Irregular Elliptic Problems in *UMD* Banach Spaces

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1. Statement of the problem

In a *UMD* Banach space E, consider a principally irregular boundary value problem in $[0,1]$ for the second order elliptic differential-operator equation

$$L(D)u := -u''(x) + Au(x) + A_1(x)u(x) = f(x), \quad (1.1)$$

$$\begin{aligned} L_1 u : &= \alpha u'(0) + \beta u'(1) + \gamma u(0) + \delta u(1) \\ &\qquad + \sum_{s=1}^{N_1} T_{1s} u(x_{1s}) = f_1, \\ L_2 u : &= \alpha u(0) - \beta u(1) + \sum_{s=1}^{N_2} T_{2s} u(x_{2s}) = f_2, \end{aligned} \quad (1.2)$$

where α, β, γ, δ are complex numbers; $x_{ks} \in [0,1]$; A, $A_1(x)$, for $x \in [0,1]$, and T_{ks} are, generally speaking, unbounded operators in E; $D := \frac{d}{dx}$.

Note that boundary conditions for equation (1.1), i.e., for

$$L(D)u := -u''(x) + Au(x) + A_1(x)u(x) = f(x),$$

with the principal part

$$L_{k0}u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1), \quad k = 1, 2,$$

where $m_k \in \{0, 1\}$, are called (Birkhoff)-**regular** boundary conditions if the number

$$\theta := (-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0.$$

Originally this definition was given for scalar equations, we just adapted the definition to abstract settings. So, our boundary conditions (1.2) are irregular for (1.1).

We find sufficient conditions for problem (1.1)–(1.2) to have the Fredholm property and we establish a coercive estimate with a defect with respect to the space variable for a solution of problem (1.1)–(1.2) in $L_p((0,1); E)$.

Furthermore, when instead of problem (1.1)–(1.2) we consider the problem

$$L(\lambda, D)u := \lambda u(x) - u''(x) + Au(x) + A_1(x)u(x) = f(x), \tag{1.3}$$

$$\begin{aligned} L_{10}u &:= \alpha u'(0) + \beta u'(1) + \gamma u(0) + \delta u(1) = f_1, \\ L_{20}u &:= \alpha u(0) - \beta u(1) = f_2, \end{aligned} \tag{1.4}$$

where λ is the spectral parameter, the coerciveness with a defect in λ is also established.

Let us mention that for regular problems we have succeeded to prove maximal L_p-regularity. For irregular problems we prove unique solvability theorems, but maximal L_p-regularity does not follow from the theorems for problem (1.3)–(1.4). Apparently, this is a phenomenon of irregular problems. One does not have maximal L_p-regularity for irregular problems even in the framework of Hilbert spaces.

2. Coerciveness with a defect on the space variable and Fredholmness

Consider problem (1.1)-(1.2). It is convenient to formulate the theorem in terms of the Fredholmness of some unbounded operator, corresponding to the problem and which acts from one Banach space into another. Let us set the operator $\mathbb{L}$ from $W_p^2((0,1); E(A), E)$ into

$$L_p((0,1); E(A^{\frac{1}{2}})) \dot{+} (E(A^2), E)_{\frac{1}{2}+\frac{1}{4p},p} \dot{+} (E(A^2), E)_{\frac{1}{4}+\frac{1}{4p},p}$$

by the equalities

$$\begin{aligned} D(\mathbb{L}) := \Big\{ u \mid u \in {} & W_p^2((0,1); E(A), E), \\ & L(D)u \in L_p((0,1); E(A^{\frac{1}{2}})), \\ & L_1 u \in (E(A^2), E)_{\frac{1}{2}+\frac{1}{4p},p}, \\ & L_2 u \in (E(A^2), E)_{\frac{1}{4}+\frac{1}{4p},p} \Big\}, \\ \mathbb{L}u := {} & (L(D)u, L_1 u, L_2 u), \end{aligned}$$

where $L(D)$, L_1, and L_2 have been defined by equalities (1.1)–(1.2).

