DUALITY AND HIDDEN SYMMETRIES IN TRANSPORT MODELS

Cristian Giardinà

MECCANICA , 27-30 August 2008

technische universiteit eindhoven

EURANDOM

Joint work with: J. Kurchan, F. Redig

Outline

Outline

1. Transport models
   - Generator, Quantum-like formalism

Outline

1. Transport models
   - Generator, Quantum-like formalism
2. Duality and Symmetry
   - Self-Duality
   - General duality

Outline

1. Transport models
   - Generator, Quantum-like formalism
2. Duality and Symmetry
   - Self-Duality
   - General duality
3. Symmetric Exclusion processes (j-SEP)
   - SU(2) ferromagnet

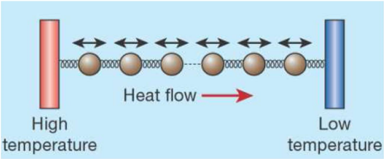
Outline

1. Transport models
   - Generator, Quantum-like formalism
2. Duality and Symmetry
   - Self-Duality
   - General duality
3. Symmetric Exclusion processes (j-SEP)
   - SU(2) ferromagnet
4. Brownian energy processes (k-BEP)
   - SU(1,1) ferromagnet
   - The KMP model

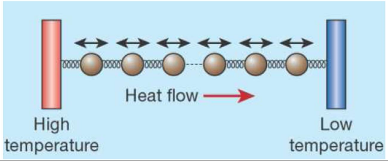
Outline

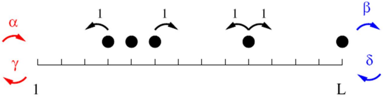
1. Transport models
   - Generator, Quantum-like formalism
2. Duality and Symmetry
   - Self-Duality
   - General duality
3. Symmetric Exclusion processes (j-SEP)
   - SU(2) ferromagnet
4. Brownian energy processes (k-BEP)
   - SU(1,1) ferromagnet
   - The KMP model
5. Consequences of duality for transport models
   - n-points correlation functions

Transport models



Transport models





Generator

State space

Ω finite (or countably infinite) set, $\eta \in \Omega$ configuration

Generator

State space

Ω finite (or countably infinite) set, $\eta \in \Omega$ configuration

Continuous time Markov process

$(\eta_t)_{t\geq 0}$ with generator L acting on $f : \Omega \to \mathbb{R}$

$$(Lf)(\eta) = \sum_{\eta' \in \Omega} c(\eta, \eta')[f(\eta') - f(\eta)]$$

Generator

State space

Ω finite (or countably infinite) set, $\eta \in \Omega$ configuration

Continuous time Markov process

$(\eta_t)_{t\geq 0}$ with generator L acting on $f : \Omega \to \mathbb{R}$

$$(Lf)(\eta) = \sum_{\eta' \in \Omega} c(\eta, \eta')[f(\eta') - f(\eta)]$$

$$L(\eta, \eta') = \begin{cases} c(\eta, \eta') & \text{if } \eta \neq \eta' \\ -\sum_{\eta'' \neq \eta} c(\eta, \eta'') & \text{if } \eta = \eta' \end{cases}$$

Generator

State space

Ω finite (or countably infinite) set, $\eta \in \Omega$ configuration

Continuous time Markov process

$(\eta_t)_{t\geq 0}$ with generator L acting on $f : \Omega \to \mathbb{R}$

$$(Lf)(\eta) = \sum_{\eta' \in \Omega} c(\eta, \eta')[f(\eta') - f(\eta)]$$

$$L(\eta, \eta') = \begin{cases} c(\eta, \eta') & \text{if } \eta \neq \eta' \\ -\sum_{\eta'' \neq \eta} c(\eta, \eta'') & \text{if } \eta = \eta' \end{cases}$$

Evolution

- L evolves expectation values $\quad \frac{d}{dt}\mathbb{E}_\eta(f(\eta_t)) = (Lf)(\eta)$

Generator

State space

Ω finite (or countably infinite) set, $\eta \in \Omega$ configuration

Continuous time Markov process

$(\eta_t)_{t\geq 0}$ with generator L acting on $f : \Omega \to \mathbb{R}$

$$(Lf)(\eta) = \sum_{\eta' \in \Omega} c(\eta, \eta')[f(\eta') - f(\eta)]$$

$$L(\eta, \eta') = \begin{cases} c(\eta, \eta') & \text{if } \eta \neq \eta' \\ -\sum_{\eta'' \neq \eta} c(\eta, \eta'') & \text{if } \eta = \eta' \end{cases}$$

Evolution

- L evolves expectation values $\frac{d}{dt}\mathbb{E}_\eta(f(\eta_t)) = (Lf)(\eta)$
- L^T evolves the probability measures $\frac{d}{dt}p_t(\eta) = (L^T p_t)(\eta)$

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Probability state

$|p(t)\rangle = \sum_{\eta \in \Omega} p_t(\eta) |\eta\rangle$ encodes the probability distribution

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Probability state

$|p(t)\rangle = \sum_{\eta\in\Omega} p_t(\eta)|\eta\rangle$ encodes the probability distribution

Master equation

$$\frac{d}{dt}|p(t)\rangle = -H|p(t)\rangle \qquad \boxed{H = -L^T}$$

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Probability state

$|p(t)\rangle = \sum_{\eta\in\Omega} p_t(\eta)|\eta\rangle$ encodes the probability distribution

Master equation

$$\frac{d}{dt}|p(t)\rangle = -H|p(t)\rangle \qquad \boxed{H = -L^T}$$

$$\frac{d}{dt}p_t(\eta)$$

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Probability state

$|p(t)\rangle = \sum_{\eta \in \Omega} p_t(\eta)|\eta\rangle$ encodes the probability distribution

Master equation

$$\frac{d}{dt}|p(t)\rangle = -H|p(t)\rangle \qquad \boxed{H = -L^T}$$

$$\frac{d}{dt}p_t(\eta) = \frac{d}{dt}\langle \eta | p(t)\rangle$$

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Probability state

$|p(t)\rangle = \sum_{\eta\in\Omega} p_t(\eta)|\eta\rangle$ encodes the probability distribution

Master equation

$$\frac{d}{dt}|p(t)\rangle = -H|p(t)\rangle \qquad \boxed{H = -L^T}$$

$$\frac{d}{dt}p_t(\eta) = \frac{d}{dt}\langle\eta|p(t)\rangle = -\langle\eta|H|p(t)\rangle$$

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Probability state

$|p(t)\rangle = \sum_{\eta\in\Omega} p_t(\eta)|\eta\rangle$ encodes the probability distribution

Master equation

$$\frac{d}{dt}|p(t)\rangle = -H|p(t)\rangle \qquad \boxed{H = -L^T}$$

$$\frac{d}{dt}p_t(\eta) = \frac{d}{dt}\langle\eta|p(t)\rangle = -\langle\eta|H|p(t)\rangle = -\sum_{\eta'}\langle\eta|H|\eta'\rangle\langle\eta'|p(t)\rangle$$

Hamiltonian

Quantum-like formalism

$\Omega \ni \eta \to |\eta\rangle$, base for an Hilbert space

Probability state

$|p(t)\rangle = \sum_{\eta\in\Omega} p_t(\eta)|\eta\rangle$ encodes the probability distribution

Master equation

$$\frac{d}{dt}|p(t)\rangle = -H|p(t)\rangle \qquad \boxed{H = -L^T}$$

$$\frac{d}{dt}p_t(\eta) = \frac{d}{dt}\langle\eta|p(t)\rangle = -\langle\eta|H|p(t)\rangle = -\sum_{\eta'}\langle\eta|H|\eta'\rangle\langle\eta'|p(t)\rangle$$
$$= -(Hp_t)(\eta)$$

Duality

Definition

$(\eta_t)_{t\geq 0}$ process on Ω with generator L,
$(\xi_t)_{t\geq 0}$ process on Ω_{dual} with generator $\mathcal{L}_{dual}$

Duality

Definition

$(\eta_t)_{t\geq 0}$ process on Ω with generator L,
$(\xi_t)_{t\geq 0}$ process on Ω_{dual} with generator $\mathcal{L}_{dual}$

η_t is dual to ξ_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall(\eta, \xi) \in \Omega \times \Omega_{dual}$$

Duality

Definition

$(\eta_t)_{t\geq 0}$ process on Ω with generator L,
$(\xi_t)_{t\geq 0}$ process on Ω_{dual} with generator $\mathcal{L}_{dual}$

η_t is dual to ξ_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

More explicitely

$$\sum_{\eta' \in \Omega} D(\eta', \xi) e^{-tL}(\eta, \eta') = \sum_{\xi' \in \Omega_{dual}} D(\eta, \xi') e^{-t\mathcal{L}_{dual}}(\xi, \xi')$$

Duality

Definition

$(\eta_t)_{t\geq 0}$ process on Ω with generator L,
$(\xi_t)_{t\geq 0}$ process on Ω_{dual} with generator $\mathcal{L}_{dual}$

η_t is dual to ξ_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall(\eta, \xi) \in \Omega \times \Omega_{dual}$$

More explicitely

$$\sum_{\eta' \in \Omega} D(\eta', \xi) e^{-tL}(\eta, \eta') = \sum_{\xi' \in \Omega_{dual}} D(\eta, \xi') e^{-t\mathcal{L}_{dual}}(\xi, \xi')$$

$$\boxed{LD = D\mathcal{L}_{dual}^T}$$

Duality

Quantum-like notation

$|\eta(t)\rangle = e^{-tH}|\eta\rangle$ prob. state at time t started from a localized state

Duality

Quantum-like notation

$|\eta(t)\rangle = e^{-tH}|\eta\rangle$ prob. state at time t started from a localized state

Duality:

$$\langle\eta(t)|D|\xi\rangle = \langle\eta|D|\xi(t)\rangle$$

Duality

Quantum-like notation

$|\eta(t)\rangle = e^{-tH}|\eta\rangle$ prob. state at time t started from a localized state

Duality:

$$\langle\eta(t)|D|\xi\rangle = \langle\eta|D|\xi(t)\rangle$$

Indeed

$$\langle\eta|e^{-tH^T}D|\xi\rangle = \langle\eta|De^{-t\mathcal{H}_{dual}}|\xi\rangle$$

