A LOCAL QUANTUM VERSION OF THE KOLMOGOROV THEOREM

David Borthwick[1], Sandro Graffi[2]

Abstract

Consider in $L^2(\mathcal{R}^l)$ the operator family $H(\epsilon) := P_0(\hbar,\omega) + \epsilon Q_0$. P_0 is the quantum harmonic oscillator with diophantine frequency vector ω, Q_0 a bounded pseudodifferential operator with symbol holomorphic and decreasing to zero at infinity, and $\epsilon \in \mathcal{R}$. Then there exists $\epsilon^* > 0$ with the property that if $|\epsilon| < \epsilon^*$ there is a diophantine frequency $\omega(\epsilon)$ such that all eigenvalues $E_n(\hbar,\epsilon)$ of $H(\epsilon)$ near 0 are given by the quantization formula $E_\alpha(\hbar,\epsilon) = \mathcal{E}(\hbar,\epsilon) + \langle\omega(\epsilon),\alpha\rangle\hbar + |\omega(\epsilon)|\hbar/2 + \epsilon O(\alpha\hbar)^2$, where α is an l-multi-index.

1 Introduction and statement of the results

Denote by $\mathcal{F}_{\rho,\sigma}$ the set of all functions $f(x,\xi) : \mathcal{R}^{2l} \to \mathcal{C}$ with finite $\|f\|_{\rho,\sigma}$ norm for some $\rho > 0$, $\sigma > 0$ (see Section 2 for the definition and examples). Any $f \in \mathcal{F}_{\rho,\sigma}$ is analytic on $\mathcal{R}^{2l}$ and extends to a complex analytic function in the region $|\Im z_i| \leq a_i|\Re z_i|$ for suitable $a_i > 0$; moreover $|f(z)| \to 0$ as $|z| \to +\infty$. Here $z := (x,\xi)$.

Let $\Phi_{\rho,\sigma}$ denote the class of semiclassical Weyl pseudodifferential operators F in $L^2(\mathcal{R}^l)$ with symbol $f(x,\xi)$ in $\mathcal{F}_{\rho,\sigma}$; namely, (notation as in [Ro])

$$\begin{aligned}(Fu)(x) &:= Op_\hbar^W(f(x,\xi))u(x) \\ &= \frac{1}{\hbar^l}\iint_{\mathcal{R}^l\times\mathcal{R}^l} e^{i\langle(x-y),\xi\rangle/\hbar} f((x+y)/2,\xi)u(y)\,dyd\xi,\ u \in \mathcal{S}(\mathcal{R}^l).\end{aligned} \tag{1.1}$$

It follows directly from the definition of $\|f\|_{\rho,\sigma}$ in (2.5) that $F \in \Phi_{\rho,\sigma}$ extends to a continuous operator in $L^2(\mathcal{R}^l)$, with

$$\|F\|_{L^2\to L^2} \leq \|f\|_{\rho,\sigma}. \tag{1.2}$$

Consider in $L^2(\mathcal{R}^l)$ the operator family $H(\epsilon) = P_0(\hbar,\omega) + \epsilon Q_0$ and assume:

(A1) $P_0(\hbar,\omega)$ is the harmonic-oscillator Schrödinger operator with frequencies $\omega \in [0,1]^l$:

$$P_0(\hbar,\omega)u = -\frac{1}{2}\hbar^2\Delta u + [\omega_1^2x_1^2 + \ldots + \omega_l^2x_1^2]u,\ \ D(P_0) = H^2(\mathcal{R}^l)\cap L^2_2(\mathcal{R}^l). \tag{1.3}$$

[1]Supported in part by NSF grant DMS-0204985. Department of Mathematics and Computer Science, Emory University, Atlanta 30322 (U.S.A.). (davidb@mathcs.emory.edu)

[2]Dipartimento di Matematica, Università di Bologna, 40127 Bologna (Italy). (graffi@dm.unibo.it)

(A2) $Q_0 \in \Phi_{\rho,\sigma}$; its symbol $q_0(x,\xi) = q_0(z)$ is real-valued for $z = (x,\xi) \in \mathcal{R}^\times\mathcal{R}^l$, and $q_0(z) = O(z^2)$ as $z \to 0$.

(A3) There exist $\tau > l-1, \gamma > 0$ such that

$$\langle\omega, k\rangle \geq \gamma|k|^{-\tau}, \quad \forall k \in \mathcal{Z}^l \setminus \{0\}, \quad |k| := |k_1| + \ldots + |k_l|, \ \omega := (\omega_1, \ldots, \omega_l). \quad (1.4)$$

Denote Ω_0 the set of all $\omega \in [0,1]^l$ fulfilling (1.4), and $|\Omega_0|$ its measure. It is well known that $|\Omega_0| = 1$.

Under the above assumptions the operator family $H(\epsilon)$ defined on $D(P_0)$ is self-adjoint with pure-point spectrum $\forall\, \epsilon \in \mathcal{R}$: $\mathrm{Spec}\,(H(\epsilon)) = \mathrm{Spec}_p\,(H(\epsilon))$. Moreover (1.4) entails in particular the rational independence of the components of ω and hence the simplicity of $\mathrm{Spec}(P_0)$ and its density in $\overline{\mathcal{R}}_+ := \mathcal{R}_+ \cup \{0\}$. Clearly, P_0 is a semiclassical pseudodifferential operator of order 2 with symbol

$$p_0(x,\xi) = \frac{1}{2}(|\xi|^2 + |\omega x|^2) = \frac{1}{2}\sum_{k=1}^{l}\omega_k I_k(x,\xi), \ I_k(x,\xi) := \frac{1}{2\omega_k}[\xi_k^2 + \omega_k^2 x_k^2], \ k = 1, \ldots, .(1.5)$$

Theorem 1.1 *Let (A1-A3) be verified; let $h^* > 0$. Then given $\eta > 0$ there exist $\epsilon^* > 0$ and, for all $\epsilon \in [-\epsilon^*, \epsilon^*]$, $\Omega^\epsilon \subset \Omega_0$ independent of ($\hbar \in [0, \hbar^*]$, η) and $\omega(\hbar, \epsilon) \in \Omega^\epsilon$, such that if $|\alpha\hbar| < \eta$ the spectrum of $H(\epsilon)$ is given by the quantization formula*

$$E_\alpha(\hbar, \epsilon) = \mathcal{E}(\hbar; \epsilon) + \langle\omega(\hbar, \epsilon), \alpha\rangle\hbar + \frac{1}{2}|\omega(\hbar, \epsilon)|\hbar + \epsilon\mathcal{R}(\alpha\hbar, \hbar; \epsilon). \quad (1.6)$$

Here:

1. $\mathcal{E}(x;\epsilon) : [0, h^*] \times [-\epsilon^*, \epsilon^*] \to \mathcal{R}$ is continuous in x and analytic in ϵ, with $\mathcal{E}(x, 0) = 0$, $\mathcal{E}(0;\epsilon) = 0$;

2. $\omega(x;\epsilon) : [0, h^*] \times [-\epsilon^*, \epsilon^*] \to \mathcal{R}$ is continuous in x and analytic in ϵ with $\omega(x;0) = \omega$.

3. $\mathcal{R}(x, y, \epsilon) : \overline{\mathcal{R}}_+^l \times [0, h^*] \times [-\epsilon^*, \epsilon^*] \to \mathcal{R}$ is continuous in $(x, y; \epsilon)$ and such that

$$|\mathcal{R}(x, y; \epsilon)| = O(|x|^2), \quad (1.7)$$

uniformly with respect to (y, ϵ).