Theorem 1. *Let the following conditions be satisfied:*

(1) *an operator A is closed, densely defined and invertible in a UMD Banach space E;*

(2) $R\{\lambda R(\lambda, A) : \arg\lambda = \pi\} < \infty$;

(3) *the embedding $E(A) \subset E$ is compact;*

(4) $\alpha\delta + \beta\gamma \neq 0$;

(5) *for any $\varepsilon > 0$ and for almost all $x \in [0,1]$*

$$\|A_1(x)u\|_{E(A^{\frac{1}{2}})} \leq \varepsilon\|Au\| + C(\varepsilon)\|u\|_{E(A^{\frac{1}{2}})}, \quad u \in D(A);$$

for each $u \in D(A)$ the function $A_1(x)u$ is measurable on $[0,1]$ in $E(A^{\frac{1}{2}})$;

(6) *for $\varepsilon > 0$ and $u \in (E(A), E)_{\frac{1}{2p},p}$, where $p \in (1,\infty)$,*

$$\|T_{ks}u\|_{(E(A^2),E)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}} \leq \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2p},p}} + C(\varepsilon)\|u\|.$$

Then,

(*a*) *for any function* $u \in D(\mathbb{L})$ *the following noncoercive estimate holds*

$$\begin{aligned}\|u''\|_{L_p((0,1);E)} + \|Au\|_{L_p((0,1);E)} \\ &\leq C\Big(\|L(D)u\|_{L_p((0,1);E(A^{\frac{1}{2}}))} \\ &+ \sum_{k=1}^{2} \|L_k u\|_{(E(A^2),E)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}} \\ &+ \|u\|_{L_p((0,1);E(A^{\frac{1}{2}}))}\Big);\end{aligned}$$

(*b*) *the operator* $\mathbb{L}$*:* $u \to \mathbb{L}u := \Big(L(D)u, L_1u, L_2u\Big)$ *from* $W_p^2((0,1);E(A),E)$ *into*

$$L_p((0,1);E(A^{\frac{1}{2}}))\dot{+}(E(A^2),E)_{\frac{1}{2}+\frac{1}{4p},p}\dot{+}(E(A^2),E)_{\frac{1}{4}+\frac{1}{4p},p}$$

is Fredholm.

3. Coerciveness with a defect of the problem with a linear parameter

Consider now problem (1.3)–(1.4), i.e.,

$$L(\lambda, D)u := \lambda u(x) - u''(x) + Au(x) + A_1(x)u(x) = f(x), \tag{3.1}$$

$$\begin{aligned} L_{10}u &:= \alpha u'(0) + \beta u'(1) + \gamma u(0) + \delta u(1) = f_1, \\ L_{20}u &:= \alpha u(0) - \beta u(1) = f_2, \end{aligned} \tag{3.2}$$

Theorem 2. *Let the following conditions be satisfied:*

(1) *an operator A is closed, densely defined and invertible in a UMD Banach space E;*

(2) $R\{\lambda R(\lambda, A) : |\arg\lambda| \geq \pi - \varphi\} < \infty$ *for some* $0 \leq \varphi < \pi$;

(3) *the embedding $E(A) \subset E$ is compact;*

(4) $\alpha\delta + \beta\gamma \neq 0$;

(5) *for any $\varepsilon > 0$ and for almost all $x \in [0, 1]$*

$$\|A_1(x)u\|_{E(A^{\frac{1}{2}})} \leq \varepsilon\|Au\| + C(\varepsilon)\|u\|_{E(A^{\frac{1}{2}})}, \ u \in D(A),$$

$$\|A_1(x)u\| \leq \varepsilon\|A^{\frac{1}{2}}u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}});$$

the function $A_1(x)u$, *for* $u \in D(A)$, *is measurable on* $[0,1]$ *in* $E(A^{\frac{1}{2}})$ *and, for* $u \in D(A^{\frac{1}{2}})$, *is measurable on* $[0,1]$ *in* E.

Then, problem (3.1)–(3.2), *for* $f \in L_p((0,1);E(A^{\frac{1}{2}}))$, $f_k \in (E(A^2),E)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}$, *where* $1 < p < \infty$, *and* $|\arg\lambda| \leq \varphi$, $|\lambda|$ *is sufficiently large, has a unique solution that belongs to the space* $W_p^2((0,1);E(A),E)$ *and, for these* λ, *the following noncoercive estimates hold for the solution of problem* (3.1)–(3.2) :

$$|\lambda|\|u\|_{L_p((0,1);E)} + \|u''\|_{L_p((0,1);E)} + \|Au\|_{L_p((0,1);E)} \leq C\Big[\|f\|_{L_p((0,1);E(A^{\frac{1}{2}}))} + \sum_{k=1}^{2}\Big(\|f_k\|_{(E(A^2),E)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{k+1}{2}-\frac{1}{2p}}\|f_k\|\Big)\Big]$$

and

$$|\lambda|^{\frac{1}{2}}\|u\|_{L_p((0,1);E)} + \|u'\|_{L_p((0,1);E)} + \|A^{\frac{1}{2}}u\|_{L_p((0,1);E)} \leq C\Big[\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{2}\Big(\|f_k\|_{(E(A),E)_{1-\frac{k}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{k}{2}-\frac{1}{2p}}\|f_k\|\Big)\Big].$$

