Long-range correlations in diffusive systems

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- **Introduction**
- **Postulates**
- **Correlation functions**
- **Equilibrium states**
- **The ABC model**
- **Conclusions**

Introduction

One of the main differences between equilibrium and non equilibrium systems, is that out of equilibrium the free energy is, in general, a non local functional of the thermodynamic variables. This implies the existence of correlations at the macroscopic scale which have been observed experimentally.

This phenomenon can be demonstrated in microscopic models but can also be derived from simple postulates characterizing the macroscopic behaviour of diffusive systems.

Postulates

1. *The macroscopic state is completely described by the local density* $\rho = \rho(t, x)$ *and the associated current* $j = j(t, x)$.

2. *The macroscopic evolution is given by the continuity equation*

$$\partial_t \rho + \nabla \cdot j = 0 \tag{2.1}$$

together with the constitutive equation

$$j = J(\rho) = -D(\rho)\nabla\rho + \chi(\rho)E \tag{2.2}$$

where the diffusion coefficient $D(\rho)$ *and the* mobility $\chi(\rho)$ *are* $d \times d$ *positive matrices.*

The transport coefficients D and χ satisfy the local Einstein relation

$$2\,D(\rho) = \chi(\rho)\, f_0''(\rho) \tag{2.3}$$

where f_0 is the equilibrium free energy of the homogeneous system.

$$f_0'\big(\rho(x)\big) = \lambda_0(x) \qquad\qquad x \in \partial\Lambda \tag{2.4}$$

We denote by $\bar\rho = \bar\rho(x)$, $x \in \Lambda$, the stationary solution, assumed to be unique, of (2.1)–(2.4).

To state the third postulate, we need some preliminaries. Consider a time dependent variation $F = F(t, x)$ of the external field so that the total applied field is $E + F$. The current then becomes $j = J^F(\rho) = J(\rho) + \chi(\rho)F$.

Given a time interval $[0, T]$, we then introduce the total power dissipated by the extra current

$$L_{[0,T]}(F) = \frac{1}{2}\int_0^T dt\, \big\langle \big[J^F(\rho^F) - J(\rho^F)\big] \cdot F \big\rangle = \frac{1}{2}\int_0^T dt\, \big\langle F \cdot \chi(\rho^F) F \big\rangle \quad (2.5)$$

where $\langle \cdot \rangle$ denotes the integration over Λ and ρ^F is the solution of the continuity equation with current $j = J^F(\rho)$.

The argument behind (2.5) is the following. Fix a point (t, x) and let $\rho(t, x)$ be the local density. A local variation dF of the external field induces the variation of current $dj = \chi\big(\rho(t, x)\big)dF$. The infinitesimal power dissipated locally is therefore $F \cdot dj = F \cdot \chi\big(\rho(t, x)\big)dF$. By integrating firstly over dF, keeping the value of $\rho(t, x)$ constant and then over dx and dt we get (2.5).

We define a *cost functional* on the set of space time trajectories as follows. Given a trajectory $\hat\rho = \hat\rho(t,x)$ we set

$$I_{[0,T]}(\hat\rho) = \inf_{F\,:\,\rho^F = \hat\rho} \, L_{[0,T]}(F) \tag{2.6}$$

namely we minimize over all the variation of the applied field F which produce the trajectory $\hat\rho$. If $\hat\rho$ solves the hydrodynamic equation (2.1)–(2.4) its cost vanishes. In view of (2.5), a computation shows that

$$I_{[0,T]}(\hat\rho) = \frac{1}{2}\int_0^T dt\, \Big\langle \big[\partial_t\hat\rho + \nabla\cdot J(\hat\rho)\big]\, K(\hat\rho)^{-1} \big[\partial_t\hat\rho + \nabla\cdot J(\hat\rho)\big] \Big\rangle \tag{2.7}$$

where the positive operator $K(\hat\rho)$ is defined on functions $u:\Lambda\to\mathbb{R}$ vanishing at the boundary $\partial\Lambda$ by $K(\hat\rho)u = -\nabla\cdot\big(\chi(\hat\rho)\nabla u\big)$.

