Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 138–145 **Birthday Issue**

On a Class of Non Self-Adjoint Quantum Nonlinear Oscillators with Real Spectrum

Emanuela CALICETI and Sandro GRAFFI [1]

Dipartimento di Matematica, Università di Bologna
40127 Bologna, Italy
E-mails: caliceti@dm.unibo.it, graffi@dm.unibo.it

This article is part of the special issue published in honour of Francesco Calogero on the occasion of his 70th birthday

Abstract

We prove reality of the spectrum for a class of $PT-$ symmetric, non self-adjoint quantum nonlinear oscillators of the form $H = p^2 + P(q) + igQ(q)$. Here $P(q)$ is an even polynomial of degree $2p$ positive at infinity, $Q(q)$ an odd polynomial of degree $2r - 1$, and the conditions $p > 2r$, $|g| < \overline{R}$ for some $\overline{R} > 0$ hold.

1 Introduction and statement of the results

Quantum nonlinear oscillators exhibiting remarkable ambiguities in the quantization procedure have recently drawn considerable attention from Francesco Calogero[1], [2], [3]. In this paper we deal with another remarkable phenomenon taking place in a different class of quantum non linear oscillators (the PT-symmetric ones, see below) namely the reality of the spectrum even though the corresponding Schrödinger operators are not self-adjoint. Consider indeed the classical Hamiltonians in the canonical coordinates $(p, q) \in \mathbb{R}^2$:

$$\mathcal{H}(p, q; g) = p^2 + P(q) + igQ(q) \tag{1.1}$$

Here $P(q)$ is a real, even polynomial of degree $2p, p \geq 1$ diverging positively at infinity, $Q(q)$ a real, odd polynomial of degree $2r-1, r \geq q \geq 1$, and g a complex number. Standard quantization of (1.1) yields the Schrödinger equation (here $\hbar = 1$)

$$-\frac{d^2\psi}{dq^2} + P(q)\psi + igQ(q)\psi = E(g)\psi \tag{1.2}$$

or, equivalently

$$H(g)\psi = E(g)\psi$$

Here $H(g)$ is the maximal operator acting in $L^2(\mathbb{R})$ generated by $-\dfrac{d^2}{dq^2} + P(q) + igQ(q)$. $H(g)$ has discrete spectrum (see e.g. [4]), but it is not self-adjoint and not even normal.

[1] INFN, Sezione di Bologna

However it has been conjectured long ago that its eigenvalues, denoted $E_n(g) : n = 1, 2, \ldots$ should be purely real if g is real. This is because the Schrödinger equation (1.2) is PT-symmetric, namely invariant under the combined application of the reflection symmetry operator $(P\psi)(x) = \psi(-x)$ and of the (antilinear) complex conjugation operation $(T\psi)(x) = \overline{\psi}(x)$. One immediately checks that $(PT)H(g) = H(g)(PT)$ when $g \in \mathbb{R}$.

$PT-$symmetric quantum mechanics (see e.g. [5],[6],[7], [8],[9],[10],[11],[12]) requires the reality of the spectrum of $PT-$symmetric operators provided PT is not spontaneously broken; actually it has been recently proved, for instance, that a large subclass of Schrödinger operators of the type (1.2) does actually have a purely real spectrum [19], [17]. Both proofs rely on rather subtle arguments of ordinary differential equations and functions of complex variables. Purpose of this paper is to provide a perturbation theoretic proof, valid for a class of polynomials P and Q. It consists in the verification of the following simple statements:

(i) All eigenvalues $E_n(0)$ of $H(0)$ are real, and stable for g suitably small (the definition of stability is recalled in the proof of Theorem 1.1 below);

(ii) There is $\overline{R} > 0$ independent of n such that the perturbation expansion for any eigenvalue $E_n(g)$ near $E_n(0)$ converges for $|g| < \overline{R}$;

(iii) For $|g| < \overline{R}$ the operator $H(g)$ has no eigenvalue other than those defined by the convergent perturbations expansions, which are real.

We will indeed prove the following

Theorem 1.1.
In the above notations, let $p > 2r$. Then assertions (i-iii) above hold true for the operator $H(g)$ defined by the maximal action of $-\dfrac{d^2}{dq^2} + P(q) + igQ(q)$ on $L^2(\mathbb{R})$.