Duality

Quantum-like notation

$|\eta(t)\rangle = e^{-tH}|\eta\rangle$ prob. state at time t started from a localized state

Duality:

$$\langle\eta(t)|D|\xi\rangle = \langle\eta|D|\xi(t)\rangle$$

Indeed

$$\langle\eta|e^{-tH^T}D|\xi\rangle = \langle\eta|De^{-t\mathcal{H}_{dual}}|\xi\rangle$$

$$\sum_{\eta'}\langle\eta|e^{-tH^T}|\eta'\rangle\langle\eta'|D|\xi\rangle = \sum_{\xi'}\langle\eta|D|\xi'\rangle\langle\xi'|e^{-t\mathcal{H}_{dual}}|\xi\rangle$$

Duality

Quantum-like notation

$|\eta(t)\rangle = e^{-tH}|\eta\rangle$ prob. state at time t started from a localized state

Duality:

$$\langle \eta(t)|D|\xi\rangle = \langle \eta|D|\xi(t)\rangle$$

Indeed

$$\langle \eta|e^{-tH^T}D|\xi\rangle = \langle \eta|De^{-t\mathcal{H}_{dual}}|\xi\rangle$$

$$\sum_{\eta'}\langle \eta|e^{-tH^T}|\eta'\rangle\langle \eta'|D|\xi\rangle = \sum_{\xi'}\langle \eta|D|\xi'\rangle\langle \xi'|e^{-t\mathcal{H}_{dual}}|\xi\rangle$$

$$\boxed{H^T D = D\mathcal{H}_{dual}}$$

Self-duality

Definition

$(\eta_t)_{t \geq 0}$ process on Ω with generator L,
$(\xi_t)_{t \geq 0}$ process on Ω, indep. copy of η_t

Self-duality

Definition

$(\eta_t)_{t\geq 0}$ process on Ω with generator L,
$(\xi_t)_{t\geq 0}$ process on Ω, indep. copy of η_t

η_t is self-dual with duality function $D : \Omega \times \Omega \to \mathbb{R}$ if

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall(\eta, \xi) \in \Omega \times \Omega$$

Self-duality

Definition

$(\eta_t)_{t\geq 0}$ process on Ω with generator L,
$(\xi_t)_{t\geq 0}$ process on Ω, indep. copy of η_t

η_t is self-dual with duality function $D : \Omega \times \Omega \to \mathbb{R}$ if

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall(\eta, \xi) \in \Omega \times \Omega$$

Equivalent to:

$$LD = DL^T$$

Self-duality

Definition

$(\eta_t)_{t\geq 0}$ process on Ω with generator L,
$(\xi_t)_{t\geq 0}$ process on Ω, indep. copy of η_t

η_t is self-dual with duality function $D : \Omega \times \Omega \to \mathbb{R}$ if

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega$$

Equivalent to:

$$LD = DL^T$$

$$H^T D = DH$$

Self-Duality and Symmetry

Theorem (I)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.

Self-Duality and Symmetry

Theorem (I)

Suppose H *is the Hamiltonian of* $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.

If $[H, S] = 0$ *then* η_t *is self-dual with self-duality fct.* $D = Q^{-1}S$.

Self-Duality and Symmetry

Theorem (I)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.

If $[H, S] = 0$ *then* η_t *is self-dual with self-duality fct.* $D = Q^{-1}S$.

If D is a self-duality fct. then $S = QD$ *is a symmetry of H.*

Self-Duality and Symmetry

Theorem (I)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ and $H^T = Q^{-1}HQ$.
If $[H, S] = 0$ then η_t is self-dual with self-duality fct. $D = Q^{-1}S$.
If D is a self-duality fct. then $S = QD$ is a symmetry of H.

Proof.

$$\langle \eta(t)|D|\xi\rangle$$

Self-Duality and Symmetry

Theorem (I)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.

If $[H,S]=0$ *then* η_t *is self-dual with self-duality fct.* $D = Q^{-1}S$.

If D is a self-duality fct. then $S = QD$ *is a symmetry of H.*

Proof.

$$\langle \eta(t)|D|\xi\rangle \;= \langle \eta|e^{-tH^T}D|\xi\rangle$$

Self-Duality and Symmetry

Theorem (I)

Suppose H *is the Hamiltonian of* $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.

If $[H, S] = 0$ *then* η_t *is self-dual with self-duality fct.* $D = Q^{-1}S$.

If D *is a self-duality fct. then* $S = QD$ *is a symmetry of* H.

Proof.

$$\langle \eta(t)|D|\xi\rangle = \langle \eta|e^{-tH^T}D|\xi\rangle = \langle \eta|e^{-tH^T}Q^{-1}S|\xi\rangle$$

Self-Duality and Symmetry

Theorem (I)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.

If $[H,S]=0$ *then* η_t *is self-dual with self-duality fct.* $D = Q^{-1}S$.

If D is a self-duality fct. then $S = QD$ *is a symmetry of H.*

Proof.

$$\begin{aligned}\langle \eta(t)|D|\xi\rangle &= \langle\eta|e^{-tH^T}D|\xi\rangle = \langle\eta|e^{-tH^T}Q^{-1}S|\xi\rangle \\ &= \langle\eta|Q^{-1}e^{-tH}QQ^{-1}S|\xi\rangle\end{aligned}$$

Self-Duality and Symmetry

Theorem (I)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.
If $[H, S] = 0$ *then* η_t *is self-dual with self-duality fct.* $D = Q^{-1}S$.
If D is a self-duality fct. then $S = QD$ *is a symmetry of H.*

Proof.

$$\begin{aligned}\langle \eta(t)|D|\xi\rangle &= \langle \eta|e^{-tH^T}D|\xi\rangle = \langle \eta|e^{-tH^T}Q^{-1}S|\xi\rangle \\ &= \langle \eta|Q^{-1}e^{-tH}QQ^{-1}S|\xi\rangle = \langle \eta|Q^{-1}Se^{-tH}|\xi\rangle\end{aligned}$$

Self-Duality and Symmetry

Theorem (I)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $H^T = Q^{-1}HQ$.
If $[H,S]=0$ *then* η_t *is self-dual with self-duality fct.* $D = Q^{-1}S$.
If D is a self-duality fct. then $S = QD$ *is a symmetry of H.*

Proof.

$$\begin{aligned}\langle \eta(t)|D|\xi\rangle &= \langle \eta|e^{-tH^T}D|\xi\rangle = \langle \eta|e^{-tH^T}Q^{-1}S|\xi\rangle \\ &= \langle \eta|Q^{-1}e^{-tH}QQ^{-1}S|\xi\rangle = \langle \eta|Q^{-1}Se^{-tH}|\xi\rangle \\ &= \langle \eta|D|\xi(t)\rangle\end{aligned}$$

□

Remarks on similarity

- In finite state space context, H and H^T are always similar.

Remarks on similarity

- In finite state space context, H and H^T are always similar.
- If μ is a reversible measure

$$\mu(\eta)H(\eta',\eta) = \mu(\eta')H(\eta,\eta')$$

then

$$Q(\eta,\eta') = \mu(\eta)\delta_{\eta,\eta'}$$

Remarks on similarity

- In finite state space context, H and H^T are always similar.
- If μ is a reversible measure

$$\mu(\eta)H(\eta',\eta) = \mu(\eta')H(\eta,\eta')$$

then

$$Q(\eta,\eta') = \mu(\eta)\delta_{\eta,\eta'}$$

- If μ is a stationary measure ($\Leftrightarrow H\mu = 0$) then

$$\tilde{H}(\eta,\eta') = H(\eta',\eta)\frac{\mu(\eta)}{\mu(\eta')}$$

is the Hamiltonian of the time reversed process.

Remarks on similarity

- In finite state space context, H and H^T are always similar.
- If μ is a reversible measure

$$\mu(\eta)H(\eta',\eta) = \mu(\eta')H(\eta,\eta')$$

then

$$Q(\eta,\eta') = \mu(\eta)\delta_{\eta,\eta'}$$

- If μ is a stationary measure ($\Leftrightarrow H\mu = 0$) then

$$\tilde{H}(\eta,\eta') = H(\eta',\eta)\frac{\mu(\eta)}{\mu(\eta')}$$

is the Hamiltonian of the time reversed process.

Similarity of H and H^T $\equiv$ Similarity of H and $\tilde{H}$.

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $\mathcal{H}_{dual}$ *is the Hamiltonian of* $(\xi_t)_{t\geq 0}$. *Assume*

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

(General) Duality and Symmetry

Theorem (II)

Suppose H *is the Hamiltonian of* $(\eta_t)_{t\geq 0}$ *and* $\mathcal{H}_{dual}$ *is the Hamiltonian of* $(\xi_t)_{t\geq 0}$. *Assume*

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ *then* η_t *is dual to* ξ_t *with duality fct.* $D = Q^{-1}CS$.

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $\mathcal{H}_{dual}$ *is the Hamiltonian of* $(\xi_t)_{t\geq 0}$. *Assume*

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ *then* η_t *is dual to* ξ_t *with duality fct.* $D = Q^{-1}CS$.
If D is a duality fct. then $S = \tilde{C}QD$ *is a symmetry of* $\mathcal{H}_{dual}$.

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $\mathcal{H}_{dual}$ *is the Hamiltonian of* $(\xi_t)_{t\geq 0}$*. Assume*

1. $H^T = Q^{-1} H Q$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ *then* η_t *is dual to* ξ_t *with duality fct.* $D = Q^{-1}CS$*.*
If D is a duality fct. then $S = \tilde{C}QD$ *is a symmetry of* $\mathcal{H}_{dual}$*.*

Proof.

$$\langle \eta(t) | D | \xi \rangle$$

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ and $\mathcal{H}_{dual}$ is the Hamiltonian of $(\xi_t)_{t\geq 0}$. Assume

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ then η_t is dual to ξ_t with duality fct. $D = Q^{-1}CS$.
If D is a duality fct. then $S = \tilde{C}QD$ is a symmetry of $\mathcal{H}_{dual}$.

