# BOREL SUMMABILITY: APPLICATION TO THE ANHARMONIC OSCILLATOR

S. GRAFFI, V. GRECCHI
*Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Italy*
and
B. SIMON
*Department of Mathematics, Princeton University, USA*

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We prove that the energy levels of an arbitrary anharmonic oscillator ($x^{2m}$ and in any finite number of dimensions) are determined uniquely by their Rayleigh-Schrödinger series via a (generalized) Borel summability method. To use this method for computations, one must make an analytic continuation which we accomplish by (a rigorously unjustified) use of Padé approximants in the case of $p^2 + x^2 + \beta x^4$. The numerical results appear to be better than with the direct use of Padé approximants.

Commun. Math. Phys. 167, 471–507 (1995)

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# Classical limit of the Quantized Hyperbolic Toral Automorphisms

**Mirko Degli Esposti, Sandro Graffi, Stefano Isola**

Dipartimento di Matematica, Università di Bologna, I-40127 Bologna, Italy

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**Abstract:** The canonical quantization of any hyperbolic symplectomorphism $A$ of the 2-torus yields a periodic unitary operator on a $N$-dimensional Hilbert space, $N = \frac{1}{h}$. We prove that this quantum system becomes ergodic and mixing at the classical limit ($N \to \infty$, $N$ prime) which can be interchanged with the time-average limit. The recovery of the stochastic behaviour out of a periodic one is based on the same mechanism under which the uniform distribution of the classical periodic orbits reproduces the Lebesgue measure: the Wigner functions of the eigenstates, supported on the classical periodic orbits, are indeed proved to become uniformly spread in phase space.

Commun. Math. Phys. 167, 471–507 (1995)

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A. Bouzouina & S. De Bièvre,
Commun. Math. Phys. 178 (1996) 83-105

[DEGI] Degli Esposti, M., Graffi, S., Isola, S.: Stochastic properties of the quantum Arnold cat in the classical limit. Commun. Math. Phys. **167**, 471–509 (1995)

S. Zelditch, Ann. Inst. Fourier 47 (1997) 305-363

[dEGI] M. d'EGLI ESPOSTI, S. GRAFFI, and S. ISOLA, Stochastic properties of the quantum Arnold cat in the classical limit, Comm. Math. Phys., 167 (1995), 471-509.

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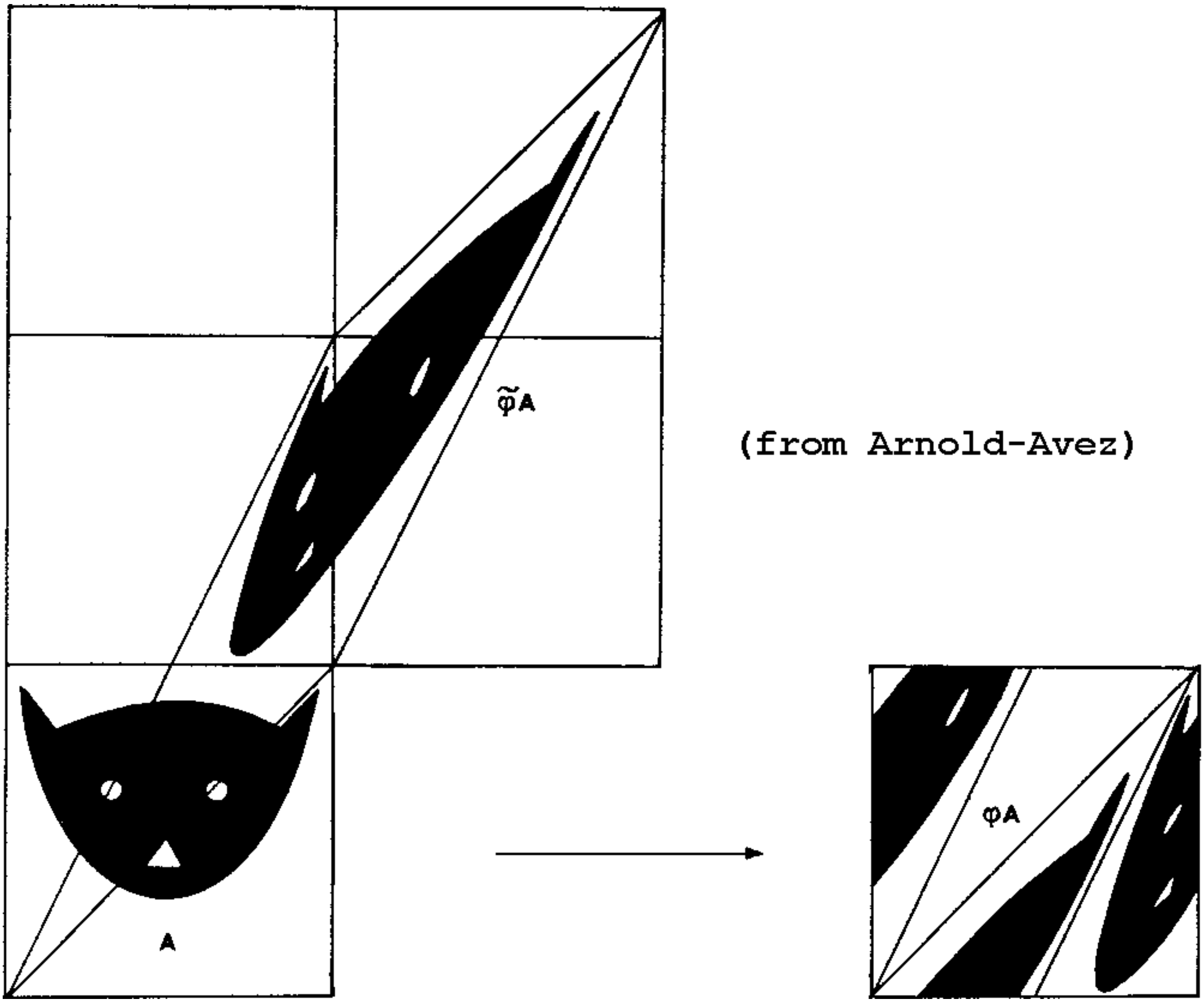


(from Arnold-Avez)


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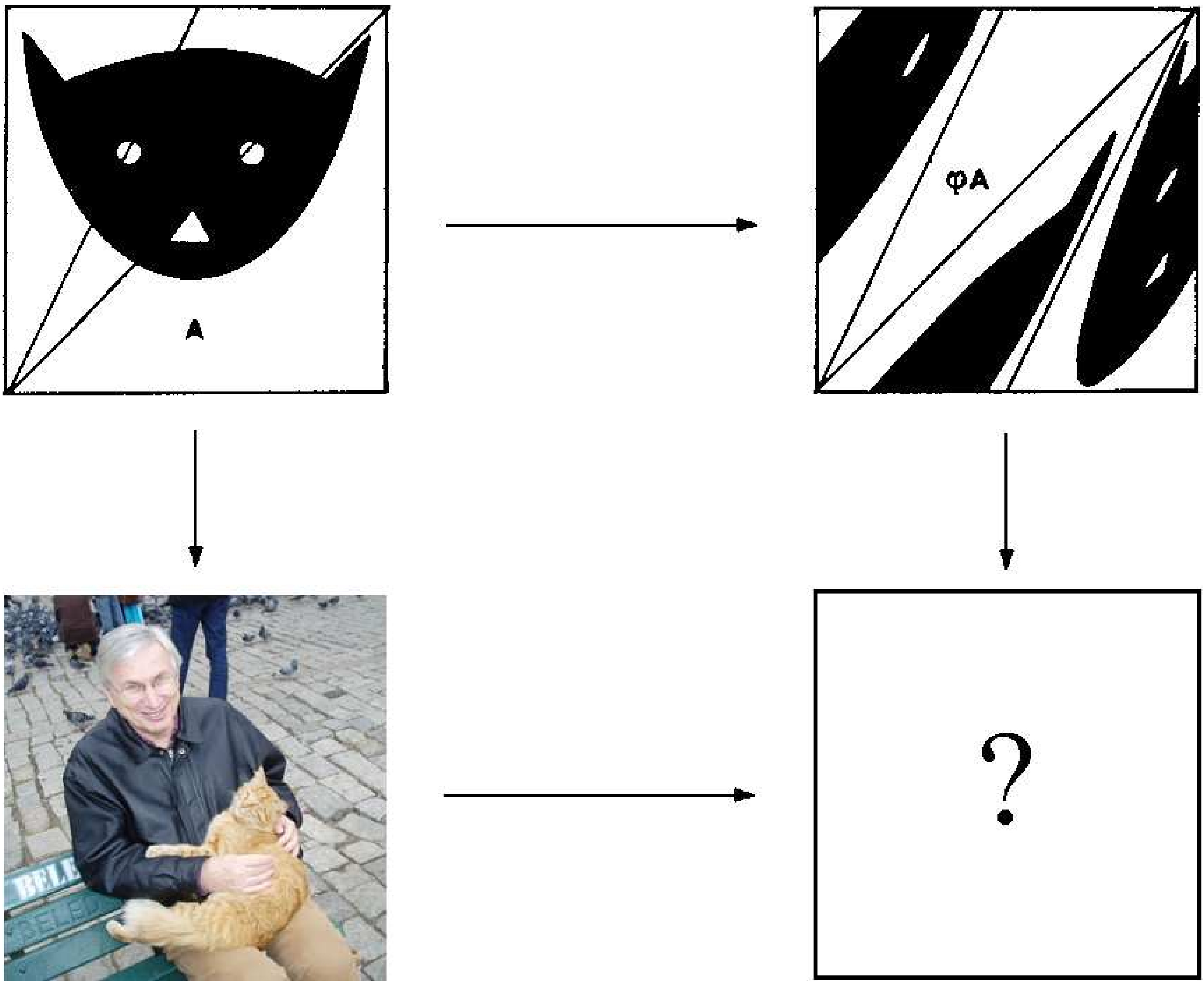
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# Toward a quantum-integrable structure of the general 1D Schrödinger equation

