A (very) short introduction to abstract parabolic problems

Davide Guidetti
Dipartimento di Matematica,
Università di Bologna
Piazza di Porta S. Donato 5,
40126 Bologna, Italy,
e-mail: guidetti@dm.unibo.it

The aim of these notes is to introduce some ideas concerning the way many parabolic systems can be seen, in a natural way, as particular cases of systems of ordinary differential equations in Banach spaces. We shall always have in mind, as models, the two following very classical examples,

$$(D)\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), \quad t \in [0,T], x \in [0,\pi], \\ u(t,x) = 0, \quad (t,x) \in [0,T] \times \{0,\pi\}, \\ u(0,x) = u_0(x), \quad x \in [0,\pi], \end{cases} \tag{1}$$

$$(N)\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), \quad t \in [0,T], x \in [0,\pi], \\ \frac{\partial u}{\partial x}(t,x) = 0, \quad (t,x) \in [0,T] \times \{0,\pi\}, \\ u(0,x) = u_0(x), \quad x \in [0,\pi], \end{cases} \tag{2}$$

with boundary conditions which are (respectively) of Dirichlet and Neumann type. In order to avoid delicate questions of integration for functions with values in Banach spaces and vector-valued Sobolev spaces, we shall consider only systems in spaces of continuous functions and derivatives in a pointwise sense.

The first subject we want to discuss is the identification of scalar valued functions of many real variables with corresponding functions of a single real variable. We take, as example, the space $C(Q)$ of continuous complex valued functions of domain

$$Q := [0,T] \times [0,\pi]. \tag{3}$$

$C(Q)$ is a Banach space with the norm

$$\|f\|_{C(Q)} := \max_{(t,x)\in Q} |f(t,x)|.$$

If $f \in C(Q)$, we can define the function $\hat{f} : [0,T] \to C([0,\pi])$,

$$[\hat{f}(t)](x) := f(t,x).$$

It is quite simple to show that $\hat{f} \in C([0,T]; C([0,\pi]))$. This is a Banach space with the norm

$$\|F\|_{C([0,T];C([0,\pi]))} := \max_{t\in[0,T]} \|F(t)\|_{C([0,\pi])}.$$

We leave to the reader the proof that the mapping $f \to \hat{f}$ is a linear isomorphism and, in fact, an isometry between $C(Q)$ and $C([0,T];C([0,\pi]))$. This space $C([0,T];C([0,\pi]))$ is just a particular case of a much more general definition: given a Banach space X, we can define the space

$$C([0,T];X) := \{F : [0,T] \to X : F \text{ is continuous } \}$$

$C([0,T];X)$ is a Banach space with the natural norm

$$\|F\|_{C([0,T];X)} := \max_{0 \le t \le T} \|F(t)\|_X.$$

If $k \in \mathbb{N}$, $C^k([0,\pi])$ is a Banach space if we equip it with the natural norm

$$\|g\|_{C^k([0,\pi])} := \max_{0 \le j \le k} \|g^{(j)}\|_{C([0,\pi])}.$$

We shall need also the space $C([0,T];C^2([0,\pi]))$. The isorphism $f \to \hat{f}$ allows to identify $C([0,T];C^2([0,\pi]))$ with

$$C^{0,2}(Q) := \{f \in C(Q) : \forall (t,x) \in Q \exists D_x f(t,x), D_x^2 f(t,x), D_x^2 f \in C(Q)\}.$$

Another interesting subspace of $C([0,T];C([0,\pi]))$ is $C^1([0,T];C([0,\pi])$. This means the class of elements F in $C([0,T];C([0,\pi]))$, which are differentiable in $[0,T]$, and such that $t \to F'(t)$ belongs to $C([0,T];C([0,\pi]))$. The definition of the derivative $F'(t)$ is the usual one: $F'(t)$ coincides with

$$\lim_{h \to 0} \frac{F(t+h) - F(t)}{h}.$$

The limit is intended in the Banach space $C([0,\pi])$: if $h \in \mathbb{R} \setminus \{0\}$ and $t+h \in [0,T]$ $\frac{F(t+h)-F(t)}{h}$ is a well defined element of $C([0,\pi])$ and

$$F'(t)(x) = \lim_{h \to 0} \frac{F(t+h)(x) - F(t)(x)}{h}, \quad \text{uniformly in } [0,\pi].$$

We leave to the reader the check of the fact that the aforementioned map $f \to \hat{f}$ allows to identify $C^1([0,T];C([0,\pi])$ with $C^{1,0}(Q)$, defined as

$$\{f \in C(Q) : \forall (t,x) \in Q \exists D_t f(t,x), D_t f \in C(Q)\}.$$

Our interest in this spaces lies in the fact that one natural choice for a class of functions containing solutions of equations of the form (1)-(2) is

$$C^{1,2}(Q) := C^{1,0}(Q) \cap C^{0,2}(Q).$$

From now on, we shall identify an element f in $C(Q)$ with the corresponding element $\hat{f}$ in $C([0,T];C([0,\pi]))$. So, we shall identify $C^{1,2}(Q$ with $C^1([0,T];C([0,\pi])) \cap C([0,T];C^2([0,\pi]))$.

Other ingredients to treat (1) and (2) are certain linear operators in the Banach space $C([0,\pi])$. Generally speaking, a linear operator in the Banach space X is a linear function $A : D(A) \to X$, with $D(A)$ linear subspace of X. We introduce the two following operators A_D and A_N in $X = C([0,\pi])$:

$$\begin{cases} D(A_D) := \{v \in C^2([0,\pi]) : v(0) = v(\pi) = 0\}, \\ \\ A_D v := D_x^2 v, \end{cases}$$

$$\begin{cases} D(A_N) := \{v \in C^2([0,\pi]) : v'(0) = v'(\pi) = 0\}, \\ \\ A_D v := D_x^2 v. \end{cases}$$

We observe that, in each case, the domain does not coincide with X. Moreover, A_D and A_N are not continuous, if we equip their domains with the norm $\|\cdot\|_{C([0,\pi])}$. However, they are closed, in the following sense:

Definition 1. *Let A be a linear operator in the Banach space X. A is closed if its graph $\{(x, Ax) : x \in D(A)\}$ is a closed subspace of $X \times X$ (equipped with the product topology).*

This means the following: given an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ with values in $D(A)$, if it converges to some element $x \in X$, and $(Ax_n)_{n\in\mathbb{N}}$ converges to some $y \in X$, it follows that $x \in D(A)$ and $Ax = y$.

Given two Banach spaces X and Y, we indicate with $\mathcal{L}(Y, X)$ the Banach space of linear, bounded operators from Y to X.

One can prove the following fact:

Proposition 1. *Let A be a closed operator in the Banach space X. If $x \in D(A)$, we set*

$$\|x\|_{D(A)} := \max\{\|x\|_X, \|Ax\|_X\}. \tag{4}$$

Then

(I) $\|\cdot\|_{D(A)}$ is a norm in $D(A)$, making it a Banach space;

(II) $A \in \mathcal{L}(D(A); X)$;

(III) $f \in C([0,T]; D(A))$ if and only if $f \in C([0,T]; X)$, $f(t) \in D(A)$ for every $t \in [0,T]$, and $t \to Af(t)$ belongs to $C([0,T]; X)$.