Proof.

$$\langle \eta(t)|D|\xi\rangle = \langle \eta|e^{-tH^T}D|\xi\rangle$$

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $\mathcal{H}_{dual}$ *is the Hamiltonian of* $(\xi_t)_{t\geq 0}$*. Assume*

1. $H^T = Q^{-1} H Q$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ *then* η_t *is dual to* ξ_t *with duality fct.* $D = Q^{-1} CS$*.*
If D is a duality fct. then $S = \tilde{C} Q D$ *is a symmetry of* $\mathcal{H}_{dual}$*.*

Proof.

$$\langle \eta(t)|D|\xi\rangle = \langle \eta|e^{-tH^T} D|\xi\rangle = \langle \eta|e^{-tH^T} Q^{-1} CS|\xi\rangle$$

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $\mathcal{H}_{dual}$ *is the Hamiltonian of* $(\xi_t)_{t\geq 0}$. *Assume*

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ *then* η_t *is dual to* ξ_t *with duality fct.* $D = Q^{-1}CS$.
If D is a duality fct. then $S = \tilde{C}QD$ *is a symmetry of* $\mathcal{H}_{dual}$.

Proof.

$$\begin{aligned}\langle \eta(t)|D|\xi\rangle &= \langle\eta|e^{-tH^T}D|\xi\rangle = \langle\eta|e^{-tH^T}Q^{-1}CS|\xi\rangle \\ &= \langle\eta|Q^{-1}e^{-tH}QQ^{-1}CS|\xi\rangle\end{aligned}$$

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ and $\mathcal{H}_{dual}$ is the Hamiltonian of $(\xi_t)_{t\geq 0}$. Assume

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ then η_t is dual to ξ_t with duality fct. $D = Q^{-1}CS$.
If D is a duality fct. then $S = \tilde{C}QD$ is a symmetry of $\mathcal{H}_{dual}$.

Proof.

$$\begin{aligned}\langle \eta(t)|D|\xi\rangle &= \langle \eta|e^{-tH^T}D|\xi\rangle = \langle \eta|e^{-tH^T}Q^{-1}CS|\xi\rangle \\ &= \langle \eta|Q^{-1}e^{-tH}QQ^{-1}CS|\xi\rangle = \langle \eta|Q^{-1}Ce^{-t\mathcal{H}_{dual}}S|\xi\rangle\end{aligned}$$

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ and $\mathcal{H}_{dual}$ is the Hamiltonian of $(\xi_t)_{t\geq 0}$. Assume

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ then η_t is dual to ξ_t with duality fct. $D = Q^{-1}CS$.
If D is a duality fct. then $S = \tilde{C}QD$ is a symmetry of $\mathcal{H}_{dual}$.

Proof.

$$\begin{aligned} \langle \eta(t)|D|\xi\rangle &= \langle \eta|e^{-tH^T}D|\xi\rangle = \langle \eta|e^{-tH^T}Q^{-1}CS|\xi\rangle \\ &= \langle \eta|Q^{-1}e^{-tH}QQ^{-1}CS|\xi\rangle = \langle \eta|Q^{-1}Ce^{-t\mathcal{H}_{dual}}S|\xi\rangle \\ &= \langle \eta|Q^{-1}CSe^{-t\mathcal{H}_{dual}}|\xi\rangle \end{aligned}$$

(General) Duality and Symmetry

Theorem (II)

Suppose H is the Hamiltonian of $(\eta_t)_{t\geq 0}$ *and* $\mathcal{H}_{dual}$ *is the Hamiltonian of* $(\xi_t)_{t\geq 0}$*. Assume*

1. $H^T = Q^{-1}HQ$ *(similarity)*
2. $HC = C\mathcal{H}_{dual}$ *(conjugacy)*

If $[\mathcal{H}_{dual}, S] = 0$ *then* η_t *is dual to* ξ_t *with duality fct.* $D = Q^{-1}CS$*.*
If D is a duality fct. then $S = \tilde{C}QD$ *is a symmetry of* $\mathcal{H}_{dual}$*.*

Proof.

$$\begin{aligned}\langle \eta(t)|D|\xi\rangle &= \langle \eta|e^{-tH^T}D|\xi\rangle = \langle \eta|e^{-tH^T}Q^{-1}CS|\xi\rangle \\ &= \langle \eta|Q^{-1}e^{-tH}QQ^{-1}CS|\xi\rangle = \langle \eta|Q^{-1}Ce^{-t\mathcal{H}_{dual}}S|\xi\rangle \\ &= \langle \eta|Q^{-1}CSe^{-t\mathcal{H}_{dual}}|\xi\rangle = \langle \eta|D|\xi(t)\rangle\end{aligned}$$

□

Exclusion processes j-SEP

State space

For $j \in \mathbb{N}/2 \quad \Omega = \times_{i=1}^{N} \Omega_i = \{0, 1, \ldots, 2j\}^N$

Exclusion processes j-SEP

State space

For $j \in \mathbb{N}/2 \quad \Omega = \times_{i=1}^{N} \Omega_i = \{0, 1, \ldots, 2j\}^N \ni \eta = (\eta_1, \ldots, \eta_N)$

Exclusion processes j-SEP

State space

For $j \in \mathbb{N}/2 \quad \Omega = \times_{i=1}^{N} \Omega_i = \{0, 1, \ldots, 2j\}^N \ni \eta = (\eta_1, \ldots, \eta_N)$

$\eta^{i,l}$ = config. obtained by removing a particle at i and adding at l

Exclusion processes j-SEP

State space

For $j \in \mathbb{N}/2 \quad \Omega = \times_{i=1}^{N} \Omega_i = \{0, 1, \ldots, 2j\}^N \ni \eta = (\eta_1, \ldots, \eta_N)$

$\eta^{i,l}$ = config. obtained by removing a particle at i and adding at l

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

Exclusion processes j-SEP

State space

For $j \in \mathbb{N}/2 \quad \Omega = \times_{i=1}^{N} \Omega_i = \{0, 1, \ldots, 2j\}^N \ni \eta = (\eta_1, \ldots, \eta_N)$

$\eta^{i,l}$ = config. obtained by removing a particle at i and adding at l

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

$$(L_{i,i+1} f)(\eta) = \eta_i (2j - \eta_{i+1})[f(\eta^{i,i+1}) - f(\eta)] \qquad \text{Bulk}$$
$$+ (2j - \eta_i)\eta_{i+1}[f(\eta^{i+1,i}) - f(\eta)]$$

Exclusion processes j-SEP

State space

For $j \in \mathbb{N}/2 \quad \Omega = \times_{i=1}^{N} \Omega_i = \{0, 1, \ldots, 2j\}^N \ni \eta = (\eta_1, \ldots, \eta_N)$

$\eta^{i,l}$ = config. obtained by removing a particle at i and adding at l

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

$$\begin{aligned}(L_{i,i+1} f)(\eta) &= \eta_i (2j - \eta_{i+1})[f(\eta^{i,i+1}) - f(\eta)] \qquad \text{Bulk} \\ &+ (2j - \eta_i)\eta_{i+1}[f(\eta^{i+1,i}) - f(\eta)]\end{aligned}$$

$$\begin{aligned}(L_1 f)(\eta) &= 2j\rho_1 (2j - \eta_1)[f(\eta^{0,1}) - f(\eta)] \qquad \text{Reservoir} \\ &+ (2j - 2j\rho_1)\eta_1 [f(\eta^{1,0}) - f(\eta)]\end{aligned}$$

Remarks

- If $j = 1/2$ then $(\eta_t)_{t\geq 0}$ is the standard boundary driven symmetric exclusion process (SEP).

Remarks

- If $j = 1/2$ then $(\eta_t)_{t\geq 0}$ is the standard boundary driven symmetric exclusion process (SEP).
- If $j > 1/2$ then $(\eta_t)_{t\geq 0}$ can be seen as the boundary driven SEP on a ladder lattice with $2j$ levels at each site

$$\eta_i(t) = \sum_{\alpha=1}^{2j} \tilde{\eta}_{\alpha,i}(t) \qquad \text{with} \quad \tilde{\eta}_{\alpha,i}(t) \quad \text{standard SEP}$$

Remarks

- If $j = 1/2$ then $(\eta_t)_{t\geq 0}$ is the standard boundary driven symmetric exclusion process (SEP).
- If $j > 1/2$ then $(\eta_t)_{t\geq 0}$ can be seen as the boundary driven SEP on a ladder lattice with $2j$ levels at each site

$$\eta_i(t) = \sum_{\alpha=1}^{2j} \tilde{\eta}_{\alpha,i}(t) \qquad \text{with} \quad \tilde{\eta}_{\alpha,i}(t) \quad \text{standard SEP}$$

- If $j \to \infty$ and $\rho \to 0$ (with ρj finite) then $\eta_{t/2j} \to \eta_t^{irw}$ system of independent random walkers.

Remarks

- If $j = 1/2$ then $(\eta_t)_{t\geq 0}$ is the standard boundary driven symmetric exclusion process (SEP).
- If $j > 1/2$ then $(\eta_t)_{t\geq 0}$ can be seen as the boundary driven SEP on a ladder lattice with $2j$ levels at each site

$$\eta_i(t) = \sum_{\alpha=1}^{2j} \tilde{\eta}_{\alpha,i}(t) \qquad \text{with} \quad \tilde{\eta}_{\alpha,i}(t) \quad \text{standard SEP}$$

- If $j \to \infty$ and $\rho \to 0$ (with ρj finite) then $\eta_{t/2j} \to \eta_t^{irw}$ system of independent random walkers.
- If $\rho_1 = \rho_N = \rho$ then the invariant measure is

$$\nu_{j,\rho} = \otimes_{i=1}^{N} \text{Bin}(2j, \rho)$$

SU(2) structure

SU(2) group

$$[J_i^o, J_i^{\pm}] = \pm J_i^{\pm} \qquad [J_i^-, J_i^+] = -2J_i^o$$

SU(2) structure

SU(2) group

$$[J_i^o, J_i^\pm] = \pm J_i^\pm \qquad [J_i^-, J_i^+] = -2J_i^o$$

Representation

$$J_i^+|\eta_i\rangle = (2j-\eta_i)|\eta_i+1\rangle$$
$$J_i^-|\eta_i\rangle = \eta_i|\eta_i-1\rangle$$
$$J_i^o|\eta_i\rangle = (\eta_i-j)|\eta_i\rangle$$

