Omega, \\ u(t,x) = 0, & (t,x) \in [0,1] \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

and

$$\int_0^1 u(t,x)\,dt = u_1(x), \qquad x \in \Omega.$$

In order to apply Theorem 9, we have to check that the operator $T : L^2(\Omega) \to L^2(\Omega)$, defined by

$$Tz = \left\{ \int_0^1 f(s)[I - S(1-s)]\,ds \right\} z,$$

for each $z \in L^2(\Omega)$, is invertible. We will show this by proving that $\|I - T\|_{\mathcal{L}(L^2(\Omega))} < 1$. Indeed, let $v \in L^2(\Omega)$ with $\|v\|_{L^2(\Omega)} = 1$ be arbitrary. We have

$$(v - Tv)(x) = \sum_{k=1}^{+\infty} \langle v, \varphi_k \rangle \left\{ 1 - \int_0^1 f(s)\big[1 - e^{-\lambda_k(1-s)}\big]\,ds \right\} \varphi_k(x).$$

Since $\|v\|_{L^2(\Omega)} = 1$ and, according to our assumptions

$$\int_0^1 f(s)\big[1 - e^{-\lambda_1(1-s)}\big]\,ds < \int_0^1 f(s)\,ds \le 1,$$

we get

$$\|v - Tv\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} |\langle v, \varphi_k\rangle|^2 \left(1 - \int_0^1 f(s)\big[1 - e^{-\lambda_k(1-s)}\big]\,ds\right)^2$$

$$\le \left(1 - \int_0^1 f(s)\big[1 - e^{-\lambda_1(1-s)}\big]\,ds\right)^2.$$

Whence we deduce

$$\|I - T\|_{\mathcal{L}(L^2(\Omega))} \le 1 - \int_0^1 f(s)\big[1 - e^{-\lambda_1(1-s)}\big]\,ds < 1.$$

In view of Theorem 9, the problem $(\mathcal{IP}_2)$ has a unique solution.

Remark 12 If a lower bound $\lambda_0 > 0$ for λ_1 is known, i.e. $\lambda_1 \ge \lambda_0$, the restriction $\int_0^1 f(s)\,ds \le 1$ on f can be relaxed to

$$\int_0^1 f(s)\big[1 - e^{-\lambda_0(1-s)}\big]\,ds < 1.$$

Remark 13 A similar result can be proved if the Dirichlet boundary condition in $(\mathcal{IP}_2)$ is replaced by the so-called Robin condition related to a *a.e. non-negative* function $\sigma \in L^\infty(\partial\Omega)$, i.e.

$$\frac{\partial u}{\partial \nu_A}(t,x) + \sigma(x)u(t,x) = 0, \qquad (t,x) \in [0,1] \times \partial\Omega,$$

the *conormal unit vector* ν_A being defined by the following formula, where $\nu(x)$ denotes the outward unit vector normal at x to $\partial\Omega$:

$$(\nu_A)_j(x) = \Big[\sum_{j=1}^{n}\Big(\sum_{i=1}^{n} a_{ij}(x)\nu_i(x)\Big)^2\Big]^{-1/2}\sum_{i=1}^{n} a_{ij}(x)\nu_i(x), \qquad j = 1,\ldots,n.$$

A second-order linear evolution equation

Let V and H be real Hilbert spaces, let V' be the topological dual of V. We assume that H is identified with its own topological dual, that $V \subseteq H \subseteq V'$ densely and continuously, and the inner product $\langle \cdot, \cdot \rangle$ on H and the duality $(\cdot, \cdot)$ between V and V' satisfy

$$(v, w) = \langle v, w \rangle$$

for each $v \in V$ and each $w \in H$. Let $A : V \to V'$ be a linear continuous symmetric operator whose restriction to H generates a C_0-semigroup of contractions on H. We denote this restriction also by A and we note that $D(A) = \{v \in V; \ Av \in H\}$. Let $f_0 \in C([0,1];\mathbb{R})$ and $f_2 \in C^1([0,1];\mathbb{R})$ be given functions, let $v_0, v_1 \in V$, $w_0, w_1 \in H$ and let μ_i be two finite Borel measures on $[0,1]$ with $\mu_i([0,1]) = 1$, $i = 1, 2$.

