Theory and simulations for weakly chaotic systems: round off and irreversibility, collisions and relaxation

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To Sandro Graffi for his 65 birthday

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Theory and simulations for weakly chaotic systems

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- 1 N body simulations and N → oo limit
- 2 Strong and weak chaos asymptotics
- 3 Finite information and round off
- Conclusions

Theory and simulations for weakly chaotic systems

Introduction

Complexity is a feature of living systems (Milnor)

1 Non linear long range interactions
2 Collective self organization (emerging properties)
3 Hyerarchical structures (networks)
4 Metastability and irreversibility

5 Information processing and storage
6 Self reproduction

Physical systems with long range forces share 1-4 (precomplex)

Life appears at the borderline between order and chaos (Kaufmann)

Information allows project coding and causes irreversibility if

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.Non linear systems classification

According to the correlations decay

A) Regular $C(t) = 1/t$

B) Weakly chaotic $C(t) = 1/t^{\alpha}$

C) Strongly chaotic $C(t) = e^{-\beta t}$

Networks

Similar classification holds

A) Hyerarchical network $L(k) = 1/k^{\alpha}$

B) Random network $L(k) = e^{-\beta k}$

Physical systems

Deterministic in Euclidean spaces (infinite information)

Symmetries in Euclidean spaces

Simple elementary units (point mass)

Environment is optional

Few scales

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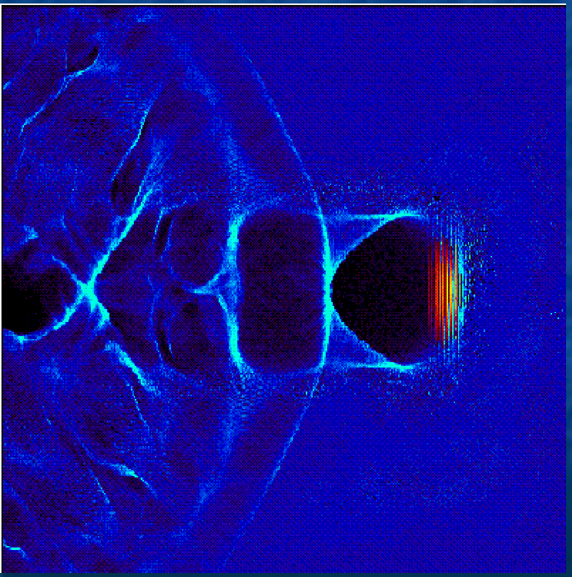
The emergence of self organized structures due to coherencence on time scales short with respect to the collisional relaxation times



Plasma wave breaking

Spiral galaxy

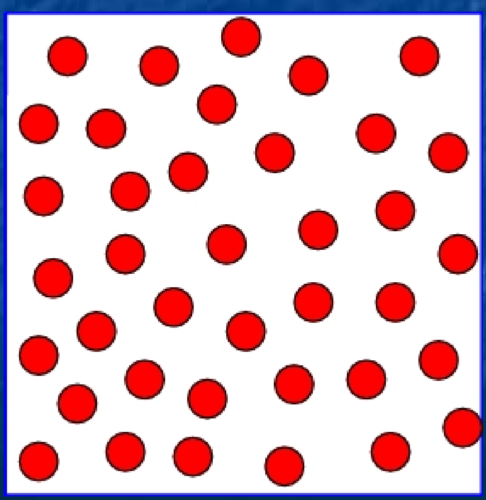
Clusters

Transtion to complexity

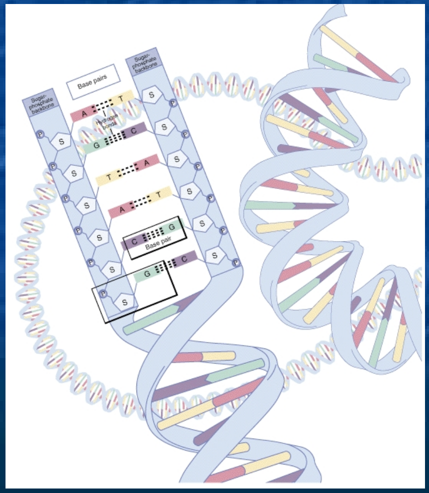
Occurs via information coding. The elementary unit is the Von Neumann automaton

Theorem I There exist self replicating automata

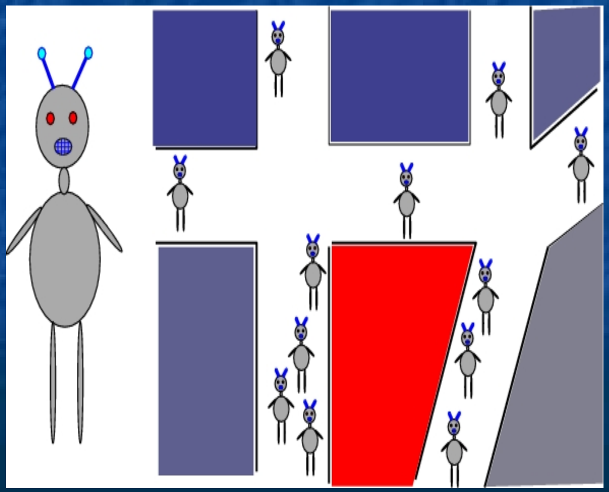
Theorem II Robust automata can be assembled with unreliable componets



Gas of atoms



Information coding



Gas of automata

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Weak chaos: predicatbility and reversibility

Return time spectra and correlations decay

Toy models

Numerical experiments: **round off arithmetics.**

Irreversibility of numerical experiments with symplectic maps.

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Information on phase space localization of a classical system is finite. Measurements perturb classical systems. Infinite accuracy requires infinite energy.

Computer simulations are close to physical reality. IrreversibilityIs intrinsic due to limited information

Langevin test particle models in $\mathbf{R}^d$ should have a small noise for round off plus a collisional noise.

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1 N body simulations and continuum limit

Such limit of N body system is still open question

- Fluid limit (T=0)
- Mean field limit (T>0)
- Kinetic limit (collisional)

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Short range forces

The Grad limit $N \to \infty$ and $\sigma \to 0$ with $N\sigma = 1/\lambda$ constant leads for hard spheres $\sigma = \pi R^2$ to the Boltzmann's equation

$$\partial f / \partial t + [\, f, \mathbf{p}^2/2+V(\mathbf{r}) \,] = J(f,f) \qquad f=f(\mathbf{r},\mathbf{p},t)$$

Kinetic

The moments of these equations provide the continuity and Navier Stokes equation after closure

$$n(\mathbf{r},t) = \int f(\mathbf{r}, \mathbf{p}, t)\, d\mathbf{p} \qquad \mathbf{P}(\mathbf{r},t)= n^{-1} \int \mathbf{p} \; f(\mathbf{r},\mathbf{p},t)\, d\mathbf{p}$$

Fluid

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Long range forces (Coulomb oscillators)

Their distinctive property is the generation of a self field.

The charge fluctuations is charged o neutral plasma generate a field self screened supposing local thermodynamical equilibrium.

$$V(r)= Q\ r^{-1}\ e^{-r/r_D} \qquad r_D = kT/(4\pi\ e^2\ n_0^2)$$

where r_D is the Debye radius.

