MECCANICA

A CONFERENCE IN HONOR OF SANDRO GRAFFI ON HIS 65TH BIRTHDAY

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Rereading Langer's influential 1937 JWKB paper: the unnecessary Langer transformation; the ambiguous $\hbar$; the power of Borel summation

In the Abstract of his influential 1937 paper, Langer attacked the JWKB analysis of the radial wave equation as "uncritical and in error." Langer "justified" Kramers' empirical substitution of $(l+\frac{1}{2})^2$ for $l(l+1)$ in the centrifugal potential, which converted the behavior of the JWKB wave function at the origin from $r^{\frac{1}{2}+[l(l+1)]^{1/2}}$ to r^{l+1}, via the transformation $r = e^x, \psi(r) = e^{x/2}u(x)$, which moved the origin to $-\infty$ and eliminated the second-order pole. The first-order-in-$\hbar$ x formulas turned out to be equivalent to first-order-in-$\hbar$ r formulas with $l(l+1)$ replaced by $(l+\frac{1}{2})^2$. Krieger and Rosenzweig (1967) pushed Langer's x solution to third order but concluded that "there is no effective [r] potential ... which will give rise to the correct result ..." Beckel and Nakhleh (1963) found a value K to substitute for $l(l+1)$ to fix the third-order (but not first-order) JWKB wave function. Fröman and Fröman (1974) extended that to 9th, and Seetharaman and Vasan (1984) to any order. In infinite order K is exactly $l(l+1)$, i.e., no modification. Hainz and Grabert (1999) got r^{l+1} in all orders beyond zeroth by the decomposition, $\hbar^2 l(l+1) = L^2 + \hbar L$ (at the end, L is set equal to $\hbar l$), with no Langer modification. The big surprise came in 2004 when Dahl and Schleich saw that Langer's exponential transformation was irrelevant, that only $r^{-1/2}$ was relevant, and that "[Langer's] analysis may, in fact, be considered as nothing more than a somewhat complicated way of solving [the radial wave equation for $r^{-1/2}\psi(r)$] by the JWKB method." These seemingly incompatible results can be understood in a unified framework by noting that for each there are *two* $\hbar$'s: the kinetic energy $\hbar$ drives the expansion; the centrifugal potential $\hbar$ is passive, implicit, intrinsic, never expanded. The solutions differ in how the "original $\hbar$" is split between expansion $\hbar$ and implicit $\hbar_i$. The different JWKB expansions are Borel summable to different analytic functions that coincide when $\hbar_i = \hbar$. In two-$\hbar$ notation, the generalization of Kramers' substitution is near trivial:

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dr^2} + V(r) + \frac{\hbar_i^2(l+\frac{1}{2})^2}{2r^2} - \frac{\hbar^2(\frac{1}{2})^2}{2r^2} - E\right) e^{\int q\,dr/\hbar} = 0.$$