Aspects of Kicked Quantum Dynamics

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August 28, 2008

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Kicked Hamiltonians

$$e^{-\frac{i}{\hbar}t(\hat{T}+\hat{V})} = \lim_{N\to\infty,\tau\to 0,N\tau=t} \left(e^{-\frac{i}{\hbar}\tau\hat{T}} e^{-\frac{i}{\hbar}\tau\hat{V}}\right)^N$$

For fixed N, τ , the rhs is the propagator from time $t = 0_-$ to time $t = N\tau_-$ of the **Kicked Hamiltonian** :

$$\hat{H}(t) = \hat{T} + \tau\hat{V}\sum_{n\in\mathbb{Z}}\delta(t - n\tau)$$

The kicked dynamics may be drastically different from the dynamics which are generated by $\hat{T} + \hat{V}$. In the 1-freedom case, the latter are classically integrable , but the former have , generically, a mixed phase space.
Path integrals for kicked dynamics are ordinary N-fold integrals.

Instances of Kicked Dynamics

($\hat{X}, \hat{P}$: canonical position & momentum operators for a point particle moving in a line)

Pendulum $\longrightarrow$ Kicked Rotor

$$\hat{T} = \frac{1}{2}\hat{P}^2 \ , \ \hat{V} = \mu \cos(\hat{X})$$

Harper $\longrightarrow$ "Kicked Harper"

$$\hat{T} = \lambda \cos(\hat{P}) \ , \ \hat{V} = \mu \cos(\hat{X})$$

Wannier-Stark $\longrightarrow$ Kicked Accelerator

$$\hat{T} = \frac{1}{2}\hat{P}^2 + \eta\hat{X} \ , \ \hat{V} = \mu \cos(\hat{X})$$

From Atoms to Rotors

In experiments, atoms move in (approximately) straight lines. However, the kicking potential is periodic in space. Quasi-momentum is then conserved. If the spatial period is 2π, then q.mom. $=$ fractional part of momentum $:= \beta$ and the Brillouin zone is $\mathbb{B}^{(\mathrm{P})} = [0, 1[$.

Bloch theory

$$L^2(\mathbb{R}) \simeq L^2(\mathbb{B}^{(\mathrm{P})}) \otimes L^2(\mathbb{T}) \quad , \quad \hat{U} \;=\; \int_{\mathbb{B}^{(\mathrm{P})}}^{\oplus} d\beta \; \hat{U}_\beta$$

Each $\hat{U}_\beta$ formally defines a rotor's dynamics. It is obtained by the replacement $X \to \theta := X \mathrm{mod}(2\pi)$, $\hat{P} \to -i\partial_\vartheta + \beta$

Example

Kicked Atom : $\hat{U} = e^{-i\mu\cos(\hat{X})} e^{-i\tau\hat{P}^2/2}$

Kicked Rotor $\hat{U}_\beta = e^{-i\mu\cos(\theta)} e^{-i\tau(-i\partial_\theta+\beta)^2/2}$

KR Resonances

A KR resonance is said to occur whenever $\hat{U}_\beta$ commutes with a momentum translation $\hat{T}^\ell$ (ℓ a strictly positive integer), where $\hat{T} : \psi(\theta) \to e^{i\theta}\psi(\theta)$. This happens if β is rational and τ is commensurate to 2π; the order of a resonance is the least $\ell > 0$ such that $\hat{T}^\ell$ commutes with $\hat{U}_\beta$.

Proposition

(Izrailev, Shepelyansky 1980; Dana, Dorofeev 06) $\hat{U}_\beta$ *commutes with* $\hat{T}^\ell$ *if, and only if, (i)* $\tau = 2\pi p/q$ *with* p, q *coprime integers, (ii)* $\ell = mq$ *for some integer* m*, (iii)* $\beta = \nu/mp + mq/2 \bmod(1)$*, with* ν *an arbitrary integer.*

Resonances with $m = 1$ and $\ell = q$ are termed primary .
At resonances, "Quasi-Position" ϑ is conserved: for primary resonances, $\vartheta \equiv \theta \bmod 2\pi/q$ and $\vartheta \in \mathbb{B}_q^{(\mathrm{x})} \equiv [0, 2\pi/q[$.