Our third postulate is then stated as follows.

3. *The nonequilibrium free energy of the system is*

$$\mathcal{F}(\rho) = \inf_{\substack{\hat\rho\,:\,\hat\rho(0)=\bar\rho \\ \hat\rho(+\infty)=\rho}} I_{[0,\infty]}(\hat\rho) \tag{2.8}$$

the functional $\mathcal{F}$ is the maximal solution of the infinite dimensional Hamilton-Jacobi equation

$$\frac{1}{2}\Big\langle \nabla \frac{\delta \mathcal{F}}{\delta \rho} \cdot \chi(\rho) \nabla \frac{\delta \mathcal{F}}{\delta \rho} \Big\rangle - \Big\langle \frac{\delta \mathcal{F}}{\delta \rho} \, \nabla \cdot J(\rho) \Big\rangle = 0 \tag{2.9}$$

where, for ρ that satisfies (2.4), $\delta \mathcal{F}/\delta \rho$ vanishes at the boundary of Λ. The arbitrary additive constant on such solution is determined by the condition $\mathcal{F}(\bar{\rho}) = 0$. Indeed, by considering the functional in (2.7) as an action functional in variables $\hat{\rho}$ and $\partial_t \hat{\rho}$ and performing a Legendre transform, the associated Hamiltonian is

$$\mathcal{H}(\rho, \Pi) = \frac{1}{2}\Big\langle \nabla \Pi \cdot \chi(\rho) \nabla \Pi \Big\rangle + \Big\langle \nabla \Pi \cdot J(\rho) \Big\rangle \tag{2.10}$$

The optimal trajectory ρ^* for the variational problem (2.8) is characterized as follows. Let

$$J^*(\rho) = -\chi(\rho) \nabla \frac{\delta \mathcal{F}}{\delta \rho} - J(\rho) \tag{2.11}$$

then ρ^* is the time reversal of the solution to

$$\partial_t \rho + \nabla \cdot J^*(\rho) = \partial_t \rho - \nabla \cdot \Big\{ D(\rho) \nabla \rho - \chi(\rho) \Big[E + \nabla \frac{\delta \mathcal{F}}{\delta \rho} \Big] \Big\} = 0 \tag{2.12}$$

with the boundary condition (2.4).

The previous claim is proven as follows. Let $\mathcal{F}$ be the maximal solution of the Hamilton-Jacobi equation and J^* as defined in (2.11). Fix a time interval $[0,T]$ and a path $\hat{\rho}(t)$, $t \in [0,T]$. We claim that

$$\begin{aligned} I_{[0,T]}(\hat{\rho}) &= \mathcal{F}\big(\rho(T)\big) - \mathcal{F}\big(\rho(0)\big) \\ &\quad + \frac{1}{2}\int_0^T dt\, \Big\langle \big[\partial_t \hat{\rho} - \nabla \cdot J^*(\hat{\rho})\big]\, K(\hat{\rho})^{-1} \big[\partial_t \hat{\rho} - \nabla \cdot J^*(\hat{\rho})\big] \Big\rangle \end{aligned} \tag{2.13}$$

as can be shown by a direct computation using (2.7), the Hamilton-Jacobi equation (2.9) and the definition (2.11) of J^*. From the identity (2.13) we immediately deduce that the optimal path for the variational problem (2.8) is the time reversal of the solution to (2.12).

