Orderings of the rationals: dynamical systems and statistical mechanics

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MECCANICA
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Number theory,

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- Giampieri, I., *A one-parameter family of analytic Markov interval maps with an intermittency transition*, Disc. Cont. Dyn. Sys. **12** (2005), 115-136
- Degli Esposti, I., Knauf, *Generalized Farey Trees, Transfer Operators and Phase Transitions*, Commun. Math. Phys. **275** (2007), 297-329
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The **child** $\frac{p}{q} \oplus \frac{p'}{q'}$ always lies between the **parents** $\frac{p}{q}$ and $\frac{p'}{q'}$:

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Connection to continued fractions

The elements of the S-B tree of **depth** d are exactly those rational numbers $x \in \mathbb{Q}_+$ whose continued fraction expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} \equiv [a_0; a_1, \ldots, a_n]$$

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Example: $\frac{8}{5}$ has depth $d = 5$ and $\frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$

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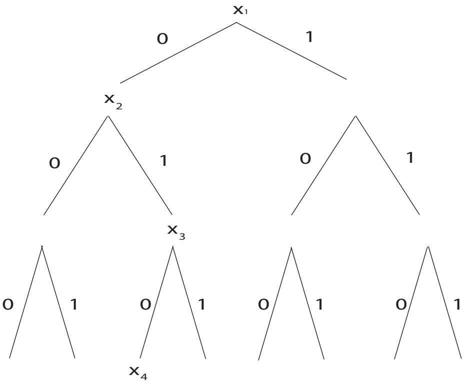

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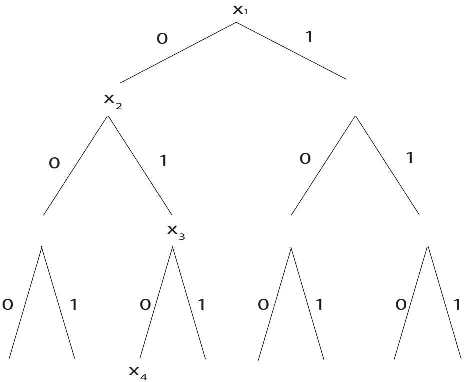

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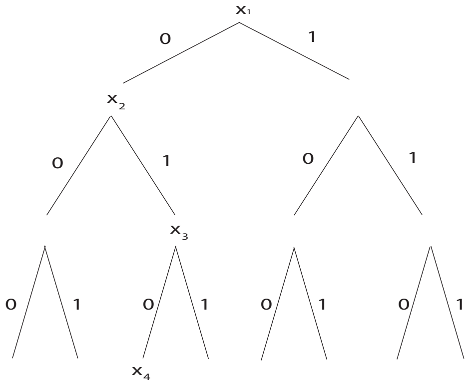

$$x = [a_0; a_1, a_2, \dots] \quad \Longleftrightarrow \quad \sigma(x) = 1^{a_0} 0^{a_1} 1^{a_2} \dots$$

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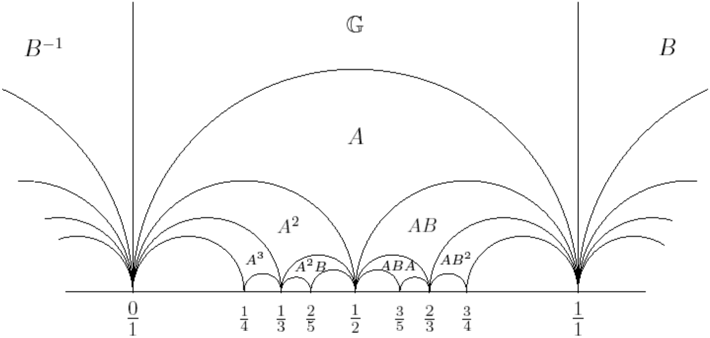
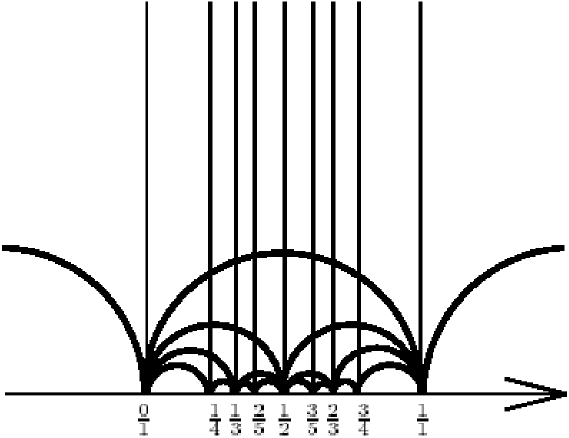
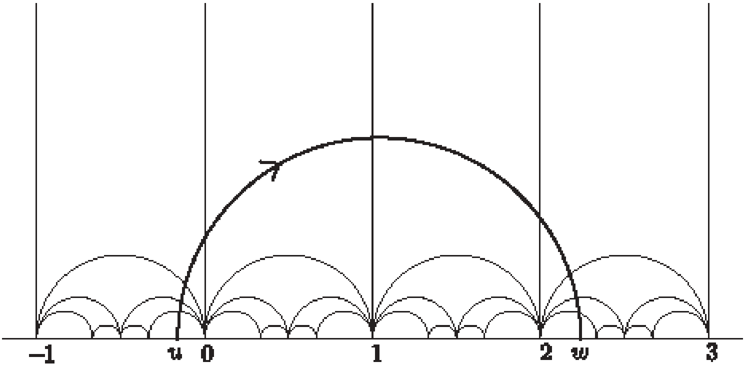