4. Principally irregular boundary value problems for elliptic equations of the second order

Let $\Omega := (0,1) \times G$, where $G \subset \mathbb{R}^r$, $r \geq 2$ be a bounded open domain with an $(r-1)$-dimensional boundary ∂G which locally admits rectification. Denote by

$$B^s_{p,q}(G) := (W^{s_0}_p(G), W^{s_1}_p(G))_{\theta,q},$$

where $0 \leq s_0, s_1$ are integers, $0 < \theta < 1$, $1 < p < \infty$, $1 < q < \infty$ and $s = (1-\theta)s_0 + \theta s_1$, and

$$W^{l,s}_{p,q}(\Omega) := W^l_p((0,1); W^s_q(G), L_q(G)),$$

where $0 \leq l, s$ are integers, $1 < p < \infty$, $1 < q < \infty$. If $p = q$ and $l = s$ then $W^{l,s}_{p,q}(\Omega) = W^l_p(\Omega)$. Finally, $L_{p,q}(\Omega) := W^{0,0}_{p,q}(\Omega) = L_p((0,1); L_q(G))$.

We consider in Ω a principally irregular boundary value problem for an elliptic differential-integral equation of the second order with differential-operator boundary conditions

$$L(x,y,D_x,D_y)u := -D_x^2 u(x,y) - \sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u(x,y)$$

$$+ \sum_{j=1}^{r} b_j(y) D_j u(x,y) + b_0(x,y) u(x,y)$$

$$+ \int_G c(x,y,z) u(x,z) dz = f(x,y), \quad (x,y) \in \Omega, \tag{4.1}$$

$$\begin{cases} L_1 u := \alpha D_x u(0,y) + \beta D_x u(1,y) + \gamma u(0,y) + \delta u(1,y) \\ \qquad + \sum_{s=1}^{N_1} T_{1s} u(x_{1s},\cdot)|_y = f_1(y), \quad y \in G, \\ L_2 u := \alpha u(0,y) - \beta u(1,y) + \sum_{s=1}^{N_2} T_{2s} u(x_{2s},\cdot)|_y \\ \qquad = f_2(y), \quad y \in G, \end{cases} \tag{4.2}$$

$$Lu := \sum_{j=1}^{r} c_j(y') D_j u(x,y') + c_0(y') u(x,y') = 0,$$

$$(x,y') \in (0,1) \times \partial G, \tag{4.3}$$

where $D_x := \frac{\partial}{\partial x}$, $D_j := -i\frac{\partial}{\partial y_j}$, $D_y := (D_1,\ldots,D_r)$; α,β, γ, δ are complex numbers, $y := (y_1,\ldots,y_r)$, $x_{ks} \in [0,1]$, T_{ks} are, generally speaking, unbounded operators in $L_q(G)$, $1 < q < \infty$. Let $m :=$ord$L$ for (4.3).

Theorem 3. *Let the following conditions be satisfied:*

(1) *(***smoothness condition***)* $a_{sj} \in W_q^3(\Omega)$, $|a_{sj}(y) - a_{sj}(z)| \le C|y-z|^\gamma$ *for some* $C > 0$ *and* $\gamma \in (0,1)$, $\forall y, z \in \overline{G}$; $b_0, \frac{\partial b_0}{\partial y} \in L_\infty(\overline{\Omega})$; $b_j \in L_\infty(\overline{G})$; $c, \frac{\partial c}{\partial y} \in L_\infty(\overline{\Omega \times G})$; $c_j, c_0 \in C^{2-m}(\partial G)$; $\partial G \in C^2$;