We come back to the operators A_D and A_N. We leave to the reader the following characterization of $C([0,T]; D(A))$, with $A \in \{A_D, A_N\}$:

$$C([0,T]; D(A)) = \begin{cases} \{g \in C^{0,2}(Q) = C([0,T]; C^2([0,\pi])) : g(t,0) = g(t,\pi) = 0 \ \forall t \in [0,T]\} \\ \text{in case } A = A_D, \\ \{g \in C^{0,2}(Q) = C([0,T]; C^2([0,\pi])) : D_x g(t,0) = D_x g(t,\pi) = 0 \quad \forall t \in [0,T]\} \\ \text{in case } A = A_N. \end{cases}$$

So, we can reformulate (1) and (2) in the following unified way:

Let $X := C([0,\pi])$, $f \in C([0,T]; X)$, $u_0 \in X$, $A \in \{A_D, A_N\}$. Determine the functions u in $C^1([0,T]; X) \cap C([0,T]; D(A))$, such that

$$\begin{cases} u'(t) = Au(t) + f(t), \quad \forall t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{5}$$

We shall call a solution with such regularity a **strict** solution of (5). We observe that, if a strict solution exists, necessarily $u_0 \in D(A)$.

More generally, it is worth considering the following abstract problem:

(AP) Let X be a Banach space, $f \in C([0,T]; X)$, $u_0 \in X$ and let A be a closed operator in X. Determine the solutions u in $C^1([0,T]; X) \cap C([0,T]; D(A))$ of (5).

Even in this more general case, we shall call solutions with such regularity strict solutions.

From now on, we shall mainly concentrate ourselves on (AP), having in mind the particular cases (1) and (2). The first step of our study is the following problem (AP0):

(AP0) Let X be a Banach space, $u_0 \in X$ and let A be a closed operator in X. Determine the solutions u in $C^1([0,\infty); X) \cap C([0,\infty); D(A))$ of

$$\begin{cases} u'(t) = Au(t), \quad \forall t \in [0,\infty), \\ u(0) = u_0. \end{cases} \tag{6}$$

In the finite dimensional case, a very standard technique of study is the Laplace transform. In order to extend this method to the (generally) infinite-dimensional case, we need some definition of integral, for vector valued functions: we shall employ the following simple generalization of Riemann's integral: let a, b be real numbers, with $a < b$, let X be a Banach space and $f \in C([a,b];X)$. We define the integral $\int_a^b f(t)dt$ as the unique element I of X, such that, for every ϵ in $\mathbb{R}^+$, there exists $\delta(\epsilon)$ in $\mathbb{R}^+$, such that, for any decomposition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$, satisfying the condition $\max_{1 \leq j \leq n}(t_j - t_{j-1}) < \delta(\epsilon)$, and for any possible choice of τ_1 in $[t_0, t_1]$, ..., τ_n in $[t_{n-1}, t_n]$,

$$\| \sum_{j=1}^{n} (t_j - t_{j-1}) f(\tau_j) - I \|_X < \epsilon.$$

The majority of classical properties of the integral can be extended to the vector valued case. In particular, there is a natural extension of the fundamental theorem of calculus: if $F(t) := \int_a^t f(s)ds$, F is differentiable and $F'(t) = f(t)$ $\forall t \in [a,b]$. Moreover, if $f \in C([a,b];X)$ and $B \in \mathcal{L}(X,Y)$, with Y second Banach space,

$$\int_a^b Bf(t)dt = B \int_a^b f(t)dt;$$

The notion of generalized integral admits a natural extension to the vector valued case. In particular, if $f \in C([a,\infty);X)$, we can define

$$\int_a^\infty f(t)dt := \lim_{t \to \infty} \int_a^t f(s)ds, \tag{7}$$

if this limit exists. One can show that, if $\int_a^\infty \|f(t)\|_X dt < \infty$, the limit in (7) exists. Moreover,

$$\| \int_a^\infty f(t)dt \|_X \leq \int_a^\infty \|f(t)\|_X dt.$$

Of course, an analogous inequality holds also in a bounded interval.

Here we take the occasion to define also the complex integral $\int_\gamma f(z)dz$, in the case of $\gamma : [a,b] \to \mathbb{C}$ of class C^1, $f : \gamma([a,b]) \to X$ continuous: we set

$$\int_\gamma f(z)dz := \int_a^b \gamma'(t) f(\gamma(t))dt.$$

We shall consider also complex integrals on piecewise regular paths and generalized complex integrals. Their definition is analogous to the case of complex valued functions. So, if $u \in C([0,\infty);X)$ and there exist $M \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$, such that $\|u(t)\|_X \leq Me^{\omega t}$ $\forall t \in [0,\infty)$, we can define, for every complex number λ, such that $Re(\lambda) > \omega$, the Laplace transform

$$\mathcal{L}u(\lambda) := \int_0^\infty e^{-\lambda t} u(t)dt.$$

It is important to say that $\mathcal{L}u$ is a holomorphic function in $\{\lambda \in \mathbb{C} : Re(\lambda) > \omega\}$. The definition of holomorphic function with values in a complex Banach space X is a natural extension of the particular case $X = \mathbb{C}$: if $f : \Omega \to X$, with Ω open subset of $\mathbb{C}$, f is holomorphic if, $\forall z \in \Omega$, there exists (in X)

$$f'(z) := \lim_{h \to 0} \frac{1}{h}[f(z+h) - f(z)].$$

Many properties of complex valued holomorphic functions can be extended to this more general situation. In particular, we have in mind properties concerning complex integrals: for example, if Ω is an open subset of $\mathbb{C}$, $f : \Omega \to X$ is holomorphic and γ_1 and γ_2 are closed, Ω-homotopic and $f : \Omega \to X$ is holomorphic

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

We shall refer to a result of this type as an application of Cauchy's theorem. We shall apply it even in case of "generalized complex integrals" (see, for example, (10)).

Concerning these elementary notions of integration and properties of holomorphis functions with values in Banach spaces, we refer (for example) to [2].

So, we assume that (6) have a strict solution u, such that, for certain $M \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$,

$$\|u(t)\|_{D(A)} + \|u'(t)\|_X \leq Me^{\omega t}, \quad \forall t \in [0, \infty).$$

Integrating by parts, we obtain

$$(\lambda - A)\mathcal{L}u(\lambda) = u_0, \quad Re(\lambda) > \omega. \tag{8}$$

To go on, we need the notion of resolvent of an operator in X.

Definition 2. *Let A be an operator in the complex Banach space X. We shall indicate with $\rho(A)$ the resolvent set of A, defined as follows:*

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A := \lambda I - A \text{ is a bijiection between } D(A) \text{ and } X, \text{ and } (\lambda - A)^{-1} \in \mathcal{L}(X)\}.$$

The set $\mathbb{C} \setminus \rho(A)$ is called the spectrum of A and is indicated with the symbol $\sigma(A)$.