"θ changes by multiples of $2\pi/q$"

Theorem

(Izrailev , Shepelyansky 1980) Identify $L^2(\mathbb{T})$ *and* $L^2(\mathbb{B}_q^{(X)}) \otimes \mathbb{C}^q$ *through* $\psi(\theta) \rightleftarrows \{\psi(\vartheta + 2\pi n/q)\}_{n=1,\dots,q}$. *Then at a primary resonance with* $\tau = 2\pi p/q$ *and* $\beta = \beta_r$,

$$\hat{U}_{\beta_r} \;=\; \int_{\mathbb{B}_q^{(X)}}^{\oplus} d\vartheta \; \hat{\mathfrak{X}}(\beta_r, \mu, \vartheta) \; ,$$

where $\hat{\mathfrak{X}}(\beta_r, \mu) : [0, 2\pi] \to \mathbb{U}(q)$ *is defined by :*

$$\mathfrak{X}_{jk}(\beta_r, \mu, \vartheta) = e^{-i\mu \cos(\vartheta + 2\pi j/q)} G_{jk}(p, q, \beta_r) \; , \qquad (1)$$

$$G_{jk}(p, q, \beta_r) = \frac{1}{q} \sum_{l=0}^{q-1} e^{-\pi i p (l+\beta_r)^2/q} \; e^{2\pi i (j-k) l/q} \; . \qquad (2)$$

Bands

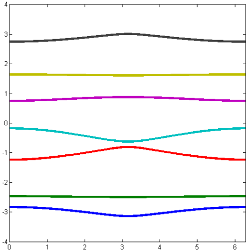


Figure: Eigenangles vs $\theta = q\vartheta$; $q = 7$, $p = 2$, $\mu = 3$.

Proposition

(IG 08) If $\mu > 0$ then at primary resonances each eigenvalue w_j $(j = 1, \ldots, q)$ of $\hat{\mathfrak{X}}(\beta, \mu, \vartheta)$ is a nonconstant analytic function of $\exp(i\vartheta)$. Hence, $\hat{U}_\beta$ has a pure AC spectrum, and the rotor's energy increases quadratically in time (ballistic transport).

Bandwidths I

Eigenvalues of free resonant rotation ($\mu = 0$, $\tau = 2\pi p/q$, $\beta = \beta_r$)

$$a_j \ = \ e^{-i\pi p(j+\beta_r)^2/q} \ , \ \ j = 1, \ldots, q \ .$$

always degenerate to various extent. Easiest case: $\beta_r = 1/2$, and q odd $\Rightarrow a_j = a_{q-j+1}$ for all $1 \leq j \leq q$. All eigenvalues a_j are degenerate, except one.

q odd and prime $\Rightarrow$ multiplicity $= 2$.

Bandwidths II

Theorem

(IG 08) If $q > 2$ is prime, p is prime to q, and $\beta = 1/2$, then, asymptotically as $\mu \to 0$,

$$\frac{dw_j}{d\vartheta} \sim \mu^{\alpha_j} s_j(p,q)\ \sin(\vartheta) \quad , \quad \alpha_j = \max\{2j-1, q-2j+1\}\ , \quad (3)$$

The coefficients s_j are exponentially small at large q:

$$|s_j(p,q)| \lesssim e^{-qA_j} \quad \text{where} \quad A_j = \gamma_j + O(q^{-1/2}\log^{3/2}(q))\ . \quad (4)$$

The numbers γ_j increase with j from $\gamma_1 \geq 0.001$ to $\gamma_{(q+1)/2} \geq 0.01$.