The electrostatic force on a charge, confined by a linear attracting field, is the sum of a **near field** and a **far field**

$$V_{near}(r) = \sum_{i,\ r_i < r_D} e^2\ |\mathbf{r}-\mathbf{r}_i|^{-1} \qquad V_{far}(r) = \sum_{i,\ r_i > r_D} e^2\ |\mathbf{r}-\mathbf{r}_i|^{-1}$$

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Electrostic case. The Hamiltonian of the system reads

$$H_{tot} = m \sum_{i=1}^{N} \left[\mathbf{p}_i^2/2m^2 + \omega_0^2\, \mathbf{r}_i^2/2 + \xi\, (2N)^{-1} \sum_{i \neq j} r_{ij}^{-1} \right]$$

$$\xi = Q^2/M$$

where M=Nm and Q= Ne are the total charge and mass, fixed as N→ oo. In this limit we assume the charge density to become continuos. After the scaling $H_{tot}/m \rightarrow H_{tot}$, $\mathbf{p}/m \rightarrow \mathbf{p}$

$$H_{tot} = \sum_{i=1}^{N} H(\mathbf{r}_i, \mathbf{p}_i) \qquad H(\mathbf{r},\mathbf{p}) = \mathbf{p}^2/2 + \omega_0^2\, \mathbf{r}^2/2 + \xi\, V(\mathbf{r})$$

The phase space distribution f(**r**,**p**,t) satisfies Liouville + Poisson (Vlasov) equation as N → oo. A proof is given by Kiessling

Theory and simulations for weakly chaotic systems

Main result In the limit N→ oo the collisional part can be ignored,

For a 2D model r^{-1}→ log r we have shown (C. Bendetti, G. Turchetti J. Phys. A **364**, 197 (2006)) by very accurate integration of the N body Hamilton's equations, that the relaxation time scales as N. It agrees with 2D Landau's Kinetic theory, which has same scaling in the 3D case.

Vlasov mean field equilibria Given any stationary distribution f= f(H) the collisions drive it to the Maxwell-Boltzamman distribution $f_{MB} = c\, e^{-H/kT}$ with a self consistent potential V.

The KV disytribution $f = c\, \delta\ (H-E)$ gives a uniformely charged cylinder of radius R.

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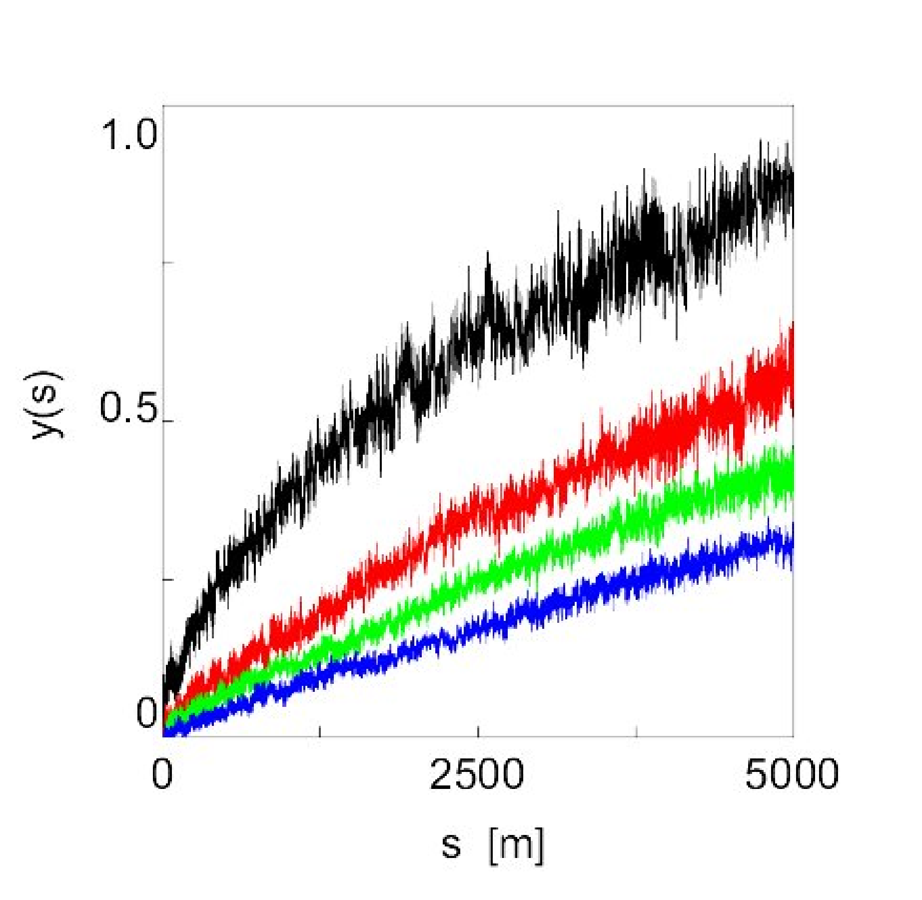
Collisions Numerical simulation with $N = (1,2,3,4) \times 10^3$ fitted with

$n = n_0 e^{-\alpha s} + n_{MB}(1 - e^{-\alpha s})$ where $s = \alpha N = 1/3$, $s = v_0 t$ and $\tau = v_0/a$

C. Benedetti



N	N α
10^3	0.31
$2\ 10^3$	0.30
$3\ 10^3$	0.33
$4\ 10^3$	0.32
$5\ 10^3$	0.32

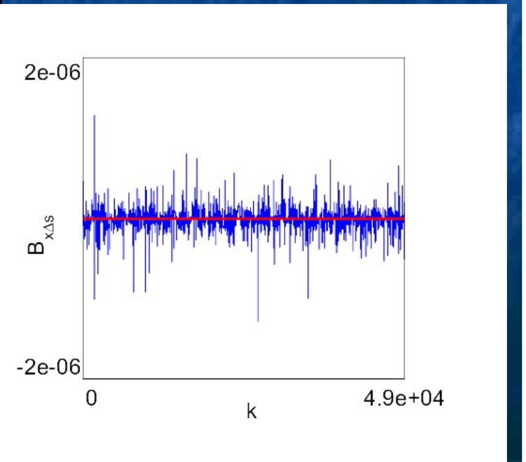
Collisions as a random process.

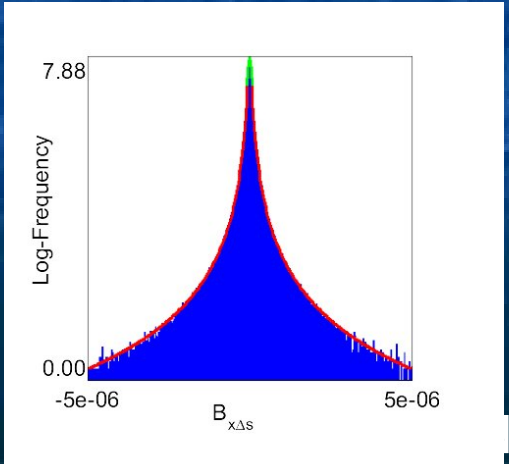
In Landau's theory collisions are assumed to be frequent, small angle, binary and independent. Letting $\mathbf{w}(s)$ be a Wiener noise the equations of motion are

$$d\,\mathbf{r} = \mathbf{p}\,ds \qquad d\,\mathbf{p} = -\frac{\partial H}{\partial \mathbf{r}}\,ds + (d\mathbf{p})_{coll}$$

$$(d\mathbf{p})_{coll} = \mathbf{K}\,ds + D^{1/2}\,d\mathbf{w}(s) \qquad K = \left\langle \frac{d(\mathbf{p})_{coll}}{ds} \right\rangle \qquad D_{ij} = \left\langle \frac{(d\mathbf{p}_i)_{coll}\;d(\mathbf{p}_j)_{coll}}{ds} \right\rangle$$