SU(2) structure

SU(2) group

$$[J_i^o, J_i^\pm] = \pm J_i^\pm \qquad [J_i^-, J_i^+] = -2J_i^o$$

Representation

$$J_i^+|\eta_i\rangle = (2j - \eta_i)|\eta_i + 1\rangle$$
$$J_i^-|\eta_i\rangle = \eta_i|\eta_i - 1\rangle$$
$$J_i^o|\eta_i\rangle = (\eta_i - j)|\eta_i\rangle$$

In a base $\{|0\rangle_i, |1\rangle_i, \ldots, |2j\rangle_i\}$ we have

$$J_i^+ = \begin{pmatrix} 0 & & & \\ 2j & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \qquad J_i^- = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 2j \\ & & & 0 \end{pmatrix} \qquad J_i^o = \begin{pmatrix} -j & & & \\ & \ddots & & \\ & & \ddots & \\ & & & j \end{pmatrix}$$

SEP and SU(2) ferromagnet

Quantum spin chain

The j-SEP generator can be read as the SU(2) ferromagnet:

$$-L^T = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

SEP and SU(2) ferromagnet

Quantum spin chain

The j-SEP generator can be read as the SU(2) ferromagnet:

$$-L^T = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

1. Bulk

$$-H_{i,i+1} = 2\vec{J}_i \cdot \vec{J}_{i+1} - 2j^2$$

SEP and SU(2) ferromagnet

Quantum spin chain

The j-SEP generator can be read as the SU(2) ferromagnet:

$$-L^T = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

1. Bulk

$$\begin{aligned} -H_{i,i+1} &= 2\vec{J}_i \cdot \vec{J}_{i+1} - 2j^2 \\ &= J_i^+ J_{i+1}^- + J_i^- J_{i+1}^+ + 2J_i^o J_{i+1}^o - 2j^2 \end{aligned}$$

SEP and SU(2) ferromagnet

Quantum spin chain

The j-SEP generator can be read as the SU(2) ferromagnet:

$$-L^T = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

1. Bulk

$$\begin{aligned} -H_{i,i+1} &= 2\vec{J}_i \cdot \vec{J}_{i+1} - 2j^2 \\ &= J_i^+ J_{i+1}^- + J_i^- J_{i+1}^+ + 2J_i^o J_{i+1}^o - 2j^2 \end{aligned}$$

Note: from the group structure we deduce symmetries

$$[H_{bulk}, J^o] = 0 \qquad [H_{bulk}, J^+] = 0 \qquad [H_{bulk}, J^-] = 0$$

SEP and SU(2) ferromagnet

Quantum spin chain

The j-SEP generator can be read as the SU(2) ferromagnet:

$$-L^T = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

1. Bulk

$$-H_{i,i+1} = 2\vec{J}_i \cdot \vec{J}_{i+1} - 2j^2$$
$$= J_i^+ J_{i+1}^- + J_i^- J_{i+1}^+ + 2J_i^o J_{i+1}^o - 2j^2$$

Note: from the group structure we deduce symmetries

$$[H_{bulk}, J^o] = 0 \qquad [H_{bulk}, J^+] = 0 \qquad [H_{bulk}, J^-] = 0$$

2. Reservoir

$$-H_1 = 2j\rho_1(J_1^+ + J_1^o - j) + (2j - 2j\rho_1)(J_1^- - J_1^o - j)$$

Self-Duality for j-SEP

Theorem

1. *The bulk j-SEP is self-dual with self-duality fct.*

$$D(\eta,\xi) = \prod_{i=1}^{N} D_i(\eta_i,\xi_i) =$$

Self-Duality for j-SEP

Theorem

1. *The bulk j-SEP is self-dual with self-duality fct.*

$$D(\eta,\xi)=\prod_{i=1}^{N} D_i(\eta_i,\xi_i)=\prod_{i=1}^{N}\frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}}=\prod_{i=1}^{N}\frac{\eta_i(\eta_i-1)\cdots(\eta_i-\xi_i+1)}{2j(2j-1)\cdots(2j-\xi_i+1)}$$

Self-Duality for j-SEP

Theorem

1. *The bulk j-SEP is self-dual with self-duality fct.*

$$D(\eta,\xi)=\prod_{i=1}^{N} D_i(\eta_i,\xi_i)=\prod_{i=1}^{N}\frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}}=\prod_{i=1}^{N}\frac{\eta_i(\eta_i-1)\cdots(\eta_i-\xi_i+1)}{2j(2j-1)\cdots(2j-\xi_i+1)}$$

2. *The boundary driven j-SEP is dual to the process $\vec{\xi}(t)=(\xi_0(t),\xi(t),\xi_{N+1}(t))$ with generator $\mathcal{L}=\mathcal{L}_1+\sum_i L_{i,i+1}+\mathcal{L}_N$*

$$(\mathcal{L}_1 f)(\vec{\xi})=\xi_1(f(\xi^{1,0})-f(\xi)) \qquad D(\eta,\vec{\xi})=\rho_1^{\xi_0}\prod_{i=1}^{N} D_i(\eta_i,\xi_i)\,\rho_N^{\xi_{N+1}}$$

Proof.

1. By theorem (I), $D_i = Q_i^{-1} S_i$

Proof.

1. By theorem (I), $D_i = Q_i^{-1} S_i$
   - Similarity $H^T = Q^{-1} H Q$: detailed balance

$$Q_i(\eta_i, \eta_i') = \mu(\eta_i)\delta_{\eta_i,\eta_i'} = \binom{2j}{\eta_i}\delta_{\eta_i,\eta_i'}$$

Proof.

1. By theorem (I), $D_i = Q_i^{-1} S_i$
   - Similarity $H^T = Q^{-1} H Q$: detailed balance
   $$Q_i(\eta_i, \eta_i') = \mu(\eta_i)\delta_{\eta_i,\eta_i'} = \binom{2j}{\eta_i}\delta_{\eta_i,\eta_i'}$$
   - Symmetry $[H_{bulk}, S] = 0$: group structure
   $$S_i = e^{J_i^+} \qquad \Rightarrow \qquad S_i(\eta_i, \xi_i) = \binom{2j-\xi_i}{\eta_i-\xi_i}$$

Proof.

1. By theorem (I), $D_i = Q_i^{-1} S_i$
   - Similarity $H^T = Q^{-1} H Q$: detailed balance

     $$Q_i(\eta_i, \eta_i') = \mu(\eta_i)\delta_{\eta_i,\eta_i'} = \binom{2j}{\eta_i}\delta_{\eta_i,\eta_i'}$$

   - Symmetry $[H_{bulk}, S] = 0$: group structure

     $$S_i = e^{J_i^+} \qquad \Rightarrow \qquad S_i(\eta_i, \xi_i) = \binom{2j - \xi_i}{\eta_i - \xi_i}$$

   - Combining

     $$D_i(\eta_i, \xi_i) = Q_i^{-1}(\eta_i, \eta_i) S_i(\eta_i, \xi_i) = \frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}}$$

Proof.

1. By theorem (I), $D_i = Q_i^{-1} S_i$
   - Similarity $H^T = Q^{-1} H Q$: detailed balance
   $$Q_i(\eta_i, \eta_i') = \mu(\eta_i)\delta_{\eta_i,\eta_i'} = \binom{2j}{\eta_i}\delta_{\eta_i,\eta_i'}$$
   - Symmetry $[H_{bulk}, S] = 0$: group structure
   $$S_i = e^{J_i^+} \qquad \Rightarrow \qquad S_i(\eta_i, \xi_i) = \binom{2j-\xi_i}{\eta_i - \xi_i}$$
   - Combining
   $$D_i(\eta_i, \xi_i) = Q_i^{-1}(\eta_i, \eta_i) S_i(\eta_i, \xi_i) = \frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}}$$
2. Follows from a direct computation: $L_1 D = D\mathcal{L}_1^T$
   Remark: Boundaries ξ_0 and ξ_{N+1} are absorbing!

□

Brownian energy process

Definition

For two sites i and j consider the generator $L_{i,j} : \mathcal{C}^\infty(\mathbb{R}^2) \to \mathcal{C}^\infty(\mathbb{R}^2)$

$$(L_{i,j}f)(x_i, x_j) = \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right)^2 f(x_i, x_j)$$

Brownian energy process

Definition

For two sites i and j consider the generator $L_{i,j} : \mathcal{C}^\infty(\mathbb{R}^2) \to \mathcal{C}^\infty(\mathbb{R}^2)$

$$(L_{i,j}f)(x_i, x_j) = \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 f(x_i, x_j)$$

If we think of (x_i, x_j) as velocities, then

- polar coordinates $$L_{i,j} = \frac{\partial^2}{\partial \theta_{ij}^2}$$
- generates a Brownian motion of the angle $\theta_{i,j} = \arctan(x_j/x_i)$
- conserves the total (kinetic) energy $r_{i,j}^2 = x_i^2 + x_j^2$

Brownian energy process

State space

$\Omega = \times_{i=1}^{N} \Omega_i = \mathbb{R}^N$

Brownian energy process

State space

$$\Omega = \times_{i=1}^{N} \Omega_i = \mathbb{R}^N \ni x = (x_1, \ldots, x_N)$$

Brownian energy process

State space

$\Omega = \times_{i=1}^{N} \Omega_i = \mathbb{R}^N \ni x = (x_1, \ldots, x_N)$

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

Brownian energy process

State space

$\Omega = \times_{i=1}^{N} \Omega_i = \mathbb{R}^N \ni x = (x_1, \ldots, x_N)$

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

$$(L_{i,i+1} f)(x) = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 f(x) \qquad \text{Bulk}$$

Brownian energy process

State space

$\Omega = \times_{i=1}^{N} \Omega_i = \mathbb{R}^N \ni x = (x_1, \ldots, x_N)$

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

$$(L_{i,i+1} f)(x) = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 f(x) \qquad \text{Bulk}$$

$$(L_1 f)(x) = \left(T_1 \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \right) f(x) \qquad \text{Reservoir}$$