Remarks

1. If H is a $PT-$ symmetric operator then $(PT)H = H^*(PT)$ so that the eigenvalues of H exist in complex conjugate pairs.

2. We recall that the eigenvalues E_n of $H(0)$ form an increasing sequence such that $\lim_{n\to\infty} E_n = +\infty$ and fulfill the estimate (see e.g.[4])

$$E_n = Bn^{2p/(p+1)} + O(n^{\frac{p-1}{p+1}}), \quad n \to \infty \tag{1.3}$$

   for some positive constant B.

3. The proof of [17] applies to the cases $P = (-iq)^m$, $Q = -i\sum_{j=1}^{m-1} a_j(iq)^{m-j}$, $m \geq 2$, g arbitrary under the condition $(j-k)a_k \geq 0 \;\forall\, k$ for at least one $1 \leq j \leq m/2$. The proof of [19] applies to $P(q) = q^2$, $Q(q) = q^{2r-1}$. Hence we see that for $m = 2p$ the present result holds for a class not contained in the previous ones.

4. The recently introduced [18] $PT-$asymmetric but $CPT-$symmetric hamiltonians represent a small perturbation of the present class.

2 Proof of the results

We have to verify Assertions (i-iii). Assertion (i) is well known: $H(0)$ is a self-adjoint operator with discrete spectrum, and Q is relatively bounded with respect to $H(0)$ with relative bound zero. This means that $D(Q) \supset D(H(0))$ and that for any $b > 0$ there is $a > 0$ such that

$$\|Qu\| \le b\|H(0)u\| + a\|u\|, \qquad \forall\, u \in D(H(0)) \tag{2.1}$$

Here $D(A)$ denotes the domain of the operator A. This bound entails that $H(g)$ defined on $D(H(0))$ is a closed operator with compact resolvent (and hence discrete spectrum) for all $g \in \mathbb{C}$. Moreover all eigenvalues of $H(0)$ are simple and stable as eigenvalues of $H(g)$ for $|g|$ suitably small, i.e. there is one and only one simple eigenvalue $E_n(g)$ of $H(g)$ near $E_n(0)$ for $|g|$ suitably small (see e.g.[23] for the formal definition).
We have thus to verify Assertions (ii) and (iii). As far as (iii) is concerned, we recall that under the present conditions the (Rayleigh-Schrödinger) perturbation expansion near any unperturbed eigenvalue $E_n(0) := E_n$ has a positive convergence radius ρ_n; a lower bound for the convergence radius is given by (see [22], formula VII.2.34):

$$\overline{R}(n) = \left[\frac{2(a + b|E_n|)}{d_n} + 2b + 1\right]^{-1} \tag{2.2}$$

Here b and a are the constants of the estimate (2.1) and d_n is half the isolation distance of the eigenvalue E_n, namely:

$$d_n := \frac{1}{2}\min\left(|E_n - E_{n-1}|, |E_n - E_{n+1}|\right) \tag{2.3}$$

In this case, by (1.3) we have

$$d_n \sim n^{\frac{p-1}{p+1}}, \qquad \frac{E_n}{d_n} \sim n, \quad n \to \infty \tag{2.4}$$

Therefore to prove Assertion (ii) it is enough to verify that, choosing $b = b(n) = \dfrac{1}{n}$ in (2.1), we can find $a(n)$ such that

$$\frac{a_n}{d_n} \le \Lambda < +\infty, \qquad n \to \infty \tag{2.5}$$

for a suitable constant $\Lambda > 0$. By (2.2) this entails that the perturbation expansions near the unperturbed eigenvalues E_n have a common convergence circle of radius $\overline{R} > 0$ independent of n, i.e. $\overline{R}(n) \ge \overline{R}\ \forall\, n$. We have indeed:

Lemma 2.1. *If $p > (4r+3)/2$, $\forall\, n$ given $b_n = K/n$ for some $K > 0$ there is $N > 0$ such that the estimate (2.1) holds with $a_n < N n^{\frac{p-1}{p+1}}$.*

