Distributional Borel Summability of Odd Anharmonic Oscillators*

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Abstract

It is proved that the divergent Rayleigh-Schrödinger perturbation expansions for the eigenvalues of any odd anharmonic oscillator are Borel summable in the distributional sense to the resonances naturally associated with the system.

1 Introduction and statement of the results

Recent work on complex operators with real spectrum (see e.g. [3, 4, 5, 12, 13, 14, 20] and references therein) in quantum mechanics and on the so-called Bessis-Zinn Justin conjecture have generated a renewed interest on spectral and perturbation theory of odd anaharmonic oscillators in quantum mechanics, namely the class of Schrödinger operators in $L^2(\mathbf{R})$ defined (on a domain to be specified later) by the action of the differential operator

$$H(\beta) = p^2 + x^2 + \beta x^{2k+1} \equiv H(0) + \beta x^{2k+1}, \quad k = 1, 2, \dots \tag{1.1}$$

Here $p = -id/dx$, β, the coupling constant, is a numerical parameter and k is fixed. The spectral and perturbation theory of the operators $H(\beta)$ (the first perturbation theory examples even introduced in quantum mechanics: see e.g.[6]) was settled long ago, from a mathematically rigorous standpoint, for non-real values of the coupling constant ([7]; see also [1, 2, 15]). The main results can be summarized as follows:

1. If $\beta \in \mathbf{C}$, $\mathrm{Im}\beta > 0$ (analogous results for $\mathrm{Im}\beta < 0$) the operator family $H(\beta)$ defined on the maximal domain $D(p^2) \cap D(x^{2k+1})$ is closed and has compact resolvents.

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2. $\forall j = 0, 1, \ldots,$ $H(\beta)$ admits one and only one eigenvalue $E_j(\beta)$ near the eigenvalue $2j+1$ of $H(0)$ for $|\beta|$ suitably small, $\mathrm{Im}\beta > 0$.

3. The function $E_j(\beta)$ is holomorphic for $\mathrm{Im}\beta > 0$, and admits a (many-valued) analytic continuation across the real axis to the (Riemann surface) sector

$$S_1(\delta) = \{\beta : |\beta| < B(\delta), -(2k-1)\pi/8 + \delta < \arg\beta < (2k+7)\pi/8 - \delta\}\,, \quad \forall \delta > 0\,.$$

4. The Rayleigh-Schrödinger perturbation expansion $\sum_{s=0}^{\infty} a_s \beta^s$ near the unperturbed eigenvalue $2j+1$ exists to all orders; it has the property $a_{2l+1} = 0$, $\forall l \in \mathbf{N}$, and is Borel (more precisely, Borel-Leroy of order $q \equiv (2k-1)/2$) summable to $E_j(\beta)$ for $\pi/8 + \eta < \arg\beta < 7\pi/8 - \eta$, $\eta > 0$ (see [4] for tests of numerical accuracy). In particular this implies that if β is purely imaginary and small as in Item 1 above the eigenvalues $E_j(\beta)$ are real.

A major problem left completely open by these results is however the meaning of the perturbation series for $\beta \in \mathbf{R}$. In this case the operator $H(\beta)$ defined on the maximal domain is not self-adjoint; it admits infinitely many self-adjoint extensions, each one with pure point spectrum (see e.g.[18], Vol.II). Now the real part of any function $E_j(\beta), \beta \in \mathbf{R}$, which has no relation with the eigenvalues of the self-adjoint extensions ([7]), admits the Rayleigh-Schrödinger perturbation expansion as an asymptotic expansion to all orders. On the other hand, it is the function $E_j(\beta)$ which has a physical meaning: any such complex eigenvalue can be indeed interpreted as a (limit) resonance of the problem, because it represents the limit of the sequence of shape resonances obtained by a general cut-off procedure of the potential at infinity as the cut-off is removed ([11]). The function $\mathrm{Re}E_j(\beta)$ is thus the natural candidate to represent the Borel sum of the original, real perturbation series; as in the Stark effect, the function $E_j(\beta)$ itself is the natural candidate to represent both location (by its real part) and width (by its imaginary part) of the resonance. However, when $\beta \in \mathbf{R}$ the coefficients of this power series have constant sign; as is well known, this prevents Borel summability because the Borel transform develops a pole on the positive real axis.