Since the optimal trajectory is the time reversal of the solution to (2.12), the applied field is

$$F = -\nabla \frac{\delta \mathcal{F}}{\delta \rho}$$

On the other hand, by (2.11) and the Hamilton-Jacobi equation (2.9),

$$\Big\langle \frac{\delta F}{\delta \rho}\, \partial_t \hat{\rho} \Big\rangle = -\Big\langle J(\hat{\rho}) \cdot \chi(\hat{\rho}) \nabla \frac{\delta \mathcal{F}}{\delta \rho} \Big\rangle = \big\langle J(\hat{\rho}) \cdot F \big\rangle$$

which is the power given to system by the applied field F. Hence

$$\mathcal{F}(\rho) - \mathcal{F}(\bar{\rho}) = \int_0^\infty dt\, \Big\langle \frac{\delta \mathcal{F}}{\delta \rho}\, \partial_t \hat{\rho} \Big\rangle = \int_0^\infty dt\, \big\langle J(\hat{\rho}) \cdot F \big\rangle$$

is the total work done by the external field.

Correlation functions

We introduce the *pressure* functional as the Legendre transform of free energy $\mathcal{F}$

$$\mathcal{G}(h) = \sup_{\rho} \left\{ \langle h\rho \rangle - \mathcal{F}(\rho) \right\}$$

By Legendre duality, the Hamilton-Jacobi equation (2.9) can then be rewritten in terms of $\mathcal{G}$ as

$$\frac{1}{2}\Big\langle \nabla h \cdot \chi\Big(\frac{\delta \mathcal{G}}{\delta h}\Big) \nabla h \Big\rangle - \Big\langle \nabla h \cdot D\Big(\frac{\delta \mathcal{G}}{\delta h}\Big) \nabla \frac{\delta \mathcal{G}}{\delta h} + \chi\Big(\frac{\delta \mathcal{G}}{\delta h}\Big) E \Big\rangle = 0 \qquad (3.1)$$

where h vanishes at the boundary of Λ.

The functional $\mathcal{G}$ is the generating functional of the correlation functions; in particular by defining

$$C(x,y) = \frac{\delta^2}{\delta h(x) \delta h(y)} \mathcal{G}(h)$$

we have, since $\mathcal{F}$ has a minimum at $\bar{\rho}$,

$$\mathcal{G}(h) = \langle h, \bar{\rho} \rangle + \frac{1}{2} \langle h, Ch \rangle + o(h^2)$$

or equivalently

$$\mathcal{F}(\rho) = \frac{1}{2} \langle (\rho - \bar{\rho}), C^{-1}(\rho - \bar{\rho}) \rangle + o((\rho - \bar{\rho})^2)$$

By expanding the Hamilton-Jacobi equation (3.1) to the second order in h, and we using that $\delta\mathcal{G}/\delta h(x) = \bar{\rho}(x) + Ch(x) + o(h^2)$, we get the following equation for C

$$\left\langle \nabla h \cdot \left[\frac{1}{2}\chi(\bar{\rho})\nabla h - \nabla(D(\bar{\rho})Ch) + \chi'(\bar{\rho})(Ch)E \right] \right\rangle = 0 \tag{3.2}$$

We now make the change of variable

$$C(x,y) = C_{eq}(x)\delta(x-y) + B(x,y)$$

where $C_{eq}(x)$ is the equilibrium covariance, given by $C_{eq}(x) = (1/2)D^{-1}(\bar{\rho}(x))\chi(\bar{\rho}(x))$. Equation (3.2) for the correlation function then gives the following equation for B

$$\mathcal{L}^{\dagger}B(x,y) = \alpha(x)\delta(x-y) \tag{3.3}$$

where $\mathcal{L}^{\dagger}$ is the formal adjoint of the elliptic operator $\mathcal{L} = L_x + L_y$, where

$$L_x = D_{ij}(\bar{\rho}(x))\partial_{x_i}\partial_{x_j} + \chi'_{ij}(\bar{\rho}(x))E_j(x)\partial_{x_i}$$

and

$$\alpha(x) = \partial_{x_i}\left[\chi'_{ij}\left(\bar{\rho}(x)\right) D^{-1}_{jk}\left(\bar{\rho}(x)\right)\bar{J}_k(x)\right]$$

where we racall $\bar{J} = J(\bar{\rho}) = -D(\bar{\rho}(x))\nabla\bar{\rho}(x) + \chi(\bar{\rho}(x))E(x)$ is the macroscopic current in the stationary profile.

