Long time semiclassical evolution

T. PAUL
C.N.R.S. and D.M.A., École Normale Supérieure, Paris

pour Sandro, 27 augusto 2008

Quantum evolution

$$\left\{ \begin{array}{rcl} i\hbar\partial_t\psi^t & = & H\psi^t \\ \psi^{t=0} & = & \psi \end{array} \right.$$

Quantum evolution

$$\left\{ \begin{array}{ccc} i\hbar\partial_t\psi^t & = & H\psi^t \\ \psi^{t=0} & = & \psi \end{array} \right. \qquad \text{OR} \qquad \left\{ \begin{array}{ccc} \dot{\mathcal{O}}^t & = & \frac{1}{i\hbar}[\mathcal{O}^t, H] \\ \mathcal{O}^{t=0} & = & \mathcal{O} \end{array} \right.$$

Quantum evolution

$$\left\{\begin{array}{lcl} i\hbar\partial_t\psi^t & = & H\psi^t \\ \psi^{t=0} & = & \psi \end{array}\right. \quad \text{OR} \quad \left\{\begin{array}{lcl} \dot{\mathcal{O}}^t & = & \frac{1}{i\hbar}[\mathcal{O}^t, H] \\ \mathcal{O}^{t=0} & = & \mathcal{O} \end{array}\right.$$

$t \leq T_\hbar \to +\infty$ as $\hbar \to 0$

Quantum evolution

$$\left\{\begin{array}{ccc} i\hbar\partial_t\psi^t & = & H\psi^t \\ \psi^{t=0} & = & \psi \end{array}\right. \quad \text{OR} \quad \left\{\begin{array}{ccc} \dot{\mathcal{O}}^t & = & \frac{1}{i\hbar}[\mathcal{O}^t, H] \\ \mathcal{O}^{t=0} & = & \mathcal{O} \end{array}\right.$$

$$t \leq T_\hbar \to +\infty \text{ as } \hbar \to 0$$

$T_\hbar \sim log(\hbar^{-1})$: unstable case
$T_\hbar \sim \frac{1}{\hbar}$: stable case.

Quantum evolution

$$\left\{\begin{array}{ccc} i\hbar\partial_t\psi^t & = & H\psi^t \\ \psi^{t=0} & = & \psi \end{array}\right. \quad \text{OR} \quad \left\{\begin{array}{ccc} \dot{\mathcal{O}}^t & = & \frac{1}{i\hbar}[\mathcal{O}^t, H] \\ \mathcal{O}^{t=0} & = & \mathcal{O} \end{array}\right.$$

$$t \leq T_\hbar \to +\infty \text{ as } \hbar \to 0$$

$T_\hbar \sim log(\hbar^{-1})$: unstable case
$T_\hbar \sim \frac{1}{\hbar}$: stable case.

Do we recover Classical Mechanics ?

Quantum evolution

$$\left\{ \begin{array}{rcl} i\hbar\partial_t\psi^t & = & H\psi^t \\ \psi^{t=0} & = & \psi \end{array} \right. \quad \text{OR} \quad \left\{ \begin{array}{rcl} \dot{\mathcal{O}}^t & = & \frac{1}{i\hbar}[\mathcal{O}^t, H] \\ \mathcal{O}^{t=0} & = & \mathcal{O} \end{array} \right.$$

$$t \leq T_\hbar \to +\infty \text{ as } \hbar \to 0$$

$T_\hbar \sim log(\hbar^{-1})$: unstable case
$T_\hbar \sim \frac{1}{\hbar}$: stable case.

Do we recover Classical Mechanics ?

Answer : not always.

Quantum evolution

$$\left\{\begin{array}{ccc} i\hbar\partial_t\psi^t & = & H\psi^t \\ \psi^{t=0} & = & \psi \end{array}\right. \quad \text{OR} \quad \left\{\begin{array}{ccc} \dot{\mathcal{O}}^t & = & \frac{1}{i\hbar}[\mathcal{O}^t, H] \\ \mathcal{O}^{t=0} & = & \mathcal{O} \end{array}\right.$$

$$t \leq T_\hbar \to +\infty \text{ as } \hbar \to 0$$

$T_\hbar \sim log(\hbar^{-1})$: unstable case
$T_\hbar \sim \frac{1}{\hbar}$: stable case.

Do we recover Classical Mechanics ?

Answer : not always.