4. $|\Omega^\epsilon - \Omega_0| \to 0$ as $\epsilon \to 0$.

The uniformity in $\hbar$ of the estimates needed to prove Theorem 1.1 yields in this particular setting a formulation of Kolmogorov's theorem equivalent to that of [BGGS]:

Corollary 1.1 *Let ϵ^*, Ω^ϵ, $\mathcal{E}(x;\epsilon)$, $\omega(x;\epsilon)$ be as above. Then $\forall\,\epsilon$ there is an analytic canonical transformation $(x,\xi)=\psi_\epsilon(I,\phi)$ of $\mathcal{R}^{2l}$ onto $\overline{\mathcal{R}}_+^l\times\mathcal{T}^l$ such that*

$$(p_\epsilon\circ\psi)(I,\phi)=\mathcal{E}(\epsilon)+\langle\omega(\epsilon),I\rangle+\epsilon\tilde{\mathcal{R}}(I,\phi;\epsilon) \tag{1.8}$$

Here $\mathcal{E}(\epsilon):=\mathcal{E}(0;\epsilon)$, $\omega(\epsilon):=\omega(0;\epsilon)\in\Omega^\epsilon$; $\tilde{\mathcal{R}}(I,\phi;\epsilon)=O(I^2)$ as $I\to 0$ uniformly in ϕ.

Remarks

1. The form (1.8) of the Hamiltonian entails that a quasi periodic-motion with diophantine perturbed frequency $\omega(\epsilon)\in\Omega^\epsilon$ exists on the perturbed torus $I=0$; equivalently, a quasi periodic motion with frequency $\omega(\epsilon)\in\Omega^\epsilon$ exists on the unperturbed torus with parametric equations $(x,\xi)=\psi_\epsilon(0,\phi)$. Making $I=\alpha\hbar$ (1.6) represents the quantization of the r.h.s. of (1.8). In the formulation of [BGGS] a quasi periodic motion with the unperturbed frequency $\omega\in\Omega$ exists on an unperturbed torus with parametric equations $(x,\xi)=\psi_\epsilon(0,\phi)$. The selection of the diophantine frequency within Ω depends here on ϵ because of the isochrony of the Hamiltonian flow generated by p_0.

2. KAM theory (see e.g. Ko, [AA], [Mo]) was first introduced in quantum mechanics in [DS] to deal with quasi-periodic Schrödinger operators. For its applications to the Floquet spectrum of non-autonomous Schrödinger operators see [BG] and references therein. Its first application to generate quantization formulas for $\hbar$ fixed goes back to [Be] for operators in $L^2(\mathcal{T}^l)$ and to [Co] for non-autonomous perturbations of the harmonic oscillators. A uniform quantum version of the Arnold version has been obtained by Popov[Po2], within a quantization different from the canonical one. The related method of the quantum normal forms also yields (much less explicit) quantization formulas with remainders of order $O(\hbar^\infty)$, $O(e^{-1/\hbar^a}), 0<a<1$, $O(e^{-1/\hbar})$ (see [Sj],[BGP],[Po1] respectively). These formulas hold for a much more general class of symbols; however they apply only to perturbations of semi-excited levels ([Sj, BGP]) or again require a quantization different from the canonical one[Po1].

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2 Proof of the results

Define an analytic action Ψ of $\mathcal{T}^l$ into $\mathcal{R}^{2l}$ through the flow of p_0:

$$\Psi : \mathcal{T}^l \times \mathcal{R}^{2l} \to \mathcal{R}^{2l}, \quad \phi, (x, \xi) \mapsto (x', \xi') = \Psi_\phi(x, \xi), \tag{2.1}$$

where

$$x'_k := \frac{\xi_k}{\omega_k} \sin \phi_k + x_k \cos \phi_k, \; \xi'_k := \xi_k \cos \phi_k - \omega_k x_k \sin \phi_k. \tag{2.2}$$

If $z := (x, \xi)$, the flow of initial datum z_0 is indeed $z(t) = \Psi_{\omega t}(z_0)$, $\omega t := (\omega_1 t, \dots, \omega_l t)$.

If $f \in L^1_{loc}(\mathcal{R}^{2l})$, the angular Fourier coefficient of order k is defined by

$$\tilde{f}_k(z) := \frac{1}{(2\pi)^l} \int_{\mathcal{T}^l} f(\Psi_\phi(z)) e^{-i\langle k, \phi\rangle} \, d\phi, \quad k \in \mathcal{Z}^l.$$

If $f \in \mathcal{C}^1$ one has, as is well known

$$f(\Psi_\phi(z)) = \sum_{k \in \mathcal{Z}^l} \tilde{f}_k(z) e^{i\langle k, \phi\rangle} \Longrightarrow f(z) = \sum_{k \in \mathcal{Z}^l} \tilde{f}_k(z).$$

Note furthermore that $f \equiv \tilde{f}_k$ for some fixed k if and only if

$$f(\Psi_\phi(z)) = e^{i\langle k, \phi\rangle} f(z). \tag{2.3}$$

Taking $f \in L^1(\mathcal{R}^{2l})$, we will consider the space Fourier transform

$$\widehat{f}(s) := \frac{1}{(2\pi)^{2l}} \int_{\mathcal{R}^{2l}} f(z) e^{-i\langle s, z\rangle} \, dz, \tag{2.4}$$

as well the space Fourier transforms of the $\tilde{f}_k$'s:

$$\widehat{\tilde{f}}_k(s) := \frac{1}{(2\pi)^{3l}} \int_{\mathcal{R}^{2l}} \int_{\mathcal{T}^l} f(\Psi_\phi(z)) e^{-i\langle k, \phi\rangle} e^{-i\langle s, z\rangle} d\phi \, dz.$$

Given $\rho > 0, \sigma > 0$, define the norm

$$\|f\|_{\rho,\sigma} := \sum_{k \in \mathcal{Z}^l} e^{\rho|k|} \int_{\mathcal{R}^{2l}} |\widehat{\tilde{f}}_k(s)| e^{\sigma|s|} \, ds. \tag{2.5}$$

Definition 2.1 *Let $\rho > 0, \sigma > 0$. Then $\mathcal{F}_{\rho,\sigma} := \{f : \mathcal{R}^{2l} \to \mathcal{C} \mid \|f\|_{\rho,\sigma} < +\infty\}$.*

Remarks.

1. If $f \in \mathcal{F}_{\rho,\sigma}$ then f is analytic on $\mathcal{R}^{2l}$, and extends to a complex analytic function on a region $\mathcal{B}_{\rho,\sigma} \subset \mathcal{C}^{2l}$ of the form $\mathcal{B}_{\rho,\sigma} := |\Im z_i| \le a_i |\Re z_i|$, with suitable a_i.