The following basic properties of $\rho(A)$ are of elementary proof (see, for example, [3], Appendix):

Theorem 1. *Let A be an operator in X. Then:*
(I) $\rho(A)$ is open in $\mathbb{C}$;
(II) if $\rho(A) \neq \emptyset$, A is closed;
(III) the function $\lambda \to (\lambda - A)^{-1}$, from $\rho(A)$ to $\mathcal{L}(X)$, is holomorphic;
(IV) if $\lambda, \mu \in \rho(A)$, the following "resolvent identity" holds:

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1};$$

(V) if $\lambda, \mu \in \rho(A)$,

$$(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - A)^{-1}(\lambda - A)^{-1}.$$

We come back to (8). Assume that, for some $\omega \in \mathbb{R}$,

$$\{\lambda \in \mathbb{C} : Re(\lambda) > \omega\} \subseteq \rho(A).$$

Then, from (8), we obtain

$$\mathcal{L}u(\lambda) = (\lambda - A)^{-1}u_0.$$

To obtain u, we can try to apply the classical Mellin inversion formula (see [4], 74): take $s > \omega$, we have, formally,

$$u(t) = \frac{1}{2\pi i} \lim_{L \to \infty} \int_{s-iL}^{s+iL} e^{\lambda t}(\lambda - A)^{-1}u_0 d\lambda. \tag{9}$$

Observe now that the existence of the limit in (9) is a problem, because it is known that, apart trivial cases, the best possible estimate of $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)}$ in $\{\lambda \in \mathbb{C} : Re(\lambda) = s\}$ is

$$\||(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq C(1 + |\lambda|)^{-1}$$

and the factor $e^{\lambda s}$ does not help, because it has a constant absolute value in $\{\lambda \in \mathbb{C} : Re(\lambda) = s\}$. To get some inspiration, we come back to the basic examples (1) and (2). Then one can verify the following:

(I) $\sigma(A_D) = \{-k^2 : k \in \{1, 2, ...\}\}$; $\sigma(A_N) = \{-k^2 : k \in \{0, 1, 2, ...\}$.

(II) If $A \in \{A_D, A_N\}$, for every $\phi_0 \in (0, \pi]$, there exists $C(\phi_0)$ in $\mathbb{R}^+$, such that, if $\lambda \in \mathbb{C} \setminus \{0\}$ and $|Arg(\lambda)| \leq \pi - \phi_0$,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(C([0,\pi])} \leq \frac{C(\phi_0)}{|\lambda|}.$$

It is worth introducing the notion of **sectorial operator**.

Definition 3. *Let A be a linear operator in the Banach space X, and let ω in $\mathbb{R}$, ϕ_0 in $(0,\pi)$ and M in $\mathbb{R}^+$. We shall write that $A \in \Sigma(\omega, \phi_0, M)$ if*

$$\{\lambda \in \mathbb{C} \setminus \{\omega\} : |Arg(\lambda - \omega)| \leq \pi - \phi_0\} \subseteq \rho(A),$$

and, for λ in this set,

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|}.$$

If ϕ_0 can be chosen in $(0, \pi/2)$, we shall say that A is sectorial.

We observe that A_D and A_N are both sectorial operators: in fact, by (I)-(II), we can take any $\omega \geq 0$, and ϕ_0 in $(0, \pi/2)$.

Assume now that A is a sectorial operator in X: we can take advantage of the fact that $\lambda \to e^{\lambda t}(\lambda - A)^{-1}u_0$ is holomorphic, to modify properly the path of integration in (9): precisely, we fix ϕ_1 in $[\phi_0, \pi/2)$, r in $\mathbb{R}^+$, and indicate with γ a piecewise regular path, describing

$$\{\lambda \in \mathbb{C} \setminus \{\omega\} : |Arg(\lambda - \omega)| = \pi - \phi_1, |\lambda - \omega| \geq r\} \cup \{\lambda \in \mathbb{C} \setminus \{\omega\} : |Arg(\lambda - \omega)| \leq \pi - \phi_1, |\lambda - \omega| = r\},$$

oriented from $\infty e^{-i(\pi - \phi_1)}$ to $\infty e^{i(\pi - \phi_1)}$. As, for $t \in \mathbb{R}^+$, $e^{\lambda t}$ tends to 0 very quickly as $|\lambda| \to \infty$ in γ, we are allowed to define, for each $t \in \mathbb{R}^+$, the following operator in $\mathcal{L}(X)$:

$$e^{tA} := \frac{1}{2\pi i} \int_\gamma e^{\lambda t}(\lambda - A)^{-1} d\lambda. \tag{10}$$

It is convenient to define also

$$e^{0A} := I. \tag{11}$$

Given u_0 in X, the function $u(t) := e^{tA}u_0$ seems to be a formal solution of (6). We are going to examine to what extent this happens. The following theorem holds:

Theorem 2. *Let $A \in \Sigma(\omega, \phi_0, M)$, with $\phi_0 \in (0, \pi/2)$, and let, for $t \geq 0$, e^{tA} be the operator defined in (10)-(11). Then:*

(I) $\forall t \in [0, \infty)$ $e^{tA} \in \mathcal{L}(X)$ and there exists C in $\mathbb{R}^+$ such that $\|e^{tA}\|_{\mathcal{L}(X)} \leq Ce^{\omega t}$, $\forall t \geq 0$;

(II) $\forall t, s$ in $\mathbb{R}^+$, $e^{tA}e^{sA} = e^{(t+s)A}$ (semigroup property);

(III) $t \to e^{tA}$ belongs to $C^\infty(\mathbb{R}^+; \mathcal{L}(X))$; $\forall t \in \mathbb{R}^+$, $\forall k \in \mathbb{N}$, the range of e^{tA} in contained in the domain $D(A^k)$ of A^k. Moreover, $A^k e^{tA} \in \mathcal{L}(X)$ and

$$D_t^k(e^{tA}) = A^k e^{tA}, \quad t \in \mathbb{R}^+.$$

(IV) In particular, if we set $u(t) := e^{tA}u_0$ ($t \geq 0, u_0 \in X$), $u'(t) = Au(t)$ $\forall t \in \mathbb{R}^+$;

(V) $\lim_{t \to 0} u(t) = u_0$ (in X) if and only if $u_0 \in \overline{D(A)}$ (the closure of $D(A)$ in X).

Proof (I) We fix a piecewise regular path γ_0, describing $\{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda)| = \pi - \phi_0, |\lambda| \geq 1\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda)| \leq \pi - \phi_0, |\lambda| = 1\}$ and observe that, $\forall t \in \mathbb{R}^+$, by Cauchy's theorem,

$$e^{tA} = \tfrac{1}{2\pi i} \int_{\gamma_0} e^{(\omega+\lambda)t}(\omega + \lambda - A)^{-1} d\lambda$$

$$= \tfrac{1}{2\pi i} \int_{t^{-1}\gamma_0} e^{(\omega+\lambda)t}(\omega + \lambda - A)^{-1} d\lambda = \tfrac{1}{2\pi i} \int_{\gamma_0} e^{\omega t + \lambda}(\omega + \tfrac{\lambda}{t} - A)^{-1} t^{-1} d\lambda$$

We deduce that

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq \frac{M}{2\pi} \int_{\gamma_0} e^{Re(\lambda)} |\lambda|^{-1} |d\lambda| \cdot e^{\omega t}.$$

(II) We fix γ, describing $\{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda - \omega)| = \pi - \phi_0, |\lambda - \omega| \geq 1\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda - \omega)| \leq \pi - \phi_0, |\lambda - \omega| = 1\}$ and a second piecewise regular path γ_1, describing $\{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda - \omega)| =$

$\pi - \phi_1, |\lambda| \geq 2\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda - \omega)| \leq \pi - \phi_1, |\lambda - \omega| = 2\}$, for some $\phi_1 \in (\phi_0, \pi/2)$. Then, by Cauchy's theorem and the resolvent identity,

$$e^{tA}e^{sA} = \frac{1}{(2\pi i)^2}\int_\gamma (\int_{\gamma_1} e^{\lambda t + \mu s}(\lambda - A)^{-1}(\mu - A)^{-1}d\mu)d\lambda$$

$$= \frac{1}{(2\pi i)^2}\int_\gamma (\int_{\gamma_1} \frac{e^{\mu s}}{\mu - \lambda}d\mu)e^{\lambda t}(\lambda - A)^{-1}d\lambda - \frac{1}{(2\pi i)^2}\int_{\gamma_1}(\int_\gamma \frac{e^{\lambda t}}{\mu - \lambda}d\lambda)e^{\mu s}(\mu - A)^{-1}d\mu$$

$$= \frac{1}{2\pi i}\int_\gamma e^{\lambda(t+s)}(\lambda - A)^{-1}d\lambda = e^{(t+s)A},$$

as

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{e^{\mu s}}{\mu - \lambda}d\mu = e^{\lambda s}, \quad \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\mu - \lambda}d\lambda = 0.$$