New phenomena : delocalization, reconstruction, ubiquity ...... contained in the (classical) infinite time.

Bambusi-Graffi-P 1998, Bouzuoina-Robert 2002 for Egorov
Haguedorn, Combescure-Robert, de Bièvre-Robert 1995-2002
for coherent states
a lot of papers in physics, including experimental

Why long time ?

Quantum Mechanics : stability, stationnary states, eigenvectors
Schrödinger (linear) equation

$$i\hbar\partial_t\psi = H\psi$$

Very different form Classical Mechanics :

$$\begin{cases} \dot{x} & = & \partial_\xi h(x,\xi) \\ \dot{\xi} & = & -\partial_x h(x,\xi) \end{cases}$$

Why long time ?

Quantum Mechanics : stability, stationnary states, eigenvectors
Schrödinger (linear) equation

$$i\hbar\partial_t\psi = H\psi$$

Very different form Classical Mechanics :

$$\begin{cases} \dot{x} = \partial_\xi h(x,\xi) \\ \dot{\xi} = -\partial_x h(x,\xi) \end{cases}$$

How to "construct" eigenvectors ?
Link with models in atomic physics (cold atoms)
How do we understand the transition Quantum/Classical ?

Why semiclassical approximation ?

Asymptotic method (very efficient)
Semiclassical limit $\subset$ Quantum Mechanics
ex. atomic systems (scalings)
systems of spins (N spins-$\frac{1}{2}$ (symmetrized) $\sim$1 spin-2N)
Corresponds to experimental situations

Why coherent states ?

Natural way of taking semiclassical limit
More precise than, e.g., Egorov theorem
Generalize to more geometrical situations (ex. spins)

Why coherent states ?

Natural way of taking semiclassical limit
More precise than, e.g., Egorov theorem
Generalize to more geometrical situations (ex. spins)
Coherent state at (q, p) and symbol-vacuum a :

$$\psi_a^{qp}(x) = \hbar^{-n/4} a\left(\frac{x-q}{\sqrt{\hbar}}\right) e^{i\frac{px}{\hbar}}$$

Main ideas

- Coherent state follows the classical flow, and $a$follows the linearized flow, *up to a certain time* $T_0(\hbar)$

Main ideas

• Coherent state follows the classical flow, and $a$follows the linearized flow, *up to a certain time* $T_0(\hbar)$

• After, quantum effects are persistent, and classical paradigm is lost

Main ideas

- Coherent state follows the classical flow, and $a$follows the linearized flow, *up to a certain time* $T_0(\hbar)$
- After, quantum effects are persistent, and classical paradigm is lost
- New "dynamics" enter the game, that we can sometimes compute

Main ideas

- Coherent state follows the classical flow, and *a*follows the linearized flow, *up to a certain time* $T_0(\hbar)$
- After, quantum effects are persistent, and classical paradigm is lost
- New "dynamics" enter the game, that we can sometimes compute
- The wave packet can reconstruct, but with (always) a singular vacuum

Main ideas

- Coherent state follows the classical flow, and *a*follows the linearized flow, *up to a certain time* $T_0(\hbar)$
- After, quantum effects are persistent, and classical paradigm is lost
- New “dynamics” enter the game, that we can sometimes compute
- The wave packet can reconstruct, but with (always) a singular vacuum
- Overlapping between quantum undeterminism and classical unpredictability

Outline

Free evolution on the circle

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi \quad \psi \in L^2(S^1)$$

$$\sigma(-\frac{\hbar^2}{2}\Delta) = \left\{\frac{\hbar^2 m^2}{2},\ m \in \mathbb{Z}\right\},\ \text{phases : } e^{it\frac{\hbar m^2}{2}}$$

Free evolution on the circle

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi \quad \psi \in L^2(S^1)$$

$$\sigma(-\frac{\hbar^2}{2}\Delta) = \left\{\frac{\hbar^2 m^2}{2},\ m \in \mathbb{Z}\right\},\ \text{phases : } e^{it\frac{\hbar m^2}{2}} \ \Rightarrow$$

Quantum Flow is $\frac{4\pi}{\hbar}$-periodic.

Free evolution on the circle

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi \quad \psi \in L^2(S^1)$$

$$\sigma(-\frac{\hbar^2}{2}\Delta) = \left\{\frac{\hbar^2 m^2}{2},\ m \in \mathbb{Z}\right\},\ \text{phases} : e^{it\frac{\hbar m^2}{2}} \ \Rightarrow$$

Quantum Flow is $\frac{4\pi}{\hbar}$-periodic.
Classical flow is NOT.