2. $F := Op^W_\hbar W(f)$ is a trace-class, self-adjoint $\hbar$-pseudodifferential operator in $L^2(\mathcal{R}^l)$ if $f \in \mathcal{F}_{\rho,\sigma}$. Let $\widehat{f}(s)$ be the Fourier transform of f. Since $\|\widehat{f}\|_{L^1} \le \|f\|_{\rho,\sigma}$, we have

$$\|F\|_{L^2 \to L^2} \le \int_{\mathcal{R}^{2l}} |\widehat{f}(s)| \, ds \equiv \|\widehat{f}\|_{L^1}, \qquad \|F\|_{L^2 \to L^2} \le \|f\|_{\rho,\sigma}. \tag{2.6}$$

3. v We introduce also the space $\mathcal{F}_\sigma$ of all functions $f:\mathcal{R}^{2l}\to\mathcal{C}$ such that

$$\|g\|_\sigma := \int_{\mathcal{R}^{2l}} |\widehat{g}(s)| e^{\sigma|s|}\,ds < +\infty.$$

Obviously if $f\in\mathcal{F}_\sigma$ then f is analytic on $\mathcal{R}^{2l}$, and extends to a complex analytic function in the multi-strip $\mathcal{S} := \{z\in C^{2l}|\,|\Im z_i| < \sigma\}$.

4. Example of $f\in\mathcal{F}_{\rho,\sigma}$: $f(x,\xi)=P(x,\xi)e^{-(|x|^2+|\xi|^2)}$, $P(x,\xi)$ any polynomial.

The starting point of the proof is represented by the first step of the Kolmogorov iteration, and is summarized in the following

Proposition 2.1 *Let $\omega\in\Omega_0$. Then, for any $0<d<\rho$, $0<\delta<\sigma$:*
1. There exists a unitary transformation $U(\omega,\epsilon,\hbar)=e^{i\epsilon W_1/\hbar}: L^2\leftrightarrow L^2$, $W_1=W_1^*$ and $\omega_1(\epsilon)\in[0,1]^l$ such that:

$$UH(\epsilon)U^{-1} = P_0(\hbar,\omega_1(\epsilon)) + \epsilon\mathcal{E}_1 I + \epsilon^2 Q_1(\epsilon,\hbar) + \epsilon R_1(\epsilon,\hbar). \tag{2.7}$$

Here: $\mathcal{E}_1=\tilde{q}_0$; $W_1=Op_\hbar^W(w_1)\in\Phi_{\rho-d,\sigma-\delta}$, $Q_1(\epsilon,\hbar)=Op_\hbar^W(q_1)\in\Phi_{\rho-d,\sigma-\delta}$ with

$$\|w_1\|_{\rho-d,\sigma-\delta}\le d^{-\tau}\|q_0\|_{\rho,\sigma}\qquad \|q_1\|_{\rho-d,\sigma-\delta}\le \delta^{-2}d^{-2\tau}\|q_0\|^2_{\rho,\sigma}. \tag{2.8}$$

2. $R_1(\epsilon)$ is a self-adjoint semiclassical pseudodifferential operator of order 4 such that $[R_1(\epsilon),P_0]=0$; $\exists\, D_1>0$ such that, for any eigenvector ψ_α of $P_0(\omega)$:

$$|\langle\psi_\alpha, R_1(\epsilon)\psi_\alpha\rangle| \le D_1(|\alpha|\hbar)^2. \tag{2.9}$$

3. $\forall\, K>0$ with $(1+K^\tau)<\dfrac{\gamma}{\epsilon\|q_0\|_{\rho,\sigma}}$ $\exists\,\Omega_1\subset\Omega_0$ closed and $d_1>1$ independent of K such that

$$|\Omega_0-\Omega_1|\le\gamma(1+1/K^{d^1}). \tag{2.10}$$

Moreover if $\omega_1\in\Omega_1$ then (1.4) holds with γ replaced by

$$\gamma_1 := \gamma - \epsilon\|q_0\|_{\rho,\sigma}(1+K^\tau). \tag{2.11}$$

Proof To prove Assertion 1 we first recall some relevant results of [BGP].

Lemma 2.1 (Lemma 3.6 of [BGP]) *Let $g\in\mathcal{F}_{\rho,\sigma}$. Then the homological equation,*

$$\{p_0,w\}+\mathcal{N}=g,\qquad \{p_0,\mathcal{N}\}=0 \tag{2.12}$$

admits the analytic solutions

$$\mathcal{N}:=\tilde{g}_0;\qquad w:=\sum_{k\neq 0}\frac{\tilde{g}_k}{i\langle\omega,k\rangle}, \tag{2.13}$$

with the property $\mathcal{N}\circ\Psi_\phi=\mathcal{N}$. Equivalently, $\mathcal{N}$ depends only on $I_1,\dots,I_l$. Moreover, for any $d<\rho$:

$$\|\mathcal{N}\|_{\rho,\sigma}\le\|g\|_{\rho,\sigma};\quad \|w\|_{\rho-d,\sigma}\le c_\Psi\frac{\|g\|_{\rho,\sigma}}{d^\tau};\qquad c_\Psi:=\left(\frac{\tau}{e}\right)^\tau\frac{1}{\gamma}. \tag{2.14}$$

Given $(g,g')\in\mathcal{F}_{\rho,\sigma}$, let $\{g,g'\}_M$ be their Moyal bracket, defined as

$$\{g,g'\}_M=g\#g'-g'\#g,$$

where $\#$ is the composition of g,g' considered as Weyl symbols. We recall that in Fourier transform representation, used throughout the paper, the Moyal bracket is (see e.g. [Fo], 3.4):

$$(\{g,g'\}_M)^\wedge(s)=\frac{2}{\hbar'}\int_{\mathcal{R}^{2n}}\widehat{g}(s^1)\widehat{g'}(s-s^1)\sin\left[\hbar(s-s^1)\wedge s^1/2\right]ds^1, \tag{2.15}$$

where, given two vectors $s=(v,w)$ and $s^1=(v^1,w^1)$, $s\wedge s^1:=\langle w,v_1\rangle-\langle v,w_1\rangle$. We also recall that $\{g,g'\}_M=\{g,g'\}$ if either g or g' is quadratic in (x,ξ).

Lemma 2.2 (Lemmas 3.1 and 3.3 of [BGP]) *Let $g\in\mathcal{F}_\sigma$, $g'\in\mathcal{F}_{\sigma-\delta}$. Then:*
1. $\forall\, 0<\delta'<\sigma-\delta$:

$$\|\{g,g'\}_M\|_{\sigma-\delta-\delta'}\le\frac{1}{e^2\delta'(\delta+\delta')}\|g\|_\sigma\|g'\|_{\sigma-\delta}. \tag{2.16}$$

2. Let $g\in\mathcal{F}_{\rho,\sigma}$ and $g'\in\mathcal{F}_{\rho,\sigma-\delta}$. Then, for any positive $\delta'<\sigma-\delta$:

$$\|\{g,g'\}_M\|_{\rho,\sigma-\delta-\delta'}\le\frac{1}{e^2\delta'(\delta+\delta')}\|g\|_{\rho,\sigma}\,\|g'\|_{\rho,\sigma-\delta}. \tag{2.17}$$

As a simple corollary of Lemmas 2.1 and 2.2, we find:

Lemma 2.3 (Lemma 3.4 of [BGP]) *Let $g\in\mathcal{F}_{\rho,\sigma}$, $w\in\mathcal{F}_{\rho,\sigma}$.*
1. Define

$$g_r:=\frac{1}{r}\{w,g_{r-1}\}_M,\qquad r\ge 1;\;\; g_0:=g.$$

Then $g_r\in\mathcal{F}_{\rho,\sigma-\delta}$ for any $0<\delta<\sigma$, and the following estimate holds

$$\|g_r\|_{\rho,\sigma-\delta}\le\left(\delta^{-2}\|w\|_{\rho,\sigma}\right)^r\|g\|_{\rho,\sigma}. \tag{2.18}$$