Brownian energy process

State space

$\Omega = \times_{i=1}^{N} \Omega_i = \mathbb{R}^N \ni x = (x_1, \ldots, x_N)$

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

$$(L_{i,i+1} f)(x) = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 f(x) \qquad \text{Bulk}$$

$$(L_1 f)(x) = \left(T_1 \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \right) f(x) \qquad \text{Reservoir}$$

Equilibrium

If $T_1 = T_N = T$ then the stationary measure is $\nu_T = \otimes_{i=1}^{N} \mathcal{N}(0, T)$

SU(1,1) structure

Definition

$$K_i^+ = \frac{1}{2} x_i^2 \qquad K_i^- = \frac{1}{2} \frac{\partial^2}{\partial x_i^2}$$

$$K_i^o = \frac{1}{4} \left(x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} x_i \right)$$

SU(1,1) structure

Definition

$$K_i^+ = \frac{1}{2} x_i^2 \qquad K_i^- = \frac{1}{2} \frac{\partial^2}{\partial x_i^2}$$

$$K_i^o = \frac{1}{4}\left(x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} x_i \right)$$

SU(1,1) group

$$[K_i^o, K_i^\pm] = \pm K_i^\pm \qquad [K_i^-, K_i^+] = 2K_i^o$$

SU(1,1) structure

Definition

$$K_i^+ = \frac{1}{2} x_i^2 \qquad K_i^- = \frac{1}{2} \frac{\partial^2}{\partial x_i^2}$$

$$K_i^o = \frac{1}{4}\left(x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} x_i \right)$$

SU(1,1) group

$$[K_i^o, K_i^\pm] = \pm K_i^\pm \qquad [K_i^-, K_i^+] = 2K_i^o$$

Representation

$$\vec{K}_i^2 = \frac{1}{2}(K_i^+ K_i^- + K_i^- K_i^+) - (K_i^o)^2 \qquad \vec{K}_i^2 |k, k_z >_i = k(k-1)|k, k_z\rangle_i$$

In our case $\vec{K}_i^2 |-\rangle_i = \frac{1}{4}\left(\frac{1}{4} - 1\right)|-\rangle_i \qquad \Rightarrow \qquad k = \frac{1}{4}$

BEP and SU(1,1) ferromagnet

Quantum spin chain

The BEP generator can be read as the SU(1,1) ferromagnet:

$$-L^{\dagger} = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

BEP and SU(1,1) ferromagnet

Quantum spin chain

The BEP generator can be read as the SU(1,1) ferromagnet:

$$-L^{\dagger} = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

1. Bulk

$$H_{i,i+1} = -4\left(2\vec{K}_i \cdot \vec{K}_{i+1} + \frac{1}{8}\right)$$

BEP and SU(1,1) ferromagnet

Quantum spin chain

The BEP generator can be read as the SU(1,1) ferromagnet:

$$-L^\dagger = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

1. Bulk

$$H_{i,i+1} = -4\left(2\vec{K}_i \cdot \vec{K}_{i+1} + \frac{1}{8}\right)$$
$$= -4\left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + \frac{1}{8}\right)$$

BEP and SU(1,1) ferromagnet

Quantum spin chain

The BEP generator can be read as the SU(1,1) ferromagnet:

$$-L^{\dagger} = H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

1. Bulk

$$H_{i,i+1} = -4\left(2\vec{K}_i \cdot \vec{K}_{i+1} + \frac{1}{8}\right)$$
$$= -4\left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + \frac{1}{8}\right)$$

2. Reservoir

$$H_1 = 2\left(T_1 K_1^- + K_1^o + \frac{1}{4}\right)$$

Origin of duality for BEP

Discrete representation

$$\mathcal{K}_i^+|\xi_i\rangle = (\xi_i + \frac{1}{2})|\xi_i + 1\rangle$$
$$\mathcal{K}_i^-|\xi_i\rangle = \xi_i|\xi_i - 1\rangle$$
$$\mathcal{K}_i^o|\xi_i\rangle = (\xi_i + \frac{1}{4})|\xi_i\rangle$$

Origin of duality for BEP

Discrete representation

$$\mathcal{K}_i^{+}|\xi_i\rangle = (\xi_i + \frac{1}{2})|\xi_i + 1\rangle$$
$$\mathcal{K}_i^{-}|\xi_i\rangle = \xi_i|\xi_i - 1\rangle$$
$$\mathcal{K}_i^{o}|\xi_i\rangle = (\xi_i + \frac{1}{4})|\xi_i\rangle$$

In a base $\{|0\rangle_i, |1\rangle_i, \ldots\}$ we have

$$\mathcal{K}_i^{+} = \begin{pmatrix} 0 & & & \\ \frac{1}{2} & \ddots & & \\ & \frac{3}{2} & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad \mathcal{K}_i^{-} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad \mathcal{K}_i^{0} = \begin{pmatrix} \frac{1}{4} & 0 & & \\ & \frac{5}{4} & \ddots & \\ & & \frac{9}{4} & \ddots \\ & & & \ddots \end{pmatrix}$$

Dual process of BEP

State space

$$\Omega_{dual} = \times_{i=1}^{N} \Omega_i^{dual} = \{0, 1, 2, \ldots\}^N$$

Dual process of BEP

State space

$$\Omega_{dual} = \times_{i=1}^{N} \Omega_i^{dual} = \{0, 1, 2, \ldots\}^N \ni \xi = (\xi_1, \ldots, \xi_N)$$

Dual process of BEP

State space

$$\Omega_{dual} = \times_{i=1}^{N} \Omega_i^{dual} = \{0, 1, 2, \ldots\}^N \ni \xi = (\xi_1, \ldots, \xi_N)$$

Hamiltonian

$$\mathcal{H}_{i,i+1}^{dual} = -4 \left(\mathcal{K}_i^+ \mathcal{K}_{i+1}^- + \mathcal{K}_i^- \mathcal{K}_{i+1}^+ - 2\mathcal{K}_i^o \mathcal{K}_{i+1}^o + \frac{1}{8} \right)$$

Dual process of BEP

State space

$$\Omega_{dual} = \times_{i=1}^{N} \Omega_i^{dual} = \{0,1,2,\ldots\}^N \ni \xi = (\xi_1,\ldots,\xi_N)$$

Hamiltonian

$$\mathcal{H}_{i,i+1}^{dual} = -4\left(\mathcal{K}_i^+\mathcal{K}_{i+1}^- + \mathcal{K}_i^-\mathcal{K}_{i+1}^+ - 2\mathcal{K}_i^o\mathcal{K}_{i+1}^o + \frac{1}{8}\right)$$

Generator

$$\begin{aligned}(\mathcal{L}_{i,i+1}^{dual} f)(\xi) &= 2\xi_i(2\xi_{i+1}+1)[f(\xi^{i,i+1}) - f(\xi)] \\ &+ (2\xi_i+1)2\xi_{i+1}[f(\xi^{i+1,i}) - f(\xi)]\end{aligned}$$

Duality for BEP

Theorem

1. *The bulk BEP* $(x(t))_{t\geq 0}$ *with generator* $L = -H^{\dagger}$ *is dual to to the process* $(\xi(t))_{t\geq 0}$ *with generator* $\mathcal{L}_{dual} = -\mathcal{H}^{T}_{dual}$ *with duality fct.*

$$D(x,\xi) = \prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

Duality for BEP

Theorem

1. *The bulk BEP* $(x(t))_{t\geq 0}$ *with generator* $L = -H^{\dagger}$ *is dual to to the process* $(\xi(t))_{t\geq 0}$ *with generator* $\mathcal{L}_{dual} = -\mathcal{H}^{T}_{dual}$ *with duality fct.*

$$D(x,\xi) = \prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

2. *The boundary driven BEP is dual to the process* $\vec{\xi}(t) = (\xi_0(t), \xi(t), \xi_{N+1}(t))$ *with generator* $\mathcal{L}_{dual} = \mathcal{L}_1 + \sum_i \mathcal{L}_{i,i+1} + \mathcal{L}_N$

$$(\mathcal{L}_1 f)(\vec{\xi}) = 2\xi_1(f(\xi^{1,0}) - f(\xi)) \qquad D(x,\vec{\xi}) = T_1^{\xi_0} D(x,\xi) T_N^{\xi_{N+1}}$$

Proof.

1. By theorem (II), $D = Q^{-1}CS$

Proof.

1. By theorem (II), $D = Q^{-1}CS$
   - Similarity $H^\dagger = Q^{-1}HQ$: H is self-adjoint $\Rightarrow Q = Id$

Proof.

1. By theorem (II), $D = Q^{-1}CS$
   - Similarity $H^{\dagger} = Q^{-1}HQ$: H is self-adjoint $\Rightarrow Q = Id$
   - Conjugancy $HC = C\mathcal{H}_{dual}$
     It is enough conjugancy of the K_i^a and $\mathcal{K}_i^a$ with $a \in \{+, -, o\}$

Proof.

1. By theorem (II), $D = Q^{-1} C S$
   - Similarity $H^{\dagger} = Q^{-1} H Q$: H is self-adjoint $\Rightarrow Q = Id$
   - Conjugancy $HC = C\mathcal{H}_{dual}$
     It is enough conjugancy of the K_i^a and $\mathcal{K}_i^a$ with $a \in \{+,-,o\}$

$$\langle x_i | K_i^+ C_i | \xi_i \rangle = \langle x_i | C_i \mathcal{K}_i^+ | \xi_i \rangle$$

$$\int dy \langle x_i | K_i^+ | y_i \rangle \langle y_i | C_i | \xi_i \rangle = \sum_{\xi_i'} \langle x_i | C_i | \xi_i' \rangle \langle \xi_i' | \mathcal{K}_i^+ | \xi_i \rangle$$

$$\frac{x_i^2}{2} C(x_i, \xi_i) = (\xi_i + \frac{1}{2}) C(x_i, \xi_i + 1) \qquad \Rightarrow \qquad C(x_i, \xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

Proof.