**André Voros**
Institut de Physique Théorique de Saclay

July 11, 2008

based on:

J. Phys. **A32** (1999) 5993–6007 [math-ph/9903045]
*Toward the exact WKB analysis of diff. eqs....* (Kyoto U. Press, 2000) 97–108
CORRIGENDUM: J. Phys. **A33** (2000) 5783–5784

J. Phys. **A33** (2000) 7423–7450 [math-ph/0005029]

Publ. RIMS **40** (2004) (2004) 973–990 [math-ph/0312024]

RIMS Kôkyûroku **1424** (2005) 214–231 [math-ph/0412041]

*Algebraic Analysis of Differential Equations (Festschrift in Honor of Takahiro KAWAI)*
(Springer, 2008) 321–334 [math-ph/0603043]

## Polynomial 1D stationary Schrödinger problem

(an **O**rdinary **D**ifferential **E**quation)

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} \quad + \quad [ \quad \underset{\{+q^N + v_1 q^{N-1} + \cdots + v_{N-1} q\}}{\overset{V(q)}{\updownarrow}} \quad + \quad \underset{\{-E\}}{\overset{\lambda}{\updownarrow}} \quad ]\right) \psi(q) = 0$$

Notations : $\vec{v} = (v_1, \cdots, v_{N-1})$ $\boxed{\text{Degree} = N}$

- Traditional view (in any dimension):
  - $N = 2$ exactly solvable (harmonic oscillator),
  - $N \neq 2$ are not:
    - Airy equation for $N = 1$ (more transcendental than $N = 2$)
    - anharmonic oscillator for $N \geq 3$ (workhorse for perturbative, semiclassical ... methods).

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      (workhorse for perturbative, semiclassical ... methods).

- Recent view (in 1 dimension):

**AN EXACTLY SOLVABLE PROBLEM IN ANY DEGREE**

by an **exact WKB method** (cf. Balian–Bloch, Zinn-Justin, Sibuya):

- semiclassical analysis using **zeta-regularization**
- **exact, selfconsistent** Bohr–Sommerfeld quantization conditions ($\approx$ **Bethe Ansatz**).

## Polynomial 1D stationary Schrödinger problem

- Initial equation:

$$\Big(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} \quad + \quad [ \quad \underset{\substack{\updownarrow \\ \{+\,q^N+\,v_1\,q^{N-1}+\cdots+\,v_{N-1}\,q\}}}{V(q)} \quad + \quad \underset{\substack{\updownarrow \\ \{-E\}}}{\lambda} \quad ]\Big)\,\psi(q)=0$$

Notations : $\quad \vec{v}=(v_1,\cdots,v_{N-1}) \quad \boxed{\text{Degree}=N}$

- **Conjugate** equations:

$$V^{[\ell]}(q) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\ell\varphi}\,V(\mathrm{e}^{-\mathrm{i}\ell\varphi/2}\,q), \qquad \lambda^{[\ell]} \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\ell\varphi}\,\lambda$$

for $\quad \ell=0,1,\cdots,L-1 \quad (\mathrm{mod}\ L) \qquad$ with $\boxed{\varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2}}$

Number of distinct conjugates : $\quad L=\begin{cases} N+2 & \text{generically} \\ \frac{N}{2}+1 & \text{for } \textit{even polynomials } V(q) \end{cases}$

## Semiclassical tools (I): Spectral functions (parity-split)

Assume confining potential $V(|q|)$ $\implies$ discrete $E$-spectrum $\mathcal{E} = \{E_k\}_{k=0,1,2,\cdots}$

- $E \to +\infty$ expansions:

Classical action: $\displaystyle\oint_{\{p^2+V(q)=E\}} \frac{p\,\mathrm{d}q}{2\pi} \sim b_\mu E^\mu$, $\boxed{\mu \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{N}}$ (*growth order*)

Semiclassical quantization condition (*Bohr–Sommerfeld expansion*):

$$\sum_\alpha b_\alpha {E_k}^\alpha \sim k + \frac{1}{2} \quad \text{for integer } k \to +\infty \quad \left(\alpha = \mu,\ \mu - \frac{1}{N},\ \mu - \frac{2}{N}, \cdots\right)$$

$$\Downarrow$$

- **(Generalized) zeta functions**

$$Z^{\pm}(s,\lambda) \stackrel{\text{def}}{=} \sum_{k \ {\text{even} \atop \text{odd}}} (E_k + \lambda)^{-s} \qquad \text{(convergent for } \operatorname{Re} s > \mu)$$

$$\text{and} \quad Z \equiv Z^+ + Z^- \quad (\textit{full}), \qquad Z^{\mathrm{P}} \equiv Z^+ - Z^- \quad (\textit{skew})$$

- **Spectral determinants** (zeta-regularized)

$$D^{\pm}(\lambda) \equiv D(\lambda \mid \mathcal{E}_{\pm}) \stackrel{\text{def}}{=} \exp[-\partial_s Z^{\pm}(s,\lambda)]_{s=0}$$

$$\text{and} \quad D \equiv D^+ D^- \quad (\textit{full}), \qquad D^{\mathrm{P}} \equiv D^+/D^- \quad (\textit{skew}, \text{ meromorphic})$$

Example of the full determinant, $D(\lambda) \equiv \det(\hat{H} + \lambda) =$ "$\prod_k (\lambda + E_k)$":

$D(\lambda)$ is an **entire** function, of order $\mu$ in $\lambda$.

$\log D(\lambda)$ has a **structure equation**:

$$\log D(\lambda) \equiv \lim_{K \to +\infty} \left\{ \sum_{k<K} \log(E_k + \lambda) + \tfrac{1}{2} \log(E_K + \lambda) \underbrace{- \sum_{\{\alpha>0\}} b_\alpha E_K^\alpha \left[\log E_K - \tfrac{1}{\alpha}\right]}_{\textit{counterterms}} \right\},$$

and a **canonical** large-$\lambda$ (*generalized Stirling*) expansion, of order $\mu$ :

$$\log D(\lambda) \sim \sum_\alpha a_\alpha \{\lambda^\alpha\}, \quad \{\lambda^\alpha\} \stackrel{\text{def}}{=} \lambda^\alpha \ (\alpha \notin \mathbb{N}),\ \{\lambda^1\} \stackrel{\text{def}}{=} \lambda(\log\lambda - 1),\ \{\lambda^0\} \stackrel{\text{def}}{=} \log\lambda\,;$$

*banned:* pure $\lambda^n$ ($n \in \mathbb{N}$) terms, including **additive constants** ($\propto \lambda^0$).