Slow decay of p.d.f. due to rare hard collisions





From the time series analysis the momentum jumps p.d.f. has a power law decay as

$$\rho\,(\Delta\,p_x) = c\,(\Delta\,p_x)^{-4} \qquad x \rightarrow y$$

can be fitted with a **Student**

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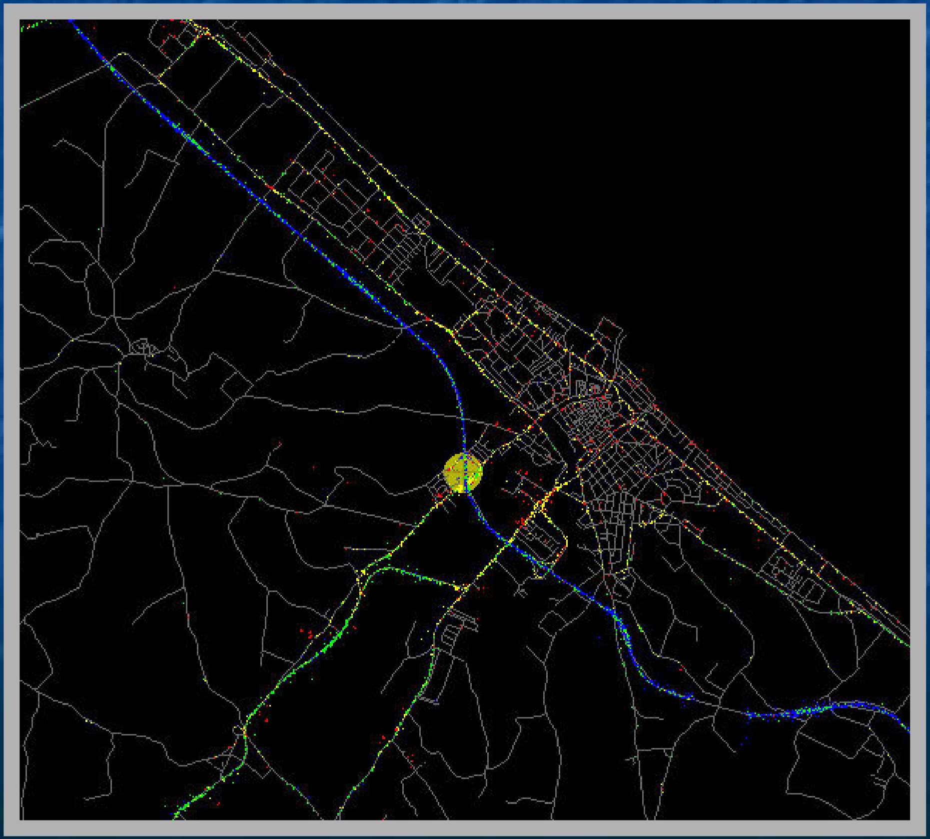
▪ The complex systems

Automata on a network: physical 1D dynamics (car following and saefty distance) cognitive dynamics (decisions at crossings) right

Space based acquisition data system (GPS) left



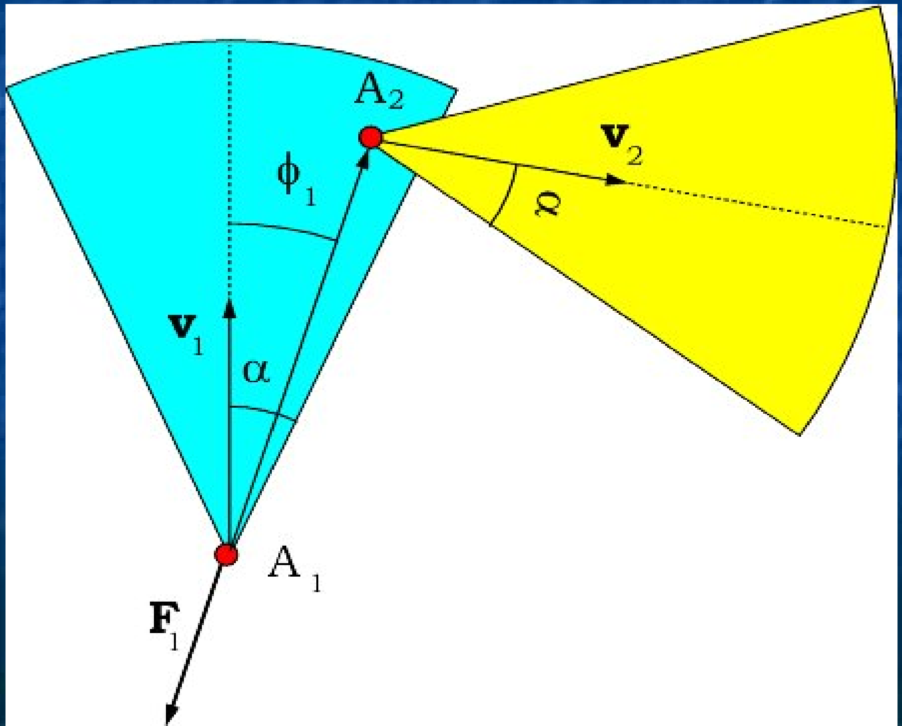
Theory and simulations for weakly chaotic system

Automata based models for pedestrian mobility

- **Model 1** Two automata interact with a long range repulsive (Coulomb) force within a sight cone. Reduced to quadratures (Turchetti, Zanlungo)

$\mathbf{F}_1 = -\omega^2 \mathbf{r}_1 + (\mathbf{r}_1 - \mathbf{r}_2) / r_{12}\, q(C_{12}) \qquad C_{12} = \mathbf{v}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) - v_1\, r_{12} \cos\alpha$

For $\alpha = 0$ the symmetry $1 \leftrightarrow 2$ is lost, and 3-rd principle breaks



Model 2 Theory fo mind

Based on recursive thinking. At order zero free uniform motion. At order 1 any automatonn sees order 0 automata and avoids collisions accordingly.
Genetic selection allows successful collision avoiding rules (Zanlungo)

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2 Strong and weak chaos asymptotics

Local and global dynamical indicators

- Lyapounov exponent λ ($\underline{\boldsymbol{x}}$) or reversibility error h($\boldsymbol{x}$) are local
- The spectrum of Poincaré recurrences $F(t, \boldsymbol{x})$ is semi-local

Limit cases: integrable and uniformly hyperbolic systems

- Weak chaos: borderline from integrability to strong chaos

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Poincaré recurrences.

Given an invertible map M defined on a set Ω **with an invariant measure μ**
The first return time in the neighborhood of a point x in Ω is given by

$$\tau(x, A) = \inf_{n>0} (\mathbf{x} \text{ in } A, \quad M^n(\mathbf{x}) \text{ in } A)$$

Kac's theorem; the average return time in A for an ergodic system is

$$< \tau_A > = 1/\mu(A)$$

The spectrum of recurrences is given by

$$F(t) = \lim_{\mu(A) \to 0} F_A(t) \qquad F_A(t) = \mu(A_{>t}) / \mu(A)$$

$$A_{>t} = (x \text{ in } A, \quad \tau(\boldsymbol{x}, A) > t < \tau_A >)$$

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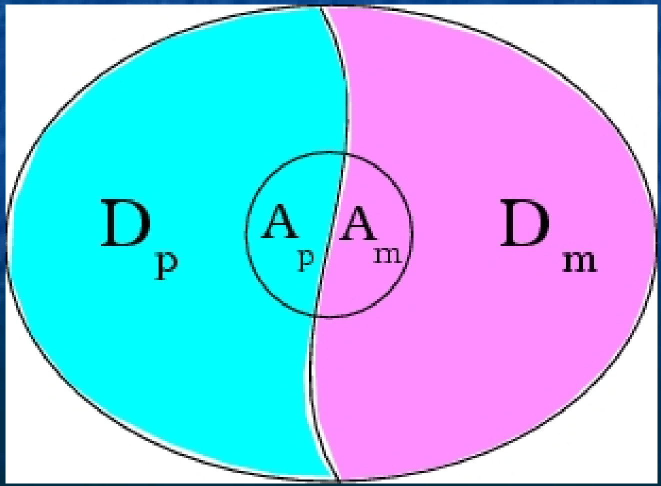
Mixing systems