Proof: Introduce from now on the standard notation $pu = -i\dfrac{du}{dx}$ so that

$$p^2 u = -\frac{d^2u}{dx^2}, \qquad H(0) = p^2 + P(q).$$

The following quadratic estimate is well known (see e.g.[13])

$$\|p^2u\| + \|P(q)u\| \le \gamma|(p^2 + P(q))u\| + \beta\|u\|, \qquad \forall\, u \in D(H(0)) \tag{2.6}$$

for some $\gamma > 0, \beta > 0$. Therefore to prove it will be enough to prove (2.1) with the stated constants a_n and b_n it will be enough to prove the further estimate

$$\|Qu\| \le b_n\|Pu\| + a_n\|u\|, \qquad \forall\, u \in D(P) \tag{2.7}$$

because we then have $\|Qu\| \le b_n\|H(0)u\| + b_n\beta\|u\| + a_n\|u\|$ and the constant $b_n\beta$ can be obviously absorbed in a_n. In turn (2.7) follows from

$$\|Qu\|^2 \le b_n^2\|Pu\|^2 + a_n^2\|u\|^2, \qquad \forall\, u \in D(P) \tag{2.8}$$

Now this L^2 inequality is clearly implied by the pointwise inequality

$$b_n^2P(q)^2 - Q(q)^2 + a_n^2 \ge 0, \qquad \forall\, q \in \mathbb{R} \tag{2.9}$$

Next we remark that, up to an additive constant which can be absorbed in the constant β of the estimate (2.6), we can limit ourselves to verify this inequality for homogeneous polynomials P and Q of degree $2p$ and $2r-1$, respectively. Since $b_n = K/n$ this last inequality reads

$$q^{4r-2} \le \frac{K^2}{n^2}q^{4p} + a_n^2, \qquad \forall\, q \in \mathbb{R} \tag{2.10}$$

Since there are only even powers, we can restrict to $q \ge 0$. Assume for the sake of simplicity $K = 1$. Remark that the inequality

$$q^{4r-2} \le n^{-2}q^{4p}$$

is fulfilled if

$$q \ge n^{\alpha}, \qquad \alpha = \frac{1}{2(p-r)+1}$$

On the other hand, if $q < n^{\alpha}$ then $q^{4r-2} < n^{(4r-2)\alpha}$; hence the inequality

$$q^{4r-2} < a_n^2$$

which yields (2.10) in this case, will be fulfilled if

$$a_n \ge n^{\frac{2r-1}{2(p-r)+1}} \tag{2.11}$$

It suffices then to have

$$\frac{2r-1}{2(p-r)+1} < \frac{p-1}{p+1}$$

or, equivalently:

$$p > 2r$$

This concludes the proof of the Lemma. An immediate consequence is the existence of the constant Λ in (2.5), and consequently of a common convergence radius $\overline{R}$. This verifies Assertion (ii).

Proof of Theorem 1.1

We have only to verify Assertion (iii). We do this by adapting the argument of [21], Theorem 2. Let us first recall that under the present assumptions $H(g)$ is a type-A holomorphic family of operators in the sense of Kato (see [22], Chapter VII.2) with compact resolvents $\forall\, g \in \mathbb{C}$. In particular:

(i) the eigenvalues $E_l(g)$ are locally holomorphic functions of g with only algebraic singularities;

(ii) the eigenvalues $E_l(g)$ are stable, namely given any eigenvalue $E(g_0)$ of H_{g_0} of (geometric) multiplicity m there are exactly m eigenvalues $E^j(g)$ of $H(g)$, $j = 1, \dots, m$ (counting geometric multiplicities) such that $\lim_{g \to g_0} E^j(g) = E(g_0)$;

(iii) the Rayleigh-Schrödinger perturbation expansion for the eigenprojections and the eigenvalues near any eigenvalue E_l of $H(0)$ are convergent.

We have seen above that all the series are convergent for all $g \in \Omega_{\overline{R}}$; $\Omega_{\overline{R}} := \{g \in \mathbb{C} : |g| \le \overline{R}\}$, where R_0 is the uniform lower bound for all convergence radii.
The first part of the argument concerns a localization of the eigenvalues of $H(g_0)$, $g_0 \in \Omega_{\overline{R}} \cap \mathbb{R}$. Since $b_n = K/n$, by (2.4) and (2.5) there exists $A > 0$ sufficently large such that

$$3b_n + \frac{b_n E_n}{d_n} + \frac{a}{d_n} \le A, \quad \forall\, n \in \mathbb{N} \tag{2.12}$$