1. By theorem (II), $D = Q^{-1}CS$
   - Similarity $H^\dagger = Q^{-1}HQ$: H is self-adjoint $\Rightarrow Q = Id$
   - Conjugancy $HC = C\mathcal{H}_{dual}$
     It is enough conjugancy of the K_i^a and $\mathcal{K}_i^a$ with $a \in \{+,-,o\}$

$$\langle x_i | K_i^+ C_i | \xi_i \rangle = \langle x_i | C_i \mathcal{K}_i^+ | \xi_i \rangle$$

$$\int dy \langle x_i | K_i^+ | y_i \rangle \langle y_i | C_i | \xi_i \rangle = \sum_{\xi_i'} \langle x_i | C_i | \xi_i' \rangle \langle \xi_i' | \mathcal{K}_i^+ | \xi_i \rangle$$

$$\frac{x_i^2}{2} C(x_i, \xi_i) = (\xi_i + \frac{1}{2}) C(x_i, \xi_i + 1) \qquad \Rightarrow \qquad C(x_i, \xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

   - Symmetry: $S = Id$

Proof.

1. By theorem (II), $D = Q^{-1}CS$
   - Similarity $H^\dagger = Q^{-1}HQ$: H is self-adjoint $\Rightarrow Q = Id$
   - Conjugancy $HC = C\mathcal{H}_{dual}$
     It is enough conjugancy of the K_i^a and $\mathcal{K}_i^a$ with $a \in \{+,-,o\}$

$$\langle x_i|K_i^+ C_i|\xi_i\rangle = \langle x_i|C_i\mathcal{K}_i^+|\xi_i\rangle$$

$$\int dy \langle x_i|K_i^+|y_i\rangle\langle y_i|C_i|\xi_i\rangle = \sum_{\xi_i'}\langle x_i|C_i|\xi_i'\rangle\langle\xi_i'|\mathcal{K}_i^+|\xi_i\rangle$$

$$\frac{x_i^2}{2}C(x_i,\xi_i) = (\xi_i + \frac{1}{2})C(x_i,\xi_i+1) \qquad \Rightarrow \qquad C(x_i,\xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i-1)!!}$$

   - Symmetry: $S = Id$
   - Combining: $D_i(x_i,\xi_i) = C_i(x_i,\xi_i)$

□

Generalization of BEP

For $k \in \mathbb{N}/4$ we consider the BEP on a ladder lattice with $4k$ levels

Generalization of BEP

For $k \in \mathbb{N}/4$ we consider the BEP on a ladder lattice with $4k$ levels

- Generator

$$L = L_1 + \sum_{i=1}^{N-1} \frac{1}{4k} \sum_{\alpha,\beta=1}^{4k} L_{(i,\alpha),(i+1,\beta)} + L_N$$

Generalization of BEP

For $k \in \mathbb{N}/4$ we consider the BEP on a ladder lattice with $4k$ levels

- Generator

$$L = L_1 + \sum_{i=1}^{N-1} \frac{1}{4k} \sum_{\alpha,\beta=1}^{4k} L_{(i,\alpha),(i+1,\beta)} + L_N$$

$$(L_{(i,\alpha),(i+1,\beta)} f)(x) = \left(x_{i,\alpha} \frac{\partial}{\partial x_{i+1,\beta}} - x_{i+1,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 f(x) \qquad \text{Bulk}$$

Generalization of BEP

For $k \in \mathbb{N}/4$ we consider the BEP on a ladder lattice with $4k$ levels

- Generator

$$L = L_1 + \sum_{i=1}^{N-1} \frac{1}{4k} \sum_{\alpha,\beta=1}^{4k} L_{(i,\alpha),(i+1,\beta)} + L_N$$

$$(L_{(i,\alpha),(i+1,\beta)} f)(x) = \left(x_{i,\alpha} \frac{\partial}{\partial x_{i+1,\beta}} - x_{i+1,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 f(x) \qquad \text{Bulk}$$

$$(L_1 f)(x) = \sum_{\alpha=1}^{4k} \left(T_1 \frac{\partial^2}{\partial x_{1,\alpha}^2} - x_{1,\alpha} \frac{\partial}{\partial x_{1,\alpha}} \right) f(x) \qquad \text{Reservoir}$$

Generalization of BEP

For $k \in \mathbb{N}/4$ we consider the BEP on a ladder lattice with $4k$ levels

- Generator

$$L = L_1 + \sum_{i=1}^{N-1} \frac{1}{4k} \sum_{\alpha,\beta=1}^{4k} L_{(i,\alpha),(i+1,\beta)} + L_N$$

$$(L_{(i,\alpha),(i+1,\beta)} f)(x) = \left(x_{i,\alpha} \frac{\partial}{\partial x_{i+1,\beta}} - x_{i+1,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 f(x) \qquad \text{Bulk}$$

$$(L_1 f)(x) = \sum_{\alpha=1}^{4k} \left(T_1 \frac{\partial^2}{\partial x_{1,\alpha}^2} - x_{1,\alpha} \frac{\partial}{\partial x_{1,\alpha}} \right) f(x) \qquad \text{Reservoir}$$

- Define new process $(z(t))_{t \geq 0}$ as $z_i(t) = \sum_{\alpha=1}^{4k} x_{i,\alpha}^2(t)$

Generalization of BEP

For $k \in \mathbb{N}/4$ we consider the BEP on a ladder lattice with $4k$ levels

- Generator

$$L = L_1 + \sum_{i=1}^{N-1} \frac{1}{4k} \sum_{\alpha,\beta=1}^{4k} L_{(i,\alpha),(i+1,\beta)} + L_N$$

$$(L_{(i,\alpha),(i+1,\beta)} f)(x) = \left(x_{i,\alpha} \frac{\partial}{\partial x_{i+1,\beta}} - x_{i+1,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 f(x) \qquad \text{Bulk}$$

$$(L_1 f)(x) = \sum_{\alpha=1}^{4k} \left(T_1 \frac{\partial^2}{\partial x_{1,\alpha}^2} - x_{1,\alpha} \frac{\partial}{\partial x_{1,\alpha}} \right) f(x) \qquad \text{Reservoir}$$

- Define new process $(z(t))_{t\geq 0}$ as $z_i(t) = \sum_{\alpha=1}^{4k} x_{i,\alpha}^2(t)$
- If $T_1 = T_N = T$ then invariant measure is $\nu_{k,T} = \otimes_{i=1}^{N} \chi_{4k}^2(T)$

Brownian energy processes k-BEP

State space

$$\Omega = \otimes_{i=1}^{N} \Omega_i = \mathbb{R}_{+}^{N}$$

Brownian energy processes k-BEP

State space

$$\Omega = \otimes_{i=1}^{N} \Omega_i = \mathbb{R}_+^N \ni z = (z_1, \ldots, z_N)$$

Brownian energy processes k-BEP

State space

$\Omega = \otimes_{i=1}^{N} \Omega_i = \mathbb{R}_+^N \ni z = (z_1, \ldots, z_N)$

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

Brownian energy processes k-BEP

State space

$\Omega = \otimes_{i=1}^{N} \Omega_i = \mathbb{R}_+^N \ni z = (z_1, \ldots, z_N)$

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

$$L_{i,i+1} = \frac{1}{k} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2 - 2(z_i - z_{i+1}) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right) \quad \text{Bulk}$$

Brownian energy processes k-BEP

State space

$\Omega = \otimes_{i=1}^{N} \Omega_i = \mathbb{R}_+^N \ni z = (z_1, \ldots, z_N)$

Generator

$$L = L_1 + \sum_{i=1}^{N-1} L_{i,i+1} + L_N$$

$$L_{i,i+1} = \frac{1}{k} z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2 - 2(z_i - z_{i+1}) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right) \quad \text{Bulk}$$

$$L_1 = 2T_1 \left(k \frac{\partial}{\partial z_1} + 2z_1 \frac{\partial^2}{\partial z_1^2} \right) - 2z_1 \frac{\partial}{\partial z_1} \qquad \text{Reservoir}$$

Hamiltonian k-BEP

$$H = \sum_i \frac{1}{k} \left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2 \right)$$

Hamiltonian k-BEP

$$H = \sum_i \frac{1}{k} \left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2 \right)$$

$$K_i^+ = \sum_\alpha K_{i,\alpha}^+ = \frac{1}{2} z_i$$

$$K_i^- = \sum_\alpha K_{i,\alpha}^- = z_i \frac{\partial^2}{\partial z_i^2} + 4k \frac{\partial}{\partial z_i}$$

$$K_i^o = \sum_\alpha K_{i,\alpha}^o = z_i \frac{\partial}{\partial z_i} + 4k$$

Hamiltonian k-BEP

$$H = \sum_i \frac{1}{k}\left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2\right)$$

$$K_i^+ = \sum_\alpha K_{i,\alpha}^+ = \frac{1}{2} z_i$$

$$K_i^- = \sum_\alpha K_{i,\alpha}^- = z_i \frac{\partial^2}{\partial z_i^2} + 4k \frac{\partial}{\partial z_i}$$

$$K_i^o = \sum_\alpha K_{i,\alpha}^o = z_i \frac{\partial}{\partial z_i} + 4k$$

$$\mathcal{H}_{dual} = \sum_i \frac{1}{k}\left(\mathcal{K}_i^+ \mathcal{K}_{i+1}^- + \mathcal{K}_i^- \mathcal{K}_{i+1}^+ - 2\mathcal{K}_i^o \mathcal{K}_{i+1}^o + 2k^2\right)$$

Hamiltonian k-BEP

$$H = \sum_i \frac{1}{k}\left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2\right)$$

$$K_i^+ = \sum_\alpha K_{i,\alpha}^+ = \frac{1}{2} z_i$$

$$K_i^- = \sum_\alpha K_{i,\alpha}^- = z_i \frac{\partial^2}{\partial z_i^2} + 4k \frac{\partial}{\partial z_i}$$

$$K_i^o = \sum_\alpha K_{i,\alpha}^o = z_i \frac{\partial}{\partial z_i} + 4k$$

$$\mathcal{H}_{dual} = \sum_i \frac{1}{k}\left(\mathcal{K}_i^+ \mathcal{K}_{i+1}^- + \mathcal{K}_i^- \mathcal{K}_{i+1}^+ - 2\mathcal{K}_i^o \mathcal{K}_{i+1}^o + 2k^2\right)$$

$$\mathcal{K}_i^+ |\xi_i\rangle = (\xi_i + 2k)|\xi_i + 1\rangle$$

$$\mathcal{K}_i^- |\xi_i\rangle = \xi_i |\xi_i - 1\rangle$$

$$\mathcal{K}_i^o |\xi_i\rangle = (\xi_i + k)|\xi_i\rangle$$

Duality for k-BEP

Theorem

1. *The bulk k-BEP* $(z(t))_{t\geq 0}$ *with generator* $L = -H^{\dagger}$ *is dual to to the process* $(\xi(t))_{t\geq 0}$ *with generator* $\mathcal{L}_{dual} = -\mathcal{H}^{T}_{dual}$ *with duality fct.*

$$D(z,\xi) = \prod_{i=1}^{N} z_i^{\xi_i} \frac{\Gamma(2k)}{2^{\xi_i}\Gamma(2k+\xi_i)}$$