finite paths $\Longrightarrow$ geodesics ending at rational points



infinite paths $\Longrightarrow$ geodesics (u, w) with $u, w \in \mathbb{R} \setminus \mathbb{Q}$



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- We have the **recursion** $\xi_{i+1} = \frac{1}{1 - \{\xi_i\} + [\xi_i]}$ (*Newmann*, 2003)

SOME (UNEXPECTED) RULES

- Reading the permuted S-B tree row by row, and each row from left to right, the i-th element is given by $\xi_1 = \frac{1}{1}$ and $\xi_i = b(i-1)/b(i)$, $i > 1$, where $b(n)$ is the number of **hyperbinary** representations of n (*Calkin and Wilf*, 2000), e.g. $8 = 2^3 = 2^2 + 2^2 = 2^2 + 2 + 2 = 2^2 + 2 + 1 + 1 \Rightarrow b(8) = 4$
- We have the **recursion** $\xi_{i+1} = \frac{1}{1-\{\xi_i\}+[\xi_i]}$ (*Newmann*, 2003)
- Reading the permuted tree genealogically starting form $\frac{1}{1}$, under each vertex $\frac{p}{q}$ there is the **set of descendants**

$$\left\{A\left(\frac{p}{q}\right), B\left(\frac{p}{q}\right)\right\} \equiv \left\{\frac{p}{p+q}, \frac{p+q}{q}\right\}$$

Statistical mechanics

Statistical mechanics

1. Random walks

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Statistical mechanics

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A more quantitative result in a minute ...

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To each of the 2^d elements of depth $d+1$ in the permuted S-B tree one can attach a **spin configuration** corresponding to its address $\sigma(a/b) \in \{0,1\}^d$ and the **energy** $E_d(\sigma) = \log b$.

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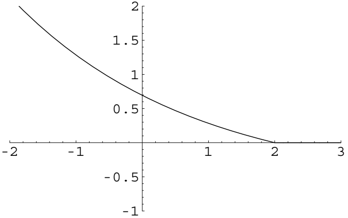

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ENERGIES OF SPIN CHAINS

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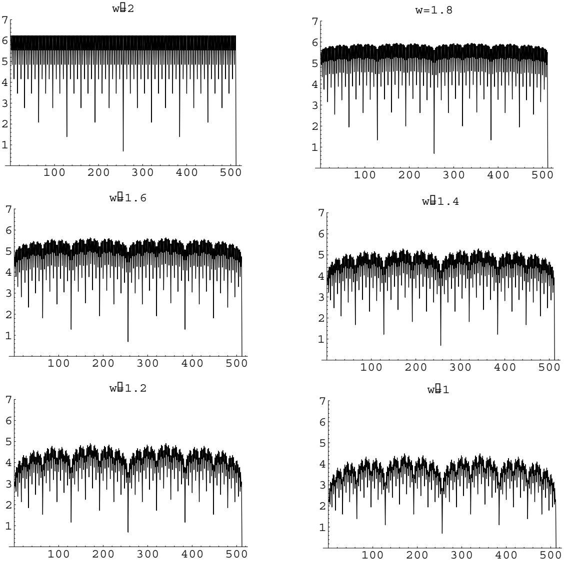

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- For $w = 1$ the phase transition at $\beta_{cr} = 2$ is of second order, but for $1 < w \leq 2$ the first derivative of $-\beta f(\beta)$ is discontinuous at β_{cr} (first order transition). Moreover, for all $w \in [1,2]$ the magnetization jumps at β_{cr} from 1 to 0.