$$2b_n + \frac{b_n(E_{n+1} - d_{n+1}}{\delta_n} + \frac{a_n}{\delta_n} \le A, \quad \forall\, n \in \mathbb{N} \tag{2.13}$$

$$2b_1 + \frac{a_1}{|E_1 - d_1|} \le A \tag{2.14}$$

Here:

$$\delta_n := \min\,(d_n, d_{n+1})$$

For any n let $\mathcal{Q}_n$ be the square centered at E_n and of side $2dn$ in the complex z plane, and recall that the eigenvalues E_n form an increasing sequence: $E_0 < E_1 < \dots$. Let $\mathcal{A} := \{z \in \mathbb{C} : \mathrm{Re} z \ge (E_1 - d_1)\} \bigcup_{n \in \mathbb{N}} \mathcal{Q}_n$. Let us show that this set has empty intersection with the sectrum of $H(g)$ for $|g| < 1/A$, i.e. $\mathcal{A} \subset \rho(H(g)) = \mathbb{C} \setminus \sigma(H(g))$. $\forall\, z \in \mathcal{A}$ there are indeed two possibilities:

$$a) \quad \exists \quad s \in \mathbb{N} \quad s.t. \quad |\mathrm{Im}\, z| \ge \delta_s \quad \text{and} \quad |\mathrm{Re}\, z - E_s| \le \delta_s$$
$$b) \quad \exists \quad s \in \mathbb{N} \quad s.t. \quad E_s + d_s \le \mathrm{Re}\, z \le E_{s+1} - d_{s+1}$$

Case a): Let $R_0(z) := (H(0) - z)^{-1}$ be the free resolvent. Then we have:

$$\begin{aligned}
\|gQR_0(z)\| &\le |g| \cdot \|QR_0(z)\| \\
&\le |g|[b_s\|[H(0) - z]R_0(z)\| + b_s|z|\,\|R_0(z)\| + a_s\,\|g(z)\|] \\
&\le |g|\left[b_s + \frac{b_s|z| + a_s}{\mathrm{dist}(z, \sigma(H(0))}\right] \\
&\le |g|\left[b_s + b_s\frac{E_s + d_s + |\mathrm{Im}\, z|}{|\mathrm{Im}\, z|} + \frac{a_s}{|\mathrm{Im}\, z|}\right] \\
&\le |g|\left[3b_s + b_s\frac{E_s}{d_s} + \frac{a_s}{d_s}\right] \le |g|A < 1
\end{aligned} \tag{2.15}$$

if $|g| < 1/A$. This formula follows by the relative boundedness condition $\|Qu\| \le b_n\|H(0)u\| + a_n\|u\|$, the fact that $\|R_0(z)\| = \dfrac{1}{\mathrm{dist}(z,\sigma(H(0))} \le \dfrac{1}{|\mathrm{Im}\, z|}$, and formula (2.12). We now prove that the resolvent

$$R_g(z) := [H(g) - z]^{-1} = R_0(z)[1 + igQR_0(z)]^{-1}$$

exists and is bounded by the uniform norm convergence of the Neumann expansion. We have indeed, by (2.15):

$$\begin{aligned} \|R_g(z)\| &= \|R_0(z)[1+igQR_0(z)]^{-1}\| = \|R_0(z)\sum_{k=0}^{\infty}[-igQR_0(z)]^k\| \le \\ &\le \|R_0(z)\|\sum_{k=0}^{\infty}|g^k|\|QR_0(z)]\|^k \le \frac{\|R_0(z)\|}{1-|g|A} \end{aligned} \tag{2.16}$$