Duality for k-BEP

Theorem

1. *The bulk k-BEP $(z(t))_{t\geq 0}$ with generator $L = -H^{\dagger}$ is dual to to the process $(\xi(t))_{t\geq 0}$ with generator $\mathcal{L}_{dual} = -\mathcal{H}_{dual}^{T}$ with duality fct.*

$$D(z,\xi) = \prod_{i=1}^{N} z_i^{\xi_i} \frac{\Gamma(2k)}{2^{\xi_i}\Gamma(2k+\xi_i)}$$

2. *The boundary driven k-BEP is dual to the process $\vec{\xi}(t) = (\xi_0(t), \xi(t), \xi_{N+1}(t))$ with generator $\mathcal{L}_{dual} = \mathcal{L}_1 + \sum_i \mathcal{L}_{i,i+1} + \mathcal{L}_N$*

$$(\mathcal{L}_1 f)(\vec{\xi}) = 8k\xi_1(f(\xi^{1,0}) - f(\xi)) \qquad D(x,\vec{\xi}) = T_1^{\xi_0} D(x,\xi) T_N^{\xi_{N+1}}$$

The KMP limit

KMP model

Observables: Energies at each site $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in \mathbb{R}_+^N$
Dynamics: Select a pair of lattices (i, j) and uniformly redistribute the energy under the constraint of conserving $\epsilon_i + \epsilon_j$.

The KMP limit

KMP model

Observables: Energies at each site $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in \mathbb{R}_+^N$
Dynamics: Select a pair of lattices (i, j) and uniformly redistribute the energy under the constraint of conserving $\epsilon_i + \epsilon_j$.

Instantaneous thermalization limit

Let $(z_i(t), z_j(t))$ be the process with generator

$$L_{i,j} = z_i z_j \left(\partial_i - \partial_j\right)^2 - 2k(z_i - z_j)\left(\partial_i - \partial_j\right)$$

The KMP limit

KMP model

Observables: Energies at each site $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in \mathbb{R}_+^N$
Dynamics: Select a pair of lattices (i, j) and uniformly redistribute the energy under the constraint of conserving $\epsilon_i + \epsilon_j$.

Instantaneous thermalization limit

Let $(z_i(t), z_j(t))$ be the process with generator

$$L_{i,j} = z_i z_j \left(\partial_i - \partial_j\right)^2 - 2k(z_i - z_j)\left(\partial_i - \partial_j\right)$$

Define $\quad (L_{i,j}^{IT} f)(z_i, z_j) = \lim_{t\to\infty} (e^{tL_{i,j}} f)(z_i, z_j) - f(z_i, z_j)$

The KMP limit

KMP model

Observables: Energies at each site $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in \mathbb{R}_+^N$
Dynamics: Select a pair of lattices (i, j) and uniformly redistribute the energy under the constraint of conserving $\epsilon_i + \epsilon_j$.

Instantaneous thermalization limit

Let $(z_i(t), z_j(t))$ be the process with generator

$$L_{i,j} = z_i z_j \left(\partial_i - \partial_j\right)^2 - 2k(z_i - z_j)\left(\partial_i - \partial_j\right)$$

$$\text{Define} \quad (L_{i,j}^{IT} f)(z_i, z_j) = \lim_{t\to\infty} (e^{tL_{i,j}} f)(z_i, z_j) - f(z_i, z_j)$$
$$= \lim_{t\to\infty} \mathbb{E}_{(z_i,z_j)}(f(x_i(t), x_j(t))) - f(z_i, z_j)$$

The KMP limit

KMP model

Observables: Energies at each site $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in \mathbb{R}_+^N$
Dynamics: Select a pair of lattices (i, j) and uniformly redistribute the energy under the constraint of conserving $\epsilon_i + \epsilon_j$.

Instantaneous thermalization limit

Let $(z_i(t), z_j(t))$ be the process with generator

$$L_{i,j} = z_i z_j \left(\partial_i - \partial_j\right)^2 - 2k(z_i - z_j)\left(\partial_i - \partial_j\right)$$

$$\text{Define} \quad (L_{i,j}^{IT} f)(z_i, z_j) = \lim_{t\to\infty} (e^{tL_{i,j}} f)(z_i, z_j) - f(z_i, z_j)$$
$$= \lim_{t\to\infty} \mathbb{E}_{(z_i,z_j)}(f(x_i(t), x_j(t))) - f(z_i, z_j)$$

Claim: $\quad L_{i,j}^{IT} = L_{i,j}^{KMP}$ for $k = 1/2$

The KMP limit

Lemma (stationary measure)

Let $(z_i(t), z_j(t))$ *be the Markov process with generator*

$$L_{i,j} = z_i z_j \left(\partial_i - \partial_j\right)^2 - 2k(z_i - z_j)\left(\partial_i - \partial_j\right)$$

and initial condition $z_i(0) + z_j(0) = E$. *Then in the limit* $t \to \infty$ *we have* $(z_i(t), z_j(t)) \xrightarrow{\mathcal{D}} (\frac{E+e}{2}, \frac{E-e}{2})$ *where*

$$f(e) = C_k(E^2 - e^2)^{2k-1} \quad \textit{for} \quad -E \le e \le E$$

The KMP limit

Lemma (stationary measure)

Let $(z_i(t), z_j(t))$ *be the Markov process with generator*

$$L_{i,j} = z_i z_j \left(\partial_i - \partial_j\right)^2 - 2k(z_i - z_j)\left(\partial_i - \partial_j\right)$$

and initial condition $z_i(0) + z_j(0) = E$. *Then in the limit* $t \to \infty$ *we have* $(z_i(t), z_j(t)) \xrightarrow{\mathcal{D}} (\frac{E+e}{2}, \frac{E-e}{2})$ *where*

$$f(e) = C_k(E^2 - e^2)^{2k-1} \quad \text{for} \quad -E \le e \le E$$

Proof.

Define $(E(t), e(t)) = (z_i(t) + z_j(t), z_i(t) - z_j(t))$.

The KMP limit

Lemma (stationary measure)

Let $(z_i(t), z_j(t))$ *be the Markov process with generator*

$$L_{i,j} = z_i z_j \left(\partial_i - \partial_j\right)^2 - 2k(z_i - z_j)\left(\partial_i - \partial_j\right)$$

and initial condition $z_i(0) + z_j(0) = E$. *Then in the limit* $t \to \infty$ *we have* $(z_i(t), z_j(t)) \stackrel{\mathcal{D}}{\to} (\frac{E+e}{2}, \frac{E-e}{2})$ *where*

$$f(e) = C_k (E^2 - e^2)^{2k-1} \quad \text{for} \quad -E \leq e \leq E$$

Proof.

Define $(E(t), e(t)) = (z_i(t) + z_j(t), z_i(t) - z_j(t))$. Then

$$L_{i,j} = (E^2 - e^2)\frac{\partial^2}{\partial e^2} - 4k\; e\frac{\partial}{\partial e}$$

How to use duality

Duality reduces the computation of n-point correlation functions for a boundary driven process to the computation of absorbtion probabilities for n (interacting) dual random walkers.

How to use duality

Duality reduces the computation of n-point correlation functions for a boundary driven process to the computation of absorbtion probabilities for n (interacting) dual random walkers.

Proposition

Let $|\xi| = \sum_{i=1}^{N} \xi_i$ be the total number of dual walkers.

How to use duality

Duality reduces the computation of n-point correlation functions for a boundary driven process to the computation of absorbtion probabilities for n (interacting) dual random walkers.

Proposition

Let $|\xi| = \sum_{i=1}^{N} \xi_i$ be the total number of dual walkers.

1. For the boundary driven j-SEP

$$\mathbb{E}(D(\eta, \vec{\xi})) = \sum_{a,b: a+b=|\xi|} \rho_1^a \rho_N^b \, \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b)$$

How to use duality

Duality reduces the computation of *n*-point correlation functions for a boundary driven process to the computation of absorbtion probabilities for *n* (interacting) dual random walkers.

Proposition

Let $|\xi| = \sum_{i=1}^{N} \xi_i$ be the total number of dual walkers.

1. For the boundary driven *j*-SEP

$$\mathbb{E}(D(\eta,\vec{\xi})) = \sum_{a,b:a+b=|\xi|} \rho_1^a \rho_N^b \, \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b)$$

2. For the boundary driven *k*-BEP

$$\mathbb{E}(D(x,\vec{\xi})) = \sum_{a,b:a+b=|\xi|} T_1^a T_N^b \, \mathbb{P}(\xi_0(\infty) = a, \xi_{N+1}(\infty) = b)$$

Proof.

Let $\nu_{\rho_1,\rho_N} = \nu$ the stationary measure of the *j*-SEP $(\eta_t)_{t\geq 0}$.

$$\mathbb{E}(D(\eta,\vec{\xi}))$$

Proof.

Let $\nu_{\rho_1,\rho_N} = \nu$ the stationary measure of the *j*-SEP $(\eta_t)_{t\geq 0}$.

$$\mathbb{E}(D(\eta,\vec{\xi})) = \lim_{t\to\infty} \sum_{\eta\in\Omega} \mathbb{E}_\eta(D(\eta_t,\vec{\xi}))\ \nu(\eta)$$

Proof.

Let $\nu_{\rho_1,\rho_N} = \nu$ the stationary measure of the j-SEP $(\eta_t)_{t\geq 0}$.

$$\mathbb{E}(D(\eta,\vec{\xi})) = \lim_{t\to\infty} \sum_{\eta\in\Omega} \mathbb{E}_\eta(D(\eta_t,\vec{\xi}))\ \nu(\eta)$$

$$= \sum_{\eta\in\Omega} \lim_{t\to\infty} \mathbb{E}_\xi(D(\eta,\vec{\xi}_t))\ \nu(\eta)$$

Proof.