Exponential spectrum for

$F(t) = e^{-t}$ generic points $F(t) = e^{-\lambda t}$ periodic points

Integrable systems

$F(t) = C\, t^{-2}$



Transition systems: $A = A_p \cup A_m$

$F(t) = p_m\, F_{Am}(k_m t) + p_p\, F_{Ap}(k_p t)$

$k = \langle t \rangle / \langle t \rangle \quad p = \mu(A)/\mu(A)$

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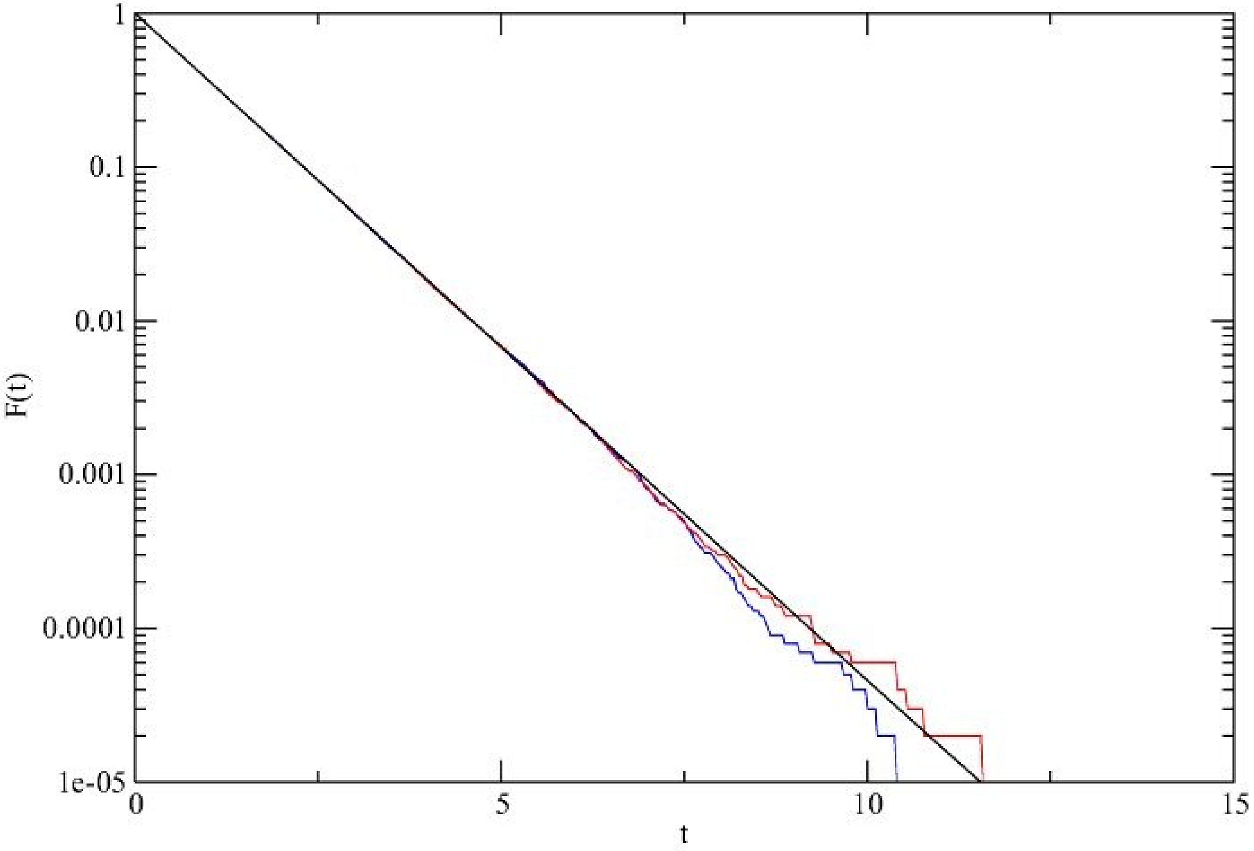
Standard map λ =8 (red), cat map (blue), e^{-t} (black)



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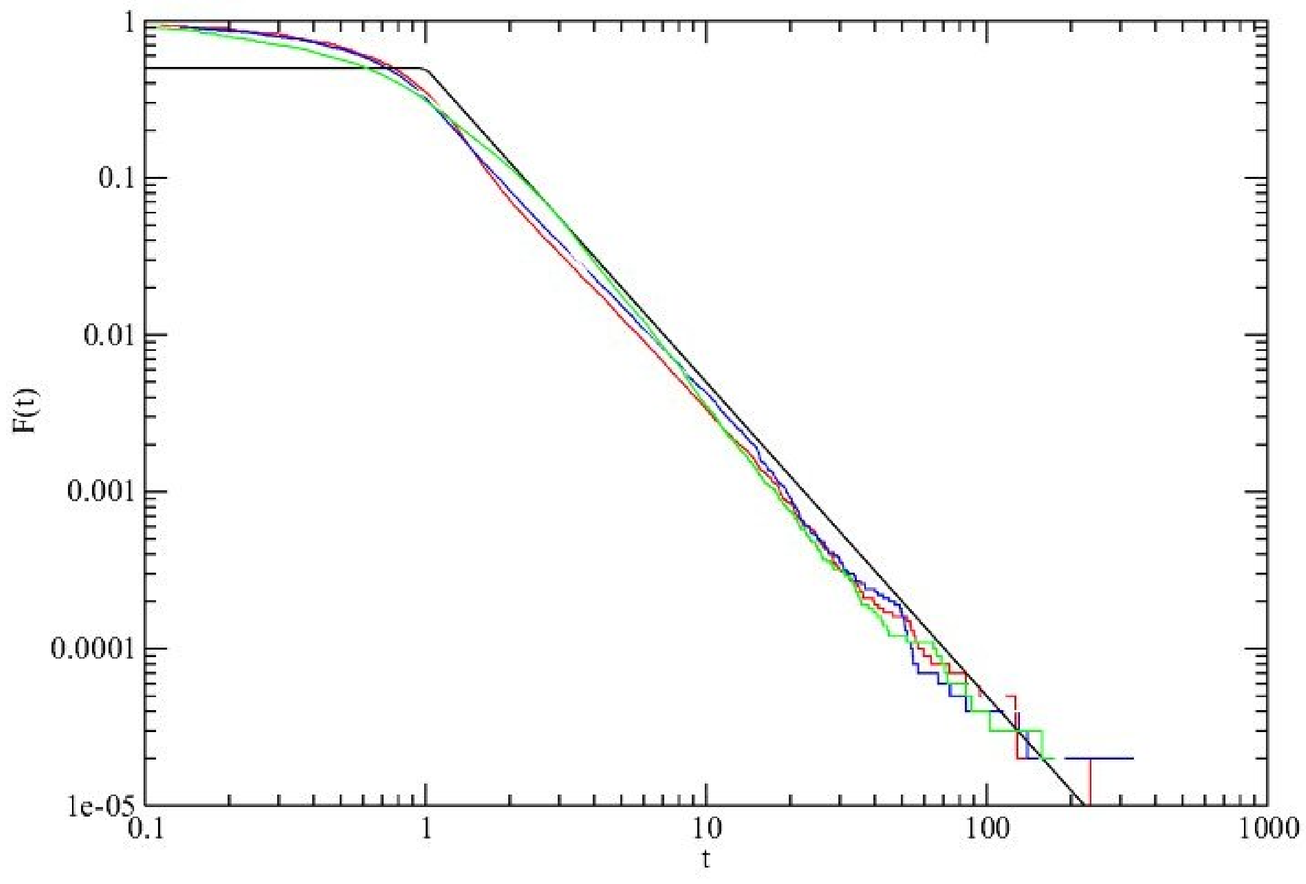
Standard map for λ =0.2, 0.5 0.9 (initial point in integrable region)
black analytical solution decay as t^{-2}