Let us consider the identification problem:

$(\mathcal{IP}_3)$ *find* $z_1 \in V$, $z_2 \in H$ *and* $v : [0,1] \to H$ *satisfying*

$$\begin{cases} v''(t) = Av(t) - 2\omega v'(t) - \omega^2 v(t) + f_0(t)z_1 + f_2(t)z_2 \\ v(0) = v_0, \ v'(0) = w_0 \end{cases} \tag{20}$$

and

$$\int_0^1 \begin{pmatrix} d\mu_1(t) & 0 \\ 0 & d\mu_2(t) \end{pmatrix} \begin{pmatrix} v(t) \\ v'(t) + \omega v(t) - f_1(t)z_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \tag{21}$$

where

$$f_1(t) = ce^{-\omega t} + \int_0^t e^{-\omega(t-s)} f_0(s)\, ds, \quad t \in [0,1]. \tag{22}$$

We emphasize that, under the hypotheses which will be imposed on both f_0 and f_1, the constant $c \in \mathbb{R}$, appearing in (22), is necessarily 0.

So $(\mathcal{IP}_3)$ can be reformulated as

$(\mathcal{IP}_4)$ *find* $z_1 \in V$, $z_2 \in H$ *and* $v : [0,1] \to H$ *satisfying*

$$\begin{cases} v'(t) = w(t) - \omega v(t) + f_1(t) z_1 \\ w'(t) = Av(t) - \omega w(t) + f_2(t) z_2 \\ v(0) = v_0, \ w(0) = w_0 \end{cases} \tag{23}$$

and

$$\int_0^1 \begin{pmatrix} d\mu_1(t) & 0 \\ 0 & d\mu_2(t) \end{pmatrix} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}. \tag{24}$$

Before passing to the statement of our main result concerning the identification problem $(\mathcal{IP}_4)$, some notations and preliminaries are needed. Let

$$X = \begin{matrix} V \\ \times \\ H \end{matrix},$$

which endowed with the usual inner product

$$\left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} \widetilde{v} \\ \widetilde{w} \end{pmatrix} \right\rangle_X = \langle v, \widetilde{v} \rangle + (w, \widetilde{w}),$$

for each $\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} \widetilde{v} \\ \widetilde{w} \end{pmatrix} \in X$, is a real Hilbert space too.

Let $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ be defined by

$$D(\mathcal{A}) = \begin{matrix} D(A) \\ \times \\ V \end{matrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} -\omega I & J \\ A & -\omega J \end{pmatrix},$$

where I is the identity on H and J is the injection of V to H. It is known that $\mathcal{A}$ generates a C_0-group $\{S(t);\ t \in \mathbb{R}\}$ in X.

Let $F(t) = \begin{pmatrix} f_1(t) & 0 \\ 0 & f_2(t) \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$, $u_0 = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \in X$ and $u_1 = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \in X$ be fixed.

The identification problem ($\mathcal{IP}_3$) can be equivalently reformulated as ($\mathcal{IP}_5$) *find* $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in X$ *and* $u : [0, 1] \to X$, $u = \begin{pmatrix} v \\ w \end{pmatrix}$, *satisfying*

$$\begin{cases} u'(t) = \mathcal{A}u(t) + F(t)z \\ u(0) = u_0 \end{cases} \tag{25}$$

and

$$\int_0^1 d\mu(t)u(t) = u_1. \tag{26}$$

Theorem 14 *Let* $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ *be as above, let* $f_i \in C^1([0,1];\mathbb{R})$, $i = 1,2$, *let* μ_i $i = 1,2$, *be positive finite Borel measures on* $[0,1]$ *with* $\mu_i([0,1]) = 1$, *and let* $v_0, w_0 \in D(A)$ *and* $v_1, w_1 \in V$. *Let us assume that*:

(H_1) $f_i(0) \geq 0$ *and* $f_i'(t) \geq 0$ *for all* $t \geq 0$, $1 = 1,2$;

(H_2) $\displaystyle\int_0^1 d\mu_i(t) \int_0^t f_i(s) e^{-\omega(t-s)} ds > 0$;