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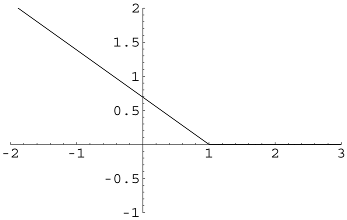
$$Z_d(\beta) = \frac{2^\beta - 1 - 2^{d(1-\beta)}}{2^\beta - 2}$$

$$\lim_{d\to\infty} Z_d(\beta) = \frac{2^\beta - 1}{2^\beta - 2}, \qquad \mathrm{Re}\,\beta > \beta_{cr} = 1$$

$$-\beta\, f(\beta) = \begin{cases} (1-\beta)\log 2 & , \text{ if } \beta < 1 \\ 0 & , \text{ if } \beta \geq 1 \end{cases}$$



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$\Longrightarrow$ The permuted S-B tree can be constructed genealogically from the root $\frac{1}{1}$ by writing under each leaf x its descendants $F^{-1}(x)$.

We plot the interval maps $\tilde{S} := \phi \circ S \circ \phi^{-1}$ and $\tilde{F} := \phi \circ F \circ \phi^{-1}$ where $\phi(x) := x/(1 + x)$ (maps the S-B tree to the Farey tree):

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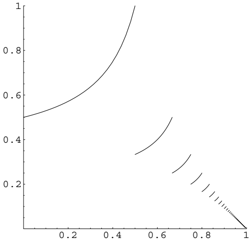
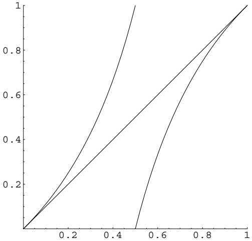
$$\tilde{S}(x) = \frac{1}{2 - \left\{\frac{x}{1-x}\right\} + \left[\frac{x}{1-x}\right]} \quad , \quad \tilde{F} = \begin{cases} \dfrac{x}{1-x} & \text{if } 0 \le x < \frac{1}{2} \\ 2 - \frac{1}{x} & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$





CONJUGATIONS AND ERGODIC PROPERTIES

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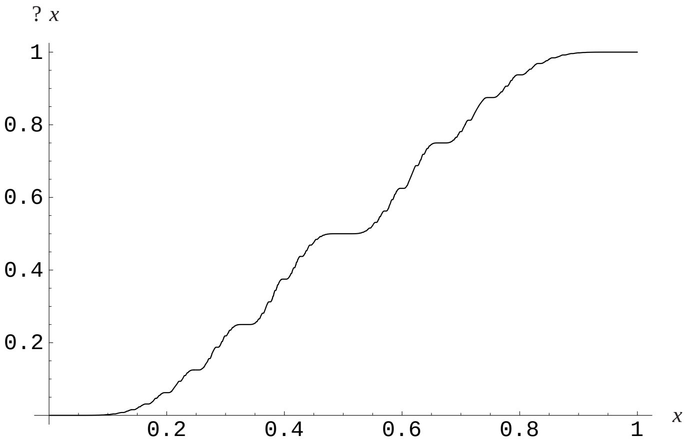
- $d?(x)$ vanishes Lebesgue-almost everywhere


? x
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
x

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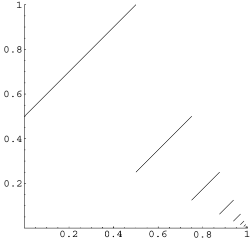

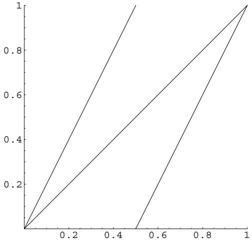

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F and S as Poincaré maps

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Considering geodesics (u, w) with $u < 0 < w$, the action of F on w can be regarded as a (factor of) the Poincaré map associated to the geodesic flow $g_t : TM \to TM$ on the modular surface $M = SL(2, \mathbb{Z}) \setminus \mathbb{H}$ (not compact but of finite hyperbolic area and gaussian curvature $K = -1$).