(2) *(***ellipticity condition*** for A) for* $y \in \overline{G}$, $\sigma \in \mathbb{R}^r$, $\arg\lambda = \pi$, $|\sigma| + |\lambda| \ne 0$,

$$\lambda + \sum_{s,j=1}^{r} a_{sj}(y)\sigma_s\sigma_j \ne 0;$$

(3) *(***Lopatinskii-Shapiro condition*** for A)* y' *is any point on* ∂G*, the vector* σ' *is tangent and* σ *is a normal vector to* ∂G *at the point* $y' \in \partial G$*. Consider the following ordinary differential problem:*

$$\Big[\lambda + \sum_{s,j=1}^{r} a_{sj}(y')\Big(\sigma_s' - i\sigma_s\frac{d}{dt}\Big)\Big(\sigma_j' - i\sigma_j\frac{d}{dt}\Big)\Big]u(t) = 0, \qquad t > 0, \; \lambda \le 0, \tag{4.4}$$

$$\sum_{j=1}^{r} c_j(y')\Big(\sigma_j' - i\sigma_j\frac{d}{dt}\Big)u(t)\Big|_{t=0} = h, \quad \textit{for } m = 1, \tag{4.5}$$

$$u(0) = h, \quad \textit{for } m = 0; \tag{4.6}$$

it is required that for $m = 1$ problem (4.4), (4.5) (for $m = 0$ problem (4.4), (4.6)) has one and only one solution, including all its derivatives, tending to zero as $t \to \infty$ for any numbers $h \in \mathbb{C}$;

(4) $\alpha\delta + \beta\gamma \neq 0$;

(5) $\forall \varepsilon > 0$, $\forall u \in B_{q,p}^{2-\frac{1}{p}}(G; Lu = 0 \text{ if } m < 2 - \frac{1}{p} - \frac{1}{q})$,

$$\|T_{ks}u\|_{B_{q,p}^{1+k-\frac{1}{p}}(G)} \leq \varepsilon \|u\|_{B_{q,p}^{2-\frac{1}{p}}(G)} + C(\varepsilon)\|u\|_{L_q(G)},$$

where $p \neq \frac{q}{q-1}$ and $p, q \in (1, \infty)$, or $p = \frac{q}{q-1}$ and $m \neq 1$.

Then,

(a) $\forall u \in W_{p,q}^{2,2}(\Omega; Lu = 0)$, *such that* $L(x, y, D_x, D_y)u \in W_{p,q}^{0,1}(\Omega; Lu = 0 \text{ if } m = 0)$, $L_1 u \in B_{q,p}^{2-\frac{1}{p}}(G; Lu = 0$ *if* $m < 2 - \frac{1}{p} - \frac{1}{q})$, *and* $L_2 u \in B_{q,p}^{3-\frac{1}{p}}(G; Lu = 0;$ $L(-\sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u + \lambda_0 u) = 0$ *if* $m < 1 - \frac{1}{p} - \frac{1}{q})$, *the following noncoercive estimate holds*:

$$\|u\|_{W_{p,q}^{2,2}(\Omega)} \leq C\Big(\|L(x, y, D_x, D_y)u\|_{W_{p,q}^{0,1}(\Omega)} + \sum_{k=1}^{2} \|L_k u\|_{B_{q,p}^{1+k-\frac{1}{p}}(G)} + \|u\|_{W_{p,q}^{0,1}(\Omega)}\Big),$$

where $L(x, y, D_x, D_y)$, L_k, and L are defined by (4.1) – (4.3).

(b) *there exists* $\lambda_0 > 0$ *such that the operator* $\mathbb{L} : u \to \mathbb{L}u := (L(x, y, D_x, D_y)u, L_1 u,\ L_2 u)$ *from* $W^{2,2}_{p,q}(\Omega; Lu = 0)$ *into* $W^{0,1}_{p,q}(\Omega; Lu = 0\ if\ m = 0) \dot{+} B^{2-\frac{1}{p}}_{q,p}(G; Lu = 0$ *if* $m < 2 - \frac{1}{p} - \frac{1}{q}) \dot{+} B^{3-\frac{1}{p}}_{q,p}(G; Lu = 0; L(-\sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u + \lambda_0 u) = 0$ *if* $m < 1 - \frac{1}{p} - \frac{1}{q})$ *is Fredholm.*