Free evolution on the circle

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi \quad \psi\in L^2(S^1)$$

$$\sigma(-\frac{\hbar^2}{2}\Delta) = \left\{\frac{\hbar^2 m^2}{2},\ m\in\mathbb{Z}\right\},\ \text{phases} : e^{it\frac{\hbar m^2}{2}} \ \Rightarrow$$

Quantum Flow is $\frac{4\pi}{\hbar}$-periodic.

Classical flow is NOT.

(except with quantized momenta ($m\hbar$) but

Free evolution on the circle

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi \quad \psi \in L^2(S^1)$$

$$\sigma(-\frac{\hbar^2}{2}\Delta) = \left\{\frac{\hbar^2 m^2}{2},\ m \in \mathbb{Z}\right\},\ \text{phases : } e^{it\frac{\hbar m^2}{2}} \ \Rightarrow$$

Quantum Flow is $\frac{4\pi}{\hbar}$-periodic.

Classical flow is NOT.

(except with quantized momenta ($m\hbar$) but
quantum period $=$ 2 $\times$ classical one (like harm. osc.))

Free evolution on the circle

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi \quad \psi\in L^2(S^1)$$

$$\sigma(-\frac{\hbar^2}{2}\Delta) = \left\{\frac{\hbar^2 m^2}{2},\ m\in\mathbb{Z}\right\},\ \text{phases}: e^{it\frac{\hbar m^2}{2}} \ \Rightarrow$$

Quantum Flow is $\frac{4\pi}{\hbar}$-periodic.
Classical flow is NOT.

(except with quantized momenta ($m\hbar$) but
quantum period $= 2\times$ classical one (like harm. osc.))
Schrödinger cats : consider fractional times : $t = \frac{p}{q}\frac{4\pi}{\hbar} \Rightarrow$

Free evolution on the circle

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi \quad \psi \in L^2(S^1)$$

$$\sigma(-\frac{\hbar^2}{2}\Delta) = \left\{\frac{\hbar^2 m^2}{2},\ m \in \mathbb{Z}\right\},\ \text{phases}:\ e^{it\frac{\hbar m^2}{2}} \Rightarrow$$

Quantum Flow is $\frac{4\pi}{\hbar}$-periodic.

Classical flow is NOT.

(except with quantized momenta ($m\hbar$) but
quantum period $= 2 \times$ classical one (like harm. osc.))
Schrödinger cats : consider fractional times : $t = \frac{p}{q}\frac{4\pi}{\hbar} \Rightarrow$

Relocalization on q sites

.

General hamiltonian on the circle

$H = h(-i\hbar\partial_x),\ h(\xi) = \xi^2 + c\xi^3 + d\xi^4 + O(\xi^5)$

coherent state : $\varphi(x) = \hbar^{-1/4} \sum e^{-\frac{m^2}{2}\hbar} e^{imx}$

We fix $t = s\frac{4\pi}{\hbar},\ s$ integer

Theorem 1 : $\exists$ function g, $\hbar$-independent s.t.

$$0 < x < 2\pi, \quad e^{-it\frac{H}{\hbar}}\varphi(x) = g(x) + O(\hbar^{\frac{1}{2}})$$

General hamiltonian on the circle

$H = h(-i\hbar\partial_x),\ h(\xi) = \xi^2 + c\xi^3 + d\xi^4 + O(\xi^5)$

coherent state : $\varphi(x) = \hbar^{-1/4} \sum e^{-\frac{m^2}{2}\hbar} e^{imx}$

We fix $t = s\frac{4\pi}{\hbar},\ s$ integer

Theorem 1 : $\exists$ function g, $\hbar$-independent s.t.

$$0 < x < 2\pi, \quad e^{-it\frac{H}{\hbar}}\varphi(x) = g(x) + O(\hbar^{\frac{1}{2}})$$

Theorem 2 : but, if $\varphi^\epsilon(x) := \hbar^{-\frac{1-\epsilon}{4}} \sum e^{-\frac{m^2\hbar^{1-\epsilon}}{2}} e^{imx}$