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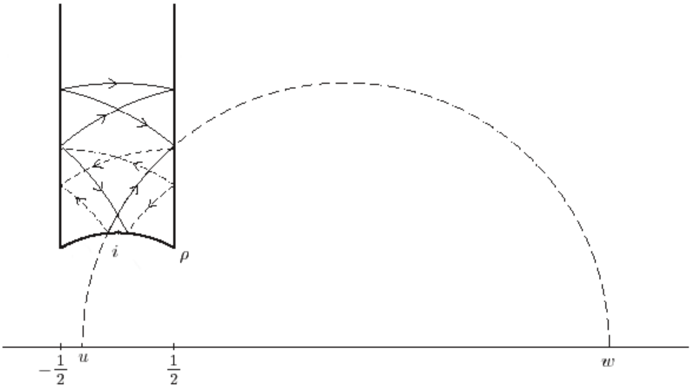

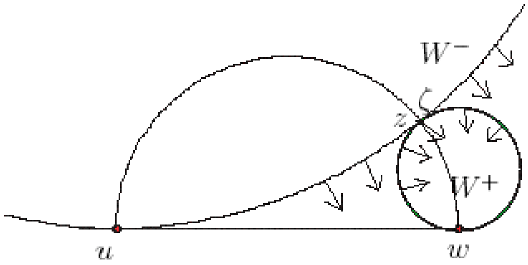

W^-
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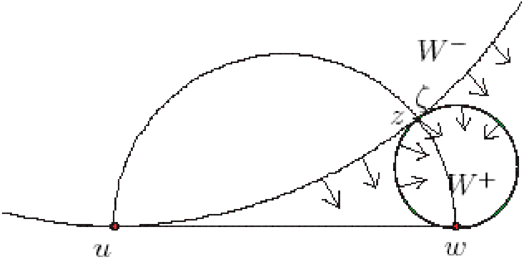

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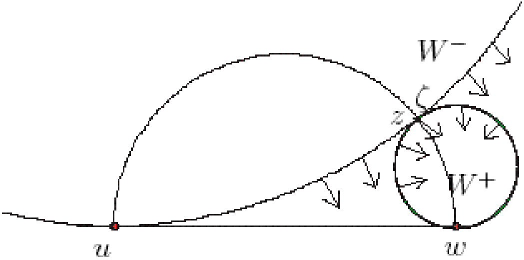

The action of S can be regarded as a Poincaré map associated to the horocycle flow $h_t : TM \to TM$, with a return time which depends on the initial point on the section ... joint work in progress with Bonanno, Degli Esposti and Knauf

Transfer operators and hyperbolic laplacian

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A **Maass wave form** of parameter $s \in \mathbb{C}$ is a $SL(2, \mathbb{Z})$-invariant fct $u : \mathbb{H} \to \mathbb{C}$, vanishing for $y \to \infty$ and satisfying $\Delta u = s(1-s)u$ where $\Delta = y^2(\partial_x^2 + \partial_y^2)$ is the hyperbolic Laplacian on $\mathbb{H}$.

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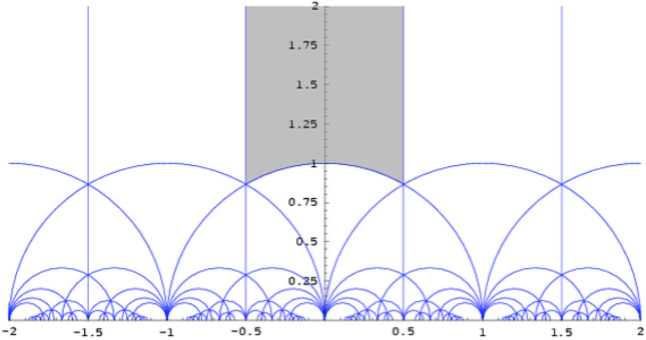

There is a **one-to-one correspondence** between the space of Maass wave forms u of parameter $s \in \mathbb{C}$ with $\mathrm{Re} s = 1/2$ and the space of real analytic solutions on $\mathbb{R}_+$ of the Lewis-Zagier eq. $L_\beta f = f$ with $\beta = 2s$, which satisfy

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(*Lewis, Zagier*, Ann. Math. **153** (2001), 191-258)

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What about the rest of the spectrum of L_β?

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Some spectral properties of $P_\beta^\pm$

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THEOREM (*Bonanno, Graffi, I.*, 2007) For each $\beta \in (0,\infty)$ the operators $P_\beta^\pm : H_\beta \to H_\beta$ are bounded, self-adjoint and isospectral. Their spectrum is $\{0\} \cup (0,1]$, with $(0,1]$ purely a.c.

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