We next derive, using the Hamilton-Jacobi equation (3.1), a recursive formula for the n-point correlation function $C_n(x_1, \ldots, x_n)$. This is defined in terms of the pressure functional $\mathcal{G}$ as

$$C_n(x_1, \ldots, x_n) = \frac{\delta^n \mathcal{G}}{\delta h(x_1) \cdots \delta h(x_n)} \Big|_{h=0}$$

so that $C_1(x) = \bar{\rho}(x)$ and C_2 is the two-point correlation function discussed above.

By expanding the functional derivative of $\mathcal{G}$ we get

$$\frac{\delta \mathcal{G}(h)}{\delta h(x_1)} = \bar{\rho}(x_1) + \sum_{n \geq 1} \frac{1}{n!} \mathcal{G}_n(h; x_1)$$

where

$$\mathcal{G}_n(h; x_1) = \int_\Lambda dx_2 \ldots dx_{n+1} h(x_2) \ldots h(x_{n+1}) C_{n+1}(x_1, x_2, \ldots, x_{n+1})$$

$$
\begin{aligned}
&\frac{1}{(n+1)!}\mathcal{L}^{\dagger}_{n+1}C_{n+1}(x_1,x_2,\ldots,x_{n+1})\\
&=\Bigg\{\frac{1}{2}\sum_{\substack{\vec{\iota}\\ N(\vec{\iota})=n-1}}\frac{1}{K(\vec{\iota})}\nabla_{x_1}\cdot\Big(\chi^{(\Sigma(\vec{\iota}))}(\bar{\rho}(x_1))C_{\vec{\iota}}(x_1,\ldots,x_n)\nabla_{x_1}\delta(x_1-x_{n+1})\Big)\\
&-\sum_{\substack{\vec{\iota}\\ N(\vec{\iota})=n,i_n=0}}\frac{1}{K(\vec{\iota})}\nabla_{x_1}\cdot\nabla_{x_1}\Big(D^{(\Sigma(\vec{\iota})-1)}(\bar{\rho}(x_1))C_{\vec{\iota}}(x_1,\ldots,x_{n+1})\Big)\\
&+\sum_{\substack{\vec{\iota}\\ N(\vec{\iota})=n,i_n=0}}\frac{1}{K(\vec{\iota})}\nabla_{x_1}\cdot\Big(\chi^{(\Sigma(\vec{\iota}))}(\bar{\rho}(x_1))C_{\vec{\iota}}(x_1,\ldots,x_{n+1})E(x_1)\Big)\Bigg\}^{sym}
\end{aligned}
$$

Equilibrium states

We define the system to be in *equilibrium* if and only if the current in the stationary profile $\bar\rho$ vanishes, i.e. $J(\bar\rho) = 0$. A particular case is that of a *homogeneous equilibrium state*, obtained by setting the external field $E = 0$ and chosing a constant chemical potential potential at the boundary, i.e. $\lambda_0(x) = \bar\lambda$. Let $\bar\rho = \text{const.}$ be the equilibrium density, i.e. $\bar\rho$ solves $\bar\lambda = f_0'(\bar\rho)$. It is then readily seen that the functional $\mathcal{F}$ defined in (2.8) is given by

$$\mathcal{F}(\rho) = \int_\Lambda dx \left\{ f_0(\rho(x)) - f_0(\bar\rho) - f_0'(\bar\rho)\left[\rho(x) - \bar\rho\right] \right\}$$

in which the first difference is the variation of the free energy f_0 while the second term is due the interaction with the reservoirs.