(H_3) $\displaystyle\sum_{i=1}^{2} \left(1 - \omega \frac{\int_0^1 d\mu_i(t) \int_0^t e^{-\omega(t-s)} f_i(s)\, ds}{\int_0^1 f_i(t)\, d\mu_i(t)}\right)^2 < \frac{1}{2}.$

Then, the identification problem $(\mathcal{IP}_5)$ *has a unique solution* (u,z), *where*

$$z = \left[\int_0^1 d\mu(t) \left(A \int_0^t S(t-s)F(s)\, ds\right)\right]^{-1} A \left(u_1 - \int_0^1 d\mu(t) S(t) u_0\right), \tag{27}$$

and $u : [0,1] \to X$ *is defined by*

$$u(t) = S(t)u_0 + \left(\int_0^t S(t-s)F(s)\, ds\right) z, \tag{28}$$

with z *given by* (27).

Remark 15 In the case of *final conditions*, i.e. if $\mu_i = \delta(1)$ and $f_i(1) = 1$, $i = 1, 2$, condition (H_3) simplifies to

$$\sum_{i=1}^{2} \left(1 - \frac{\omega}{f_i(1)} \int_0^1 f_i(s) e^{-\omega(1-s)}\, ds\right)^2 < \frac{1}{2}.$$

Theorem 16 *Let $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ be as above, let $f_0 \in C([0,1];\mathbb{R})$ and $f_2 \in C^1([0,1];\mathbb{R})$, $i = 1, 2$, let μ_i $i = 1, 2$, be positive finite Borel measures on $[0,1]$ with $\mu_i([0,1]) = 1$, let $v_0, w_0 \in D(A)$ and let $v_1, w_1 \in V$. Let us assume that:*

(H_1) $f_i'(t) \geq 0$ *for all* $t \geq 0$, $i = 0, 2$;

(H_2) $e^{\omega t} f_0(t) - \omega \int_0^t e^{\omega s} f_0(s)\, ds \geq 0, \quad t \in [0,1]$;

(H_3) $\int_0^1 d\mu_1(t) \int_0^t (t-s) e^{-\omega(t-s)} f_0(s)\, ds > 0$;

(H_4) $\int_0^1 d\mu_2(t) \int_0^t e^{-\omega(t-s)} f_2(s)\, ds > 0$;

$$(H_5)\quad \left(1-\omega\frac{\int_0^1 d\mu_1(t)\int_0^t (t-s)e^{-\omega(t-s)}f_0(s)\,ds}{\int_0^1 d\mu_1(t)\int_0^t e^{-\omega(t-s)}f_0(s)\,ds}\right)^2 + \left(1-\omega\frac{\int_0^1 d\mu_2(t)\int_0^t e^{-\omega(t-s)}f_2(s)\,ds}{\int_0^1 f_2(t)\,d\mu_2(t)}\right)^2 < \frac{1}{2}.$$

Then, the identification problem $(\mathcal{IP}_3)$ *has a unique solution* (v,z)*, where*

$$z=\left[\int_0^1 d\mu(t)\left(A\int_0^t S(t-s)F(s)\,ds\right)\right]^{-1} A\left(v_1-\int_0^1 d\mu(t)S(t)v_0\right), \qquad (29)$$

and v *is the projection on* V *of the function* $u:[0,1]\to H$*, defined by*

$$u(t)=S(t)u_0+\left(\int_0^t S(t-s)F(s)\,ds\right)z, \qquad (30)$$

z *being given by* (29) *and*

$$F(t)=\begin{pmatrix} \int_0^t e^{\omega s}f_0(s)\,ds & 0 \\ 0 & f_2(t) \end{pmatrix}.$$

Remark 17 In the case of *final conditions*, i.e. if $\mu_i=\delta(1)$, $i=1,2$, conditions $(H_3)\sim(H_5)$ simplify to $\int_0^1 (1-s)e^{-\omega(1-s)}f_0(s)\,ds>0$, $f_2(1)>0$ and

$$\left(1-\omega\frac{\int_0^1 (1-s)e^{-\omega(1-s)}f_0(s)\,ds}{\int_0^1 e^{-\omega(1-s)}f_0(s)\,ds}\right)^2+\left(1-\omega\frac{\int_0^t e^{-\omega(1-s)}f_2(s)\,ds}{f_2(1)}\right)^2<\frac{1}{2}.$$