Quantum Brownian Motion, Quantum Friction, Decoherence

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Introduction & Contents

Some central problems of general theoretical physics

1) Meaning of QT, decoherence ↔ events; emergence of space-time & causal structure

2) Origin of atomistic structure (in "quantization")

3) Foundation of TD in stat. mech. of open systems; irreversibility

4) Foundations of transport theory – diffusive motion, friction, decoherence.

4) is focus of lecture

Contents

1. Models

One (or several) NR particle(s) interacting w. quantum gas; e.g., a bosonic atom gas, or phonons, photons at temperature $T>0$; $T \simeq 0$.

Particle moves

i) in continuum $\mathbb{E}^3$;
Hilbert space $\mathcal{H}_p = L^2(\mathbb{R}^3, dx)$

ii) on lattice $\mathbb{Z}^3$;

$$\mathcal{H}_p = \ell_2(\mathbb{Z}^3).$$

Particle Hamiltonian

$$H_p = -\frac{1}{2M}\Delta + V(X),$$

where Δ is

i) cont. Laplacian; or

ii) discrete Laplacian: ε

Quantum gas:

Hilbert space $\mathcal{H}_g \simeq \mathcal{F}_\beta,$

$\beta = (k_B T)^{-1}$, $\mathcal{F}_{\beta=\infty} \simeq$ Fock sp.

Hamiltonian

$$\text{i) } H_g = \int dx \Big\{ \frac{1}{2m} (\nabla\psi^*)(x)(\nabla\psi)(x) + \lambda \int dy \, [\psi^*(y)\psi(y) - \rho] \times \varphi(y-x)[\psi^*(x)\psi(x) - \rho] \Big\}$$

ψ, ψ^* satisfy CCR;

$\lambda > 0$, φ of positive type.

$$H_I = g\int dx\, W(X-x)\left[\psi^*(x)\psi(x) - \rho\right]$$

when $\lambda = 0$: $g\,W$ repulsive.

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ii) $$H_g = \bigoplus_{X \in \mathbb{Z}^3} \left(|X\rangle\langle X|\right)\int dk\, a_X^*(k)\omega(k)a_X(k)$$

a_X, a_X^*: CCR, $\omega(k) \overset{e.g.}{=} |k|$;

$$H_I = g\sum_X |X\rangle\langle X| \int dk \left[a_X^*(k)\Phi(k) + h.c.\right]$$

- One phonon reservoir; or
- at each site $X \in \mathbb{Z}^3$, an independent phonon res.

2. Mean-field limit, (i).

$$M_{part.} \longmapsto M_N = NM$$

$$V(X) \longmapsto V_N(X) = NV(X)$$

$$\rho \longmapsto \rho_N = \frac{N}{g^2}\rho \quad : G.D.$$

$$\lambda \longmapsto \lambda_N = N^{-1}\lambda$$

$$1 \leq N < \infty, \quad N^{-1} \leftrightarrow \hbar$$

Hamiltonian $H = H_p + H_g + H_I$

$$\longmapsto H_N =: NH^{(N)}$$

Schrödinger Eq.:

$$i\partial_t \Psi_t = H_N \Psi_t, \quad \Psi_t \in \mathcal{H}_p \otimes \mathcal{H}_g$$

$$\Longleftrightarrow \frac{i}{N}\partial_t \Psi_t = \Big(-\frac{1}{N^2}\frac{\Delta}{2M} + V(X) + \\ + N^{-1}H_g + N^{-1}H_I\Big)\Psi_t$$

$$\psi^{\#}(x) =: \sqrt{N}\left(b_N^{\#}(x) + \frac{\sqrt{\rho}}{g}\right)$$

Then

$$[b_N^{\#}, b_N^{\#}] = 0,\ [b_N(x), b_N^{*}(y)] = \frac{1}{N}\delta(x-y)$$

$N \to \infty \leftrightarrow$ "class. limit"

We find that

$$H^{(N)} = -\frac{1}{N^2}\frac{\Delta}{2M} + V(X) + \mathcal{H}^{(N)}(X),$$

$$\mathcal{H}^{(N)}(X) = \int dx \Big\{ \frac{1}{2m} \nabla b_N^{*}(x) \cdot \nabla b_N(x)$$

$$+ \Big(g\, W(X-x) + \lambda \int dy \Big[b_N^{*}(y) b_N(y)$$

$$+ \frac{\sqrt{\rho}}{g}\big(b_N^{*}(y) + b_N(y)\big)\Big] \varphi(y-x)\Big) \times$$

$$\Big[b_N^{*}(x) b_N(x) + \frac{\sqrt{\rho}}{g}\big(b_N^{*}(x) + b_N(x)\big)\Big] \Big\}$$

3 variants of model (i):

(1) B-model: $\lambda = 0$, $g \to 0$;
 gW repulsive;

(2) C-model: $\lambda = 0$, $g > 0$;
 gW rep.