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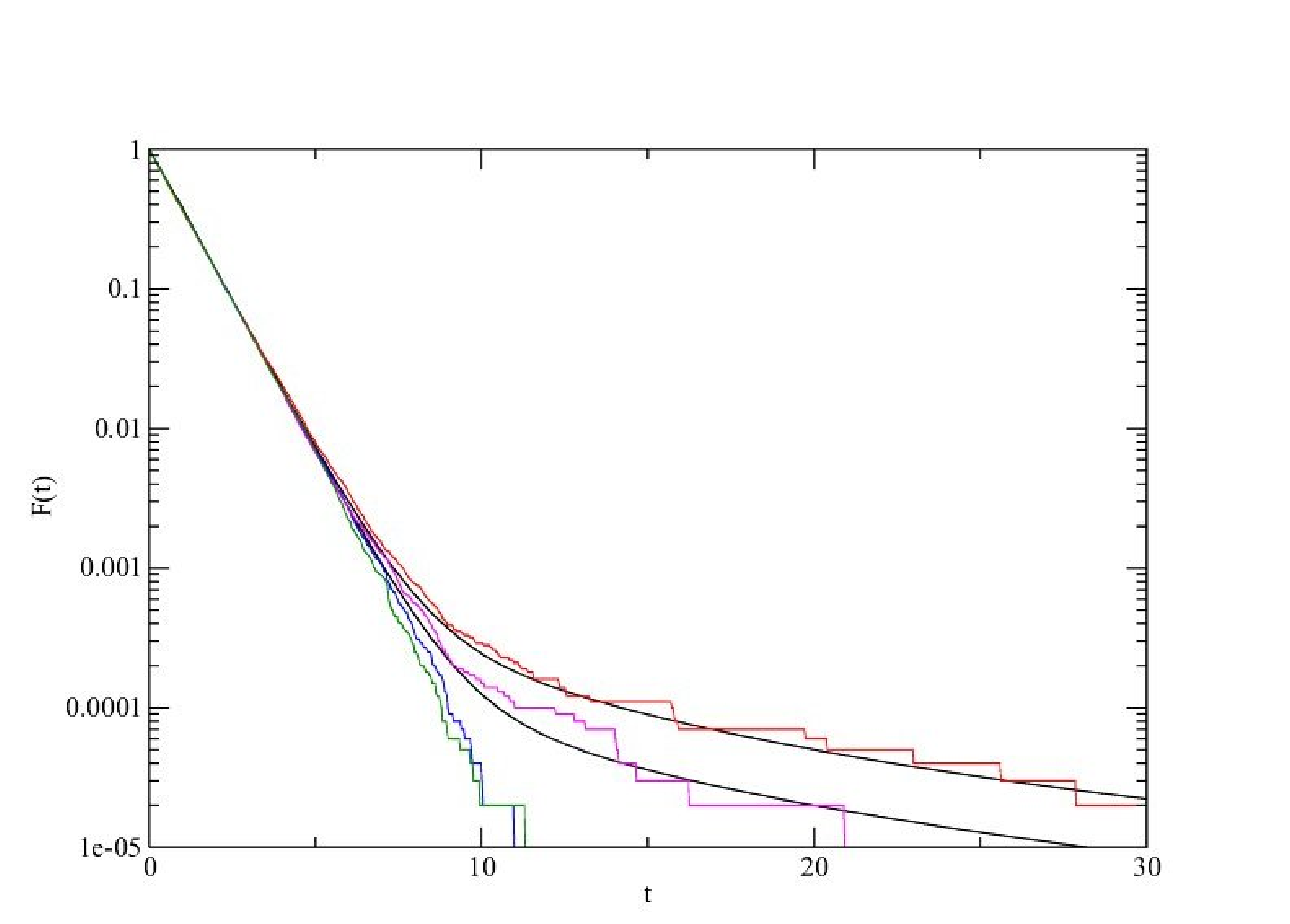
Standard map at the edge of the chaotic region

λ =2,3,4,5 (red, blue, purple, green). Black curve $F(t) = p\, e^{-t} + (1-p)\, t^{-2}$



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3 Finite information and round off

- The reversibility error.

- Iterating forward and backwards a map one does not come back to the initial points.

- The round off causes an error since it acts as a noise and renders the map irreversible

Theory and simulations for weakly chaotic systems

- The computer arithmetics (D. Knuth *The art of comp. Progr.* Vol 2)

The base b excess q representation of a real number x is x_* where

$$x_* = (e,f) = f\, b^{e-q} = x\,[1+\delta_p(x)] \qquad |f| < 1 \qquad |\delta_p| < b^{1-p}$$

where f is a signed fraction and $0 \leqslant e < 2q$.

In a computer b=2, q=32 and $0 \leqslant e \leqslant 63$ and $f = n\, 2^{-24}$ where
$0 < n < 2^{24}$ in the 4 bytes representation (simple precision).
Three bytes used for f and one byte for e and in base 10 representation

$$x_* = + 0.d_1 d_2 \ldots d_7\ \ 10^{+E} \qquad E < 32$$

The arithmetic operations involve round off

$$z = x+y \rightarrow x_* \oplus y_* = (ex, f) = z\,[1+\delta_p(z)] \qquad f = \textit{round}\,(f_x + f_y b^{e_y - e_x})$$

Orbits and pseudo-orbits

The round off acts as a random perturbation and breaks the Reversibility. Supposing M(x) is an invertible map $M^{-1} \circ M = I$.

Letting $M_*(x_*) = \text{round}\,(\,M(x_*)\,)$ and $M_*^{-1}(x_*) = \text{round}\,(\,M^{-1}(x_*)\,)$

$$M_*^{-1} \circ M_* = I + \varepsilon$$

The reversibility error at a point $x = x_*(1+\delta\ )$ is defined as

$$\varepsilon\quad(n) = |\, M_*^{-n} \circ M_*^{n}(x_*) - x_* \,|$$

This is basically the same as the round off error on the orbit

Theory and simulations for weakly chaotic systems

- Let ξ be the round off error on the map be

$$M_*(x_*) = M(x_*)\,(\,1 + \xi\ (x_*)\,) \qquad |\xi\,| < c\, b^{-p} \qquad |x-x^*| < b^{1-p}$$

The round off error on the trajectory setting $x_{*n} = M^n(x_*)$

$$\eta\ (n) = |M^n(x) - M_*^n(x_*)| < |DM^n(x_*)|\ |x-x_*| + x_{*n}\ \xi\ (x_{*n-1})$$

$$+ \sum_{1 \le k \le n-1} D M^k (x_{*n-k})\ x_{*n-k}\ \xi\ (x_{*n-k-1}) + O(|\xi\,|^2)$$

If the map is ergodic it is not hard to prove that

$$\lim_{n \to \infty} n^{-1} \operatorname{Ln} \eta\ (n) < \lambda$$

maximum Lyapounov exponent

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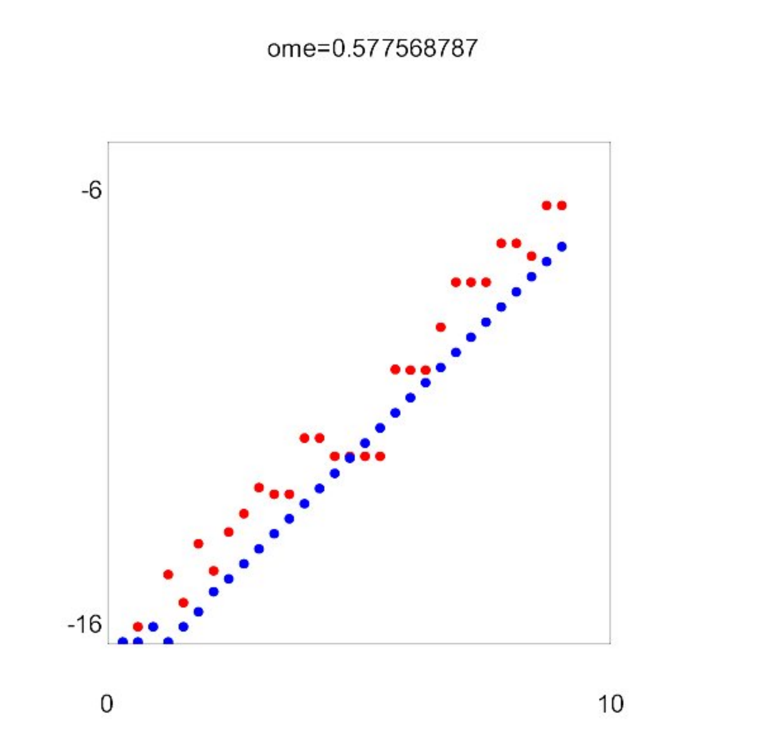
The simplest examples are the maps on the torus T^1

i) $$M(x) = x + \omega \quad \text{Mod } 1$$

Iterating $n=b^k$ times (b base), we have p-n digits after round off of $x+ b^n$

Figure: $\text{Log}_{10}\,\eta$ vs $\log_{10} n$

$$\eta\ (b^k) = b^{-(p-k)}$$



ii) $$M(x) = q\,x \text{ Mod } 1 \quad q \in Z$$

Choosing b=q at every step one digit is lost

$$\eta\ (k) = q^{-(p-k)}$$

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- The reversibility error $\varepsilon(n)$ is about the same as the round off error $\eta(2n)$ for the same initial point

$$\varepsilon(n) \simeq \eta(2n)$$

For an integrable map (i.e. translation on the torus)

$$\text{Log}\ \eta(n) = \log n - p\ \text{Log}\ b$$

For an hyperbolic map

$$\text{Log}\ \eta(k) = k\lambda - p\ \text{Log}\ b$$

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Other model maps

Elliptic maps

$\boldsymbol{x}' = R\left(2\pi \nu + 2\pi \,^2 \boldsymbol{x}^2\right)$ $\boldsymbol{x}=(x, y)$ rotation in R^2