## **Semiclassical tools (II): recessive WKB solutions** (cf. Sibuya)

Exact solution $\psi_\lambda(q)$, recessive for $q \to +\infty$ (*canonical* WKB specification):

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(q')\,\mathrm{d}q' , \qquad \Pi_\lambda(q) \stackrel{\mathrm{def}}{=} (V(q)+\lambda)^{1/2} \quad \text{(classical momentum)}$$

$$\int_q^{+\infty} \Pi_\lambda(q')\,\mathrm{d}q' : \textit{improper} \text{ action integral} \qquad (\Pi_\lambda(q') \sim {q'}^{N/2}).$$

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$$\text{Trick:} \qquad = \Bigg[ \underbrace{\int_q^{+\infty} (V(q')+\lambda)^{1/2-s}\,\mathrm{d}q'}_{I_q(s,\lambda)} \Bigg]_{s \rightsquigarrow 0} \qquad (\text{convergent for } \operatorname{Re} s > \mu)$$

(analytical continuation in $s$), fine if $I_q(s,\lambda)$ is **regular** at $s=0$. But in general,

$$(V(q)+\lambda)^{-s+1/2} \sim \sum_\rho \beta_\rho(s)\, q^{\rho - Ns} \quad \left(\rho = \frac{N}{2}, \frac{N}{2}-1, \ldots\right) \quad (q \to +\infty)$$

$$\Rightarrow \quad I_q(s,\lambda) \sim -\sum_\rho \beta_\rho(s) \frac{q^{\rho+1-Ns}}{\rho+1-Ns} \qquad \text{(singular expansion)}$$

and $I_q(s,\lambda)$ has at most a simple pole at $s=0$, of residue $\boxed{\frac{1}{N}\beta_{-1}(s=0)}$

$$\beta_{-1}(s)\ (\text{“}\mathbf{residual}\text{” } \mathbf{polynomial}) \begin{cases} \equiv 0 : & \textbf{N}\text{ormal case} \\ \not\equiv 0 : & \textbf{A}\text{nomaly case} \end{cases}$$

**Semiclassical interpretation of improper action integral**

QUANTUM

zeta function

$$Z(s,\lambda) = \mathrm{Tr}\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+V(q)+\lambda\right)^{-s}$$

$$= \sum_k (E_k+\lambda)^{-s}$$

determinant

$$D(\lambda) = \text{``}\overset{\textit{formally}}{\prod_k (\lambda+E_k)}\text{''}$$

$$D(\lambda) \overset{\text{def}}{=} \exp\left\{-\partial_s Z(s,\lambda)|_{s=0}\right\}$$

**Semiclassical interpretation of improper action integral**

$$\text{QUANTUM} \quad \underset{\text{correspondence}}{\longleftrightarrow} \quad \text{CLASSICAL}$$

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$$Z_{\mathrm{cl}}(s,\lambda) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^2} \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\left(p^2+V(q)+\lambda\right)^{-s}$$

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$$D_{\mathrm{cl}}(\lambda) \stackrel{\mathrm{def}}{=} \exp\left\{-\partial_s Z_{\mathrm{cl}}(s,\lambda)|_{s=0}\right\}$$

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zeta function | zeta function

$$Z(s,\lambda) = \mathrm{Tr}\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+V(q)+\lambda\right)^{-s} \qquad Z_{\mathrm{cl}}(s,\lambda) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^2}\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\left(p^2+V(q)+\lambda\right)^{-s}$$

$$= \sum_k (E_k+\lambda)^{-s} \qquad = \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\,\Gamma(s)}\int_{-\infty}^{+\infty}(V(q)+\lambda)^{1/2-s}\mathrm{d}q$$

determinant | determinant

$$D(\lambda) \stackrel{\textit{formally}}{=} \text{“}\prod_k(\lambda+E_k)\text{”} \qquad D_{\mathrm{cl}}(\lambda) \stackrel{\textit{formally}}{=} \exp\ \text{“}\int_{-\infty}^{+\infty}(V(q)+\lambda)^{1/2}\,\mathrm{d}q\text{”}$$

$$D(\lambda) \stackrel{\mathrm{def}}{=} \exp\{-\partial_s Z(s,\lambda)|_{s=0}\} \qquad D_{\mathrm{cl}}(\lambda) \stackrel{\mathrm{def}}{=} \exp\{-\partial_s Z_{\mathrm{cl}}(s,\lambda)|_{s=0}\}$$

## CANONICAL normalization of recessive solution

$$\log D_{\rm cl}(\lambda) = \int_{-\infty}^{+\infty} (V(q)+\lambda)^{1/2}\,\mathrm{d}q.$$

*(Assuming $N>2$ for simplicity :)*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\log D_{\rm cl}(\lambda) = \tfrac12 \int_{-\infty}^{+\infty} (V(q)+\lambda)^{-1/2}\,\mathrm{d}q$$

&

$$\log D_{\rm cl}(\lambda) \ \sim\ \text{CANONICAL} \quad \text{for } \lambda\to+\infty$$

$$(\iff \quad \text{without pure} \quad \lambda^0 \quad \text{terms in large-}\lambda \text{ expansion})$$

specify improper action integral completely:

$$\int_q^{+\infty} \Pi_\lambda(q')\,\mathrm{d}q' \stackrel{\rm def}{=} \mathrm{FP}_{s=0}\, I_q(s,\lambda) \ + 2(1-\log 2)\,\beta_{-1}(0)/N\ ,$$

consistently with *additivity*:

$$\int_q^{+\infty} \Pi_\lambda(q')\,\mathrm{d}q' = \int_q^{q''} \Pi_\lambda(q')\,\mathrm{d}q' + \int_{q''}^{+\infty} \Pi_\lambda(q')\,\mathrm{d}q' \qquad \text{for finite } q,\, q''$$

($\int_{q'}^{q''}(V(q)+\lambda)^{1/2}\,\mathrm{d}q \sim (q''-q')\lambda^{1/2} + O(\lambda^{-1/2})$ being canonical for $q'$, $q''$ finite).

Simplest examples:

$$\int_0^{+\infty} (q^4+vq^2)^{1/2}\,\mathrm{d}q \ = \ -\tfrac13\, v^{3/2} \qquad \mathbf{N}$$

$$\int_0^{+\infty} (q^N+\lambda)^{1/2}\,\mathrm{d}q \ = \ -(2\sqrt{\pi})^{-1}\Gamma(1+\tfrac1N)\,\Gamma(-\tfrac12-\tfrac1N)\,\lambda^\mu \qquad (N\neq 2) \quad \boxed{\mu \stackrel{\rm def}{=} \frac12+\frac1N} \qquad \mathbf{N}$$

$$\int_0^{+\infty} (q^2+\lambda)^{1/2}\,\mathrm{d}q \ = \ -\tfrac14\,\lambda(\log\lambda-1) \qquad \mathbf{A}$$

Even quartic oscillator ($v,\lambda\ge 0$ for simplicity):

$$\int_0^{+\infty} (q^4+vq^2+\lambda)^{1/2}\,\mathrm{d}q =$$

$$(v\ge 2\sqrt\lambda): \qquad = \tfrac13 (v+2\sqrt\lambda)^{1/2}[2\sqrt\lambda\, K(k) - vE(k)], \quad k = \left(\frac{v-2\sqrt\lambda}{v+2\sqrt\lambda}\right)^{1/2};$$

$$(v\le 2\sqrt\lambda): \qquad = \tfrac13 \lambda^{1/4}[(2\sqrt\lambda+v)K(\tilde k) - 2vE(\tilde k)], \quad \tilde k = \frac{(2\sqrt\lambda - v)^{1/2}}{2\,\lambda^{1/4}}.$$

**Semiclassical interpretation of improper action integral**

$$\text{QUANTUM} \quad \xleftrightarrow[\text{correspondence}]{\qquad\qquad} \quad \text{CLASSICAL}$$

zeta function

$$\begin{aligned} Z(s,\lambda) &= \mathrm{Tr}\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+V(q)+\lambda\right)^{-s} \\ &= \sum_k (E_k+\lambda)^{-s} \end{aligned}$$

zeta function

$$\begin{aligned} Z_{\mathrm{cl}}(s,\lambda) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\left(p^2+V(q)+\lambda\right)^{-s} \\ &= \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\,\Gamma(s)}\int_{-\infty}^{+\infty}(V(q)+\lambda)^{1/2-s}\mathrm{d}q \end{aligned}$$

determinant

$$D(\lambda) \;=\; \text{“}\overset{\textit{formally}}{\prod_k}(\lambda+E_k)\text{”}$$