(III). As

$$\int_\gamma \|\lambda^k e^{\lambda t}(\lambda - A)^{-1}\|_{\mathcal{L}(X)}|d\lambda| < \infty \quad \forall k \in \mathbb{N}, \forall t \in \mathbb{R}^+,$$

(by virtue of the rapid decay of $|e^{\lambda t}|$), one can verify that $t \to e^{tA}$ belongs to $C^\infty(\mathbb{R}^+; \mathcal{L}(X))$ and, for every k,

$$D_t^k(e^{\cdot A})(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^k(\lambda - A)^{-1}d\lambda.$$

We observe also that, as

$$\|A(\lambda - A)^{-1}\|_{\mathcal{L}(X)} = \| - 1 + \lambda(\lambda - A)^{-1}|_{\mathcal{L}(X)} \leq 1 + M$$

if $\lambda \in \mathbb{C} \setminus \{\omega\}$ and $|Arg(\lambda - \omega)| \leq \pi - \phi_0$, we have that the integral in (10) converges also in $\mathcal{L}(E; D(A))$. In particular, for $k = 1$,

$$(e^{\cdot A})'(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}(\lambda - A + A)(\lambda - A)^{-1}d\lambda = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}d\lambda + \frac{1}{2\pi i}\int_\gamma e^{\lambda t}A(\lambda - A)^{-1}d\lambda$$

$$= A\frac{1}{2\pi i}\int_\gamma e^{\lambda t}(\lambda - A)^{-1}d\lambda = Ae^{tA}.$$

The argument can be iterated: for example,

$$(e^{\cdot A})''(t) = A\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda(\lambda - A)^{-1}d\lambda = A^2e^{tA}.$$

(IV) follows immediately from (III).

Concerning (V), the "only if" part is quite obvious, as $e^{tA}u_0 \in D(A)\ \forall t \in \mathbb{R}^+$. We assume first that $u_0 \in D(A)$. Then, if $\lambda \in \rho(A) \setminus \{0\}$,

$$(\lambda - A)^{-1}u_0 = \lambda^{-1}(\lambda - A)^{-1}(\lambda - A + A)u_0 = \lambda^{-1}u_0 + \lambda^{-1}(\lambda - A)^{-1}Au_0.$$

Now we indicate with γ a piecewise regular path, describing

$$\{\lambda \in \mathbb{C} \setminus \{\omega\} : |Arg(\lambda - \omega')| = \pi - \phi_0, |\lambda - \omega'| \geq 1\}$$

$$\cup\{\lambda \in \mathbb{C} \setminus \{\omega'\} : |Arg(\lambda - \omega')| \leq \pi - \phi_0, |\lambda - \omega'| = 1\},$$

with $\omega' := \omega \vee 0$. So we have, for $t \in \mathbb{R}^+$,

$$\begin{aligned} e^{tA}u_0 &= \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{-1}u_0 d\lambda + \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{-1}(\lambda - A)^{-1}Au_0 d\lambda \\ &= u_0 + \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{-1}(\lambda - A)^{-1}Au_0 d\lambda. \end{aligned} \tag{12}$$

As $\|e^{\lambda t}\lambda^{-1}(\lambda - A)^{-1}Au_0\|_X \leq C|\lambda|^{-2}$, we can pass to the limit, as $t \to 0$, in the last integral in (12), to obtain

$$\lim_{t\to 0} e^{tA}u_0 = u_0 + \frac{1}{2\pi i}\int_\gamma \lambda^{-1}(\lambda - A)^{-1}Au_0 d\lambda = u_0.$$

This follows from the fact that, by Cauchy's theorem, for every $r \geq 1$,

$$\int_\gamma \lambda^{-1}(\lambda - A)^{-1} A u_0 d\lambda = \int_{\gamma_r} \lambda^{-1}(\lambda - A)^{-1} A u_0 d\lambda$$

with γ_r describing $\{\lambda \in \mathbb{C} \setminus \{\omega\} : |Arg(\lambda - \omega')| = \pi - \phi_0, |\lambda - \omega'| \geq r\} \cup \{\lambda \in \mathbb{C} \setminus \{\omega'\} : |Arg(\lambda - \omega')| \leq \pi - \phi_0, |\lambda - \omega'| = r\}$ and $\lim_{r \to \infty} \int_{\gamma_r} \|\lambda^{-1}(\lambda - A)^{-1} A u_0\|_X |d\lambda| \to 0$ $(r \to \infty)$.

Now we take u_0 in $\overline{D(A)}$. We fix ϵ in $\mathbb{R}^+$ and take u_1 in $D(A)$. Then, $\forall t \geq 0$, applying (I),

$$\|e^{tA} u_0 - u_0\|_X \leq \|e^{tA}(u_0 - u_1)\|_X + \|e^{tA} u_1 - u_1\|_X + \|u_1 - u_0\|_X$$

$$\leq (Ce^{\omega t} + 1)\|u_1 - u_0\|_X + \|e^{tA} u_1 - u_1\|_X.$$

We choose u_1 in such a way that $(Ce^{\omega t} + 1)\|u_1 - u_0\|_X \leq \frac{\epsilon}{2}$ for every $t \in [0, 1]$ (say). As $u_1 \in D(A)$, there exists $\delta \in (0, 1]$, such that, if $0 \leq t \leq \delta$, $\|e^{tA} u_1 - u_1\|_X \leq \frac{\epsilon}{2}$. We deduce that $\|e^{tA} u_0 - u_0\|_X \leq \epsilon$.

□

We leave to the reader the proof of the following simple result:

Lemma 1. *Let A be a sectorial operator in X. Then:*

(I) $\forall \lambda_0 \in \mathbb{C}$, $A - \lambda_0$ is a sectorial operator. Moreover, $\forall t \geq 0$

$$e^{t(A - \lambda_0)} = e^{-\lambda_0 t} e^{tA};$$

(II) $\forall \lambda \in \rho(A)$, $\forall t \in [0, \infty)$, $(\lambda - A)^{-1} e^{tA} = e^{tA}(\lambda - A)^{-1}$.

(III) If $t \in [0, \infty)$ and $x \in D(A)$, $Ae^{tA}x = e^{tA}Ax$.