(3) F-model: $\lambda > 0$, gW arb.

Formal mean-field lim, $N \to \infty$.

$\frac{i}{N} \nabla_X \to P$, $b^{\#}_N(x) \longrightarrow \beta^{\#}(x)$,

with Poisson brackets:

$\{P_i, X^j\} = \delta_i^j$, $\{\beta^{\#}, \beta^{\#}\} = 0$,

$\{\beta(x), \bar{\beta}(y)\} = i\delta(x-y)$

Phase space: $\Gamma = \mathbb{R}^6 \times H^1(\mathbb{R}^3)$

Classical Hamilton funct.

$$\mathcal{H}(P,X;\bar{\beta},\beta) = \frac{P^2}{2M} + V(X) + \int dx \left\{ \frac{1}{2m} |\nabla\beta(x)|^2 + \left(gW(X-x) + \lambda \int dy \left[|\beta(y)|^2 + 2\frac{\sqrt{\rho}}{g} \mathrm{Re}\,\beta(y) \right] \varphi(y-x) \right) \left[|\beta(x)|^2 + 2\frac{\sqrt{\rho}}{g} \mathrm{Re}\,\beta(x) \right] \right\}$$

Classical Eqs. of motion:

$$u := X_t - x \leftrightarrow x = X_t - u.$$

$$\gamma_t(u) := \beta_t(X_t - u)$$

$$\frac{\partial \beta_t(x)}{\partial t} = \dot{\gamma}_t(u) + \dot{X}_t \cdot \nabla\gamma_t(u).$$

Then we find:

$$(1)\quad \begin{aligned} \dot{X}_t &= M^{-1}P_t\,, \\ \dot{P}_t &= -(\nabla V)(X_t) - \int du\; g(\nabla W)(u) \\ &\quad \times \big[|\gamma_t(u)|^2 + 2g^{-1}\sqrt{\rho}\,\mathrm{Re}\,\gamma_t(u)\big] \end{aligned}$$

$$\hookleftarrow F(\{\dot{X}_s\}_{s\leq t})$$

and

$$(2)\quad \begin{aligned} i\dot{\gamma}_t(u) &= \sqrt{\rho}\,W(u) - i\dot{X}_t\cdot\nabla\gamma_t(u) \\ &\quad + \Big(-\frac{\Delta}{2m} + gW(u)\Big)\gamma_t(u) \\ &\quad + \lambda\int dv\,\big[|\gamma_t(v)|^2 + 2g^{-1}\sqrt{\rho}\,\mathrm{Re}\,\gamma_t(v)\big] \\ &\quad * \varphi(v-u)\big[\gamma_t(u) + g^{-1}\sqrt{\rho}\big] \end{aligned}$$

$V\in C^\infty$, $V\geq 0$, W,φ smooth & of short range, $\lambda\geq 0$, φ of positive type; $\gamma_{t=0}\in H^1(\mathbb{R}^3)$:

Then (1) & (2) have global-in-time solutions $\rightarrow$ canonical flow, Φ_t, on Γ!

"<u>Egorov Theorem</u>" (rigorous for B- and C-model; – ?)

Let Q denote Wick quantization: $P \mapsto \frac{i}{N}\nabla_x$, $\beta^{\#} \mapsto b^{\#}_N(x)$, followed by Wick ordering;

F: (gauge-inv.) polynomial funct. on Γ.

Then

$$e^{itNH^{(N)}} Q(F) e^{-itNH^{(N)}}\Big|_{\mathcal{D}}$$

$$= Q(F \circ \Phi_t)\Big|_{\mathcal{D}} + \text{"}O\left(\frac{1}{N}\right)\text{"}$$

Hepp's variant:

$|X, P; \gamma\rangle_N$: coherent state.