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identities

$$D_{\mathrm{cl}}^{-}(\lambda) \equiv \Pi_\lambda(0)^{-1/2}\exp\int_0^{+\infty}\Pi_\lambda(q)\,\mathrm{d}q \equiv [\psi_\lambda]_{\mathrm{WKB}}(0)$$

$$D_{\mathrm{cl}}^{+}(\lambda) \equiv \Pi_\lambda(0)^{+1/2}\exp\int_0^{+\infty}\Pi_\lambda(q)\,\mathrm{d}q \equiv -[\psi'_\lambda]_{\mathrm{WKB}}(0)$$

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**basic** identities

$$\boxed{D^-(\lambda)\equiv\psi_\lambda(0)}$$

$$\boxed{D^+(\lambda)\equiv-\psi'_\lambda(0)}$$

identities

$$D^-_{\mathrm{cl}}(\lambda) \equiv \Pi_\lambda(0)^{-1/2}\exp\int_0^{+\infty}\Pi_\lambda(q)\,\mathrm{d}q \equiv [\psi_\lambda]_{\mathrm{WKB}}(0)$$

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# The Wronskian identity

- Exact solution $\psi_\lambda(q)$, recessive for $q \to +\infty$ (WKB specification):

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(q')\mathrm{d}q' \,, \quad \Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q)+\lambda)^{1/2} \quad \text{(classical momentum)}$$

- Adjacent conjugate solution, recessive for $q \to +\,\mathrm{e}^{-\mathrm{i}\varphi/2}\infty$ :

$$\Psi_\lambda(q) \stackrel{\text{def}}{=} \psi^{[1]}_{\lambda^{[1]}}(\mathrm{e}^{\mathrm{i}\varphi/2}\, q)$$

- $q \to +\infty$ expansions **fully known**, e.g.:

$$\psi_\lambda(q) \sim \mathrm{e}^{\mathcal{C}}\, q^{-N/4-\beta_{-1}(\vec{v})} \exp\left\{-\sum_{\{\sigma>0\}} \beta_{\sigma-1}(\vec{v})\,\frac{q^\sigma}{\sigma}\right\}, \qquad \mathcal{C} = \tfrac{1}{N}\left[-2\log 2\ \beta_{-1}(\cdot) + \partial_s \tfrac{\beta_{-1}(\cdot)}{1-2s}\right]_{s=0}$$

$\vdots$

$\Longrightarrow$ **Explicit** Wronskian (evaluated in $q \to +\infty$ limit):

$$\boxed{\psi'_\lambda(q)\Psi_\lambda(q) - \Psi'_\lambda(q)\psi_\lambda(q) \equiv 2\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi/4}\,\mathrm{e}^{\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}}$$

- Add **basic exact identities**:

$D^+(\lambda) \equiv -\psi'_\lambda(0)$ $\Updownarrow$ $D^-(\lambda) \equiv \psi_\lambda(0)$

$$\boxed{-\,\mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}^{[1]}_-) + \mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}^{[1]}_+)\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}}$$

# Exact quantization condition?

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

# Exact quantization condition?

$$e^{+i\varphi/4}\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\,e^{+i\varphi\beta_{-1}(\vec{v})/2}$$

Degenerate cases:

- $N = 2:$ $$\left[-\frac{d^2}{dq^2} + (q^2+\lambda)\right]\psi(q) = 0 \qquad \text{(harmonic oscillator)}$$

$$\boxed{\varphi = \pi \qquad \beta_{-1} = \lambda/2}$$

$$e^{+i\pi/4}\, D^+(-\lambda)\, D^-(\lambda) - e^{-i\pi/4}\, D^+(\lambda)\, D^-(-\lambda) \equiv 2\,i\,e^{+i\pi\lambda/4}$$

# Exact quantization condition

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

## Degenerate cases:

- $N=2$ : $$\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+(q^2+\lambda)\right]\psi(q)=0 \qquad \text{(harmonic oscillator)}$$

$$\boxed{\varphi=\pi \qquad \beta_{-1}=\lambda/2}$$

$$\mathrm{e}^{+\mathrm{i}\pi/4}\, D^+(-\lambda)\, D^-(\lambda) - \mathrm{e}^{-\mathrm{i}\pi/4}\, D^+(\lambda)\, D^-(-\lambda) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\pi\lambda/4}$$

unknowns **real**, splits:

$$\cos\pi/4\,[D^+(-\lambda)\,D^-(\lambda)-D^+(\lambda)\,D^-(-\lambda)] = -2\sin\pi\lambda/4$$
$$\sin\pi/4\,[D^+(-\lambda)\,D^-(\lambda)+D^+(\lambda)\,D^-(-\lambda)] = +2\cos\pi\lambda/4$$

$$D^+(\lambda) \qquad D^-(-\lambda) \qquad = 2\cos\pi(\lambda-1)/4$$

*zeros* : $\ldots,\ -9,\ -5,\ -1,\quad +3,\ +7,\ +11,\ldots$

$$\Longrightarrow \qquad D^+(\lambda)=\frac{2^{-\lambda/2}\,2\sqrt{\pi}}{\Gamma(\frac{1+\lambda}{4})} \qquad D^-(\lambda)=\frac{2^{-\lambda/2}\sqrt{\pi}}{\Gamma(\frac{3+\lambda}{4})}$$

- Same for $\left[-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+(q^N+\Lambda\,q^{\frac{N}{2}-1})\right]\psi(q)=0$ (zero-energy generalized eigenvalue problem),

likewise exactly solvable (**supersymmetric**): with $\boxed{\nu \stackrel{\mathrm{def}}{=} \frac{1}{N+2}}$

$$D_N^+(\Lambda) = -\frac{2^{-\Lambda/N}(4\nu)^{\nu(\Lambda+1)+1/2}\,\Gamma(-2\nu)}{\Gamma(\nu(\Lambda-1)+1/2)} \qquad D_N^-(\Lambda) = \frac{2^{-\Lambda/N}(4\nu)^{\nu(\Lambda-1)+1/2}\,\Gamma(2\nu)}{\Gamma(\nu(\Lambda+1)+1/2)}$$

also for $\det[-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+(q^N+\Lambda\,q^{\frac{N}{2}-1})]$ over the whole real line:

$$= D_N^+(\Lambda)D_N^-(\Lambda) \quad \text{if } N\equiv 2 \pmod 4; \qquad = \frac{1}{\sin\pi\nu}\cos\pi\nu\Lambda \quad \text{if } N\equiv 0 \pmod 4.$$

## Exact quantization condition?

$$e^{+i\varphi/4}\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(e^{-i\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\, e^{+i\varphi\beta_{-1}(\vec{v})/2}$$

## Exact quantization condition?

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$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

## Exact quantization condition?

For even spectrum $\mathcal{E}_+$: $\lambda = -E_{2n}$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

# Exact quantization condition?

For even spectrum $\mathcal{E}_+$: $\quad \lambda = -E_{2n} \quad \Longleftrightarrow \quad D(\lambda \mid \mathcal{E}_+) = 0$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, \boxed{D(\lambda \mid \mathcal{E}_+)}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, \boxed{D(\lambda \mid \mathcal{E}_+)}\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

## Exact quantization condition?

For even spectrum $\mathcal{E}_+$: $\lambda = -E_{2n} \iff D(\lambda \mid \mathcal{E}_+) = 0$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

# Exact quantization condition

For even spectrum $\mathcal{E}_+$: $\lambda = -E_{2n} \iff D(\lambda \mid \mathcal{E}_+) = 0$

$$e^{+i\varphi/4} D(e^{-i\varphi}\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - e^{-i\varphi/4} D(\lambda \mid \mathcal{E}_+)\, D(e^{-i\varphi}\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,i\, e^{+i\varphi\beta_{-1}(\vec{v})/2}$$

$$e^{+i\varphi/4} D(\lambda \mid \mathcal{E}_+)\, D(e^{i\varphi}\lambda \mid \mathcal{E}_-^{[-1]}) - e^{-i\varphi/4} D(e^{i\varphi}\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,i\, e^{-i\varphi\beta_{-1}(\vec{v})/2}$$

$$\Longrightarrow \qquad \frac{D(e^{-i\varphi}\lambda \mid \mathcal{E}_+^{[+1]})}{D(e^{+i\varphi}\lambda \mid \mathcal{E}_+^{[-1]})} = -\,e^{i\,[-\varphi/2+\varphi\beta_{-1}(\vec{v})]} \qquad \left(\varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2}\right)$$

$$2\arg D(-\,e^{-i\varphi} E \mid \mathcal{E}_+^{[+1]}) - \varphi\,\beta_{-1}(\vec{v}) = \pi\left[k + \frac{1}{2} + \frac{N-2}{2(N+2)}\right] \qquad \text{for } k = 2n \geq 0$$