2. Let $g\in\mathcal{F}_{\rho,\sigma}$, and w be the solution of the homological equation (2.12). Define the sequence $p_{r0}:r=0,1,\dots$ as follows:

$$p_{00}:=p_0;\qquad p_{r0}:=\frac{1}{r}\{w,p_{r-10}\}_M,\;r\ge 1.$$

Then, for any $0<d<\rho, 0<\delta<\sigma$, $p_{r0}\in\mathcal{F}_{\rho-d,\sigma-\delta}$ and fulfills the following estimate

$$\|p_{r0}\|_{\rho-d,\sigma-\delta}\le 2\left(\delta^{-2}\|w\|_{\rho-d,\sigma}\right)^{r-1}\|g\|_{\rho-d,\sigma},\;\; k\ge 1.$$

Proof of Proposition 2.1

With $U_1 = e^{i\epsilon W_1/\hbar}$, W_1 continuous and self-adjoint, we have in general:

$$U_1(P_0 + \epsilon Q_0)U_1^{-1} = P_0 + \epsilon P_1 + \epsilon^2 Q_1, \tag{2.19}$$

$$P_1 := Q_0 + [W_1, P_0]/i\hbar, \tag{2.20}$$

$$Q_1 := \epsilon^{-2}\left(U_1(P_0 + \epsilon Q_0)U_1^{-1} - P_0 - \epsilon(Q_0 + [W_1, P_0]/i\hbar)\right). \tag{2.21}$$

We start by looking for $W_1 \in \mathcal{F}_{\rho,\sigma}$ such that the first order term yields an operator $N_1 \in \mathcal{F}_{\rho,\sigma}$ commuting with P_0:

$$Q_0 + [W_1, P_0]/i\hbar = N_1, \quad [N_1, P_0] = 0. \tag{2.22}$$

Denoting by w_1, $\mathcal{N}_1$ the (Weyl) semiclassical symbols of W_1, N_1, respectively, eq.(2.22) is equivalent to a classical homological equation in $\mathcal{F}_{\rho,\sigma}$

$$\{p_0, w_1\}_M + \mathcal{N}_1 = q_0, \qquad \{p_0, \mathcal{N}_1\}_M = 0. \tag{2.23}$$

However p_0 is quadratic in (x,ξ). Therefore the Moyal bracket $\{p_0, w_1\}_M$ coincides with the Poisson bracket $\{p_0, w_1\}$ and the above equation becomes

$$\{p_0, w_1\} + \mathcal{N}_1 = q_0, \qquad \{p_0, \mathcal{N}_1\} = 0. \tag{2.24}$$

The existence of $w_1 \in \mathcal{F}_{\rho-d,\sigma}$, $\mathcal{N}_1 \in \mathcal{F}_{\rho,\sigma}$ with the stated properties now follows by direct application of Lemma 2.1.

We now prove the second estimate in (2.8). We have:

$$Q_1 = \int_0^1\int_0^s e^{is_1\epsilon W_1/\hbar}[[P_0 + \epsilon Q_0, W_1], W_1]e^{-is_1\epsilon W_1/\hbar}\, ds_1 ds,$$

and we can estimate

$$\|[[P_0 + \epsilon Q_0, W_1], W_1]\|_{L^2\to L^2} \le \|\{\{p_0 + \epsilon q_0, w_1\}_M, w_1\}_M\|_{\rho-d,\sigma-\delta}.$$

It follows, by Lemma 2.3 and Lemma 2.1, that

$$\|Q_1\|_{L^2\to L^2} \le \|\{\{p_0 + \epsilon q_0, w_1\}_M, w_1\}_M\|_{\rho-d,\sigma-\delta} \le \delta^{-2}d^{-2\tau}\|q_0\|^2_{\rho,\sigma}.$$

This proves the second estimate of (2.8).

To prove the Assertion 2 set:

$$\mathcal{E}_1 \;:=\; \mathcal{N}_1(0); \quad \omega_1(\epsilon) = \omega + \epsilon(\nabla_I \mathcal{N}_1)(0), \tag{2.25}$$

$$\mathcal{R}_1(I,\epsilon) \;=\; \mathcal{N}_1(I) - \langle(\nabla_I\mathcal{N}_1)(0), I\rangle - \mathcal{E}_1, \tag{2.26}$$

and define

$$R_1(\epsilon) := Op_\hbar^W(\mathcal{R}_1(I,\epsilon)). \tag{2.27}$$

Then clearly $R_1(\epsilon)$ is a self-adjoint semiclassical, tempered pseudodifferential operator of order 4, vanishing to 4-th order at the origin, and with the property $[R_1(\epsilon), P_0] = 0$. Therefore formula (2.9) follows directly by Proposition A.1.

As far as Assertion 3 is concerned, set:

$$\begin{aligned} \mathcal{T}_k(\alpha) &:= \{\omega \in [0,1]^k : |\langle \omega, k\rangle| \le \alpha\}, && (2.28)\\ \Omega_1 &:= \Omega_0 - \bigcup_{|k|\ge K} \mathcal{T}_k\left(\frac{\gamma_1}{|k|^\tau}\right). && (2.29)\end{aligned}$$

As in [BG], Lemma 5.6, we have:

$$|\mathcal{T}_l(\alpha)| \le \frac{4}{k}\alpha.$$

Hence if $\tau > l-1$ we can write

$$\left|\bigcup_{|k|\ge K} \mathcal{T}_k\left(\frac{\gamma_1}{|k|^\tau}\right)\right| \le \sum_{|k|\ge K} \frac{\gamma_1}{|k|^{\tau+1}} < \frac{\gamma_1}{K^{d_1}}.$$

Since $|\langle \omega_1(\epsilon), k\rangle| \ge \gamma_1/|k|^\tau$ by construction when $|k| \le K$, the proposition is proved.

3 Iteration

The above result represents the starting point for the iteration. To ensure convergence, we first preassign the values of the parameters involved in the iterative estimates. Keeping ϵ, K, γ, ρ and σ fixed define, for $p \ge 1$:

$$\begin{aligned} \sigma_p &:= \frac{\sigma}{4p^2}, \quad s_p := s_{p-1} - \sigma_p, \quad \rho_p := \frac{\rho}{4p^2}, \quad r_p := r_{p-1} - \rho_p, && (3.1)\\ \gamma_p &:= \gamma_{p-1} - \frac{4\epsilon_p}{1+K_p^\tau}, \quad K_p := pK. && (3.2)\end{aligned}$$

where ϵ_p is defined in (3.15) below. The initial values of the parameter sequences are chosen as follows:

$$\gamma_0 := \gamma; \quad s_0 := \sigma; \quad r_0 := \rho, \quad \epsilon_0 = 0. \tag{3.3}$$

We then have:

Proposition 3.1 *let $\omega \in \Omega_0$. There exist $\epsilon^*(\gamma) > 0$ and, $\forall\, p \ge 1$, a closed set $\Omega_p^\gamma \subset \Omega_0$ such that, if $|\epsilon| < \epsilon^*(\gamma) > 0$ and $\omega_p(\hbar;\epsilon) \in \Omega_p^\gamma$:*

1. One can construct two sequences of unitary transformations $\{X_p\}$, $\{Y_p\}$ *in* $L^2(\mathcal{R}^l)$ *with the property*

$$X_p(P_0(\omega)+\epsilon Q_0)X_p^{-1} = \qquad (3.4)$$
$$P_0(\omega_p(\hbar;\epsilon))+\epsilon\mathcal{E}_p(\hbar;\epsilon)I+e^{2^p}Q_p+$$
$$\epsilon^{2^p}R_p(\hbar;\epsilon)+\epsilon\sum_{s=2}^{p}Y_sR_{s-1}(\hbar)Y_s^{-1}\epsilon^{2^{s-2}}.$$