Kicked Accelerator.

$$\hat{U} \;=\; e^{-i\mu\cos(\hat{X})}e^{-i(\tau\hat{P}^2/2+\eta\hat{X})}$$

translation invariance recovered via $\psi(X,t)\rightarrow\psi(X,t)e^{-it\eta X}$ ("Falling frame"). Rotor Evolution from t-th kick to $t+1$-th kick at quasi-momentum β:

$$\begin{aligned}\hat{U}_{\beta,t} \;&=\; e^{-i\mu\cos(\theta)}e^{-i\frac{\tau}{2}(-i\partial_\theta+\phi_t)^2}\\ \phi_t \;&=\; \beta+\frac{\eta}{2}+\eta t\;. \end{aligned} \tag{5}$$

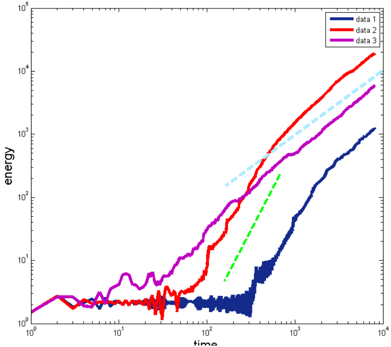
The Problem of Dynamical Localization

For $\psi\in\mathcal{H}_1$ denote $\psi(t)=\hat{U}_{\beta,t}\hat{U}_{\beta,t-1}\dots\hat{U}_{\beta,0}\psi$. Is $||\psi(t)||_{\mathcal{H}_1}$ bounded in time?

Destruction of localization

$(\eta\tau/2\pi$ irrational$)$

η increases through ■ ■ ■. Dashed lines: linear and quadratic growth.

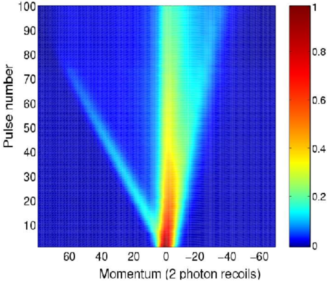


Ballistic growth at intermediate times is due to Quantum Accelerator modes

Quantum Accelerator Modes

QAMs were first discovered in experiments at Oxford. There, η is gravity.

M.K. Oberthaler, R.M. Godun, M.B. d'Arcy, G.S. Summy and K. Burnett, PRL **83**, 4447, (1999)



Pseudoclassical Action

Near Resonance: $\tau = 2\pi \frac{p}{q} + \epsilon$

$$\hat{U}_t \psi(\theta) = \sum_{s=0}^{q-1} G_s \, e^{-i\frac{\epsilon}{2}(-i\partial_\theta + \beta)^2} \psi(\theta - 2\pi s/q - \tau\phi_t) =$$

$$\frac{1}{\sqrt{2\pi i \epsilon}} \sum_{m\in\mathbb{Z}} \sum_{s=0}^{q-1} G_s \int_0^{2\pi} d\theta' \, e^{-\frac{i}{\epsilon} S(\theta,\theta',s,m,t)} \psi(\theta')$$

Action ($\tilde{k} := \epsilon k$):

$$S(\theta, \theta', s, m, t) = -\tilde{k}\cos(\theta) + \frac{1}{2}(\theta - \theta' - 2\pi s/q - 2\pi m - \tau\phi_t)^2$$

Propagation over t kicks: sum over paths. Each path is specified by $(\theta_0, \theta_1, \ldots, \theta_t)$, $(m_0, \ldots, m_t)$, $(s_0, \ldots, s_t)$.

Pseudoclassical Asymptotics

$$\epsilon \to 0 \; ; \; k \to \infty \; ; \; \tilde{k} = k\epsilon = const.$$

Stationary Phase selects paths with $(m_0, \ldots, m_t)$ and $(s_0, \ldots, s_t)$ arbitrary, and rays $(\theta_0, \theta_1, \ldots, \theta_t)$ that obey:

$$\begin{aligned} \theta_{t+1} &= \theta_t + I_t + \tau\phi_t + 2\pi s_t/q \mod 2\pi \;, \\ I_{t+1} &= I_t + \tilde{k}\, \sin(\theta_{t+1}) \;. \end{aligned}$$

$q = 1$: the Pseudoclassical Limit

S Fishman , IG, L Rebuzzini PRL 89 (2002) 0841011; J Stat Phys 110 (2003) 911; A Buchleitner, MB d'Arcy, S Fishman, SA Gardiner, IG, ZY Ma, L Rebuzzini and GS Summy, PRL 96 (2006) 164101