## Exact quantization condition

For odd spectrum $\mathcal{E}_-$: $\quad \lambda = -E_{2n+1} \iff D(\lambda \mid \mathcal{E}_-) = 0$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\mathrm{e}^{-\mathrm{i}\varphi}\lambda \mid \mathcal{E}_+^{[1]})\, D(\lambda \mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{-\mathrm{i}\varphi}\lambda \mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D(\lambda \mid \mathcal{E}_+)\, D(\mathrm{e}^{\mathrm{i}\varphi}\lambda \mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D(\mathrm{e}^{\mathrm{i}\varphi}\lambda \mid \mathcal{E}_+^{[-1]})\, D(\lambda \mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\Longrightarrow \qquad \frac{D(\mathrm{e}^{-\mathrm{i}\varphi}\lambda \mid \mathcal{E}_-^{[+1]})}{D(\mathrm{e}^{+\mathrm{i}\varphi}\lambda \mid \mathcal{E}_-^{[-1]})} = -\,\mathrm{e}^{\mathrm{i}\,[+\varphi/2+\varphi\beta_{-1}(\vec{v})]} \qquad \left(\varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2}\right)$$

$$2\arg D(-\,\mathrm{e}^{-\mathrm{i}\varphi} E \mid \mathcal{E}_-^{[+1]}) - \varphi\,\beta_{-1}(\vec{v}) = \pi\left[k + \frac{1}{2} - \frac{N-2}{2(N+2)}\right] \qquad \text{for } k = 2n+1 > 0$$

## Complete set of exact quantization conditions

(for all conjugate, $\substack{\text{even}\\ \text{odd}}$ spectra $\mathcal{E}_\pm^{[\ell]}$)

$$\frac{1}{\mathrm{i}}\left[\log D(-\,\mathrm{e}^{-\mathrm{i}\varphi} E \mid \mathcal{E}_\pm^{[\ell+1]}) - \log D(-\,\mathrm{e}^{+\mathrm{i}\varphi} E \mid \mathcal{E}_\pm^{[\ell-1]})\right] - (-1)^\ell \varphi\,\beta_{-1}(\vec{v})$$

$$= \pi\left[k + \frac{1}{2} \pm \frac{N-2}{2(N+2)}\right] \qquad \text{for} \quad k = \substack{0,2,4,\cdots\\ 1,3,5,\cdots} \qquad \ell = 0, 1, \cdots, L-1 \pmod{L}$$

+ structure equations:

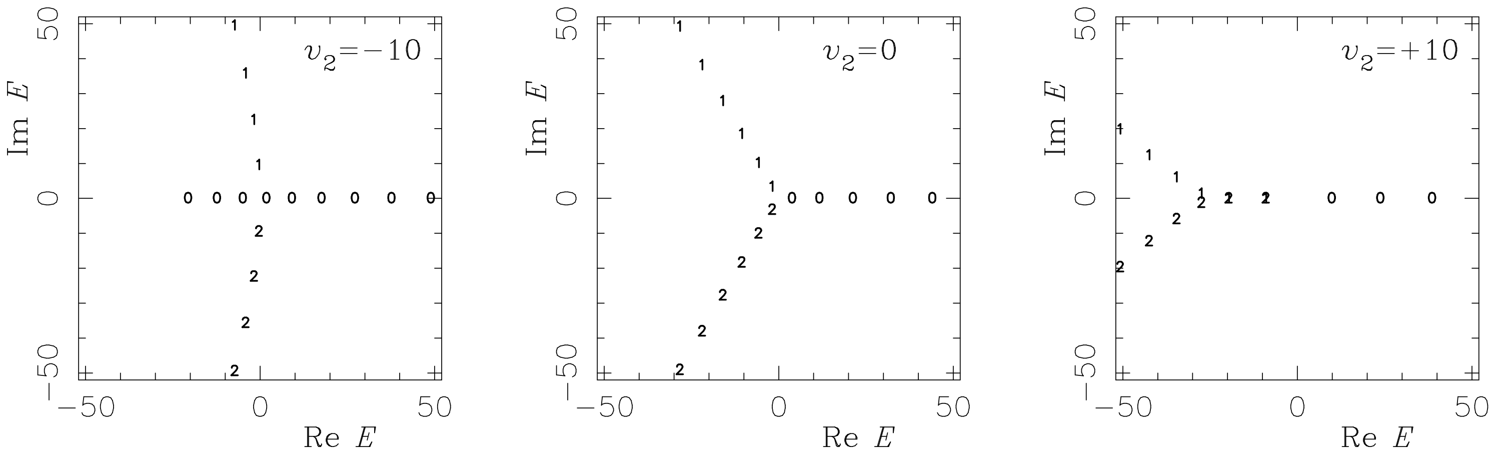
$$\log D(\lambda \mid \mathcal{E}_\pm^{[\ell]}) \equiv \lim_{K\to+\infty}\left\{ \sum_{k<K} \log(E_k^{[\ell]} + \lambda) + \tfrac{1}{2}\log(E_K^{[\ell]} + \lambda) - \sum_{\{\alpha>0\}} \tfrac{1}{2}\, b_\alpha^{[\ell]} \left[E_K^{[\ell]}\right]^\alpha (\log E_K^{[\ell]} - 1/\alpha) \right\}$$

$$\left(k, K \ \substack{\text{even}\\ \text{odd}}\right)$$

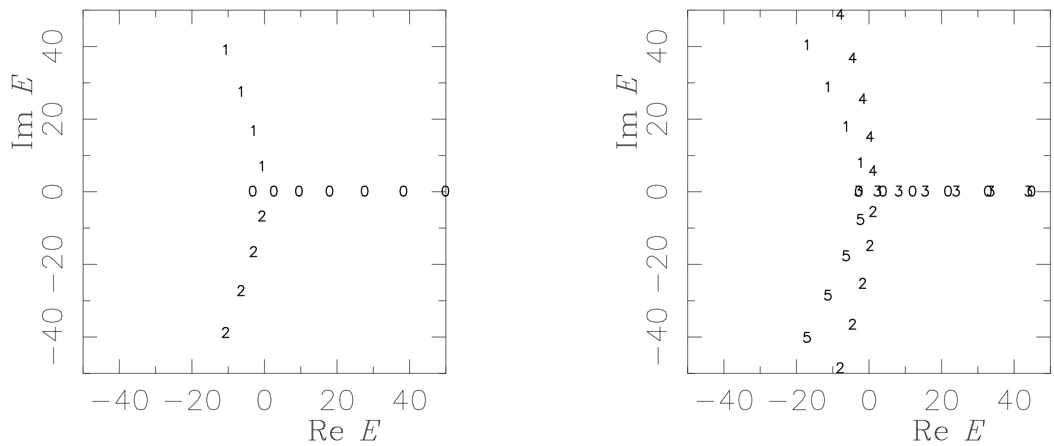
altogether define a formally **complete** set of **fixed-point conditions**

$(\mathcal{M}^\pm\{\mathcal{E}_\pm^{[\ell]}\} = \{\mathcal{E}_\pm^{[\ell]}\}$ for some mappings $\mathcal{M}^\pm)$.

The three conjugate (odd, and rotated) spectra for $V(q) = q^4 + v_2 q^2$.




Same for $V(q) = q^4 - 5\,q^2$; $V(q) \approx q^4 + q^3 - 4.625\,q^2 - 2.4375\,q$.