We next show that also for a non homogenous equilibrium, characterized by a non constant stationary profile $\bar\rho(x)$ such that $J(\bar\rho) = 0$ the free energy functional $\mathcal{F}$ can be explicitly computed. Let

$$f(\rho, x) = \int_{\bar\rho(x)}^{\rho} dr \int_{\bar\rho(x)}^{r} ds\, f_0''(s) = f_0(\rho) - f_0(\bar\rho(x)) - f_0'\big(\bar\rho(x)\big)\left[\rho - \bar\rho(x)\right]$$

we claim that the maximal solution of the Hamilton-Jacobi equation (2.9) is

$$\mathcal{F}(\rho) = \int_\Lambda dx\, f\big(\rho(x), x\big) \tag{4.1}$$

Indeed from the previous expression we get

$$\frac{\delta \mathcal{F}}{\delta \rho(x)} = f_0'(\rho(x)) - f_0'(\bar{\rho}(x))$$

so that, by an integration by parts,

$$\begin{aligned} &\frac{1}{2}\Big\langle \nabla\left[f_0'(\rho) - f_0'(\bar\rho)\right] \cdot \chi(\rho) \nabla \left[f_0'(\rho) - f_0'(\bar\rho)\right]\Big\rangle \\ &\qquad + \Big\langle \left[f_0'(\rho) - f_0'(\bar\rho)\right] \nabla \cdot \Big[D(\rho)\nabla\rho - \chi(\rho)E\Big]\Big\rangle \\ &= \frac{1}{2}\Big\langle \nabla\left[f_0'(\rho) - f_0'(\bar\rho)\right] \cdot \chi(\rho)\Big[\nabla f_0'(\bar\rho) - 2E\Big]\Big\rangle = 0 \end{aligned}$$

where we used (2.3) and $(1/2)\nabla f_0'(\bar\rho) - E = \chi(\bar\rho)^{-1} J(\bar\rho) = 0$.

In remains to show that $\mathcal{F}$, as defined in (4.1), is the maximal solution to the Hamilton-Jacobi equation (2.9). Recalling (2.7), simple computations show that

$$\begin{aligned} I_{[0,T]}(\hat\rho) \;=\; & \mathcal{F}\big(\hat\rho(T)\big) - \mathcal{F}\big(\hat\rho(0)\big) \\ & + \frac{1}{2}\int_0^T dt \,\Big\langle \left[\partial_t \hat\rho - \nabla\cdot J(\hat\rho)\right] K(\hat\rho)^{-1} \left[\partial_t\hat\rho - \nabla \cdot J(\hat\rho)\right]\Big\rangle \end{aligned} \tag{4.2}$$

which clearly implies the maximality of $\mathcal{F}$.

the condition $J(\bar{\rho}) = 0$ is equivalent to either one of the following statements.

– There exists a function $\lambda : \Lambda \to \mathbb{R}$ such that

$$2\,E(x) = \nabla\lambda(x)\,, \quad x \in \Lambda \qquad \lambda(x) = \lambda_0(x)\,, \quad x \in \partial\Lambda \tag{4.3}$$

– The system is *macroscpically reversible* in the sense that for each profile ρ we have $J^*(\rho) = J(\rho)$.

We emphasize that the notion of macroscopic reversibility does not imply that an underlying microscopic model satisfies the detailed balance condition.

We also note that macroscopic reversibility $J(\rho) = J^*(\rho)$ implies the invariance of the Hamiltonian $\mathcal{H}$ in (2.10) under the time reversal symmetry, $(\rho, \Pi) \mapsto \big(\rho, \delta\mathcal{F}/\delta\rho - \Pi\big)$, where $\mathcal{F}$ is the maximal solution of the Hamilton-Jacobi equation (2.9).

