On elliptic operators with unbounded diffusion coefficients in L^2 spaces with respect to invariant measures

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Given an elliptic operator $\mathcal{A}$ with coefficients defined in $\mathbb{R}^N$, a (probability) measure μ is said to be a (infinitesimal) invariant measure of the operator $\mathcal{A}$ if

$$\int_{\mathbb{R}^N} \mathcal{A}u d\mu = 0,$$

for any smooth and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$.

Elliptic operators with unbounded coefficients have been widely studied in these last decades. It is well known that they "generate" semigroups $\{T(t)\}$ which, in general, are neither strongly continuous nor analytic in $BUC(\mathbb{R}^N)$. The picture changes drastically when these operators are studied in L^p spaces with respect to the invariant measure (say in $L^p_\mu(\mathbb{R}^N)$), whenever such a measure exists. Indeed, $\{T(t)\}$ extends naturally in these latter spaces with a strongly continuous semigroup. Unfortunately, the characterization of its infinitesimal generator is still an open problem and only a few results are available in the literature. Of course, this lack of information on the domain of the infinitesimal generator prevents us from obtaining optimal regularity results for the solutions to elliptic and parabolic problems associated with the operator $\mathcal{A}$, when $f \in L^p_\mu(\mathbb{R}^N)$.

Here, we restrict ourselves to the Hilbert case $p = 2$ and, under suitable assumptions both of algebraic and growth type, we prove that the realization in $L^2_\mu(\mathbb{R}^N)$ of the operator

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^{N} q_{ij}(x) D_{ij}\varphi(x) + \sum_{j=1}^{N} b_j(x) D_j\varphi(x), \qquad x \in \mathbb{R}^N,$$

with its maximal domain, generates a strongly continuous semigroup. Moreover, we discuss some nice properties of the associated semigroup.