**Homogeneous case** $\quad V(q) = q^N \quad \boxed{N \neq 2}$

- All conjugate spectra **identical**: $\quad \mathcal{E}_{\pm}^{[\ell]} \equiv \mathcal{E}_{\pm}$
- **Residue polynomial** $\beta_{-1}(s;\lambda)|_{\vec{v}=\vec{0}} \equiv 0 \quad$ [ **except** $N=2$: $\quad \beta_{-1}(s;\lambda) \equiv \lambda(-s+\frac{1}{2})$ ]
- Wronskian identity (**W.I.**):

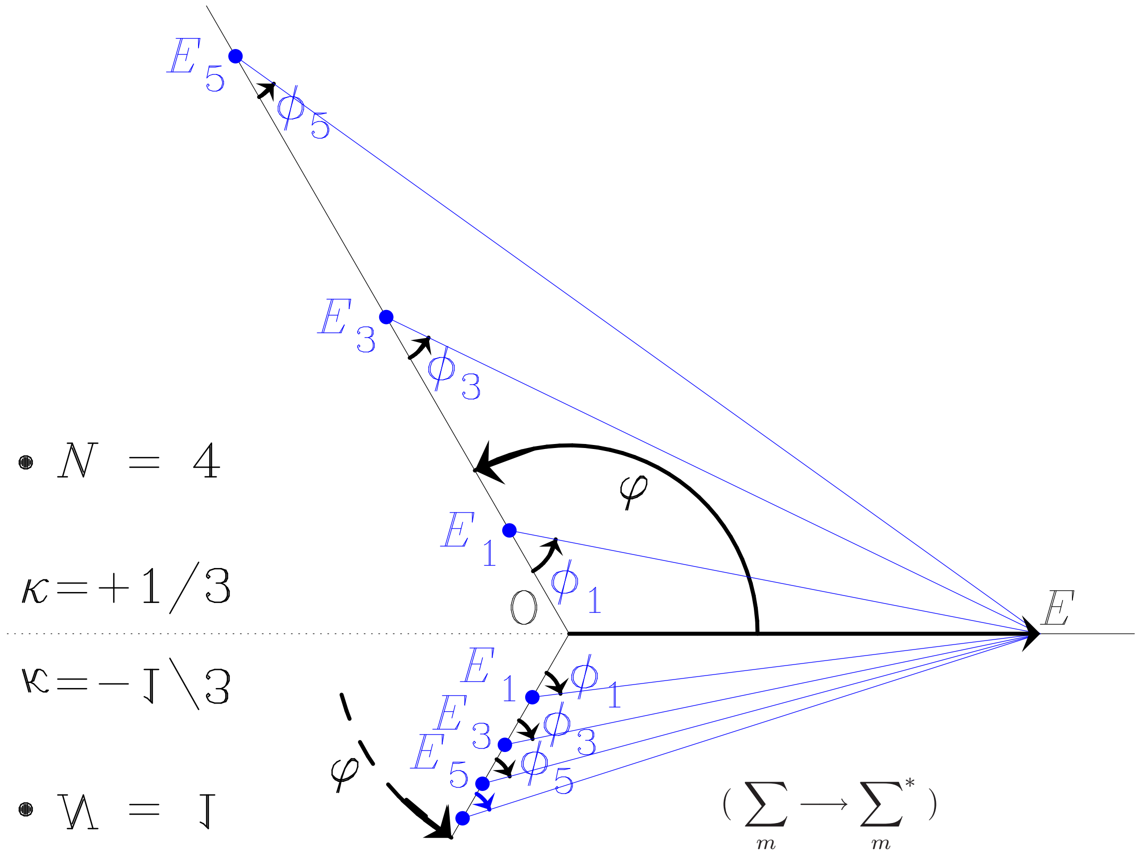
$$\mathrm{e}^{+\mathrm{i}\varphi/4}\, D^{+}(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda)\, D^{-}(\lambda) - \mathrm{e}^{-\mathrm{i}\varphi/4}\, D^{+}(\lambda)\, D^{-}(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda) \equiv 2\,\mathrm{i} \qquad \boxed{\varphi = \frac{4\pi}{N+2}}$$

- **Exact quantization condition**:

$$\begin{aligned}
2\,\Sigma_{+}(E_k) &= k + \frac{1}{2} + \frac{\kappa}{2} & k &= 0, 2, 4, \ldots \\
2\,\Sigma_{-}(E_k) &= k + \frac{1}{2} - \frac{\kappa}{2} & k &= 1, 3, 5, \ldots \\
\kappa &\stackrel{\text{def}}{=} \frac{N-2}{N+2} & & \\
\Sigma_{\pm}(E) &\stackrel{\text{def}}{=} \frac{1}{\pi} \sum_{m \ \substack{\text{even} \\ \text{odd}}} \underbrace{\arg(E_m - \mathrm{e}^{-\mathrm{i}\varphi}\,E)}_{\phi_m(E)} & & (N > 2)
\end{aligned}$$

$$+ \text{ boundary condition} \qquad b_\mu E_k^{\mu} \sim k + \frac{1}{2} \qquad \text{for } k \to +\infty$$

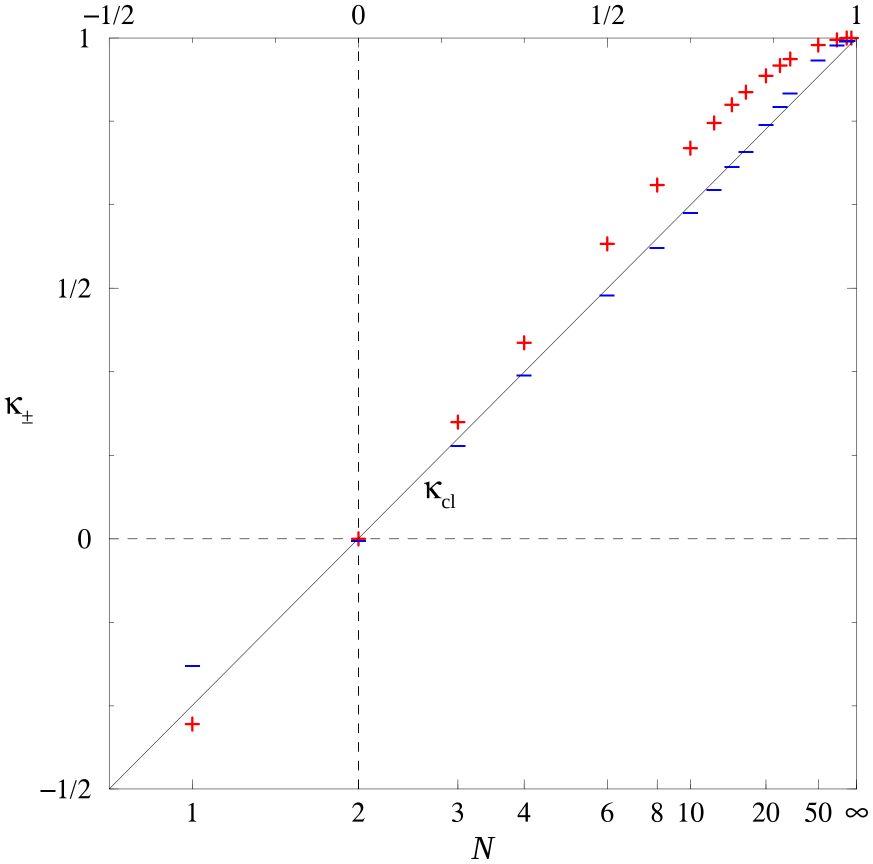
$\iff$ fixed-point equations $\mathcal{M}^{\pm}\{\mathcal{E}_{\pm}\} = \{\mathcal{E}_{\pm}\}$
(mappings $\mathcal{M}^{\pm}$ proved **globally contractive** for $N > 2$, by *A. Avila*).



## Numerical tests

Ex. $\kappa_-(120) \approx 0.9980$

Horizontal scale $\kappa_{\rm cl} = \kappa \overset{\rm def}{=} \frac{N-2}{N+2}$ $\kappa_+(400) \approx 0.99975$



# Exact wave-function analysis

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + [V(q)+\lambda]\right)\psi(q) = 0$$

and, e.g., $\psi(q)$ recessive for $q \to +\infty$ ($\lambda$ arbitrary, input).