The notion of distributional Borel summmability (more precisely, in this case, Borel-Leroy of order q, as recalled in the statement of Theorem 1.1 below) was introduced in [8] exactly to deal with this kind of situations, and its validity was proved in ([10, 9]) for the perturbation expansions of the double well oscillator and of the Stark effect, respectively. In this last case the distributional Borel summability puts into one-to-one correspondence the perturbation series near the Hydrogen bound states with the real part (location) of the resonances. Here the analogous result is proved for the odd anharmonic oscillators, namely:

Theorem 1.1 *Let $q = (2k-1)/2$, $\beta \in \mathbf{R}$, $j \in \mathbf{N}$, and $f_j(\beta) \equiv \mathrm{Re}E_j(\beta)$, $g_j(\beta) \equiv \mathrm{Im}E_j(\beta)$. Then the Rayleigh-Schrödinger perturbation expansion is Borel-Leroy summable of order q in the distributional sense to $f_j(\beta)$ for $|\beta|$ suitably small, i.e.:*

(i) Set

$$B_j(t) \equiv \sum_{s=0}^{\infty} \frac{a_s}{\Gamma(qs+1)} t^s \tag{1.2}$$

Then $B_j(t)$ is holomorphic in some circle $|t| < \Lambda_j$; moreover $B_j(t)$ admits a holomorphic continuation to the intersection of some neighbourhood of $\mathbf{R}_+$ with $\mathbf{C}^+ \equiv \{t \in \mathbf{C} \,:\, \mathrm{Im}t > 0\}$.

(ii) The boundary value distributions $B_j(t \pm i0)$ exist $\forall t \in \mathbf{R}_+$ and the following representation holds:

$$f_j(\beta) = \frac{1}{q|\beta|} \int_0^{\infty} PP(B_j(t)) e^{-(t/|\beta|)^{1/q}} \left(\frac{t}{|\beta|}\right)^{-1+1/q} dt \tag{1.3}$$

where $PP(B_j(t)) = \dfrac{1}{2}(B_j(t+i0) + \overline{B_j(t+i0)})$.

(iii) $f_j(\beta) = f_j(-\beta)$, $g_j(\beta) = -g_j(-\beta)$.

Remark 1.2 *1.* As for the ordinary Borel sum, the representation (1.3) is unique among all real functions admitting the prescribed formal power series expansion and fulfilling suitable analyticity requirements and remainder estimates (the Nevanlinna conditions: see below for their definition and verification in the distributional case).

2. The symmetry property $f_j(\beta) = f_j(-\beta)$ is a consequence of the property $a_{2l+1} = 0$, $\forall l$, which in turn follows from the odd symmetry of the perturbation x^{2k+1}.

3. The distributional Borel summability procedure actually determines also the imaginary part of the functions $E_j(\beta)$, $\beta \in \mathbf{R}$, i.e. also the width of the resonances. The discussion of this aspect is postponed after the proof of Theorem 1.1.

The proof of Theorem 1.1 requires the verification of the analogous of the Nevanlinna criterion as stated and proved in Theorem 4 of [8]. This is accomplished in two steps.

In the first one (details in Sect.2) it is proved that the eigenvalues $E_j(\beta)$, $\mathrm{Im}\beta > 0$, admit a (many-valued) analytic continuation to a sector wider than the one obtained in [7] , namely $-(2k-1)\pi/4 < \arg\beta < (2k+3)\pi/4$. To do this we apply to this situation the Hunziker-Vock technique ([16]), developed after [7], to establish eigenvalue stability. The second one (Section 3) consists in extending this analyticity to a suitable Nevanlinna disk, as required by the criterion for distributional Borel summability. We do this by adapting to the present situation the techniques introduced in [10, 9] to deal with the double well oscillators and the Stark effect.