General hamiltonian on the circle

$H = h(-i\hbar\partial_x), \; h(\xi) = \xi^2 + c\xi^3 + d\xi^4 + O(\xi^5)$

coherent state : $\varphi(x) = \hbar^{-1/4} \sum e^{-\frac{m^2}{2}\hbar} e^{imx}$

We fix $t = s\frac{4\pi}{\hbar}, \; s$ integer

Theorem 1 : $\exists$ function g, $\hbar$-independent s.t.

$$0 < x < 2\pi, \quad e^{-it\frac{H}{\hbar}}\varphi(x) = g(x) + O(\hbar^{\frac{1}{2}})$$

Theorem 2 : but, if $\varphi^\epsilon(x) := \hbar^{-\frac{1-\epsilon}{4}} \sum e^{-\frac{m^2\hbar^{1-\epsilon}}{2}} e^{imx}$

$$0 < x < 2\pi, \quad e^{-it\frac{H}{\hbar}}\varphi(x) = C\hbar^{-\frac{\epsilon}{2}} e^{-\frac{x}{sc\hbar^\epsilon}} + O(\hbar^{\frac{1}{2}})$$

General hamiltonian on the circle

$H = h(-i\hbar\partial_x),\ h(\xi) = \xi^2 + c\xi^3 + d\xi^4 + O(\xi^5)$

coherent state : $\varphi(x) = \hbar^{-1/4} \sum e^{-\frac{m^2}{2}\hbar} e^{imx}$

We fix $t = s\frac{4\pi}{\hbar},\ s$ integer

Theorem 1 : $\exists$ function g, $\hbar$-independent s.t.

$$0 < x < 2\pi, \quad e^{-it\frac{H}{\hbar}}\varphi(x) = g(x) + O(\hbar^{\frac{1}{2}})$$

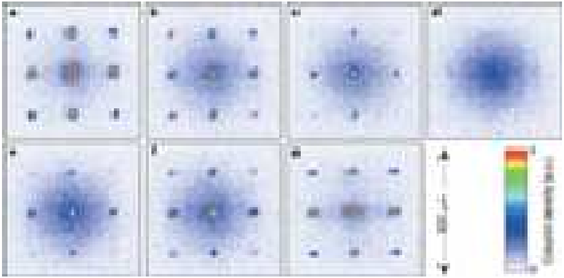
Theorem 2 : but, if $\varphi^\epsilon(x) := \hbar^{-\frac{1-\epsilon}{4}} \sum e^{-\frac{m^2\hbar^{1-\epsilon}}{2}} e^{imx}$

$$0 < x < 2\pi, \quad e^{-it\frac{H}{\hbar}}\varphi(x) = C\hbar^{-\frac{\epsilon}{2}} e^{-\frac{x}{sc\hbar^\epsilon}} + O(\hbar^{\frac{1}{2}})$$

Less localization permits relocalization, because of less sensitivity to non-linear classical effects (thanks to Heisenberg inequalities).

Cold atoms

Hamiltonian $H = \frac{1}{2}\hat{n}(\hat{n} - 1)$
$\hat{n}$ is a "number" operator, i.e. it has linear spectrum
$H \sim$ Laplacian on the circle



(I. Bloch, 2002)

H only an approximation $H = \frac{1}{2}\hat{n}(\hat{n} - 1) + \hat{n}^3 + \dots$

The case of a stable periodic trajectory

X $(n+1)$-dimensional manifold
$H: C_0^\infty(X) \to C^\infty(X)$ semiclassical elliptic pseudo-differential operator with leading symbol, $H(x,\xi)$
γ periodic trajectory of $H(x,\xi)$ elliptic and non-degenerate.
on $\mathbb{R}^n \times S^1$ $P_i = \hbar^2 D_{x_i}^2 + x_i^2$ and $\zeta = \hbar D_t$

Theorem

Quantum Birkhoff Normal Form
There exists a semiclassical Fourier integral operator $A_\varphi : C_0^\infty(X) \to C^\infty(\mathbb{R}^n \times S^1)$ *such that microlocally on a neighborhood,* $\mathcal{U}$*, of* $p = \tau = 0$

$$A_\varphi^* = A_\varphi^{-1}$$

and

$$A_\varphi H A_\varphi^{-1} = H'(P_1, ..., P_n, \zeta, \hbar) + H''$$

the symbol of H'' *vanishing to infinite order on* $p = \tau = 0$.