Restrict to half-line $[Q, +\infty)$ ($Q$ a parameter):

$$\begin{aligned} V_Q(q) &\stackrel{\text{def}}{=} [V(q) - V(Q)] \qquad \text{for } q \in [Q, +\infty) \\ D_Q^{\pm}(\lambda) &\stackrel{\text{def}}{=} \det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + V_Q(q) + \lambda\right)^{\pm} \qquad \left[\begin{smallmatrix}\text{Neumann}\\ \text{Dirichlet}\end{smallmatrix} \text{ boundary conditions at } q = Q\right] \end{aligned}$$

Translated **basic identities**:

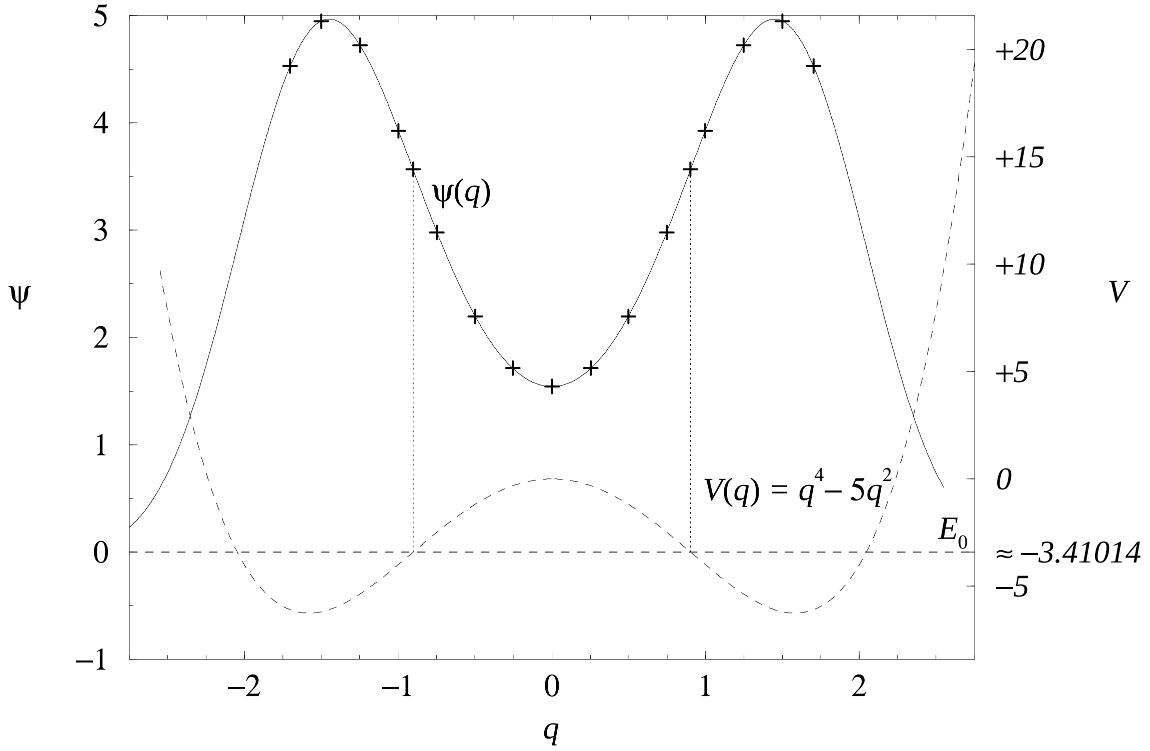
$$\psi_\lambda(Q) \equiv D_Q^-(\lambda + V(Q)), \qquad \psi'_\lambda(Q) \equiv -D_Q^+(\lambda + V(Q))$$

hence $\psi_\lambda(Q)$ follows by solving a parametric fixed-point problem

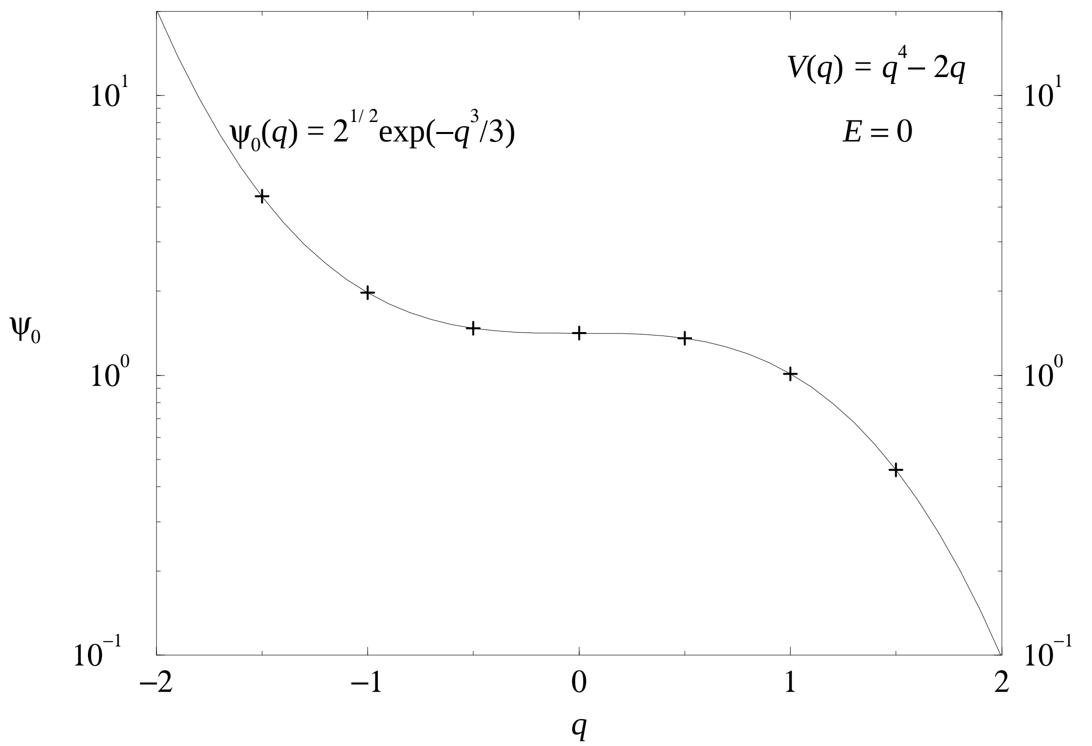
$$\mathcal{M}_Q^-\{\mathcal{E}_{Q,-}^{[\ell]}\} = \{\mathcal{E}_{Q,-}^{[\ell]}\}$$

for the Dirichlet spectrum $\mathcal{E}_{Q,-}$ of the potential $V_Q$.

Ground-state eigenfunction $\psi(q)$ for the potential $V(q) = q^4 - 5q^2$.




Non-square-integrable solution $\psi_0(q)$ for the potential $V(q) = q^4 - 2q$ at energy $E = 0$.



## “ODE/IM correspondence”

(Dorey–Tateo, Suzuki, Bazhanov–Lukyanov–Zamolodchikov,...)

A dictionary between *some* 2D exactly solvable models and *some* 1D Schrödinger equations: e.g.,

| **O**rdinary **D**ifferential **E**quations | $\longleftrightarrow$ | **I**ntegrable **M**odels |
|---|---|---|
| **1D Schrödinger equation** with **homogeneous potential** $q^{2M}$ | | **2D 6-vertex model with twist** $\phi = \pi/(2M+2)$ |
| Spectral parameter $\lambda$ | | Spectral parameter $\nu$ |
| Degree of potential $2M$ | $\mathrm{e}^{2\pi \mathrm{i}/(2M+2)} = -\,\mathrm{e}^{-2\mathrm{i}\eta}$ | Anisotropy $\eta$ |
| Stokes multiplier $C(\lambda)$ | | Transfer matrix $T(\nu)$ |
| $D^{-}(\lambda) = \psi_\lambda(0)$ | | $Q(\nu)$ operator |
| Exact quantization conditions | | Bethe Ansatz equations |

(transposed from Dorey–Dunning–Tateo, *The ODE/IM correspondence* `[hep-th/0703066]`)

That 2D 6-vertex model is also connected with the 1D quantum XXZ chain, of Hamiltonian

$$\mathcal{H}(\eta) = -\tfrac{1}{2} \sum_{j=1}^{N} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \cos 2\eta\, \sigma_j^z \sigma_{j+1}^z)$$

with twisted boundary condition

$$\sigma_{N+1}^x \pm \mathrm{i}\sigma_{N+1}^y = \mathrm{e}^{2\mathrm{i}\phi}(\sigma_1^x \pm \mathrm{i}\sigma_1^y), \qquad \sigma_{N+1}^z = \sigma_1^z.$$