□

We come back to our basic models (1) and (2), taking $A = A_D$ (A_N). By Theorem 2 (III), $\forall u_0 \in C([0, \pi])$, $e^{\cdot A} u_0 \in C^\infty(\mathbb{R}^+; C([0, \pi]))$. Moreover, for every $k \in \mathbb{N}$, $\forall t \in \mathbb{R}^+$, $e^{tA} u_0 \in D(A^k)$ and it is not difficult to see that even $A^k e^{\cdot A} u_0 \in C^\infty(\mathbb{R}^+; C([0, \pi]))$. We observe that, for every $k \in \mathbb{N}$, $D(A^k)$ is a subspace of $C^{2k}([0, \pi])$. This implies that the function

$$u(t, x) := [e^{tA} u_0](x)$$

belongs, in fact, to $C^\infty(\mathbb{R}^+ \times [0, \pi])$. So, we have obtained the well known regularization property of parabolic equations: starting (at time $t = 0$) from any continuous function u_0, we have, that, for $t > 0$, u is of class C^∞. We observe also that, by (III), u solves the first equation in (1) ((2)) with $f(t, x) \equiv 0$ in $(0, T] \times [0, \pi]$ and $u(t, x) = 0$ $(D_x u(t, x) = 0)$ $\forall (t, x) \in (0, T] \times \{0, \pi\}$. Concerning the initial condition at $t = 0$, by Theorem 2(V), u is continuous in $[0, T] \times [0, \pi]$ if $u_0 \in \overline{D(A)}$. In case $A = A_N$, $\overline{D(A)} = C([0, \pi])$, while in case $A = A_D$,

$$\overline{D(A_D)} = \{u_0 \in C([0, \pi]) : u(0) = u(\pi) = 0\}.$$

These remarks make the following definition quite natural: we consider the abstract nonhomogeneous problem (5).

Definition 4. *Let A be a closed operator in X, $f \in C([0, T]; X)$ and $u_0 \in X$. A* **classical** *solution of (5) is an element u of $C([0, T]; X) \cap C^1((0, T]; X) \cap C((0, T]; D(A))$, satisfying the first equation in $(0, T]$.*

It is clear that a necessary solution in order that a classical solution exist is that $u_0 \in \overline{D(A)}$. So, from now on, we shall always assume that in (5) $u_0 \in \overline{D(A)}$.

A simple consequence of Theorem 2 and the foregoing discussion are the following

Corollary 1. *Let A be a sectorial operator in X. Let $u_0 \in \overline{D(A)}$. Then, if we set $u(t) := e^{tA} u_0$ and take $f(t) \equiv 0$, for every T in $\mathbb{R}^+$ u is a classical solution of (5).*

Moreover, if we set

$$C^{1,2}((0, T] \times [0, \pi]) := C^1((0, T]; C([0, \pi])) \cap C((0, T]; C^2([0, \pi])).$$

Corollary 2. *(I) In the particular case of systems (1) and (2), a classical solution is an element of* $C([0,T]\times[0,\pi])\cap C^{1,2}((0,T]\times[0,\pi])$.

(II) Take $f(t,x)\equiv 0$*. Then (2) has a classical solution for every* u_0 *in* $C([0,\pi])$.

(II) Instead, (1) has a classical solution only for every u_0 *in* $C([0,\pi])$*, such that* $u_0(0)=u_0(\pi)=0$.

The following important result gives very precise information about the shape of a classical solution:

Theorem 3. *Let* A *be a sectorial operator in* X *and let* $f\in C([0,T];X)$*,* $u_0\in\overline{D(A)}$*. Assume that (5) has a classical solution* u*. Then the following "variation of parameter formula" holds:*

$$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)ds. \tag{13}$$

Proof We start by observing that the integral in (13) can be intended in a generalized sense as $\lim_{\tau\to t}\int_0^\tau e^{(t-s)A}f(s)ds$. In fact, the function $s\to e^{(t-s)A}f(s)$ is continuous in $[0,t)$ by Theorem 2 (III), and is bounded in $[0,t]$ by Theorem 2 (I).

Now, we fix t in $(0,T]$ and consider the mapping $v:(0,t)\to X$, $v(s):=e^{(t-s)A}u(s)$. Then, $\forall s\in(0,t)$,

$$v'(s)=-Ae^{(t-s)A}u(s)+e^{(t-s)A}u'(s)=-Ae^{(t-s)A}u(s)+e^{(t-s)A}Au(s)+e^{(t-s)A}f(s)=e^{(t-s)A}f(s),$$

because, as $u(s)\in D(A)$,

$$Ae^{(t-s)A}u(s)=e^{(t-s)A}Au(s),$$

by Lemma 1 (II). So, if $0<s_1<s_2<t$, we have

$$e^{(t-s_2)A}u(s_2)-e^{(t-s_1)A}u(s_1)=\int_{s_1}^{s_2}e^{(t-s)A}f(s)ds. \tag{14}$$

Letting $s_1\to 0$ and $s_2\to t$, we easily see that the second term in (14) converges to $\int_0^t e^{(t-s)A}f(s)ds$. As $u(s_1)\to u_0$ $(s_1\to 0)$, $e^{(t-s_1)A}u(s_1)\to e^{tA}u_0$ $(s_1\to 0)$. Finally,

$$e^{(t-s_2)A}u(s_2)=e^{(t-s_2)A}(u(s_2)-u(t))+e^{(t-s_2)A}u(t)$$

We have

$$\|e^{(t-s_2)A}(u(s_2)-u(t))\|_X\le Ce^{\omega(t-s_2)}\|u(s_2)-u(t)\|_X\to 0 \quad (s_2\to t).$$

Finally, as $u(t)\in D(A)$, by Theorem 2 (V),

$$e^{(t-s_2)A}u(t)\to u(t) \quad (s_2\to 0).$$

We have deduced that

$$u(t)-e^{tA}u_0=\int_0^t e^{(t-s)A}f(s)ds.$$

□

Given $f\in C([0,T];X)$ and u_0 in X, we shall call the function

$$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)ds$$

the **mild** solution of (5). Now we look for sufficient conditions, ensuring that the mild solution is really a classical or strict solution. Concerning strict solutions, we have the following necessary condition:

Proposition 2. *Assume that* $f\in C([0,T];X)$*,* $u_0\in X$ *and (5) has a strict solution. Then* $u_0\in D(A)$ *and* $Au_0+f(0)\in\overline{D(A)}$.

Proof We have already observed that $u_0\in D(A)$. Let u be the mentioned strict solution. Then

$$Au_0+f(0)=u'(0)=\lim_{t\to 0}t^{-1}[u(t)-u(0)].$$

So $Au_0+f(0)\in\overline{D(A)}$.

□

Before stating and proving the main result concerning the existence of strict and classical solution, we introduce the follwing important estimate, concerning $\|Ae^{tA}\|_{\mathcal{L}(X)}$:

Proposition 3. *Assume that the assumptions of Theorem 2 are fulfilled. Then, there exists C in $\mathbb{R}^+$, such that, for every $t \in \mathbb{R}^+$,*

$$\|Ae^{tA}\|_{\mathcal{L}(X)} \leq Ce^{\omega t}(1 + t^{-1}).$$

Proof Continuing to employ the notation in the proof of Theorem 2 (I), we obtain

$$Ae^{tA} = D_t e^{tA} = \tfrac{1}{2\pi i}\int_\gamma \lambda e^{\lambda t}(\lambda - A)^{-1}d\lambda = \tfrac{1}{2\pi i}\int_{\gamma_0} e^{(\omega+\lambda)t}(\omega+\lambda)(\omega+\lambda - A)^{-1}d\lambda$$

$$= \tfrac{1}{2\pi i}\int_{t^{-1}\gamma_0} e^{(\omega+\lambda)t}(\omega+\lambda)(\omega+\lambda - A)^{-1}d\lambda = \tfrac{1}{2\pi i}\int_{\gamma_0} e^{\omega t+\lambda}(\omega+\tfrac{\lambda}{t})(\omega+\tfrac{\lambda}{t} - A)^{-1}t^{-1}d\lambda,$$

so that

$$\|Ae^{tA}\|_{\mathcal{L}(X)} \leq \frac{M}{2\pi}e^{\omega t}\int_{\gamma_0} e^{Re(\lambda)}(\frac{|\omega|}{|\lambda|} + t^{-1})|d\lambda|,$$

which implies the conclusion.