Let $\nu_{\rho_1,\rho_N} = \nu$ the stationary measure of the j-SEP $(\eta_t)_{t\geq 0}$.

$$\mathbb{E}(D(\eta,\vec{\xi})) = \lim_{t\to\infty} \sum_{\eta\in\Omega} \mathbb{E}_\eta(D(\eta_t,\vec{\xi}))\,\nu(\eta)$$
$$= \sum_{\eta\in\Omega} \lim_{t\to\infty} \mathbb{E}_\xi(D(\eta,\vec{\xi}_t))\,\nu(\eta)$$

$$\left(\text{Use} \qquad D(\eta,\vec{\xi}) = \rho_1^{\xi_0} \prod_{i=1}^{N} \frac{\eta_i(\eta_i-1)\cdots(\eta_i-\xi_i+1)}{2j(2j-1)\cdots(2j-\xi_i+1)}\,\rho_N^{\xi_{N+1}}\right)$$

Proof.

Let $\nu_{\rho_1,\rho_N} = \nu$ the stationary measure of the j-SEP $(\eta_t)_{t\geq 0}$.

$$\begin{aligned}\mathbb{E}(D(\eta,\vec{\xi})) &= \lim_{t\to\infty} \sum_{\eta\in\Omega} \mathbb{E}_\eta(D(\eta_t,\vec{\xi}))\,\nu(\eta)\\ &= \sum_{\eta\in\Omega} \lim_{t\to\infty} \mathbb{E}_\xi(D(\eta,\vec{\xi}_t))\,\nu(\eta)\end{aligned}$$

$$\left(\text{Use} \qquad D(\eta,\vec{\xi}) = \rho_1^{\xi_0} \prod_{i=1}^{N} \frac{\eta_i(\eta_i-1)\cdots(\eta_i-\xi_i+1)}{2j(2j-1)\cdots(2j-\xi_i+1)}\,\rho_N^{\xi_{N+1}}\right)$$

$$= \sum_{\eta\in\Omega} \sum_{a,b:a+b=|\xi|} \rho_1^a\rho_N^b\,\mathbb{P}(\xi_0(\infty)=a,\xi_{N+1}(\infty)=b)\,\nu(\eta)$$

Proof.

Let $\nu_{\rho_1,\rho_N} = \nu$ the stationary measure of the j-SEP $(\eta_t)_{t\geq 0}$.

$$\mathbb{E}(D(\eta,\vec{\xi})) = \lim_{t\to\infty} \sum_{\eta\in\Omega} \mathbb{E}_\eta(D(\eta_t,\vec{\xi}))\,\nu(\eta)$$

$$= \sum_{\eta\in\Omega} \lim_{t\to\infty} \mathbb{E}_\xi(D(\eta,\vec{\xi}_t))\,\nu(\eta)$$

$$\left(\text{Use} \qquad D(\eta,\vec{\xi}) = \rho_1^{\xi_0} \prod_{i=1}^{N} \frac{\eta_i(\eta_i-1)\cdots(\eta_i-\xi_i+1)}{2j(2j-1)\cdots(2j-\xi_i+1)}\,\rho_N^{\xi_{N+1}}\right)$$

$$= \sum_{\eta\in\Omega} \sum_{a,b:a+b=|\xi|} \rho_1^a\rho_N^b\,\mathbb{P}(\xi_0(\infty)=a,\xi_{N+1}(\infty)=b)\,\nu(\eta)$$

$$= \sum_{a,b:a+b=|\xi|} \rho_1^a\rho_N^b\,\mathbb{P}(\xi_0(\infty)=a,\xi_{N+1}(\infty)=b)$$

□

Density/Energy profile

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0)$ $\Rightarrow$ 1 walker $(X_t)_{t \geq 0}$ with $X_0 = i$

site i $\nearrow$

Density/Energy profile

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \quad \Rightarrow \quad$ 1 walker $(X_t)_{t \geq 0}$ with $X_0 = i$
site i $\nearrow$

1. For j-SEP $\quad D(\eta, \vec{\xi}) = \frac{\eta_i}{2j}$

$$\mathbb{E}\left(\frac{\eta_i}{2j}\right) = \rho_1 \, \mathbb{P}_i(X_\infty = 0) + \rho_N \, \mathbb{P}_i(X_\infty = N+1)$$
$$= \rho_1 \left(1 - \frac{i}{N+1}\right) + \rho_N \left(\frac{i}{N+1}\right)$$

Density/Energy profile

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \quad \Rightarrow \quad$ 1 walker $(X_t)_{t \geq 0}$ with $X_0 = i$

site $i \nearrow$

1. For j-SEP $\quad D(\eta, \vec{\xi}) = \frac{\eta_i}{2j}$

$$\mathbb{E}\left(\frac{\eta_i}{2j}\right) = \rho_1 \, \mathbb{P}_i(X_\infty = 0) + \rho_N \, \mathbb{P}_i(X_\infty = N+1)$$
$$= \rho_1 \left(1 - \frac{i}{N+1}\right) + \rho_N \left(\frac{i}{N+1}\right)$$

2. For k-BEP $\quad D(x, \vec{\xi}) = \frac{x_i^2}{4k}$

$$\mathbb{E}\left(\frac{x_i^2}{4k}\right) = T_1 \left(1 - \frac{i}{N+1}\right) + T_N \left(\frac{i}{N+1}\right)$$

Density/Energy correlation

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ $\Rightarrow$ 2 walkers $(X_t, Y_t)_{t\geq 0}$ with $(X_0, Y_0) = (i, l)$

site i $\nearrow$ site l $\nearrow$

Density/Energy correlation

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \quad \Rightarrow \quad$ 2 walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, l)$

site $i \nearrow$ site $l \nearrow$

1. For j-SEP $\quad D(\eta, \vec{\xi}) = \frac{\eta_i}{2j}\frac{\eta_l}{2j}$

$$\mathbb{E}\left(\frac{\eta_i}{2j}\frac{\eta_l}{2j}\right) = \rho_1^2\, \mathbb{P}_{i,l}(X_\infty = 0, Y_\infty = 0) + \rho_N^2\, \mathbb{P}_{i,l}(X_\infty = N+1, Y_\infty = N+1)$$

$$= \rho_1 \rho_N \left(\mathbb{P}_{i,l}(X_\infty = 0, Y_\infty = N+1) + \mathbb{P}_{i,l}(X_\infty = N+1, Y_\infty = 0)\right)$$

Density/Energy correlation

If $\vec{\xi} = (0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0)$ $\Rightarrow$ 2 walkers $(X_t, Y_t)_{t\geq 0}$ with $(X_0, Y_0) = (i, l)$

site i ↗ site l ↗

1. For j-SEP $D(\eta,\vec{\xi}) = \frac{\eta_i}{2j}\frac{\eta_l}{2j}$

$$\mathbb{E}\left(\frac{\eta_i}{2j}\frac{\eta_l}{2j}\right) = \rho_1^2\, \mathbb{P}_{i,l}(X_\infty = 0, Y_\infty = 0) + \rho_N^2\, \mathbb{P}_{i,l}(X_\infty = N+1, Y_\infty = N+1)$$
$$= \rho_1\rho_N\left(\mathbb{P}_{i,l}(X_\infty = 0, Y_\infty = N+1) + \mathbb{P}_{i,l}(X_\infty = N+1, Y_\infty = 0)\right)$$

$$\mathbb{E}\left(\frac{\eta_i}{2j};\frac{\eta_l}{2j}\right) = -\frac{2i(N-l)}{N^3}(\rho_1-\rho_N)^2$$

Density/Energy correlation

If $\vec{\xi} = (0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0) \quad \Rightarrow \quad$ 2 walkers $(X_t, Y_t)_{t\geq 0}$ with $(X_0, Y_0) = (i, l)$

site $i \nearrow$ site $l \nearrow$

1. For j-SEP $\quad D(\eta,\vec{\xi}) = \frac{\eta_i}{2j}\frac{\eta_l}{2j}$

$$\mathbb{E}\left(\frac{\eta_i}{2j}\frac{\eta_l}{2j}\right) = \rho_1^2\,\mathbb{P}_{i,l}(X_\infty = 0, Y_\infty = 0) + \rho_N^2\,\mathbb{P}_{i,l}(X_\infty = N+1, Y_\infty = N+1)$$
$$= \rho_1\rho_N\left(\mathbb{P}_{i,l}(X_\infty = 0, Y_\infty = N+1) + \mathbb{P}_{i,l}(X_\infty = N+1, Y_\infty = 0)\right)$$

$$\mathbb{E}\left(\frac{\eta_i}{2j};\frac{\eta_l}{2j}\right) = -\frac{2i(N-l)}{N^3}(\rho_1-\rho_N)^2$$

2. For k-BEP $\quad D(x,\vec{\xi}) = x_i^2 x_l^2$

$$\mathbb{E}\left(\frac{x_i^2}{4k};\frac{x_l^2}{4k}\right) = \frac{2i(N-l)}{N^3}(T_1-T_N)^2$$

Conclusions

Conclusions

1. Duality Theorems
   - Relation to symmetry

Conclusions

1. Duality Theorems
   - Relation to symmetry
2. Application to transport model
   - "Fermionic" model: j-SEP
   - "Bosonic" model: k-BEP

Conclusions

1. Duality Theorems
   - Relation to symmetry
2. Application to transport model
   - “Fermionic” model: j-SEP
   - “Bosonic” model: k-BEP
3. Long-range correlations
   With $0 < y_i < y_l < 1, \quad y_i = \lim_N i/N,\ y_l = \lim_N l/N$
   - $$\lim_{N\to\infty} N\mathbb{E}\left(\frac{\eta_{i/N}}{2j};\frac{\eta_{l/N}}{2j}\right) = -2y_i(1-y_l)(\rho_1-\rho_N)^2$$
   - $$\lim_{N\to\infty} N\mathbb{E}\left(\frac{x^2_{i/N}}{4k};\frac{x^2_{l/N}}{4k}\right) = 2y_i(1-y_l)(T_1-T_N)^2$$