2 Analytic continuation of the complex eigenvalues

Let $k \in \mathbf{N}$ be fixed and $\beta \in \mathbf{C} - \{0\}$; $H(\beta)$ will denote the operator in $L^2(\mathbf{R})$ defined by: $D(H(\beta)) = D(p^2) \cap D(x^{2k+1})$ and

$$H(\beta)u = (p^2 + x^2 + \beta x^{2k+1})u \ , \quad \forall u \in D(H(\beta)). \tag{2.1}$$

In [7] it was proved that, for $\mathrm{Im}\beta > 0$, $H(\beta)$ represents a holomorphic family of type A of operators with compact resolvents and, for $|\beta| < B$, non-empty (discrete) spectrum. The norm resolvent convergence of $H(\beta)$ to the harmonic oscillator

$$H(0) = p^2 + x^2 \ , \quad D(H(0)) = D(p^2) \cap D(x^2) \tag{2.2}$$

as $|\beta| \to 0$, $\mathrm{Im}\beta > 0$, yielded the stability of the eigenvalues of $H(0)$ with respect to the family $H(\beta)$ in the following sense: for any fixed $j \in \mathbf{N}$ and $\forall\delta > 0$, there exists $B_j(\delta) \equiv B(\delta) > 0$ such that for $|\beta| < B(\delta)$, $\mathrm{Im}\beta > 0$, $H(\beta)$ has exactly one eigenvalue $E_j(\beta)$ such that $|E_j(\beta) - (2j+1)| < \delta$, and therefore $E_j(\beta) \to (2j+1)$ as $|\beta| \to 0$, $\mathrm{Im}\beta > 0$. Moreover such eigenvalues are analytic functions of β, for $|\beta| < B(\delta)$, $\mathrm{Im}\beta > 0$, and they admit a (many-valued) analytic continuation across the real axis to the sector

$$S_1(\delta) = \left\{\beta \in \mathbf{C} : |\beta| < B(\delta), -(2k-1)\frac{\pi}{8} + \delta < \arg\beta < (2k+7)\frac{\pi}{8} - \delta\right\}. \tag{2.3}$$

Finally, there exist constants $C, \eta > 0$ such that the corresponding Rayleigh-Schrödinger perturbation expansion is Borel summable to $E_j(\beta)$ in the sector $|\beta| < C$, $\pi/8 + \eta < \arg\beta < 7\pi/8 - \eta$. The main result in this section consists in extending the analyticity of the eigenvalues of $H(\beta)$ to the wider sector $-(2k-1)\pi/4 + \delta < \arg\beta < (2k+3)\pi/4 - \delta$, as stated in the following

Theorem 2.1 *The eigenvalues $E_j(\beta)$ of $H(\beta)$, $\mathrm{Im}\beta > 0$, which exist for $|\beta| < B$, admit a (many-valued) analytic continuation across the real axis to any sector*

$$S(\delta) = \left\{ \beta \in \mathbf{C} : |\beta| < B(\delta), -(2k-1)\frac{\pi}{4} + \delta < \arg\beta < (2k+3)\frac{\pi}{4} - \delta \right\} , \forall \delta > 0. \tag{2.4}$$

In order to prove this theorem we need some preliminary results based on the standard method of dilation analyticity (see e.g.[18], Vol.IV, §XIII.10). More precisely we introduce the operator

$$H(\beta,\theta) \equiv e^{-2\theta}p^2 + e^{2\theta}x^2 + \beta e^{(2k+1)\theta}x^{(2k+1)} \equiv e^{-2\theta}K(\beta,\theta) \tag{2.5}$$

which, for $\theta \in \mathbf{R}$, is unitarily equivalent to $H(\beta)$, $\mathrm{Im}\beta > 0$, via the dilation operator $U(\theta)$ defined by

$$(U(\theta)u)(x) = e^{\frac{\theta}{2}}u(e^\theta x) , \quad \forall u \in L^2(\mathbf{R}).$$