Case b) Analogous computations yield

$$\begin{aligned} \|gR_0(z)\| &\le |g|\left[b_s + b_s\frac{E_{s+1}-d_{s+1}+|\mathrm{Im}\, z|}{\mathrm{dist}(z,\sigma(H(0))} + \frac{a_s}{\delta_s}\right] \\ &\le |g|\left[2b_s + b_s\frac{E_{s+1}-d_{s+1}}{\delta_s} + \frac{a_s}{\delta_s}\right] \le |g|A < 1 \end{aligned}$$

provided $|g| < 1/A$. Here we have used (2.13). Now the same argument of case a) shows that $z \in \rho(H(g))$ if $|g| < 1/A$. Finally let us prove that $\mathbb{C}\setminus\bigcup_{n\in\mathbb{N}}\mathcal{Q}_n \subset \rho(H(g))$ if $|g| < 1/A$. We only need to show that $z \in \rho(H(g))$ if $|g| < 1/A$ and $\mathrm{Re}\, z \le E_1 - d_1$. Once more we have:

$$\begin{aligned} \|gR_0(z)\| &\le |g|\left[b_1 + b_1\frac{|z|}{\mathrm{dist}(z,\sigma(H(0))} + \frac{a_1}{\mathrm{dist}(z,\sigma(H(0))}\right] \\ &\le |g|\left[2b_1 + \frac{a_1}{|E_1-d_1|}\right] \le |g|A \le 1 \end{aligned}$$

for $|g| < 1/A$. Here we have used (2.14) and the inequalities

$$|z| \le \mathrm{dist}\,(z,\sigma(H(0)), \quad |E_1 - d_1| \le \mathrm{dist}\,(z,\sigma(H(0))$$

The results so far obtained allow us to assert that if $E(g_0)$ is an eigenvalue of $H(g_0)$ with $g_0 \in \mathbb{R}$, $|g_0| < 1/A$, then $E(g_0) \in \bigcup_{n\in\mathbb{N}}\mathcal{Q}_n$. Since the open squares $\mathcal{Q}_n$ are disjoint, there exists $n_0 \in \mathbb{N}$ such that $E(g_0) \in \mathcal{Q}_{n_0}$. Moreover, if $g \mapsto E(g)$ is a continuous function defined on any subset D of the circle $\{g : |g| < 1/A\}$ containing g_0, then $E(g) \in \mathcal{Q}_{n_0}$. Now let m_0 denote the multiplicity of the eigenvalue $E(g_0)$. Then for g close to g_0 there are m_0 eigenvalues (counting multiplicities) $E^l(g)$, $l = 1, \ldots, m_0$ of $H(g)$ such that $\lim_{g\to g_0} E^l(g) = E(g_0)$. Each function $E^l(g)$ represents a branch of one or several holomorphic functions which have at most algebraic singularities at $g = g_0$ (see [22], Thm VII.1.8). Assume without loss $g_0 > 0$. Let us follow one of such branches from g_0 to 0, dropping from now on the index l. As remarked above, $E(g)$ is contained in $\mathcal{Q}_{n_0}$ as $g \to g_0^-$. Suppose that the holomorphic function $E(g)$ is defined on the interval $]g_1, g_0[$ with $g_1 > 0$. We will show that it can be analytically continued up to $g = 0$ (in fact, up to $g = -1/A$). Since $E(g) \in \mathcal{Q}_{n_0}$, $E(g)$ is bounded on $]g_1, g_0[$. Thus, by the stability property of the eigenvalues of holomorphic operator families, $E(g)$ must converge to an eigenvalue $E(g_1)$ of $H(g_1)$ as $g \to g_1^+$, and $E(g_1) \in \mathcal{Q}_{n_0}$. Repeating the argument starting from $E(g_1)$, we can continue $E(g)$ to a holomorphic function in the interval $]g_2, g_1]$ with at most an algebraic singularity at g_1. In this way we build a piecewise holomorphic function $E(g)$ defined on $]-1/A, 1/A[$ such that $E(g)$ is an eigenvalue of $H(g)$. In particular $E(0)$ coincides with E_{n_0}, which is the only eigenvalue of $H(0)$ inside $\mathcal{Q}_{n_0}$. Since E_{n_0} is simple, $E(g)$ is the only eigenvalue of $H(g)$ close to E_{n_0} for g small. Thus $E(g)$ must be real for $g \in \mathbb{R}$, $|g|$ small, because if it is complex also its conjugate $\overline{E}(g)$ enjoys the same property, which is ruled out by the stability. Moreover, $E(g)$ is the sum of the perturbation expansion around E_{n_0}, and therefore is a holomorphic function for $|g| < \overline{R}$. Let from now on $|g| < T := \min\,(\overline{R}, 1/A)$. Then the holomorphy implies that the real-valuedness for $|g|$ small extends to all $|g| < T$, $g \in \mathbb{R}$. This concludes the proof of Theorem 1.1.