Creation of Schrödinger cat states, due to the interaction with transverse degrees of freedom.

Finite time c.s. propagation

Definition

Let $(q,p) \in \mathbb{R}^{en}$ and $a \in \mathcal{S}(\mathbb{R}^n)$. Then :

$$\psi_a^{qp}(x) := \hbar^{-\frac{n}{4}} a\left(\frac{x-q}{\sqrt{\hbar}}\right) e^{i\frac{px}{\hbar}}$$

example : $a(\eta) = e^{-\frac{\eta^2}{2}}$
but need of general "symbol (vacuum)".

Finite time c.s. propagation

Definition

Let $(q, p) \in \mathbb{R}^{en}$ and $a \in \mathcal{S}(\mathbb{R}^n)$. Then :

$$\psi_a^{qp}(x) := \hbar^{-\frac{n}{4}} a\left(\frac{x-q}{\sqrt{\hbar}}\right) e^{i\frac{px}{\hbar}}$$

example : $a(\eta) = e^{-\frac{\eta^2}{2}}$
but need of general "symbol (vacuum)".
$\forall a \ \psi_a^{qp}$ is (micro)localized at the point (q, p) (in phase-space).

Theorem

Let H such that $e^{it\frac{H}{\hbar}}$ *is unitary* $\forall t$ *and* $\psi_a^{qp} \in \mathcal{D}(H)$. *Let* $d\Phi_{qp}^t$ *the derivative of the flow starting at the point* (q,p). *Let us suppose that*

$$\exists \mu(q,p) > 0, \textit{Hölder continuous}, s.t. |d\Phi_{qp}^t| \leq Ce^{\mu(q,p)|t|}$$

Then $\exists M(t)$ *unitary* ($\hbar$*-independent*) *such that :*

$$||e^{it\frac{H}{\hbar}}\psi_a^{qp} - e^{i\frac{I(t)}{\hbar}}\psi_{M(t)a}^{\Phi^t(q,p)}||_{L^2} \leq C\hbar^{\frac{1}{2}}e^{3\mu(q,p)|t|}$$

Theorem

Let H such that $e^{it\frac{H}{\hbar}}$ is unitary $\forall t$ and $\psi_a^{qp} \in \mathcal{D}(H)$. Let $d\Phi_{qp}^t$ the derivative of the flow starting at the point (q,p). Let us suppose that

$$\exists \mu(q,p) > 0, \text{Hölder continuous}, s.t. |d\Phi_{qp}^t| \leq Ce^{\mu(q,p)|t|}$$

Then $\exists M(t)$ unitary ($\hbar$-independent) such that :

$$||e^{it\frac{H}{\hbar}}\psi_a^{qp} - e^{i\frac{I(t)}{\hbar}}\psi_{M(t)a}^{\Phi^t(q,p)}||_{L^2} \leq C\hbar^{\frac{1}{2}}e^{3\mu(q,p)|t|}$$

In particular $= O(\hbar^{\epsilon})$

Theorem

Let H such that $e^{it\frac{H}{\hbar}}$ is unitary $\forall t$ and $\psi_a^{qp} \in \mathcal{D}(H)$. Let $d\Phi_{qp}^t$ the derivative of the flow starting at the point (q,p). Let us suppose that

$$\exists \mu(q,p) > 0, \textit{Hölder continuous}, s.t. |d\Phi_{qp}^t| \leq Ce^{\mu(q,p)|t|}$$

Then $\exists M(t)$ unitary ($\hbar$-independent) such that :

$$||e^{it\frac{H}{\hbar}}\psi_a^{qp} - e^{i\frac{l(t)}{\hbar}}\psi_{M(t)a}^{\Phi^t(q,p)}||_{L^2} \leq C\hbar^{\frac{1}{2}}e^{3\mu(q,p)|t|}$$

In particular $= O(\hbar^\epsilon)$ for $t < \frac{1-\epsilon}{6\mu(q,p)} log(D\hbar^{-1})$,

Theorem

Let H such that $e^{it\frac{H}{\hbar}}$ is unitary $\forall t$ and $\psi_a^{qp} \in \mathcal{D}(H)$. Let $d\Phi^t_{qp}$ the derivative of the flow starting at the point (q,p). Let us suppose that

$$\exists \mu(q,p) > 0, \text{Hölder continuous}, s.t. |d\Phi^t_{qp}| \leq Ce^{\mu(q,p)|t|}$$