2. X_p *and* Y_p *have the form*

$$X_p = U_1U_2\cdots U_p; \qquad (3.5)$$

$$Y_s = U_pU_{p-1}\cdots U_s. \qquad (3.6)$$

Here $U_p(\omega,\epsilon,\hbar)=\exp[i\epsilon^{2^{p-1}}W_p/\hbar] : L^2 \leftrightarrow L^2$, $W_p = W_p^*$

$$W_p = Op_\hbar^W(w_p)\in\Phi_{r_p,s_p}, \quad Q_p(\epsilon,\hbar)=Op^W(q_p)\in\Phi_{r_p,s_p}, \qquad (3.7)$$

$$\|w_p\|_{r_p,s_p}\le\rho_p^{-2\tau}\|q_{p-1}\|_{r_{p-1},s_{p-1}} \quad \|q_p\|_{r_p,s_p}\le\rho_p^{-2\tau}\sigma_p^{-2}\|q_{p-1}\|^2_{r_{p-1},s_{p-1}}, \qquad (3.8)$$

$$\mathcal{E}_p(\hbar;\epsilon)=\sum_{s=0}^{p}\mathcal{N}_s(\hbar)\epsilon^{2^s}, \quad \mathcal{N}_s(\hbar)=(\tilde{q}_s)_0(\hbar). \qquad (3.9)$$

3. $R_s(\epsilon)$ *is a self-adjoint semiclassical pseudodifferential operator of order 4;* $[R_s(\epsilon),P_0]=0$*; there exist* $D_p>0,\overline{D}_p>0$ *such that, for any eigenvector* ψ_α *of* $P_0(\omega)$*:*

$$|\langle\psi_\alpha,R_p(\epsilon)\psi_\alpha\rangle|\le\ D_p(|\alpha|\hbar)^2, \qquad (3.10)$$

$$|\langle\psi_\alpha,\sum_{s=2}^{p}Y_sR_{s-1}Y_s^{-1}\epsilon^{2^{s-2}}\psi_\alpha\rangle|\le\overline{D}_p(|\alpha|\hbar)^2. \qquad (3.11)$$

4. $\forall K_{p-1}>0$ *such that*

$$(1+K_{p-1}^\tau)<\frac{\gamma_{p-1}}{\epsilon\|q_{p-1}\|_{r_{p-1},s_{p-1}}}, \qquad (3.12)$$

$\exists\ \Omega_p\subset\Omega_{p-1}$ *closed and* $d_p>1$ *independent of* K_p *such that*

$$|\Omega_p-\Omega_{p-1}|\le\frac{\gamma_{p-1}}{1+1/(K_{p-1})^{d_p}}. \qquad (3.13)$$

Moreover if $\omega_p(\epsilon)\in\Omega_p$ *then (1.4) holds with* γ *replaced by*

$$\gamma_p \ := \ \gamma_{p-1}-\epsilon_p(1+K_{p-1}^\tau) \qquad (3.14)$$

$$\epsilon_p \ := \ \epsilon^{2^{p-1}}\|q_{p-1}\|_{r_{p-1},s_{p-1}} \qquad (3.15)$$

Proof

We proceed by induction. For $p=1$ the assertion is true because we can take W_1, Q_1, R_1, ω_1, Ω_1^ϵ, K_1 as in Proposition 2.1. To go from step $p-1$ to step p we consider the operator

$$\begin{aligned} X_{p-1}(P_0(\omega)+\epsilon Q_0)X_{p-1}^{-1} :=& \\ & P_0(\omega_{p-1}(\hbar;\epsilon))+\epsilon\mathcal{E}_{p-1}(\hbar;\epsilon)I+e^{2^{p-1}}Q_{p-1} \\ & +\epsilon^{2^{p-1}}R_{p-1}(\hbar;\epsilon)+\epsilon\sum_{s=2}^{p-1}Y_sR_{s-1}(\hbar)Y_s^{-1}\epsilon^{2^{s-2}}. \end{aligned}$$

We have to determine and estimate the unitary map U_p transforming it into the form (3.4) via the definitions (3.5). With $U_p=e^{i\epsilon W_p/\hbar}$, W_p continuous and self-adjoint, we have at the p-th iteration step

$$\begin{aligned} U_p(P_0(\omega_{p-1}+\epsilon^{2^{p-1}}Q_{p-1})U_p^{-1} &= P_0(\omega_p)+\epsilon^{2^{p-1}}P_p+\epsilon^{2^p}Q_p, \\ P_p &:= Q_{p-1}+[W_p,P_0]/i\hbar, \\ Q_p := \epsilon^{-2}\left(U_p(P_0(\omega_{p-1})+\epsilon Q_0)U_1^{-1}\right.&\left.-P_0(\omega_{p-1})-\epsilon(Q_{p-1}+[W_p,P_0]/i\hbar)\right). \end{aligned}$$

(the explicit dependence of the frequencies on $(\hbar,\epsilon)$ has been omitted). We will look therefore for $W_p\in\Phi_{r_p,s_p}$ and an operator $N_p\in\Phi_{r_p,s_p}$ such that

$$Q_p+[W_p,P_0]/i\hbar=N_p,\quad [N_p,P_0]=0. \tag{3.16}$$

Denoting w_p, $\mathcal{N}_p$ the (Weyl) semiclassical symbols of W_p, N_p, respectively, eq.(3.16) is again equivalent to the classical homological equation in $\mathcal{F}_{\rho,\sigma}$

$$\{p_0,w_p\}_M+\mathcal{N}_p=q_p,\qquad \{p_0,\mathcal{N}_p\}_M=0$$

which once more becomes

$$\{p_0,w_p\}+\mathcal{N}_p=q_p,\qquad \{p_0,\mathcal{N}_p\}=0.$$

The existence of $w_p\in\mathcal{F}_{r_p,s_p}$, $\mathcal{N}_p\in\mathcal{F}_{r_p,s_p}$ with the stated properties now follows by direct application of Lemma 2.1. Expanding $\mathcal{N}_p$ as in the proof of Proposition 2.1 and taking into account the definitions (3.5) we immediately check that $X_pX_{p-1}(P_0(\omega)+\epsilon Q_0)X_{p-1}^{-1}X_p$ has the form (3.4). The estimate of Q_p and the small denominator estimates follow by exactly the same argument of Proposition 2.1. The estimate (3.10) is proved exactly as (2.9). It remains to prove the estimate (3.11). By the inductive assumption, it is enough to prove the existence of $D_p'>0$ such that

$$|\langle\psi_\alpha,U_pR_{p-1}U_p^{-1}\psi_\alpha\rangle|\le D_p'(|\alpha|\hbar)^2.$$