Multiples of $2\pi/q$ drop out. Time dependence is removed by changing variable to:

$$J_t = I_t + \frac{\eta}{2} + \delta\beta + \tau\eta t$$

(Difference linearly grows with time)

$$J_{t+1} = J_t + \tau\eta + \tilde{k}\sin(\theta_{t+1}) ,$$
$$\theta_{t+1} = \theta_t + J_t .$$

Rays are trajectories of a classical dynamical system on $\mathbb{T} \times \mathbb{R}$. Stable Periodic Orbits of the map on $\mathbb{T} \times \mathbb{T} \rightsquigarrow$ Stable Accelerating Rays $\rightsquigarrow$ Quantum Accelerator Modes.

$q > 1$: Near Higher-Order Resonances

IG, L Rebuzzini PRL 100 (2008) 234103

$$J_{t+1} = J_t + \tau\eta + \tilde{k}\sin(\theta_{t+1}) + \delta_t ,$$
$$\theta_{t+1} = \theta_t + J_t ,$$
$$\delta_t = \frac{2\pi}{q}(s_{t+1} - s_t) .$$

Rays are not trajectories of a unique classical system anymore. There is a ray for each choice of an integer string $\mathbf{s} := (s_0, \ldots, s_t)$: so rays exponentially proliferate with the number t of kicks. Each ray contributes an amplitude:

$$\frac{1}{\sqrt{q^t \epsilon |\det(\mathfrak{M}_t)|}} e^{\frac{i}{\epsilon} S_{\mathbf{s},\mathbf{m}} + i\Phi_{\mathbf{s},\mathbf{m}}} .$$

Stable Rays?

$\mathfrak{M}_t$ is the stability matrix :

$$\mathfrak{M}_t = \begin{vmatrix} 2+\tilde{k}\cos(\theta_0) & -1 & 0 & \dots & \dots & 0 \\ -1 & 2+\tilde{k}\cos(\theta_1) & -1 & 0 & \dots & 0 \\ 0 & -1 & 2+\tilde{k}\cos(\theta_2) & -1 & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & -1 & 2+\tilde{k}\cos(\theta_{t-1}) & -1 \\ 0 & 0 & \dots & \dots & -1 & 2+\tilde{k}\cos(\theta_t) \end{vmatrix}$$

Herbert-Jones-Thouless formula:

$$\log(|\det(\mathfrak{M})|) \;=\; t\int dn(E)\log(|E|) \;=\; t\times \text{Lyapunov exponent}$$

As t increases, most sequences δ_t are random and so are θ_t : $\Rightarrow$ LE positive (localization) $\Rightarrow$ Each such ray yields an exponentially small contribution.

Stable Rays

Distinguished individual contributions from rays, whose matrices $\mathfrak{M}$ have extended states.

How to find stable rays

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$$J_{t+1} = J_t + \tau\eta + \tilde{k}\,\sin(\theta_{t+1}) + \delta_t\,,$$
$$\theta_{t+1} = \theta_t + J_t\,,$$
$$\delta_t = \frac{2\pi}{q}(s_{t+1} - s_t)\,.$$

whenever δ_t is a periodic sequence of period T, T-fold iteration of the above equations defines a dynamical system on the 2-torus. Each stable periodic orbit of that system defines a stable ray that gives rise to an accelerator mode.

Acceleration

$$\frac{1}{\epsilon}\left\{\frac{2\pi}{T}\frac{\mathfrak{j}}{\mathfrak{p}} - \tau\eta - \frac{1}{T}\sum_{t=0}^{T-1}\delta_t\right\}\,.$$

Summary

Kicked dynamics in gravity:

- Quantum Accelerator Modes exist near resonances of arbitrary order;
- they are exposed by small- ϵ-asymptotics, which is similar in nature to quasi-classical approximation (short-wave asymptotics);
- They are associated with stable periodic orbits of :
  - a single classical dynamical system ("pseudoclassical limit") in the "spinless" case $q = 1$;
  - a infinite hierarchy of classical dynamical systems in the case $q > 1$. No pseudoclassical limit proper exists in this case .

Partial similarity to the case of multi-component wave equations : Littlejohn & Flynn 1991.