Then

$$e^{-itNH^{(N)}} |X_0, P_0; \gamma_0\rangle_N \simeq |X_t, P_t; \gamma_t\rangle_N + \text{"}O\left(\tfrac{1}{N}\right)\text{"}$$

Applications to friction & decoherence!

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3. Quantum Brownian Motion

Model ii): Particle hopping on lattice $\mathbb{Z}^3$; $V(X) \equiv 0$. Indep. thermal reservoirs of phonons at each site $x \in \mathbb{Z}^3$; creationannihilation ops. $a_x^*(k), a_x(k)$, dispersion $\omega(k) = |k|$; inv. temp. $\beta = (k_B T)^{-1}$; ($\nearrow$ p. 6).

Rep. of lattice translations:

$$T_z |x\rangle\langle y| T_z^{-1} = |x+z\rangle\langle y+z|$$

$$T_z a_x^{\#}(k) T_z^{-1} = a_{x+z}^{\#}(k)$$

ρ_R^β : Gibbs state of free thermal res. at inv. temp. β;

ρ_R^β is translation inv.

Effective time evol. of particle:

$$Z_t^g(P) := \rho_R^\beta\left[e^{itH_g}(P \otimes 1)e^{-itH_g}\right],$$

$P = P(x,y)$ a density matrix on $\mathcal{H}_p$.

Momentum-space rep.:

$$P(X,y) = (2\pi)^{-3} \int_{\mathbb{T}^3 \times \mathbb{T}^3} dk\, dK\, e^{ik\cdot(X-y)} \times e^{-iK\cdot\left(\frac{X+y}{2}\right)} \hat{P}(k,K)$$

$$T_Z \hat{P}(k,K) T_Z^{-1} = e^{-iK\cdot Z} \hat{P}(k,K)$$

Z_t^g commutes w. transl. $T_Z \Rightarrow$ preserves fibres w. fixed K!

"Phase space density" of part.:

$$\nu(k;Z) := (2\pi)^{-3/2} \int_{\mathbb{T}^3} dK\, e^{-iK\cdot Z} \hat{P}(k,K)$$

Time evol. of ν in kinetic lim.

$$\partial_t \nu_t(k;Z) = \left(\nabla\omega(k) \cdot \nabla_Z \nu_t\right)(k;Z)$$

(BE)

$$+ \int_{\mathbb{T}^3} dk' \left[c^\beta(k',k)\, \nu_t(k',Z) - c^\beta(k,k')\, \nu_t(k,Z) \right]$$

det. by detailed balance

Solu. ν_t of (BE) $\simeq Z^g_{g^{-2}t}(\nu)$, as $g \to 0$. $\rightsquigarrow$ Construct expansion around kinetic lim convergent for small g; ($\nearrow$ W. De R.)

<u>Results.</u>

(1) <u>Equipartition.</u>

$$tr_{\mathcal{H}_p}\left(Z^g_t(P)F(\varepsilon)\right) \xrightarrow[t\to\infty]{} (Z^\beta_g)^{-1}\int dk\left[e^{-\beta\varepsilon(k)} + o(g^0)\right]F(\varepsilon(k))$$

"return to equilibrium" for fns. of particle momentum

(2) <u>Decoherence of part. veloc.</u>

$$tr_{\mathcal{H}_p}\left[P\rho^\beta_R\left(\alpha^g_{t_1}(\nabla\varepsilon)\alpha^g_{t_2}(\nabla\varepsilon)\right)\right] \xrightarrow[\min(t_1,t_2)\to\infty]{}$$

$\rightarrow \psi^{\beta}(t_1 - t_2)$, w. $\int \psi^{\beta}(s)\, ds < \infty$

"$\Rightarrow$" $\langle 0 | X(t)^2 | 0 \rangle \underset{t \to \infty}{\sim} D_g\, t$,

D_g : diffusion constant; $\overset{\dots}{\overset{(1)}{>}} 0!$

(3) Central limit theorem.