We only have to prove that the operator $U_pR_{p-1}U_p^{-1}$ is an $\hbar$-pseudodifferential operator of order 4 fulfilling the hypotheses of Proposition A.1, assuming by the inductive argument the validity of these properties for R_{p-1}. On the other hand, $U_p = \exp{(i\epsilon^{2^{p-1}}W_p/\hbar)}$, and W_p is an $\hbar$-pseudodifferential operator of order 0. We can therefore apply the semiclassical Egorov theorem (see e.g. [Ro], Chapter 4) to assert that $U_pR_{p-1}U_p^{-1}$ is again an $\hbar$-pseudodifferential operator. Denote $\sigma(x,\xi;\epsilon;\hbar)$ the Weyl symbol of $U_pR_{p-1}U_p^{-1}$, and consider its expansion

$$\sigma(x,\xi;\epsilon;\hbar) = \sigma_0(x,\xi;\epsilon) + \sum_{j=2}^{M}\hbar^j\sigma_j(x,\xi;\epsilon) + O(h^{M+1}).$$

It is clearly enough to prove that the principal symbol $\sigma_0(x,\xi;\epsilon)$ has order 4. Denote by

$$\phi(x,\xi;\epsilon) := \exp{[\epsilon^{2^p}\mathcal{L}_{w_p}]}(x,\xi)$$

the Hamiltonian flow on $\mathcal{R}^{2l}$ generated by the Hamiltonian vector field $\mathcal{L}_{w_p}$ at time ϵ^{2^p}; here $w_p^0(x,\xi)$ is the principal symbol of W_p. Then $\sigma_0(x,\xi;\epsilon) = \mathcal{R}_{p-1}^0(\phi(x,\xi;\epsilon))$ where $\mathcal{R}_{p-1}^0(x,\xi)$ is in turn the principal symbol of R_{p-1}. Now

$$\phi(x,\xi;\epsilon) = (x + \int_0^{\epsilon^{2^p}} \nabla_\xi w_p(x,\xi;\eta)\,d\eta, \xi - \int_0^{\epsilon^{2^p}} \nabla_x w_p(x,\xi;\eta)\,d\eta).$$

By Assumption A2 and the inductive hypothesis we know that $w_p(z) = O(|z|^2)$ as $|z|\to 0$. Hence we can write $\phi(z) = z + \epsilon r(z)$ where $r(z) = O(z), z\to 0$. This concludes the proof of Proposition 3.1.

Proof of Theorem 1.1

Applying the estimates on q_p in Propositions 2.1 and 3.1 iteratively, we have

$$\|q_p\|_{r_p,s_p} \le \left(\frac{4p^2}{\rho}\right)^{2\tau p}\cdot\left(\frac{4p^2}{\sigma}\right)^{2p}\|q_0\|^{2^p}, \tag{3.17}$$

whence

$$|\epsilon|^{2^p}\|Q_p\|_{L^2\to L^2} \le |\epsilon|^{2^p}(4p^2)^{2p(\tau+1)}\rho^{-2\tau p}\sigma^{-2p}\|q_0\|^{2^p} \to 0 \quad \text{as } p\to\infty, \tag{3.18}$$

for all $|\epsilon| \le \epsilon^*$ provided $\epsilon^* > 0$ is small enough. At the p-th iteration the frequency is given by

$$\omega_p(\hbar;\epsilon) = \omega + \sum_{s=1}^{p}\nabla_I\mathcal{N}_s(\hbar)\epsilon^{2^s}. \tag{3.19}$$

Since $\|\nabla_z f(z)\|_{\rho-d,\sigma-\delta} \le \dfrac{1}{d\delta}\|f(z)\|_{\rho,\sigma}$, by (3.17) we have

$$\sum_{s=1}^{p}|\nabla_I\mathcal{N}_s(\hbar)\epsilon^{2^s}| \le \sum_{s=1}^{p}|\epsilon|^{2^s}(4s^2)^{2s(\tau+1)}\rho^{-2\tau s}\sigma^{-2s}\|q_0\|^{2^s}. \tag{3.20}$$

Hence the series (3.19) converges as $p \to \infty$ for $|\epsilon| < \epsilon^*$ if ϵ^* is small enough, uniformly with respect to $\hbar) \in [0, \hbar^*]$. In the same way, the estimate (3.17) entails, by the definition (3.14), the existence of $\lim_{p\to\infty} \gamma_p := \gamma_\infty$. Let $\omega(\hbar;\epsilon) := \lim_{p\to\infty}\omega_p(\hbar\ \epsilon)$. Then $\omega(\hbar;\epsilon)$ is diophantine with constant γ_∞ by Proposition 3.1. In the same way:

$$\mathcal{E}(\hbar;\epsilon) = \sum_{s=1}^{\infty} \mathcal{N}_s(\hbar)\epsilon^{2^s}, \quad |\epsilon| < \epsilon^*.$$

Finally, let $\mathcal{R}(\alpha\hbar,\epsilon)$ be an asymptotic sum of the power series $\sum_{s=2}^{\infty} Y_s R_{s-1} Y_s^{-1} \epsilon^{2^s-2}$. Then the validity of (1.7) follows by its validity term by term. This concludes the proof of Theorem 1.1.

Proof of Corollary 1.1

It is enough to illustrate the specialization of the argument of Propositions 2.1 and 3.1 to the $\hbar = 0$ case. Denoting by $e^{\epsilon\mathcal{L}_{w_1}}$ the canonical flow at time ϵ generated by the Hamiltonian vector field generated by the symbol w_1, we have:

$$e^{\epsilon\mathcal{L}_{w_1}}(p_0 + \epsilon q_0)(x,\xi) = (p_0 + \epsilon p_1 + \epsilon^2 q_1^0)(x,\xi), \tag{3.21}$$

$$p_1 := q_0 + \{w_1, p_0\}, \tag{3.22}$$

$$q_1^0 := \epsilon^{-2}\left(e^{\epsilon\mathcal{L}_{w_1}}(p_0 + \epsilon q_0)(x,\xi) - p_0 - \epsilon(q_0 + \{w_1, p_0\})\right). \tag{3.23}$$

Remark that $e^{\epsilon\mathcal{L}_{w_1}}(p_0 + \epsilon q_0)(x,\xi)$ is the principal symbol of $U_1(P_0 + \epsilon Q_0)U_1^{-1}$ by the semiclassical Egorov theorem; p_1 is the full, and hence principal, symbol of P_1 because p_0 is quadratic. Likewise, q_1^0 is the principal symbol of Q_1. Hence the classical definitions (3.21,3.22,3.23) correspond to the principal symbols of the semiclassical pseudodifferential operators $U_1(P_0+\epsilon Q_0)U_1^{-1}$, P_1, Q_1 defined in (2.19,2.20,2.21). Therefore we can take over the homological equation (2.24) and apply Lemma 2.1 once more. This yields the same w_1 and $\mathcal{N}_1$ of Proposition 2.1. To prove the estimate (2.8) for q_1^0 we write

$$q_1^0 = \int_0^1 e^{s\epsilon\mathcal{L}_{w_1}}\{\{p_0 + \epsilon q_0, w_1\}, w_1\}\, ds$$

Now as in [BGGS], Lemma 1, note that if $|\epsilon| < \epsilon^*$ and $z = (x,\xi) \in \mathcal{B}_{\rho-d,\sigma-\delta}$ then $e^{s\mathcal{L}_{w_1}}z \in \mathcal{B}_{\rho,\sigma}$ for $0 \le s \le 1$ because (Lemma 2.1) $\epsilon\|\nabla w_1\|_{\rho-d,\sigma} \le \epsilon(\tau/e)c_\psi d^{-\tau}\|q_0\|_{\rho,\sigma}$. Therefore we can apply Lemma 2.3, valid a fortiori for the Poisson bracket, and, as in the proof of Proposition 2.1, get the estimate corresponding to the second one of (2.8):

$$\|q_1^0\|_{\rho-d,\sigma-\delta} \le \|\{\{p_0 + \epsilon q_0, w_1\}, w_1\}\|_{\rho-d,\sigma-\delta} \le \delta^{-2}d^{-2\tau}\|q_0\|^2_{\rho,\sigma}. \tag{3.24}$$