□

In order to state and prove the main regularity result, we intruduce the classes of Hölder continuous function: let X be a Banach space, let $T \in \mathbb{R}^+$ and $\alpha \in (0,1)$. We shall indicate with $C^\alpha([0,T];X)$ the class of functions $f:[0,T] \to X$, satisfying the following condition: there exists $C \in \mathbb{R}^+$, such that

$$\|f(t) - f(s)\|_X \leq C|t-s|^\alpha, \quad \forall s,t \in [0,T].$$

$C^\alpha([0,T];X)$ becomes a Banach space if we equip it with the natural norm

$$\|f\|_{C^\alpha([0,T];X)} := \max\{\|f\|_{C([0,T];X)}, \sup_{0\leq s,t\leq T, s\neq t}|t-s|^{-\alpha}\|f(t)-f(s)\|_X\}.$$

Theorem 4. *Let A be a sectorial operator in the Banach space X. We consider problem (IV). Then:*
(I) if $f \in C^\alpha([0,T];X)$, for some α in $(0,1)$, and $u_0 \in \overline{D(A)}$, the mild solution is classical;
(II) if $f \in C^\alpha([0,T];X)$, for some α in $(0,1)$, $u_0 \in D(A)$ and $Au_0 + f(0) \in \overline{D(A)}$, the mild solution is strict.

Proof We start by showing that, if $u_0 \in \overline{D(A)}$, the mild solution u is continuous in $[0,T]$. It is not difficult to show that (even in case $u_0 \in X$), it is continuous in $(0,T]$. It is easy to see that $\|\int_0^t e^{(t-s)}f(s)ds\|_X \to 0$ $(t \to 0)$. Next, $\|e^{tA}u_0 - u_0\|_X \to 0$ $(t \to 0)$, in force of Theorem 2, (V).

We pass to study the differentiability of u in $(0,T]$. The difficulty lies in the singularity of e^{tA} for $t=0$. We fix ϵ in $(0,T)$, and consider the function

$$u_\epsilon(t) := e^{tA}u_0 + \int_0^{t-\epsilon} e^{(t-s)A}f(s)ds,$$

with domain $[\epsilon, T]$. As e^{tA} is smooth in $\mathbb{R}^+$, we can differentiate it and get, applying Theorem 2,

$$u_\epsilon'(t) = Ae^{tA}u_0 + e^{\epsilon A}f(t-\epsilon) + \textstyle\int_0^{t-\epsilon} Ae^{(t-s)A}f(s)ds$$

$$= Ae^{tA}u_0 + e^{tA}f(t) + e^{\epsilon A}[f(t-\epsilon) - f(t)] + \textstyle\int_0^{t-\epsilon} Ae^{(t-s)A}[f(s) - f(t)]ds,$$

as $\int_0^{t-\epsilon} Ae^{(t-s)A}f(t)ds = (e^{tA} - e^{\epsilon A})f(t)$. By Theorem 2,

$$\|e^{\epsilon A}[f(t-\epsilon) - f(t)]\|_X \leq C\epsilon^\alpha.$$

Moreover, by Proposition 3,

$$\|Ae^{(t-s)A}[f(s) - f(t)]\|_X \leq C(t-s)^{\alpha-1}.$$

So, we can see that

$$\lim_{\epsilon \to 0} u'_\epsilon(t) = Ae^{tA}u_0 + e^{tA}f(t) + \int_0^t Ae^{(t-s)A}[f(s) - f(t)]ds,$$

uniformly in $[\delta, T]$, for every δ in $(0, T)$. We conclude that, as $\lim_{\epsilon \to 0} u_\epsilon(t) = u(t)$, uniformly in $[\delta, T]$, for every δ in $(0, T)$, u is differentiable in $(0, T]$, and

$$u'(t) = Ae^{tA}u_0 + e^{tA}f(t) + \int_0^t Ae^{(t-s)A}[f(s) - f(t)]ds. \tag{15}$$

In fact, one can verify from (15) that $u \in C^1((0, T]; X)$.

Next, we have

$$\int_0^t e^{(t-s)A}f(s)ds = \int_0^t e^{(t-s)A}[f(s) - f(t)]ds + \int_0^t e^{sA}f(t)ds.$$

By Theorem 2, (III), $s \to e^{(t-s)A}[f(s) - f(t)]$ belongs to $C([0, t); D(A))$. Moreover, $\|Ae^{(t-s)A}[f(s) - f(t)]\|_X \leq C(t-s)^{\alpha-1}$. So, $\int_0^t e^{(t-s)A}[f(s) - f(t)]ds \in D(A)$ and

$$A\int_0^t e^{(t-s)A}[f(s) - f(t)]ds = \int_0^t Ae^{(t-s)A}[f(s) - f(t)]ds, \quad t \in (0, T]. \tag{16}$$

Moreover, let us fix λ_0 in $\rho(A)$. Then, by Lemma 1 (I), $\forall t \in \mathbb{R}^+$,

$$\int_0^t e^{sA}ds = \int_0^t e^{\lambda_0 s}e^{s(A-\lambda_0)}ds,$$

with the integral converging in $\mathcal{L}(X)$. We observe that, by Theorem 2(III), for $t \in \mathbb{R}^+$,

$$D_t((A - \lambda_0)^{-1}e^{t(A-\lambda_0)}) = e^{t(A-\lambda_0)}.$$

So, if $0 < \epsilon < t$, integrating by parts, we have

$$\int_\epsilon^t e^{sA}ds = \int_\epsilon^t e^{\lambda_0 s}e^{s(A-\lambda_0)}ds$$

$$= [e^{\lambda_0 s}(A - \lambda_0)^{-1}e^{s(A-\lambda_0)}]_{s=\epsilon}^{s=t} - \lambda_0 \int_\epsilon^t e^{\lambda_0 s}(A - \lambda_0)^{-1}e^{s(A-\lambda_0)}ds$$

$$= (A - \lambda_0)^{-1}[e^{\lambda_0 t}e^{t(A-\lambda_0)} - e^{\lambda_0 \epsilon}e^{\epsilon(A-\lambda_0)} - \lambda_0 \int_\epsilon^t e^{\lambda_0 s}e^{s(A-\lambda_0)}ds].$$

and, by the result of Lemma 1 (II),

$$\int_\epsilon^t e^{sA}f(t)ds$$

$$= (A - \lambda_0)^{-1}[e^{\lambda_0 t}e^{t(A-\lambda_0)}f(t) - \lambda_0 \int_\epsilon^t e^{\lambda_0 s}e^{s(A-\lambda_0)}f(t)ds] - e^{\lambda_0 \epsilon}e^{\epsilon(A-\lambda_0)}(A - \lambda_0)^{-1}f(t).$$

Letting $\epsilon \to 0$, we obtain, using the fact that $(A - \lambda_0)^{-1}f(t) \in D(A)$ and Theorem 2 (V),

$$\int_0^t e^{sA}f(t)ds = (A - \lambda_0)^{-1}[e^{tA}f(t) - \lambda_0 \int_0^t e^{sA}f(t)ds] - (A - \lambda_0)^{-1}f(t).$$

So $\int_0^t e^{sA}f(t)ds \in D(A)$ and

$$A\int_0^t e^{sA}f(t)ds = (A - \lambda_0)\int_0^t e^{sA}f(t)ds + \lambda_0 \int_0^t e^{sA}f(t)ds = e^{tA}f(t) - f(t). \tag{17}$$

By (16) and (17), $u(t) \in D(A)$ $\forall t \in (0,T]$ and

$$Au(t) = Ae^{tA}u_0 + \int_0^t Ae^{(t-s)A}[f(s) - f(t)]ds + e^{tA}f(t) - f(t) = u'(t) - f(t).$$

We have also that $t \to Au(t)$ belongs to $C((0,T];X)$. In this way, we have proved (I).