So far we have assumed the Einstein relation and we have shown that -for equilibrium systems- it implies (4.1). Conversely, we now show that macroscopic reversibility and (4.1) implies the Einstein relation (2.3). By writing explicitely $J(\rho) = J^*(\rho)$ we obtain

$$-\big[\chi(\rho)R(\rho) - 2D(\rho)\big]\nabla\rho + \chi(\rho)\big[R(\bar\rho) - 2\chi^{-1}(\bar\rho)D(\bar\rho)\big]\nabla\bar\rho = 0 \qquad (4.4)$$

where R is the second derivative of f_0 in the case of one-component systems while $R_{ij} = \partial_{\rho_i}\partial_{\rho_j} f_0$ for multi-component systems. In (4.4) we used, besides (4.1) $J(\bar\rho) = 0$ to eliminate E. Note that $J(\bar\rho) = 0$ follows from the Hamilton-Jacobi equation and $J(\rho) = J^*(\rho)$ without further assumptions. Since $\nabla\rho$ and $\nabla\bar\rho$ are arbitrary the Einstein relation $2D = \chi R$ follows from (4.4).

We have defined the macroscopic reversibility as the identity between the currents $J(\rho)$ and $J^*(\rho)$. We emphasize that this is not equivalent to the identity between $\nabla \cdot J(\rho)$ and $\nabla \cdot J^*(\rho)$. Indeed, we next show that there exists a non reversible system, i.e. satisfying $J(\bar{\rho}) \neq 0$, such that the optimal trajectory for the variational problem (2.8) is the time reversal of the solution to the hydrodynamic equation (2.1)–(2.5).

Let $\Lambda = [0,1]$, $D(\rho) = \chi(\rho) = 1$, $\lambda_0(0) = \lambda_0(1) = \bar{\lambda}$, and a constant external field $E \neq 0$. In this case hydrodynamic evolution of the density is given by the heat equation independently of the field E. The stationary profile is $\bar{\rho} = \bar{\lambda}$, the associated current is $J(\bar{\rho}) = E \neq 0$. By a computation analogous to the one leading to (3.2), we easily get that

$$\mathcal{F}(\rho) = \int_0^1 dx \left[\rho(x) - \bar{\rho}\right]^2$$

and the optimal trajectory for the variational problem (2.8) is the time reversal of the solution to the heat equation. On the other hand $J(\rho) = -\nabla\rho + E$ while $J^*(\rho) = -\nabla\rho - E$

The ABC Model

We here consider - both from a microscopic and macroscopic point of view - a model with two conservation laws. Given an integer $N \geq 1$ let $\mathbb{Z}_N = \{1, \ldots, N\}$ be the discrete ring with N sites so that $N + 1 \equiv 1$. The microscopic space state is given by $\Omega_N = \{A, B, C\}^{\mathbb{Z}_N}$ so that at each site $x \in \mathbb{Z}_N$ the occupation variable, denoted by η_x, take values in the set $\{A, B, C\}$; one may think that A, B stand for two different species of particles and C for an empty site. Note that this state space takes into account an exclusion condition: at each site there is at most one specie of particles.

We first consider a weakly asymmetric dynamics that fits in the framework discussed in Section 2 that is defined by choosing the following transition rates. If the occupation variables across the bond $\{x, x+1\}$ are (ξ, ζ), they are exchanged to (ζ, ξ) with rate $(1/2)\exp\{(E_\xi - E_\zeta)/N\}$ for fixed constant external fields E_A, E_B, E_C.

the hydrodynamic equations for the densities of A and B particles are given by

$$\partial_t \begin{pmatrix} \rho_A \\ \rho_B \end{pmatrix} = \frac{1}{2}\Delta \begin{pmatrix} \rho_A \\ \rho_B \end{pmatrix} - \nabla \cdot \begin{pmatrix} \rho_A(1-\rho_A) & -\rho_A\rho_B \\ -\rho_A\rho_B & \rho_B(1-\rho_B) \end{pmatrix} \begin{pmatrix} E_A - E_C \\ E_B - E_C \end{pmatrix} \tag{5.1}$$

of course the density of C particles is then $\rho_C = 1 - \rho_A - \rho_B$.