Now, writing:

$$\psi_\epsilon^1(x,\xi) \quad = \quad e^{\epsilon\mathcal{L}_{w_1}}(x,\xi), \quad \mathcal{E}_1 := \mathcal{N}_1(0); \tag{3.25}$$

$$\omega_1(\epsilon) = \omega+\epsilon(\nabla_I\mathcal{N}_1)(0), \tag{3.26}$$

$$\tilde{\mathcal{R}}_1(I,\epsilon) = \mathcal{N}_1(0)-\langle(\nabla_I\mathcal{N}_1)(0),I\rangle-\mathcal{E}_1, \tag{3.27}$$

we can sum up the above argument by writing (compare with (2.7))

$$\psi^1_\epsilon\circ(p_0+\epsilon q_0)=\mathcal{E}_1+\langle\omega_1(0;\epsilon),I\rangle+\epsilon^2q_1(I,\phi)+\epsilon\mathcal{R}^0_1(I,\epsilon) \tag{3.28}$$

where $\mathcal{R}^0_1$ is the principal symbol of R_1. Morover, Assertion 3 of Proposition 2.1 holds without change.

Let us now specialize the iterative argument of Proposition 3.1. First, the parameters defined in (3.1,3.2,3.3) remain unchanged. Then:

1. The construction of the two sequences of canonical transformations

$$\chi^p_\epsilon = \psi^1_\epsilon\circ\psi^2_\epsilon\cdots\circ\psi^p_\epsilon,\quad p=1,2,\ldots \tag{3.29}$$

$$\zeta^s_\epsilon = \psi^p_\epsilon\circ\psi^{p-1}_\epsilon\cdots\circ\psi^s_\epsilon,\quad p=1,2,\ldots \tag{3.30}$$

$$\psi^s_\epsilon(x,\xi) = e^{\epsilon\mathcal{L}_{w^0_s}}(x,\xi) \tag{3.31}$$

such that

$$\psi^p_{\epsilon,I_0}\circ(p_0+\epsilon q_0)= \tag{3.32}$$
$$\langle\omega_p(0,\epsilon),I\rangle+\mathcal{E}_p(\epsilon)+e^{2^p}q^0_p+\epsilon^{2^p}\mathcal{R}^0_p+\epsilon\sum_{s=2}^p\psi^s_\epsilon\circ\mathcal{R}^0_{s-1}\epsilon^{2^s-2}.$$

follows as in the above argument valid for $p=1$. Here w^0_s, q^0_p, $\mathcal{R}^0_s$ are the principal symbols of the semiclassical pseudodifferential operators W_s, Q_p and R_s, once reexpressed on the (x,ξ) canonical variables via, with ω_p in place of ω_1. Morover:

$$\mathcal{E}_p(\epsilon)=\sum_{s=0}^p\mathcal{N}_s(0)\epsilon^{2^s},\quad \mathcal{N}_s(0)=(\tilde{q}^0_s)_0(0). \tag{3.33}$$

$$\omega_p(\epsilon)=\omega+\sum_{s=0}^p\omega_s(0)\epsilon^{2^s},\quad \omega_s(0)=\nabla_I\mathcal{N}_s(0) \tag{3.34}$$

2. The estimates (3.8) are a fortiori valid with w^0_p, q^0_p in place of w_p, w_p; as a consequence, (3.13) holds unchanged together with the definitions (3.12,3.14,3.15). Hence the uniform estimate (3.17) allows us to set $\hbar=0$ in (3.19,3.20).

3. Finally, remark that $\mathcal{R}^0_s(I)=O(I^2), s=1,\ldots,p$. Now the estimate $\psi^\epsilon_s\mathcal{R}_s(I)=O(I^2)$ as $I\to 0$ follows by exactly the same argument of Proposition 3.1 after rexpression on the canonical variables (x,ξ).

Appendix

To establish the remainder estimate (1.7) the key fact is that vanishing of a symbol at the origin $(x, \xi) = 0$ implies bounds on harmonic oscillator matrix elements that are uniform in $\hbar$. No analyticity of the symbol is required for this result, so we will state and prove it in somewhat greater generality, using the following semiclassical symbol class defined in Shubin [Sh]:

$$\Sigma^{m,\mu} = \{f \in C^\infty(\mathcal{R}^{2l} \times (0, c]) : |\partial_z^\gamma f(z, \hbar)| \leq C_\gamma \langle z \rangle^{m-|\gamma|} \hbar^\mu\},$$

where $z = (x, \xi)$, here considered a real variable, and $\langle z \rangle = \sqrt{1 + |z|^2}$. For future reference we note that Proposition A.2.3 of [Sh] gives the result:

$$\forall f \in \Sigma^{0,\mu}, \quad \|Op_\hbar^W(f)\|_{L^2} \leq C(f)\hbar^\mu, \tag{A.1}$$

for all $\hbar \in (0, c]$.

The matrix elements in question are most easily computed in Bargmann space, with the remainder operator written as a Toeplitz operator. Since these are anti-Wick ordered, we first must consider the translation from Weyl symbols to anti-Wick (for these notions, see e.g. [BS]). Denoting by $Op_\hbar^{AW}(f)$ the anti-Wick quantization of a symbol $f \in \Sigma^{m,\mu}$, the correspondence is given by the action of the heat kernel on the symbol:

$$Op_\hbar^{AW}(f) = Op_\hbar^W(e^{\hbar\Delta/4} f), \tag{A.2}$$

where $\Delta = \Delta_z = \partial_x \cdot \partial_x + \partial_\xi \cdot \partial_\xi$. To begin, we show that the Weyl symbol of an anti-Wick operator is given by formal expansion of the heat kernel up to a remainder.

Lemma A1 *For $f, g \in \Sigma^{m,\mu}$, suppose that $Op_\hbar^{AW}(g) = Op_\hbar^W(f)$. Then for all $n \geq 1$,*

$$f - \sum_{k=0}^{n-1} \frac{1}{k!} \left(\frac{\hbar}{4}\Delta\right)^k g \in \Sigma^{m-2n,\mu+n}.$$

Proof. According to (A.2),

$$f(z, \hbar) = \frac{1}{(\pi\hbar)^l} \int e^{-|z-w|^2/\hbar} g(w) dw.$$

In this expression we will expand $g(w)$ in a Taylor series centered at $w = z$:

$$g(w, \hbar) = \sum_{|\alpha|<2n} \frac{1}{\alpha!} \partial^\alpha g(z, \hbar)(w - z)^\alpha + r(w, z, \hbar),$$

where

$$r(w, z) = \sum_{|\alpha|=2n} c'_\alpha (w - z)^\alpha \int_0^1 (1 - t)^{2n-1} \partial^\alpha g(z + t(w - z))\, dt.$$

Thus,

$$f(z,\hbar) = \sum_{|\alpha|<2n} c_\alpha \partial^\alpha g(z,\hbar) + r(z,\hbar),$$

where

$$c_\alpha = \frac{1}{(\pi\hbar)^l}\frac{1}{\alpha!}\int w^\alpha e^{-|w|^2/\hbar}\,dw,$$

and

$$r(z,\hbar) = \sum_{|\alpha|=2n} c''_\alpha \hbar^{-l}\int\int_0^1 (w-z)^\alpha e^{-|z-w|^2/\hbar}(1-t)^{2n-1}\partial^\alpha g(z+t(w-z))\,dt\,dw.$$

Note that $c_\alpha = 0$ for $|\alpha|$ odd, and for any integer k

$$\sum_{|\alpha|=2k} c_\alpha \partial^\alpha g = \frac{1}{k!}\left(\frac{\hbar}{4}\Delta\right)^k g.$$

The lemma is thus reduced to the claim that $r(z,\hbar) \in \Sigma^{m-2n,\mu+n}$.