$$P_X = |X\rangle\langle X|, \text{ proj on } \delta_X,.$$

$$\mu_t^g(X) := tr_{\mathcal{H}_p}\left(Z_t^g(P_0)\, P_X\right) \geqslant 0.$$

$$\sum_{X \in \mathbb{Z}^3} \mu_t^g(X) = 1.$$

Then

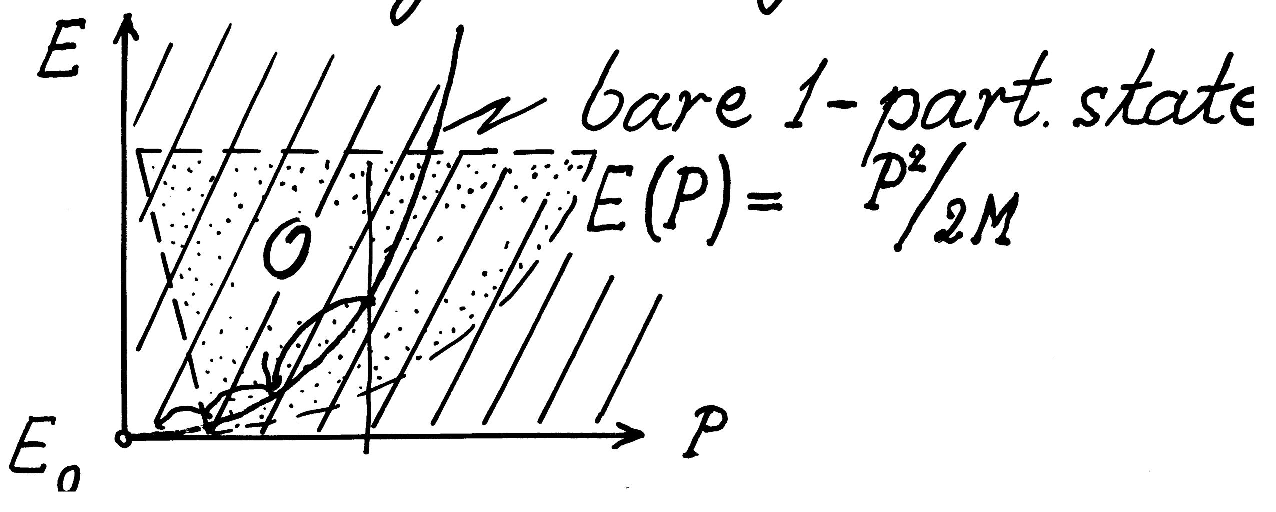
$$\mu_t^g(X) \underset{t \to \infty}{\sim} c\, t^{-3/2} \exp\left[-\frac{|X|^2}{2 D_g t}\right]$$

Extension of results to model

i) - one NR res. - feasible.

4. Quantum Friction

(E.g.,) model i) at $T=0$;
(continuum; one reservoir of NR bosonic atoms exhibiting BEC; $\lambda = 0$, i.e., B- or C-model); $V(X) \equiv 0 \rightarrow$
System is translation-inv.
$\Longrightarrow$ H commutes w. total momentum operator, P.
spec(H,P) given by



(1) Theorem (DeR, F, Pizzo)

Let g be c.c. (in H_I). Then, for $|g|$ small enough, dep. on $\mathcal{O}$..., there do not exist any dressed 1-particle states w. $(E(P), P) \in \mathcal{O}$, (i.e., bare 1-part. states become unstable after interactions turned on).

$\rightarrow$ no ballistic motion!

Quantum friction by emission of Cerenkov radiation.

Proof based on multi-scale virial theorem / $D_g(T=0) = 0$!

(2) Model i) in mean-field limit (↗ part 2).

Eqs. of motion (1) & (2) (↗ p.11) for B- (and C-) model, i.e., $\lambda = 0$, with $V(X) \equiv 0$.

Initial conditions: $X_{t=0} = X_0$, $P_{t=0} = P_0$; $\gamma_{t=0} \overset{e.g.}{=} 0$ (pure BEC).

<u>Theorem</u> (FSSS)

$$\left.\begin{array}{l} P_t \to 0,\ X_t \to X_\infty \\ \gamma_t \to \gamma_{X_\infty} \end{array}\right\} \text{as } t \to \infty,$$

with $\|\gamma_t\|_2 \sim \text{const.}\ t \to \infty$, as $t \to \infty$.

γ_{X_∞}: splash around X_∞.