The functional $I_{[T_1,T_2]}$ in (2.7) with $D = (1/2)\mathbb{1}$ and mobility

$$\chi(\rho_A, \rho_B) = \begin{pmatrix} \rho_A(1-\rho_A) & -\rho_A\rho_B \\ -\rho_A\rho_B & \rho_B(1-\rho_B) \end{pmatrix} \tag{5.2}$$

is the dynamical large functional associated to this model. The free energy is the maximal solution of the Hamilton-Jacobi equation (2.9) which can be easily computed. Namely,

$$\begin{aligned} \mathcal{F}^0_{m_A,m_B}(\rho_A, \rho_B) = \int dx \Big[\rho_A \log \frac{\rho_A}{m_A} + \rho_B \log \frac{\rho_B}{m_B} \\ + (1-\rho_A-\rho_B) \log \frac{1-\rho_A-\rho_B}{1-m_A-m_B}\Big] \end{aligned} \tag{5.3}$$

where $\int dx\, \rho_A = m_A$ and $\int dx\, \rho_B = m_B$. If E_A, E_B and E_C are not all equal, this model is a nonequilibrium model nevertheless, in view of the periodic boundary conditions, its free energy is independent of the external field;

We next discuss a different choice of the weakly asymmetric perturbation which, as we shall see, does not satisfy the Einstein relation (2.3). This choice is the one referred to in the literature as the *ABC model*. The transition rates are the following. If the occupation variables across the bond $\{x, x+1\}$ are (ξ, ζ), they are exchanged to (ζ, ξ) with rate $(1/2)\exp\{V(\xi,\eta)/N\}$ where $V(A,B) = V(B,C) = V(C,A) = -\beta$ and $V(B,A) = V(C,B) = V(A,C) = \beta$ for some $\beta > 0$. Therefore the A-particles prefer to jump to the left of the B-particles but to the right of the C-particles while the B-particles prefer to jump to the the left of the C-particles, i.e. the preferred sequence is ABC and its cyclic permutations.

the hydrodynamic equations are

$$\partial_t \begin{pmatrix} \rho_A \\ \rho_B \end{pmatrix} + \nabla \cdot \begin{pmatrix} J_A(\rho_A, \rho_B) \\ J_B(\rho_A, \rho_B) \end{pmatrix} = 0 \tag{5.4}$$

where

$$J(\rho_A, \rho_B) = \begin{pmatrix} J_A(\rho_A, \rho_B) \\ J_B(\rho_A, \rho_B) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\nabla\rho_A + \beta\rho_A(1 - 2\rho_B - \rho_A) \\ -\frac{1}{2}\nabla\rho_B + \beta\rho_B(2\rho_A + \rho_B - 1) \end{pmatrix} \tag{5.5}$$

The asymmetric term in the hydrodynamic equation (5.4) is not of the form $\nabla \cdot \big(\chi(\rho)E\big)$ as in (2.2). Hence the Einstein relation (2.3) does not hold.

The appropriate cost functional is however still given by (2.7) with J as in (5.5) and χ as in (5.2) The solution of the Hamilton-Jacobi equation (2.9) then gives the free energy. In the case of equal densities $\int dx\, \rho_A = \int dx\, \rho_B = 1/3$, a straightforward computation shows that for any positive β the solution is given by the functional

$$\mathcal{F}^{\beta}_{\frac{1}{3},\frac{1}{3}}(\rho_A, \rho_B) = \mathcal{F}^{0}_{\frac{1}{3},\frac{1}{3}}(\rho_A, \rho_B)$$
$$+2\beta \int_0^1 dx \int_0^1 dy\, y \Big\{\rho_A(x)\rho_B(x+y) + \rho_B(x)[1-\rho_A(x+y)-\rho_B(x+y)]$$
$$+ [1-\rho_A(x)-\rho_B(x)]\rho_A(x+y)\Big\}$$

where $\mathcal{F}^{0}_{\frac{1}{3},\frac{1}{3}}$ is the functional in (5.3) with $m_A = m_B = 1/3$.