To see this, we change variables by $w' = (w-z)/\sqrt{\hbar}$ to write

$$r(z,\hbar) = \sum_{|\alpha|=2n} c''_\alpha \hbar^n \int\int_0^1 w^\alpha e^{-|w|^2}(1-t)^{2n-1}\partial^\alpha g(z+tw\sqrt{\hbar})\,dt\,dw.$$

We must estimate the derivatives:

$$\partial^\gamma r(z,\hbar) = \sum_{|\alpha|=2n} c''_\alpha \hbar^n \int\int_0^1 w^\alpha e^{-|w|^2}(1-t)^{2n-1}\partial^\beta g(z+tw\sqrt{\hbar})\,dt\,dw,$$

where $|\beta| = 2n + |\gamma|$. This integral for $\partial^\gamma r$ we then split into two pieces according to the domain of the w-integral, $I'_{\alpha,\beta} : |w| < |z|/2$ and $I''_{\alpha,\beta} : |w| > |z|/2$. The assumption $g \in \Sigma^{m,\mu}$ implies an estimate

$$|I'_{\alpha,\beta}| \le C\langle z\rangle^{m-2n-|\gamma|}\hbar^{n+\mu}. \tag{A.3}$$

The second term is taken care of by the exponential factor in $|w|$:

$$|I''_{\alpha,\beta}| < C_l \hbar^l \langle z\rangle^{-l}, \quad \forall l.$$

Therefore $\partial^\gamma r$ satisfies an estimate of the form (A.3) for any γ, and hence $r \in \Sigma^{m-2n,\mu+n}$. □

Our application of Lemma A.1 will be specifically to operators of order 4:

Lemma A.2 *For $g \in \Sigma^{4,0}$,*

$$Op^W_\hbar(g) = Op^{AW}_\hbar(g) - \frac{\hbar}{4}Op^{AW}_\hbar(\Delta g) + R(\hbar),$$

where $\|R(\hbar)\|_{L^2} \leq C\hbar^2$.

Proof. Let $\sigma(A)$ denote the Weyl symbol of the $\hbar$-pseudodifferential operator A. Applying Lemma A.1 with $n = 2$ gives

$$\sigma(Op_\hbar^{AW}(g)) = g + \frac{\hbar}{4}\Delta g + r_1,$$

and

$$\frac{\hbar}{4}\sigma(Op_\hbar^{AW}(\Delta g)) = \frac{\hbar}{4}\Delta g + r_2,$$

where $r_1, r_2 \in \Sigma^{0,2}$. Noting that

$$Op_\hbar^{W}(g) - Op_\hbar^{AW}(g) + \frac{\hbar}{4}\sigma(Op_\hbar^{AW}(\Delta g)) = Op_\hbar^{W}(r_1 - r_2),$$

the bound on $R(\hbar)$ follows from (A.1). $\square$

The point of introducing anti-Wick symbols is to exploit the Bargmann space representation of the harmonic oscillator. The Bargmann space is (see e.g. [BS])

$$\mathcal{H}_\hbar = L^2_{hol}(\mathcal{C}^l, e^{-|z|^2/\hbar}\, dz d\bar{z}).$$

The Bargmann transform is an isomorphism $\mathcal{B} : L^2(\mathcal{R}^l) \to \mathcal{H}_\hbar$, defined so as to intertwine anti-Wick operators with Toeplitz operators:

$$\mathcal{B} \circ Op_\hbar^{AW}(f) \circ \mathcal{B}^{-1} = T_\hbar(f).$$

The Toeplitz operator $T_\hbar(f) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ is defined for $f \in \Sigma^{m,\mu}$ by

$$T_\hbar(f) = \Pi_\hbar M(f),$$

where $M(f)$ denotes the multiplication operator on $L^2(\mathcal{C}^l, e^{-|z|^2/\hbar}\, dz d\bar{z})$ (identifying $\mathcal{R}^{2l} = \mathcal{C}^l$ by $z = x + i\xi$), and $\Pi_\hbar : L^2(\mathcal{C}^l, e^{-|z|^2/\hbar}\, dz d\bar{z}) \to \mathcal{H}_\hbar$ is orthogonal projection onto the holomorphic subspace.

The main result of this Appendix is the following matrix element estimate:

Proposition A.1 *Let $\{\psi_\alpha\}$ be the normalized eigenstates of the standard harmonic oscillator on $L^2(\mathcal{R}^l)$. Suppose $f \in \Sigma^{4,0}$ satisfies*

$$f(z, \hbar) = \sum_{|\gamma|=4} z^\gamma g_\gamma(z, \hbar),$$

where $\sup |\partial^\beta g_\gamma| \leq M$ *for all* $|\beta| \leq 2$. *Then*

$$|\langle \psi_\alpha, Op_\hbar^{W}(f)\psi_\alpha\rangle| \leq CM(|\alpha|\hbar)^2$$

for all $\alpha, \hbar$, where C depends only on the dimension.

Proof. Under the Bargmann transform the harmonic oscillator eigenstates have a particularly convenient form:

$$(\mathcal{B}^{-1}\psi_\alpha)(z) = (\pi^l \hbar^{|\alpha|+l}\alpha!)^{-1/2} \cdot z^\alpha.$$

Using Lemma A.1 we write

$$Op^W(f) = Op_\hbar^{AW}(f) - \frac{\hbar}{4} Op_\hbar^{AW}(\Delta f) + R(\hbar), \tag{A.4}$$

where $|\langle R(\hbar)\rangle| \leq C\hbar^2$.

Consider the matrix element of the first term on the right-hand side of (A.4). In Bargmann space this becomes

$$\langle \psi_\alpha, Op_\hbar^{AW}(f)\psi_\alpha\rangle = \frac{1}{\pi^l \hbar^{|\alpha|+l}\alpha!}\int \bar{z}^\alpha f(z,\hbar) z^\alpha e^{-|z|^2/\hbar}\, dz d\bar{z}.$$

Writing f as a sum over $z^\gamma g_\gamma$ with $|\gamma| = 4$, the estimate for a particular γ is straightforward:

$$\begin{aligned} |\langle \psi_\alpha, Op_\hbar^{AW}(z^\gamma g_\gamma)\psi_\alpha\rangle| &\leq M\frac{1}{\pi^l \hbar^{|\alpha|+l}\alpha!}\int |z^\alpha|^2\, |z|^4 e^{-|z|^2/\hbar}\, dz d\bar{z} \\ &= M\hbar^2(|\alpha|+l)(|\alpha|+l+1). \end{aligned}$$

The second term on the right in (A.4) is handled in a similar way. By assumption we can write $\Delta f = \sum_{|\eta|=2} z^\eta h_\eta(z,\hbar)$, where $\sup|h_\eta| \leq 12M$. The estimate then proceeds exactly as above (noting that there is an extra factor of $\hbar$ in front of this term). □

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