$X' = X + \nu + Y$ mod 1 map on cylinder $T \times R$
$Y' = Y$

$x = (Y/\pi)^{1/2} \cos(2\pi X)$ change of coordinates
$y = (Y/\pi)^{1/2} \sin(2\pi X)$ from R^2 to $T \times R$

Hyperbolic maps

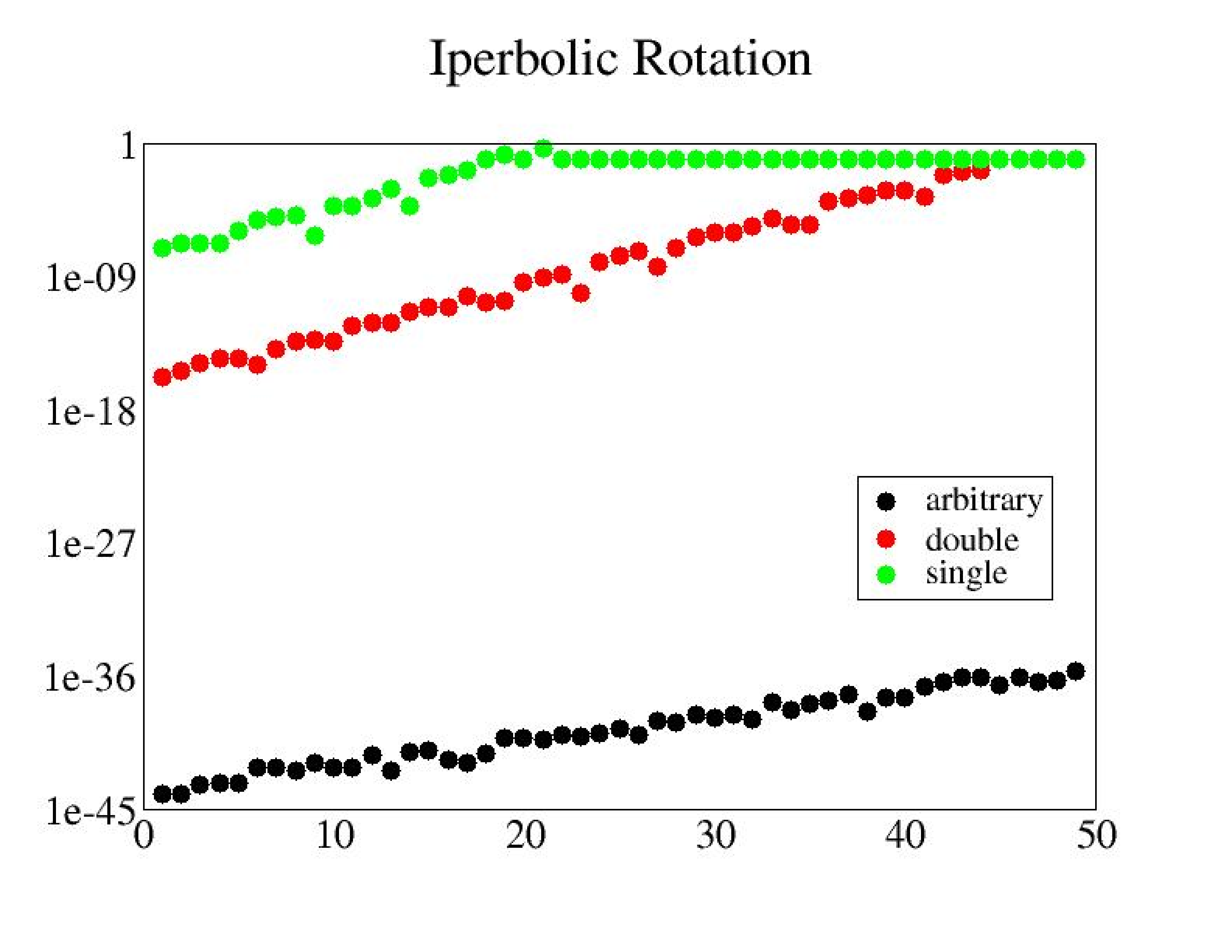
$\boldsymbol{x}' = R_H\left(2\pi \nu + 2\pi \,^2 \boldsymbol{x}^2\right)$ hyperbolic rotation in R^2