Let us now consider in the same cylindrical domain $\Omega := (0,1)\times G$ a principally regular boundary value problem for an elliptic differential-integral equation of the second order with differential-operator boundary conditions and with a linear parameter λ

$$\begin{aligned} L(x,y,D_x,D_y)u := \lambda u(x,y) - D_x^2 u(x,y) \\ - \sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u(x,y) \\ + \sum_{j=1}^{r} b_j(y) D_j u(x,y) + b_0(x,y)u(x,y) \\ + \int_G c(x,y,z)u(x,z)dz = f(x,y), \end{aligned} \tag{4.7}$$

$$\begin{cases} L_1 u := \alpha D_x u(0,y) + \beta D_x u(1,y) \\ \qquad +\gamma u(0,y) + \delta u(1,y) = f_1(y), \ y \in G, \\ L_2 u := \alpha u(0,y) - \beta u(1,y) = f_2(y), \quad y \in G, \end{cases} \tag{4.8}$$

$$\begin{aligned} Lu := \sum_{j=1}^{r} c_j(y') D_j u(x,y') + c_0(y')u(x,y') = 0, \\ (x,y') \in (0,1)\times \partial G. \end{aligned} \tag{4.9}$$

Let $m :=$ord$L$ for (4.9).

Theorem 4. *Let conditions* (1)–(4) *of Theorem 3 be satisfied*; *moreover, conditions* (2) *and* (3) *hold in the angle* $|\arg\lambda| \geq \pi - \varphi$, *where* $0 \leq \varphi < \pi$.

Then, problem (4.7) – (4.9) *for*

$$f \in W^{0,1}_{p,q}(\Omega; Lu = 0 \quad if \quad m = 0),$$

$f_1 \in B^{2-\frac{1}{p}}_{q,p}(G; Lu = 0$ *if* $m < 2 - \frac{1}{p} - \frac{1}{q})$, *and* $f_2 \in B^{3-\frac{1}{p}}_{q,p}(G; Lu = 0;\ L(-\sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u + \lambda_0 u) = 0$ *if* $m < 1 - \frac{1}{p} - \frac{1}{q})$, *where* $1 < q, p < \infty$ *and* $|\arg\lambda| \leq \varphi$, $|\lambda|$ *is sufficiently large, has a unique solution that belongs to the space* $W^{2,2}_{p,q}(\Omega; Lu = 0)$ *and, for these* λ, *the following noncoercive estimates hold for the solution*

$$|\lambda| \|u\|_{L_{p,q}(\Omega)} + \|u\|_{W^{2,2}_{p,q}(\Omega)} \leq C\Big(\|f\|_{W^{0,1}_{p,q}(\Omega)} + \sum_{k=1}^{2}\Big(\|f_k\|_{B^{1+k-\frac{1}{p}}_{q,p}(G)} + |\lambda|^{\frac{k+1}{2}-\frac{1}{2p}} \|f_k\|_{L_q(G)}\Big)\Big)$$

and

$$|\lambda|^{\frac{1}{2}} \|u\|_{L_{p,q}(\Omega)} + \|u\|_{W^{1,1}_{p,q}(\Omega)} \leq C\Big(\|f\|_{L_{p,q}(\Omega)} + \sum_{k=1}^{2}\Big(\|f_k\|_{B^{k-\frac{1}{p}}_{q,p}(G)} + |\lambda|^{\frac{k}{2}-\frac{1}{2p}} \|f_k\|_{L_q(G)}\Big)\Big).$$

Examples of T_{ks} (at least for $\partial G \in C^\infty$) satisfying condition (6) of Theorem 3:

$$(T_{ks}u)(y) := \int_G \sum_{|\ell|\le 1} T_{ks\ell}(x,y) \frac{\partial^{|\ell|} u(x)}{\partial x_1^{\ell_1} \cdots \partial x_r^{\ell_r}} dx,$$

where $T_{ks\ell} \in L_{t'}(G \times G)$, $\frac{1}{t'} + \frac{1}{t} = 1$, $t = \min(q, q')$, $\frac{1}{q'} + \frac{1}{q} = 1$; $T_{1s\ell}(x,y)$ are twice continuously differentiable with respect to y variable and $\frac{\partial}{\partial y_j} T_{1s\ell} \in L_{t'}(G \times G)$, $\frac{\partial^2}{\partial y_i \partial y_j} T_{1s\ell} \in L_{t'}(G \times G)$. For $T_{2s\ell}(x,y)$, we increase the smoothness by one and claim also that $\frac{\partial^3}{\partial y_m \partial y_i \partial y_j} T_{2s\ell} \in L_{t'}(G \times G)$. So, we consider, in particular, differential-integral boundary conditions.