Concerning (II), we assume that $u_0 \in D(A)$ and $Au_0 + f(0) \in \overline{D(A)}$. From (15), we have that, if $0 < t \leq T$,

$$u'(t) = e^{tA}(Au_0 + f(0)) + e^{tA}(f(t) - f(0)) + \int_0^t Ae^{(t-s)A}[f(s) - f(t)]ds.$$

As $Au_0 + f(0) \in \overline{D(A)}$, by Theorem 2(V) we have that $e^{tA}(Au_0 + f(0)) \to Au_0 + f(0)$ $(t \to 0)$. Moreover,

$$\|e^{tA}(f(t) - f(0)) + \textstyle\int_0^t Ae^{(t-s)A}[f(s) - f(t)]ds\|_X \leq C_0(\|f(t) - f(0)\|_X + \int_0^t (t-s)^{\alpha-1}ds)$$

$$\leq C_1 t^\alpha \to 0 \quad (t \to 0).$$

So

$$\lim_{t \to 0} u'(t) = Au_0 + f(0).$$

We define

$$U(t) = \begin{cases} u'(t) & \text{if } t \in (0,T], \\ Au_0 + f(0) & \text{if } t = 0. \end{cases}$$

Then if $0 < t \leq T$,

$$u(t) - u(0) = \lim_{\epsilon \to 0} (u(t) - u(\epsilon)) = \lim_{\epsilon \to 0} \int_\epsilon^t U(s)ds = \int_0^t U(s)ds,$$

so that

$$\frac{u(t) - u(0)}{t} = \frac{1}{t}\int_0^t U(s)ds \to U(0) = Au_0 + f(0) \quad (t \to 0),$$

because U is continuous in 0. So $u \in C^1([0,T];X)$. Moreover,

$$Au(t) = u'(t) - f(t) \to Au_0 \quad (t \to 0).$$

We deduce that $u \in C([0,T];D(A))$.

The proof is complete.

□

We apply Theorem 4 to problem (1)-(2). If $\alpha \in (0,1)$, we set

$$C^{\alpha,0}([0,T] \times [0,1]) := \{f \in C([0,T] \times [0,1]) | \sup_{0 \leq s < t \leq T, 0 \leq x \leq 1} (t-s)^{-\alpha}|f(t,x) - f(s,x)| < \infty\}.$$

It is clear that $C^{\alpha,0}([0,T] \times [0,1])$ can be identified with $C^\alpha([0,T];C([0,1]))$. Then we have:

Corollary 3. *(I) A classical solution of (1) ((2)) is an element u of $C([0,T] \times [0,\pi])$, such that, for every $\delta \in (0,T)$, the restriction of u to $[\delta,T] \times [0,\pi]$ belongs to $C^{1,2}([\delta,T] \times [0,\pi])$, it satisfies the first equation in (1) ((2)) in $(0,T] \times [0,\pi]$, $u(t,x) = 0$ ($\frac{\partial u}{\partial x}(t,x) = 0$) $\forall (t,x) \in (0,T] \times \{0,\pi\}$, and $u(0,\cdot) = u_0$.*

Let $f \in C^{\alpha,0}([0,T];C([0,\pi]))$, for some $\alpha \in (0,1)$. Then:

(II) if $u_0 \in C([0,\pi])$, (2) has a unique classical solution;

(III) if $u_0 \in C([0,\pi])$ and $u_0(0) = u_0(\pi) = 0$, (1) has a unique classical solution;

(IV) if $u_0 \in C^2([0,\pi])$ and $D_x u_0(0) = D_x u_0(\pi) = 0$, (2) has a unique strict solution;

V) if $u_0 \in C^2([0,\pi])$, $u_0(0) = u_0(\pi) = 0$ and $D_x^2 u_0(0) + f(0,0) = D_x^2 u_0(\pi) + f(0,\pi) = 0$, (1) has a unique strict solution.

We leave to the reader the proof of Corollary 3. We recall only that $D(A_N)$ is dense in $C([0,\pi])$, while the closure of $D(A_D)$ in the same space is $\{g \in C([0,\pi]) : g(0) = g(\pi) = 0\}$.

We observe that Theorem 4 is, to some extent, unsatisfying: in fact, the assumption that f is Hölder continuous is not necessary, in order to get (for example) the conclusion that u is a strict solution. For some applications (in particular to nonlinear problems) it is desirable to have at disposal theorems establishing linear and topological isomorphisms between spaces of solutions and spaces of data. In these circumstances, we speak of "maximal regularity". We conclude this note with a couple of results in this direction.

We start by introducing certain intermediate spaces between the basic Banach space X and the domain $D(A)$ of the operator A, which is sectorial in X. We recall that, by Proposition 1, $D(A)$ is a Banach space with the natural norm (4). If $x \in X$, we have, by Proposition 3,

$$\|Ae^{tA}x\|_X \leq Ct^{-1}, \quad t \in (0,1], \tag{18}$$

but, in the particular case $x \in D(A)$, the following better estimate holds:

$$\|Ae^{tA}x\|_X = \|e^{tA}Ax\|_X \leq C, \quad t \in (0,1], \tag{19}$$

by Theorem 2 (I). It is natural to consider classes of elements satisfying estimates which are intermediate between (18) and (19): let $\theta \in (0,1)$. We set

$$D_A(\theta) := \{x \in X | \exists C \in \mathbb{R}^+ : \|Ae^{tA}x\|_X \leq Ct^{\theta-1} \quad \forall t \in (0,1]\}.$$

One can easily verify that $D_A(\theta)$ is a Banach space with the norm

$$\|x\|_{D_A(\theta)} := \max\{\|x\|, \sup_{0<t\leq 1} t^{1-\theta}\|Ae^{tA}x\|_X\}. \tag{20}$$

The following has a simple proof:

Lemma 2. *Let A be a sectorial operator in the Banach space X and let $\theta \in (0,1)$. Then:*
(I) $D_A(\theta)$ is a Banach space with the norm defined in (20);
(II) $D(A) \hookrightarrow D_A(\theta) \hookrightarrow X$ (continuous embeddings).

One can show that $D_A(\theta)$ is an interpolation space between X and $D(A)$, in the following sense: let Y be another Banach spaces and B a sectorial operator in Y. Assume that $S \in \mathcal{L}(X,Y)$ and the restriction $S_{|D(A)}$ of S to $D(A)$ belongs to $\mathcal{L}(D(A), D(B))$. Then, $\forall \theta \in (0,1)$, the restriction $S_{|D_A(\theta)}$ of S belongs to $\mathcal{L}(D_A(\theta), D_B(\theta))$.

In the following we shall indicate with $B([0,T];Y)$, with Y Banach space, the class of bounded functions from $[0,T]$ to Y.