$x' = (q+1)x + y$ mod 1
$y' = qx + y$ mod 1 *hyperbolic automorphism of* T^2

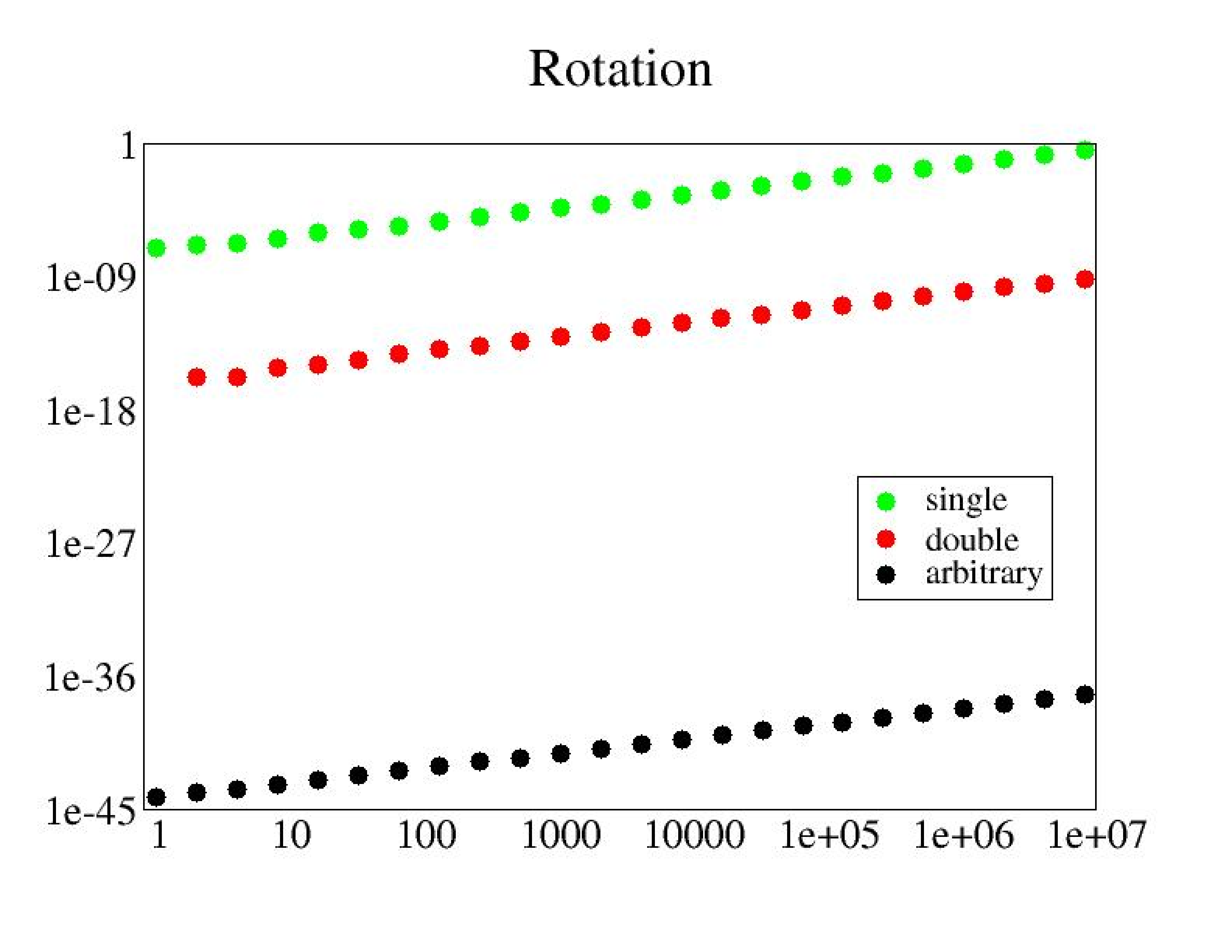
Small perturbations

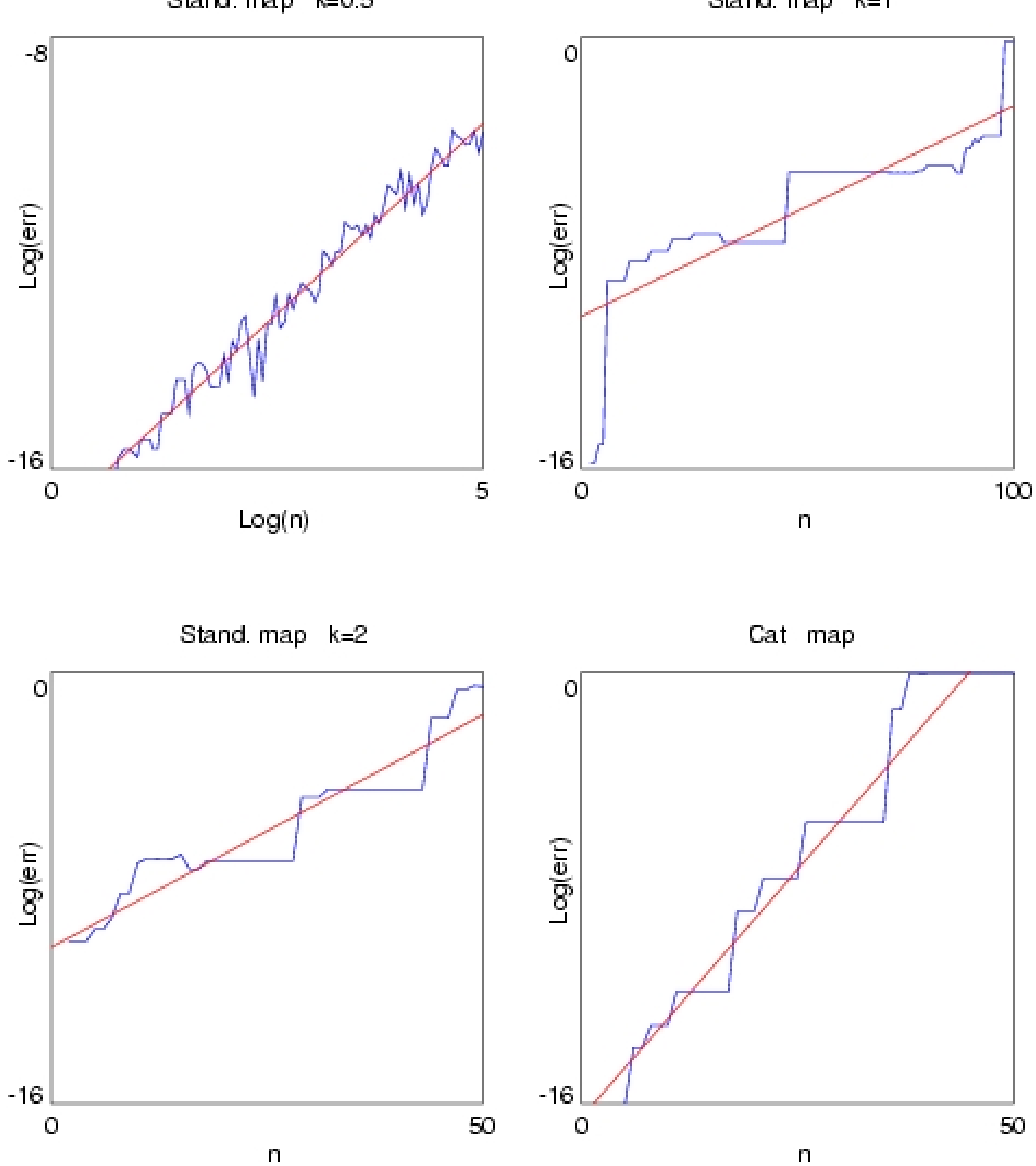
of these maps (Cirikov and Henon maps)

Nonlinearity, noise and information



Nonlinearity, noise and information





-8
-16
Log(err)
0
5
Log(n)
0
-16
Log(err)
0
100
n
Stand. map k=2
0
-16
Log(err)
0
50
n
Cat map
0
-16
Log(err)
0
50
n

Theory and simulations for weakly chaotic systems

- Pdf of the pseudo-orbit and orbit distance

Given any smooth function f(x) we consider the random variable

$$\Delta_f(x,n) = f(M^n(x)) - f(M_*^n(x))$$

function of the random process ξ since $M_*(x) = M(x) + \varepsilon\,\xi$.

Let ρ be the pdf of this process

$$F(t) = E\,(\Delta_f(x,n) < t) \qquad \rho\,(t) = F'(t)$$

The characteristic function of $\Delta_f(x,n)$ for $n \to \infty$ using the fidelity theorem is

$$\lim_{n\to\infty} E\,(e^{ik\Delta_f(x,n)}) = \lim_{n\to\infty} \int \exp\big(ikf(M^n(x))\big)\,\exp\big(ikf(M_*^n(x))\big)\,dm(x)\,d\theta_1(\xi)\dots d\theta_n(\xi) =$$

$$= \int e^{ik\,f(M(x))}\,d\mu\,(x) \int e^{ik\,f(M(x))}\,d\mu_\varepsilon(x)$$

If the map is ergodic and ξ stationary the last means can be written as limit

Theory and simulations for weakly chaotic systems

After the limit $n \to \infty$ the limit $\varepsilon \to 0$ can be taken. In this case the distribution $\rho_\varepsilon(t)$ has a limit $\rho(t) = \rho(-t)$. The simmetry follows from

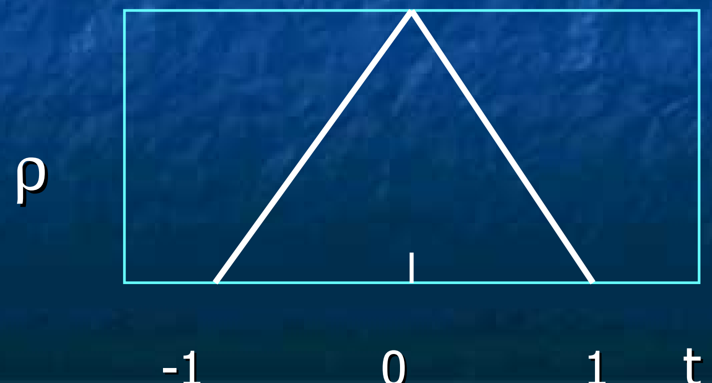
$$E(\Delta_f(\infty, \varepsilon)) = \int f(x)\, d\mu(x) - \int f(x)\, d\mu_\varepsilon(x) = \int t\, \rho_\varepsilon(t)\, dt$$

whose $\varepsilon \to 0$ limit vanishes. As the possible simplest example we consider