Proof based on simple techniques of non-linear analysis, (diff. inequalities, etc.) Hamiltonian friction

5. Decoherence

Model i), as above, w. Bose gas of density $\propto \rho \cdot N$, $1 < N < \infty$ "large". Coherent states of exc. BEC, $|\gamma\rangle_N$, are eigenstates of annihilation ops. $b_N(x)$:

$$b_N(x)|\gamma\rangle_N = \gamma(X_t - x)|\gamma\rangle_N,$$

$$|{}_N\langle\gamma|\gamma'\rangle_N| = \exp\left[-\frac{N}{2}\|\gamma-\gamma'\|_2^2\right]$$

For B- and C-model ($\lambda=0$), by Hepp's theorem,

$$e^{-itNH^{(N)}}|X_0, P_0; \gamma_0\rangle_N = |X_t, P_t; \gamma_t\rangle_N + \text{"}O\left(\frac{1}{N}\right)\text{"},$$

where $\{X_t, P_t; \gamma_t\}$ is solu. of eqs. (1) & (2) with initial cond $\{X_0, P_0; \gamma_0\}$.

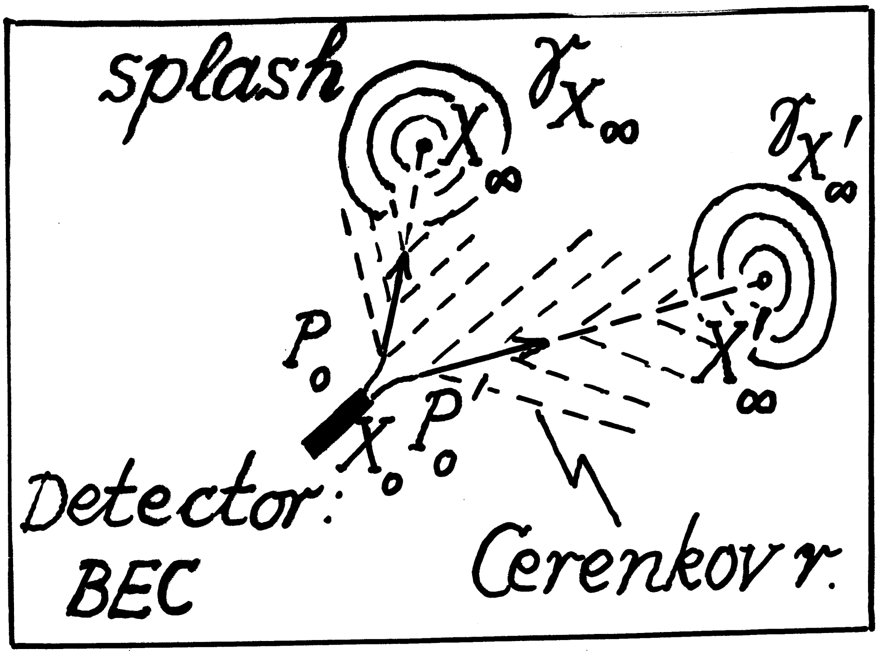
By theorem in part 4, for $X_0 = X_0'$, but $P_0 \neq P_0'$, $\gamma_0 = \gamma_0' = 0$, it follows that $X_\infty \neq X_\infty'$.

Then $\|\gamma_t - \gamma_t'\|_2^2 = O(|X_\infty - X_\infty'|)$, for $|X_\infty - X_\infty'| \leq O(t)$. Thus, for $t \gg |X_\infty - X_\infty'|$,

$$|{}_N\langle X_t, P_t; \gamma_t | X_t', P_t', \gamma_t'\rangle_N| \sim \exp[-\text{const.} N |X_\infty - X_\infty'|]$$

→ decoherence, for large N and t large enough.

→ Particle-momentum measurement:



Has applications to "quantum theory of experiments", (next time).

6. Conclusions

Have sketched a mathematically coherent approach to decoherence, friction & diffusion in phys. realistic models of "open quantum systs."

→ Some understanding of irreversible behavior & of meaning of QM.

But: Sandro, we need you to solve rem. problems!

E.g.:

(1) Diffusion of quantum particle coupled to single reservoir at $T>0$ ($\nearrow$ DeR.)

(2) Details of dynamical picture of friction at $T \simeq 0$, for finite N

(3) Stationary states when const. ext. force turned on $\longrightarrow$ Ohm's law ...

(4) Non-perturbative results: large g; Anderson model ...