Then $\exists M(t)$ unitary ($\hbar$-independent) such that :

$$||e^{it\frac{H}{\hbar}}\psi_a^{qp} - e^{i\frac{I(t)}{\hbar}}\psi^{\Phi^t(q,p)}_{M(t)a}||_{L^2} \leq C\hbar^{\frac{1}{2}}e^{3\mu(q,p)|t|}$$

In particular $= O(\hbar^\epsilon)$ for $t < \frac{1-\epsilon}{6\mu(q,p)} log(D\hbar^{-1})$, where D is a (dimensional) constant $D = \sup_{t\in\mathbb{R}} ||H^3(t)a||_{L^2}/\mu$.

$M(t)$ "quantization" of the linearized flow
$I(t)$ Lagrangian action along the flow

Long time c.s. propagation

For simplicity (q, p) periodic and t multiple of the period.

Theorem

$\exists S(x), S(0) = dS(0) = d^2S(0) = 0$ *such that*

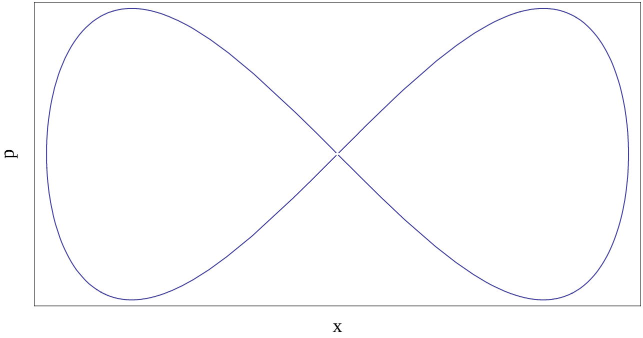
$$e^{it\frac{H}{\hbar}}\psi_a^{qp}(x) \sim e^{i\frac{I(t)}{\hbar}}\psi_{M(t)a}^{qp}(x)e^{i\frac{S(q-x)}{\hbar}}, \ |t| \leq \frac{1-\epsilon}{2\mu(q,p)}log(\hbar^{-1})$$

Need a change of phase.
In fact $S = S_{qp}$ is the generating function (minus its quadratic part) of the *unstable manifold* of the flow at (q, p).
$\Rightarrow$ Egorov theorem up to times $\sim \frac{2}{3}\frac{1}{\mu}log(\hbar^{-1})$ and
$\Rightarrow$ Egorov theorem *wrong* for longer times.

Homoclinic junction

Consider a “8” :

e.g. $H = -\hbar^2 \Delta + x^2(x^2 - 1)$



Consider as initial datum a c.s. of symbol a pined up at the fixed point ψ_a

Theorem

let H be as before and let $0 < \gamma < \frac{1}{5}$
$\exists t_0$ *such that, if* $t_\hbar := log\frac{1}{\hbar} - t_0$. *then*

$$e^{-i\frac{t_\hbar H}{\hbar}}\psi_a = e^{i(S^+ + \pi/2)/\hbar}\psi_{b_+} + e^{i(S^- + \pi/2)/\hbar}\psi_{b_-} + O(\hbar^{\gamma/2})$$

where

$$b_\pm(\eta) := \int_0^{\pm\infty} a(1/\mu)\frac{1}{\mu}\rho(\mu\hbar^\gamma)e^{i\eta\mu}d\mu$$

and ρ *is a cut-off function, that is*
$\rho \in C^\infty,\ \rho(y) = 1,\ -1 \leq y \leq 1, \rho(y) = 0, |y| > 2.$

The new "vacuum" is singular at the origin : $b(x) \sim log(x),\ x \sim 0$.

$$U\alpha(\eta) := e^{i(S^{+}+\pi/2)/\hbar}\int_0^{+\infty}\alpha(1/\mu)\frac{1}{\mu}\rho(\mu\hbar^{\gamma})e^{i\eta\mu}d\mu +$$
$$e^{i(S^{-}+\pi/2)/\hbar}\int_0^{-\infty}\alpha(1/\mu)\frac{1}{\mu}\rho(\mu\hbar^{\gamma})e^{i\eta\mu}d\mu.$$

Theorem

let $C>0$ *and let* $n \leq C\frac{log\frac{1}{\hbar}}{loglog\frac{1}{\hbar}}$. *Then*

$$e^{-i\frac{nt_{\hbar}H}{\hbar}}\psi_a = \psi_{U^n a} + O(\hbar^{\gamma/2}(log\frac{1}{\hbar})^{n/2}).$$

That is : the semiclassical revival is valid for times of the order

$$t \sim C\frac{log^2\frac{1}{\hbar}}{loglog\frac{1}{\hbar}}.$$

Morality

The quantum (semiclassical) flow is periodic with period

$$\sim \log(\hbar^{-1})$$

.