The following theorem holds (see [3], Chapter 4):

Theorem 5. *Consider the problem (5), with A sectorial operator in the Banach space X.*
(I) Assume that, for some $\theta \in (0,1)$, $f \in C([0,T];X) \cap B([0,T]; D_A(\theta))$, $u_0 \in D(A)$, and $Au_0 \in D_A(\theta))$. Then the mild solution u of (5) is strict. Moreover, $D_t u$ and Au are bounded with values in $D_A(\theta)$.
(II) Assume that, for some $\theta \in (0,1)$, $f \in C^\theta([0,T];X)$, $u_0 \in D(A)$ and $Au_0 + f(0) \in D_A(\theta)$. Then the mild solution u of (5) is strict. Moreover, $D_t u$ and u belong to $C^\theta([0,T];X)$.

One can verify that the conditions of f and u_0 in Theorem 5 are also necessary to obtain solutions with the declared regularity; in case (I), this can be easily shown.

In order to apply Theorem 5 to (1)-(2), one has to characterize the spaces $D_{A_N}(\theta)$ and $D_{A_D}(\theta)$. The following theorem holds:

Theorem 6. *Let $\theta \in (0,1) \setminus \{1/2\}$. Then:*
(I)

$$D_{A_D}(\theta) = \{g \in C^{2\theta}([0,\pi]) : g(0) = g(\pi) = 0\};$$

(II)

$$D_{A_N}(\theta) = \begin{cases} C^{2\theta}([0,\pi]) \text{ if } 0 < \theta < 1/2, \\ \\ \{g \in C^{2\theta}([0,\pi]) : D_x g(0) = D_x g(\pi) = 0\} \text{ if } 1/2 < \theta < 1. \end{cases}$$

If $\alpha \in (0,1)$, we set

$$C^{0,\alpha}([0,T]\times[0,\pi]) := \{f \in C([0,T]\times[0,\pi]) : \sup_{0\le t\le T, 0\le x<y\le\pi} (y-x)^{-\alpha}|f(t,y)-f(t,x)| < \infty\}.$$

As an application of Theorem 6, we have:

Corollary 4. *Consider problem (2). Let $\alpha \in (0,1)$. Then:*

(I) the following conditions are necessary and sufficient in order that the mild solution u of (2) is strict and $D_t u$ and $D_x^2 u$ belong to $C^{0,\alpha}([0,T]\times[0,\pi])$: $f \in C^{0,\alpha}([0,T]\times[0,\pi])$, $u_0 \in C^{2+\alpha}([0,\pi])$, $D_x u_0(0) = D_x u_0(\pi) = 0$;

(II) the following conditions are necessary and sufficient in order that the mild solution u of (2) is strict and $D_t u$ and $D_x^2 u$ belong to $C^{\alpha,0}([0,T]\times[0,\pi])$: $f \in C^{\alpha,0}([0,T]\times[0,\pi])$, $u_0 \in C^2([0,\pi])$, $D_x u_0(0) = D_x u_0(\pi) = 0$, $D_x^2 u_0 + f(0,\cdot) \in C^\alpha([0,\pi])$.

Proof It follows from Theorem 6, taking $\theta = \frac{\alpha}{2}$.

□

We conclude this short exposition with some extension of (1)-(2) to higher space dimensions.

We consider a bounded open subset Ω of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2. Let

$$\mathcal{A}(x, D_x) := \sum_{1\le i,j\le n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n a_j(x)\frac{\partial}{\partial x_j} + a_0(x)$$

be a strongly elliptic second order differential operator, with coefficients $a_{ij} \in C^1(\overline{\Omega})$ $\forall i,j \in \{1,\dots,n\}$, $a_j \in C(\overline{\Omega})$ $\forall i,j \in \{0,\dots,n\}$. This means the following: that, $\forall i,j \in \{1,\dots,n\}$, a_{ij} is real valued and there exists $N \in \mathbb{R}^+$, such that

$$\sum_{1\le i,j\le n} a_{ij}(x)\xi_i\xi_j \ge N|\xi|^2, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n.$$

We can consider the two following operators A_D and A_N in the Banach space $C(\overline{\Omega})$:

$$\begin{cases} D(A_D) := \{u \in \cap_{1\le p<\infty} W^{2,p}(\Omega) : u_{|\partial\Omega} = 0\}, \\ A_D u := \mathcal{A}(x, D_x)u, \end{cases} \tag{21}$$

$$\begin{cases} D(A_N) := \{u \in \cap_{1\le p<\infty} W^{2,p}(\Omega) | \frac{\partial u}{\partial \nu_{\mathcal{A}}} := \sum_{i=1}^n \sum_{i=1}^n a_{ij}\nu_j \frac{\partial u}{\partial x_i} \equiv 0 \text{ in } \partial\Omega\}, \\ A_N u := \mathcal{A}(x, D_x)u, \end{cases} \tag{22}$$

where we have indicated with $\nu(x)$ ($x \in \partial\Omega$) the unit vector which is orthogonal to $\partial\Omega$ in x, pointing outside Ω. The following theorem holds:

Theorem 7. *(I) The operators A_D and A_N, defined in (21)-(22), are sectorial in $C(\overline{\Omega})$. Their domain is contained in $C^{2-\epsilon}(\overline{\Omega})$ for every $\epsilon \in (0,2]$.*

(II) If $\theta \in (0,1) \setminus \{1/2\}$,

$$D_{A_D}(\theta) = \{g \in C^{2\theta}(\overline{\Omega}) : g_{|\partial\Omega} = 0\};$$

(II)

$$D_{A_N}(\theta) = \begin{cases} C^{2\theta}(\overline{\Omega}) \text{ if } 0 < \theta < 1/2, \\ \{g \in C^{2\theta}(\overline{\Omega}) : \frac{\partial u}{\partial \nu_{\mathcal{A}}} = 0 \text{ in } \partial\Omega\} \text{ if } 1/2 < \theta < 1. \end{cases}$$

For proofs, see [3], Chapter 3. Theorem 7 allows to extend (to some extent) Corollary 4 to higher space dimensions. We observe that, contrary to what expected, in case $n > 1$, the domains of A_D and A_N are not contained in $C^2(\overline{\Omega})$, but only (as a consequence of Sobolev embedding theorem) in $C^{2-\epsilon}(\overline{\Omega})$, for every $\Omega \in (0,2]$.

As application of Theorems 5 and 7, we deduce the following

Corollary 5. *Consider the problem*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mathcal{A}(x,D_x)(t,x) + f(t,x), \quad t \in [0,T], x \in \overline{\Omega}, \\ \\ u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \overline{\Omega}. \end{cases} \tag{23}$$

with Ω *and* $\mathcal{A}(x,D_x)$ *satisfying the previous conditions. Let* θ *in* $(0,1)$. *Then the following conditions are necessary and sufficient, in order that (23) have a (unique) solution in* $C^{1,0}([0,T];C(\overline{\Omega})) \cap C([0,T];D(A_D))$, *with* $D_t u$ *and* $A_D u$ *in* $C^{\theta,0}([0,T];C(\overline{\Omega}))$:

(I) $f \in C^{\theta,0}([0,T];C(\overline{\Omega}))$;

(II) $u_0 \in D(A_D)$, $A_D u_0 + f(0,\cdot) \in C^\theta(\overline{\Omega})$ *and* $A_D u_0(x) + f(0,x) = 0$ $\forall x \in \partial\Omega$.

References

[1] A. Lorenzi, *An Introduction to Identification Problems via Functional Analysis*, Inverse and Ill-Posed Problems Series (2001), VSP.

[2] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser (1995).

[3] H. F. Weinberger, *A first course in partial differential equations*, Wiley (1965).