Acknowledgment

We thank Francesco Cannata for his interest in this work and many useful suggestions.

References

[1] F. Calogero and S. Graffi, *On the quantization of a nonlinear oscillator*, Phys.Lett. **313 A**, 381-386 (2003)

[2] F. Calogero, *On the quantization of two other nonlinear oscillators*, Phys.Lett. **319 A**, 240-3245 (2003)

[3] F. Calogero, *On the quantization of another two nonlinear oscillators*, J.Nonlinear Math.Phys. **11**, 1-6 (2004)

[4] F.A. Berezin, M.S. Shubin, *The Schrödinger Equation*, Kluwer 1991

[5] Z. Ahmed, *P-, T-, PT-, and CPT-invariance of Hermitian Hamiltonians* Phys.Lett. **A310**, 39-142 (2003)

[6] C.M. Bender, S. Boettcher, and P. N. Meisinger, *PT-Symmetric Quantum Mechanics* Journal of Mathematical Physics **40**,2201-2229 (1999)

[7] C.M. Bender, M.V. Berry, and A. Mandilara *Generalized PT Symmetry and Real Spectra.* J.Phys. A: Math. Gen. **35**, L467-L471 (2002)

[8] C.M. Bender, D.C. Brody, and H.F. Jones *Must a Hamiltonian be Hermitian?*, American Journal of Physics, in press

[9] F. Cannata, G. Junker and J. Trost, *Schrödinger operators with complex potential but real spectrum*, Phys Lett **A246** 219-226 (1998)

[10] M. Znojil, F. Cannata, B. Bagchi and R. Roychoudhury, *Supersymmetry without Hermiticity within PT symmetric quantum mechanics.* Phys. Lett. **B483**, 284 (2000)

[11] F. Cannata, M.V. Ioffe and D.N. Nishniadinze, *Two-dimensional SUSY Pseudo-Hermiticity without Separation of Variables.* Phys. Lett. **A310**, 344-352 (2003)

[12] G. Levai and M. Znojil, *Systematic search for PT symmetric potentials with real energy spectra* J. Phys. A: Math. Gen. **33** (2000) 7165.

[13] B. Simon, *Anharmonic oscillator: a singular perturbation theory*, Ann.Phys. **57**, 76-130 (1970)

[14] R. Kretschmer and L. Szymanovski, *Pseudo-Hermiticity in infinite dimensional Hilbert spaces* quant-ph/0305123 (2003)

[15] A. Mostafazadeh, *Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian* J.Math.Phys. **43**, 205-212 (2002); *Pseudo-Hermiticity versus PT-symmetry. II. A complete characterization of non-Hermitian Hamiltonians with a real spectrum*, ibidem, 2814-2816 (2002); *Pseudo-Hermiticity versus PT-symmetry III: Equivalence of pseudo-Hermiticity and the presence of antilinear symmetries* 3944-3951 (2002)

[16] S. Weigert, *Completeness and Orthonormality in PT-symmetric Quantum Systems* quant-ph/0306040 (2003) M. Znojil, *What is PT symmetry?* (quant-ph/0103054)

[17] K.C. Shin, *On the reality of the eigenvalues for a class of PT-symmetric oscillators.* Comm. Math. Phys. **229**, 543-564 (2002)

[18] E. Caliceti, F.Cannata, M.Znojil, A.Ventura, *Construction of* $PT-$*asymmetric non Hermitian hamiltonians with* CPT *symmetry*, preprint 2004

[19] P. Dorey, C.Dunning, R.Tateo, *Spectral Equivalences, Bethe ansatz equations, and reality properties in PT-symmetric quantum mechanics.* J.Phys. A **34** (2001), 5679-5704, (2001).

[20] P.E. Dorey, C. Dunning and R. Tateo, *Supersymmetry and the spontaneous breakdown of PT symmetry*, J.Phys. A **34**, L391-L400 (2001)

[21] E. Caliceti, S. Graffi and J. Siöstrand, *Multiple wells in the semiclassical limit*, J. Phys. A: Math. Gen. **38** (2005) 185-193.

[22] T. Kato, *Perturbation Theory for Linear Operators*, 2nd Edition, Springer-Verlag, 1976

[23] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. IV, Academic Press 1978