Morality

The quantum (semiclassical) flow is periodic with period

$$\sim \log(\hbar^{-1})$$

.

The vacuum "evolves" not according the (linearized) classical flow, but follows a new dynamical system :

Morality

The quantum (semiclassical) flow is periodic with period

$$\sim \log(\hbar^{-1})$$

.

The vacuum "evolves" not according the (linearized) classical flow, but follows a new dynamical system :

$$(q,p) \to (pq^2, \frac{1}{q}) = (qp.q, (qp)^{-1}.p).$$

The Harper case

$$h^{HARPER}(p,q) := cos(p) - cos(q)$$

By a simple change of variable it can be unitary transform into

$$h(p,q) := \pi^2(cos((p+q)/2\pi) - cos((p-q)/2\pi))$$

with $h(p,q) \sim pq$ near zero.
Let us, once again, consider a coherent state at the origin.

The coherent state will relocalize on a net of points, growing by two at each period (quantum random walk).

Theorem

Let Œ^n (for Œdipus) the set of paths Γ on $\mathbb{Z}^2$ starting at $(0,0)$ and containing no line of length greater than one. Let us denote $\Gamma(n)$ the extremity of Γ and Γ_i a vertex of Γ. Let $t_\hbar = log\frac{1}{\hbar}\hbar$. Then

$$e^{-i\frac{nt_\hbar H}{\hbar}}\psi_a = \sum_{\Gamma\in Œ^n} e^{iS_\Gamma/\hbar}\psi^{a_\Gamma}_{\Gamma(n)} + O(\hbar^{\gamma/2}(log\frac{1}{\hbar})^{n/2}),$$

where $S_\Gamma = \frac{1}{2}\int_{\tilde{\Gamma}} pdq - qdp$, where $\tilde{\Gamma}$ is the path in $\mathbb{R}^2$ consisting in segment joining the points of Γ and

$$a^\Gamma = \Pi_{i=1}^n V^{\Gamma_i} a := V_\Gamma a$$

where

$$V^{\Gamma_i} a(\eta) = \int_0^\infty e^{i\eta\mu} a(1/\mu)\rho(\mu\hbar^\gamma)\frac{d\mu}{\mu}.$$

if the segment (Γ_{i-1},Γ_i) is horizontal right oriented, ...

Another way of saying the same result is the following "path integral" type result

Corollary

let $n \leq C \frac{\log \frac{1}{\hbar}}{\log\log \frac{1}{\hbar}}$ *and let consider the matrix elements*

$$U((0,0);(q,p)) := < \psi^{a}_{(0,0)}, e^{-i\frac{nt_{\hbar}H}{\hbar}} \psi^{b}_{(p,q)} >$$

will have a leading order behaviour only when $(p,q) = (i,j) \in \mathbb{Z}^2$ *and*

$$U((0,0);(i,j) = \sum_{\Gamma \in Œ,\ \Gamma(n)=(i,j)} e^{iS_{\Gamma}/\hbar} < a, V_{\Gamma} b > + O(\hbar^{\gamma/2} (\log \frac{1}{\hbar})^{n/2}).$$

the sum has to be understood as zero when there is no path satisfying $\Gamma(n) = (i,j)$.

Another application : Jaynes-Cummings model

$$H = \sum \epsilon_j s_j^z + \omega b^* b + g \sum \left(b^* s_j^- + b s_j^+ \right)$$

Reduction to one (big) spin

$$H = \epsilon s^z + \omega b^* b + g \left(b^* s^- + b s^+ \right)$$

This is an integrable system with a degenerate torus containing an unstable fixed point at zero.
Periods as before correspond to oscillations between the number of bosons and fermions (Babelon, Douçot, P, in preparation).