Extension to $\mathcal{PT}$-symmetric potentials:

| Specific potentials $V(q)$ | Specific integrable models |
|---|---|
| $-(\mathrm{i}q)$ | $N=2$ SUSY point of sine-Gordon |
| $-(\mathrm{i}q)^2$ | free-fermion point |
| $-(\mathrm{i}q)^3$ | Yang–Lee |
| $-(\mathrm{i}q)^4$ | $\mathbb{Z}_4$ parafermions |
| $-(\mathrm{i}q)^6$ | 4-state Potts |

(Dorey–Dunning–Tateo, *Ordinary Differential Equations and Integrable Models* `[hep-th/0010148]`)

## Open issues

- Inhomogeneous polynomial potentials:
  - contractivity of fixed-point mapping? (Numerically OK near $\vec{v} = \vec{0}$)
  - correspondence with integrable models (generalized Bethe Ansatz).
- More general problems:
  - rational potentials (e.g., centrifugal term)
  - all Heun equations
  - higher-order equations/systems, higher-dimensional Schrödinger equations, ...
- **Consistency with perturbative regime.**

# Toward singular quantum perturbation theory

(here $N > M \geq 0$)

$$\hat{H}(v) = -\mathrm{d}^2/\mathrm{d}q^2 + q^N + vq^M \ (\textit{coupled} \text{ problem}) \approx v^{2/(M+2)}\left[-\mathrm{d}^2/\mathrm{d}q^2 + q^M + gq^N\right]$$

$$\hat{H}_0(v) = -\mathrm{d}^2/\mathrm{d}q^2 + vq^M \qquad (\textit{uncoupled} \text{ problem}) \approx v^{2/(M+2)}\left[-\mathrm{d}^2/\mathrm{d}q^2 + q^M\right]$$

hence: relate $\det^{\pm}(\hat{H}(v)+\lambda)$ to $\det^{\pm}(\hat{H}_0+\lambda)$ for $v \to +\infty \Leftrightarrow g \to 0^+$ ?

$g \to 0$ : *a most singular limit!* E.g., in exact quantization condition

$$2 \arg D(-\mathrm{e}^{-\mathrm{i}\varphi}\lambda_k \mid \mathcal{E}_+^{[+1]}) - \varphi\,\beta_{-1}(\vec{v}) = \pi\left[k + \frac{1}{2} + \frac{N-2}{2(N+2)}\right] \qquad \text{for } k = 2n,$$

- the degree, hence the angle $\varphi$ as well, jump ($N \to M$);
- the anomaly type, hence $\beta_{-1}(\vec{v})$ as well, may jump (e.g., $\mathbf{N} \to \mathbf{A}$ for $q^2 + gq^4$).

### Main theoretical estimate

$$\det{}^{\pm}(\hat{H}(v)+\lambda) \sim \left[\frac{\det_{\mathrm{cl}}(\hat{H}(v)+\lambda)}{\det_{\mathrm{cl}}(\hat{H}_0(v)+\lambda)}\right]^{1/2} \det{}^{\pm}(\hat{H}_0(v)+\lambda)$$

### Practical implication

There only remains to compute *two improper actions*,

$$\left(\tfrac{1}{2}\log\det_{\mathrm{cl}}(\hat{H}(v)+\lambda) = \right) \int_0^{+\infty} \Pi_\lambda(q,v)\,\mathrm{d}q = \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2}\,\mathrm{d}q \quad \text{(coupled)},$$

$$\left(\tfrac{1}{2}\log\det_{\mathrm{cl}}(\hat{H}_0(v)+\lambda) = \right) \int_0^{+\infty} \Pi_{0,\lambda}(q,v)\,\mathrm{d}q = \int_0^{+\infty} (vq^M + \lambda)^{1/2}\,\mathrm{d}q \qquad \text{(uncoupled)}.$$

$$(N > M \geq 0)$$

## Binomial $\Pi(q)^2 = uq^N + vq^M$

• **Exact** evaluation of improper action integral:

$$\int_0^{+\infty} (uq^N + vq^M)^{1/2}\,\mathrm{d}q = \frac{\Gamma(\frac{M+2}{2(N-M)})\,\Gamma(-\frac{N+2}{2(N-M)})}{(N-M)\,\Gamma(-1/2)}\,u^{-\frac{M+2}{2(N-M)}}\,v^{\frac{N+2}{2(N-M)}}$$

when the RHS factor is finite, i.e., in **N**ormal case: $\dfrac{N+2}{2(N-M)} \notin \mathbb{N}$.

## Trinomial $\Pi(q)^2 = q^N + vq^M + \lambda$

• **Asymptotic** evaluation of improper action integral for $v \to +\infty$:

$$\int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2}\,\mathrm{d}q \sim \int_0^{+\infty} (q^N + vq^M)^{1/2}\,\mathrm{d}q \qquad \left(= C_{N,M}\,v^{\frac{N+2}{2(N-M)}} \qquad [\mathbf{N}]\right)$$

$$+ \int_0^{+\infty} (vq^M + \lambda)^{1/2}\,\mathrm{d}q \qquad \left(= C'_M \begin{cases} v^{-\frac{1}{M}}\lambda^{\frac{1}{2}+\frac{1}{M}} & M \neq 2 \\ v^{-\frac{1}{2}}\lambda(1-\log\lambda) & M = 2 \end{cases}\right)$$

$$+ \ \delta_{M,2} \times \left(C''_N\,\lambda v^{-\frac{1}{2}}(\log v + 2\log 2)\right).$$

• **Exactly computable case:** $N = 4$ (in complete elliptic integrals, $k =$ modulus)

$$\int_0^{+\infty} (q^4 + vq^2 + \lambda)^{1/2}\,\mathrm{d}q \equiv$$

$$\begin{cases} \frac{1}{3}\,\lambda^{1/4}[(2\sqrt{\lambda}+v)K(\tilde{k}) - 2vE(\tilde{k})], & \tilde{k} = \dfrac{(2\sqrt{\lambda}-v)^{1/2}}{2\,\lambda^{1/4}} \qquad (v \leq 2\sqrt{\lambda}) \\ \frac{1}{3}(v+2\sqrt{\lambda})^{1/2}[2\sqrt{\lambda}K(k) - vE(k)], & k = \left(\dfrac{v-2\sqrt{\lambda}}{v+2\sqrt{\lambda}}\right)^{1/2} \qquad (v \geq 2\sqrt{\lambda}) \end{cases}$$

$$\sim -\tfrac{1}{3}\,v^{3/2} + 0\,v^{1/2}\log v + 0\,v^{1/2} - \tfrac{1}{4}\,\lambda v^{-1/2}\,[\log(\lambda/v^2) - 4\log 2 - 1].$$

## Samples of end results ($g \to 0^+$ limit, $E$ fixed)

$$\frac{\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^M + gq^N - E)}{\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^M - E)} \sim g^{-\frac{4}{N(N+2)}\beta_{-1}(0)} \times \qquad [\mathbf{A}]$$

$$\exp\ 2\int_0^{+\infty} (q^N + vq^M)^{1/2}\,\mathrm{d}q \times$$

$$\exp\left\{\delta_{M,2}\,\frac{1}{N-2}\,[\log g - N\log 2]\,E\right\}$$

(with $\int_0^{+\infty}(q^N + vq^M)^{1/2}\,\mathrm{d}q \propto g^{-(M+2)/2(N-M)}$).

Basic example: $N=4,\ M=2$

$$\det\Big(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^2 + gq^4 - E\Big) \sim \exp\left\{-\frac{2}{3g} + \Big[\tfrac12\log g - 2\log 2\Big]E\right\} \underbrace{\det\Big(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^2 - E\Big)}_{2^{E/2}\sqrt{2\pi}/\,\Gamma(\frac12(1-E))}$$

$$\sum_{k=0}^{\infty}\frac{1}{E_k(g)-E} \sim -\Big[\tfrac12\log g - 2\log 2\Big] - \tfrac12\Big[\log 2 + \psi(\tfrac12(1-E))\Big]$$

$$Z_g(1) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty}\frac{1}{E_k(g)} \sim -\tfrac12\log g + \tfrac12(\gamma + 5\log 2) \qquad (g \to 0^+).$$