$M(x) = qx \bmod 1$ q integer $m(x) = \mu(x) = x$

$$\int e^{ik M(x)}\, dx = 2 k^{-1} e^{ik/2} \sin k/2$$

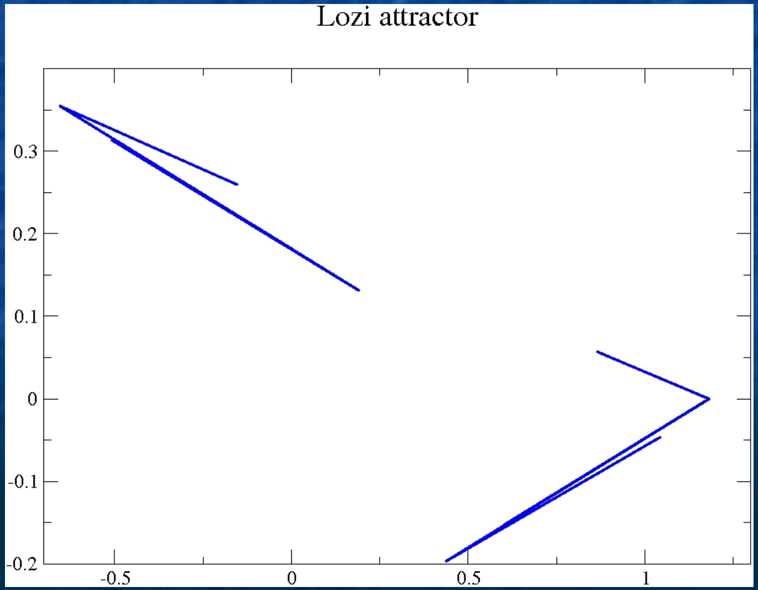
$$\rho(t) = (1-|t|)\, \Theta(1-|t|)$$

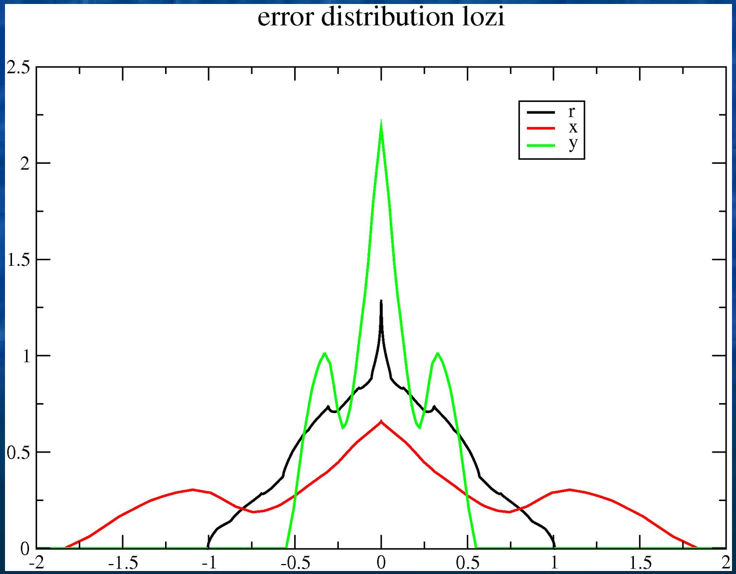


Theory and simulations for weakly chaotic systems

Numerical investigations were performed on strange attractors generated by Baker's, Lozi and Henon map. The triangular distribution changes into

Simmetric distributions peaked at t=0 which reflect the attractor nature and its topology. The R x C_{antor} structure of Baker's attractor reflects into a continuous-singular measure F(t) for the orbit-pesudorbit fluctuations.





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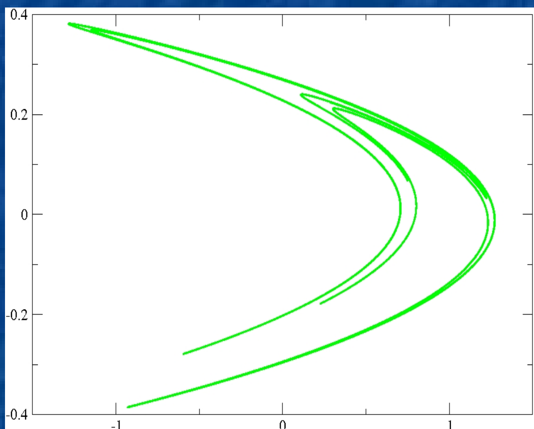
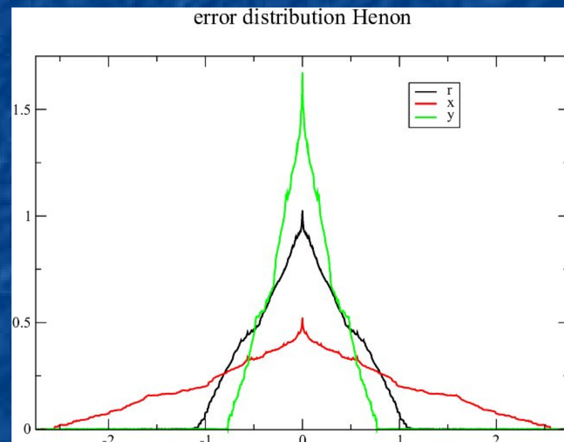
Hénon attractor and p.d.f. ρ (t), f(x)=x

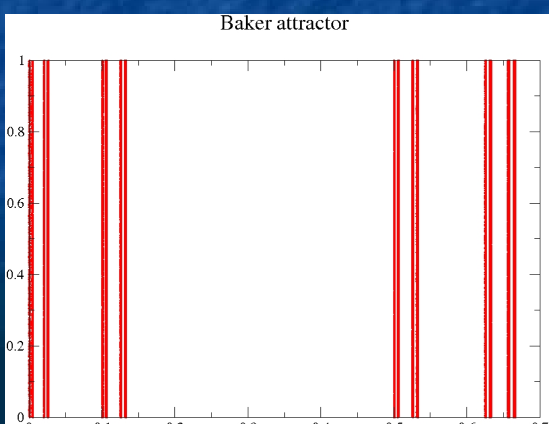


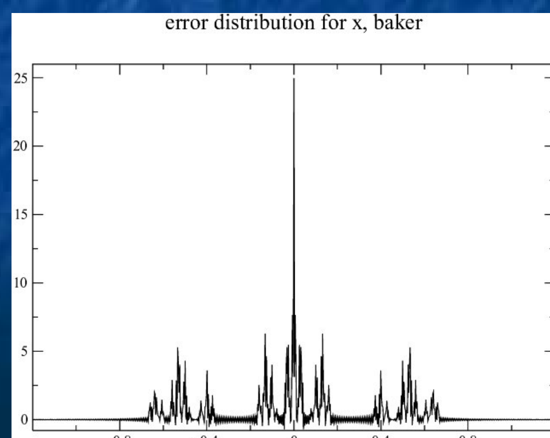


The error p.d.f r(t) has A similar structure for the different attractors.

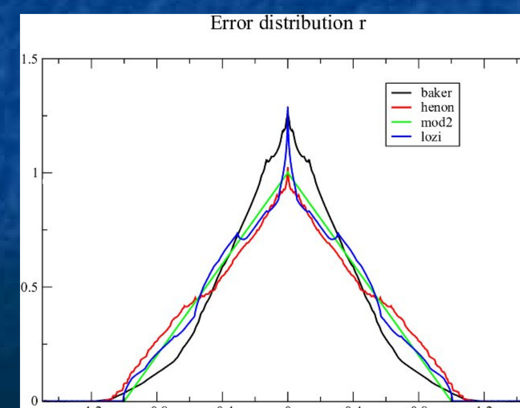
The choice f(x,y)=r $r=(x^2+y^2)^{1/2}$ mediates the smooth structure of leaves and the transverse Cantor structure (see baker's)

Baker's attractor and p.df. ρ (t) and f(x)=x





Comparison of ρ (t)



Theory and simulations for weakly chaotic systems

The other face of information (Dr. Jeckil and Mr Hide)

Coding allows to write projects. Energetically writing a code is cheap compared to assembling the whole structure

In the physical world the information is finite. Position determinacy is limited by the atomic size.
Measurements disturb also the classical state.

A computer simulation, based on finite information, is close to physics.

Computer round off is equivalent to add noise in the equations defined on R^{2d}

Finite information in dyn. sys. = irreversibility

Theory and simulations for weakly chaotic systems

Conclusions

- The theory of dynamical systems has provided the theoretical foundation of non linear phenomena and a way to approach non equilibrium statistical mechanics

- Complex systems require the inclusion of information theory in order to describe the cognitive properties of the elementary units, wich are Von Neumann automata

- The finite information content of the physical world can be described by introducing some background noise. The effect is similar to finite digital computation with round off artithmetics