What is the classical symbol of an operator ?

$$B\psi_a^{qp} \sim b(q,p)\psi_a^{qp} : \ \textit{locality}$$

What is the classical symbol of an operator ?

$$B\psi_a^{qp} \sim b(q,p)\psi_a^{qp} : \ \textit{locality}$$

$$B^t\psi_a^{qp} := e^{-i\frac{tH}{\hbar}} B e^{+i\frac{tH}{\hbar}} \psi_a^{qp}$$

What is the classical symbol of an operator ?

$$B\psi_a^{qp} \sim b(q,p)\psi_a^{qp} : \ \textit{locality}$$

$$B^t\psi_a^{qp} := e^{-i\frac{tH}{\hbar}} B e^{+i\frac{tH}{\hbar}} \psi_a^{qp} \sim b(\Phi^t(q,p))\psi_a^{qp}, \ t \leq \frac{1}{2\mu}\log(\hbar^{-1})$$

What is the classical symbol of an operator ?

$$B\psi_a^{qp} \sim b(q,p)\psi_a^{qp}: \ \textit{locality}$$

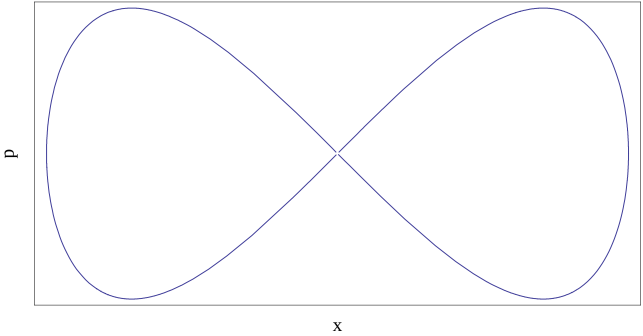
$$B^t\psi_a^{qp} := e^{-i\frac{tH}{\hbar}}Be^{+i\frac{tH}{\hbar}}\psi_a^{qp} \sim b(\Phi^t(q,p))\psi_a^{qp}, \ t \leq \frac{1}{2\mu}\log(\hbar^{-1})$$

for larger t not true anymore, but : possibility of defining the symbol as an operator on the horocyclic leaf,

$$\textit{link with non} - \textit{commutative geometry}.$$

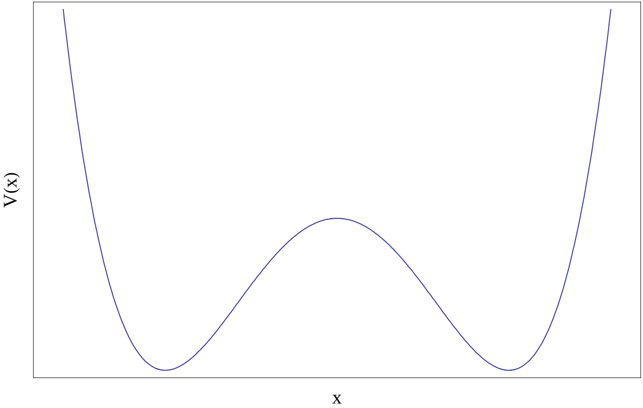
Quantum undeterminism vs. sensitivity to initial conditions

Consider again the "8"



Quantum undeterminism vs. sensitivity to initial conditions

corresponding to a potential :



Quantum undeterminism vs. sensitivity to initial conditions

the origin 0 is a fixed point, but for all point "y" on the unstable manifold :

$$\Phi^{-\infty}(y) = 0$$

Quantum undeterminism vs. sensitivity to initial conditions

the origin 0 is a fixed point, but for all point "y" on the unstable manifold :

$$\Phi^{-\infty}(y) = 0$$

"equivalent" to

$$\Phi^{+\infty}(0) = y, \ \forall y$$

Quantum undeterminism vs. sensitivity to initial conditions

the origin 0 is a fixed point, but for all point "y" on the unstable manifold :

$$\Phi^{-\infty}(y) = 0$$

"equivalent" to

$$\Phi^{+\infty}(0) = y, \ \forall y$$

undeterminism ?

Quantum undeterminism vs. sensitivity to initial conditions

the origin 0 is a fixed point, but for all point "y" on the unstable manifold :

$$\Phi^{-\infty}(y) = 0$$

"equivalent" to

$$\Phi^{+\infty}(0) = y, \ \forall y$$

undeterminism ?

as time $\rightarrow \infty$ quantum undeterminism and classical unpredictability merge.

BUON COMPLEANNO, SANDRO !