In [7] it was proved that, when defined on $D(p^2) \cap D(x^{2k+1})$, $H(\beta,\theta)$ represents a holomorphic family of type A of operators with compact resolvents for $-(2k-1)\pi/8 < \arg\beta < (2k+7)\pi/8$, $\mathrm{Im}\theta = (\pi/2 - \arg\beta)/(2k+3)$. This was obtained by means of a quadratic estimate for the operator $p^2 + e^{4\theta}x^2 + i|\beta|x^{2k+1}$ (which corresponds to $K(\beta,\theta)$ for $\arg\beta + (2k+3)\mathrm{Im}\theta = \pi/2$), valid for $-\pi/2 < 4\mathrm{Im}\theta < \pi/2$. Now, a first step in the proof of Theorem 2.1 consists in proving an analogous quadratic estimate for the operator

$$K(\beta,\theta) = p^2 + e^{4\theta}x^2 + |\beta|e^{i\arg\beta + (2k+3)\theta}x^{2k+1} \tag{2.6}$$

under two more general conditions:

$$\begin{cases} 0 < \arg\beta + (2k+3)\mathrm{Im}\theta < \pi \\ 0 < \arg\beta + (2k-1)\mathrm{Im}\theta < \pi \end{cases} \tag{2.7}$$

Remark 2.2 The first of the (2.7) corresponds to require the positivity of the imaginary part of the coefficient of x^{2k+1}; as for the second one, if we denote $\alpha = \arg\beta + (2k+3)\mathrm{Im}\theta$ the argument of the coefficient of x^{2k+1}, it is equivalent to require that the coefficient $\gamma \equiv e^{4\theta}$ of x^2 is in the half-plane $-\pi + \alpha < \arg\gamma < \alpha$.

Lemma 2.3 *Let $\alpha \in]0,\pi[$ and $\Omega \subset \mathbf{C}$ be a compact subset of the half-plane $-\pi + \alpha < \arg\gamma < \alpha$. Then there exist $a, b > 0$ such that*

$$\|p^2u\|^2 + |\gamma|^2\|x^2u\|^2 + |\beta|^2\|x^{2k+1}u\|^2 \le a\|(p^2 + \gamma x^2 + |\beta|e^{i\alpha}x^{2k+1})u\|^2 + b\|u\|^2 , \tag{2.8}$$

$\forall u \in D(p^2) \cap D(x^{2k+1})$, $\gamma \in \Omega$, $0 < |\beta| \le 1$, a and b independent of γ in Ω and α in a closed interval comtained in $]0,\pi[$.

Proof. We shall prove the following estimate, equivalent to (2.8):

$$\|p^2u\|^2+|\sigma|^2\|x^2u\|^2+|\beta|^2\|x^{2k+1}u\|^2\le a\|(e^{-i\alpha}p^2+\sigma x^2+|\beta|x^{2k+1})u\|^2+b\|u\|^2\ , \quad (2.9)$$

$\forall u\in D(p^2)\cap D(x^{2k+1})$, with $\sigma=\gamma e^{-i\alpha}$ varying in a compact subset of the half-plane $-\pi<\arg\sigma<0$. As quadratic forms on $D(p^2)\cap D(x^{2k+1})\otimes D(p^2)\cap D(x^{2k+1})$ we have:

$$\begin{aligned}
&(e^{i\alpha}p^2+\overline{\sigma}x^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+\sigma x^2+|\beta|x^{2k+1})\\
&=(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})+|\sigma|^2x^4+\mathrm{Re}\sigma(e^{i\alpha}p^2+|\beta|x^{2k+1})x^2\\
&+i\mathrm{Im}\sigma(e^{i\alpha}p^2+|\beta|x^{2k+1})x^2+\mathrm{Re}\sigma x^2(e^{-i\alpha}p^2+|\beta|x^{2k+1})-i\mathrm{Im}\sigma x^2(e^{-i\alpha}p^2+|\beta|x^{2k+1})\\
&=\left|\frac{\mathrm{Re}\sigma}{\sigma}\right|(e^{i\alpha}p^2+|\beta|x^{2k+1}\pm|\sigma|x^2)(e^{-i\alpha}p^2+|\beta|x^{2k+1}\pm|\sigma|x^2)\\
&+\left(1-\left|\frac{\mathrm{Re}\sigma}{\sigma}\right|\right)\left[(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})+|\sigma|^2x^4\right]\\
&+i\mathrm{Im}\sigma(e^{i\alpha}p^2x^2-e^{-i\alpha}x^2p^2)\\
&\ge\left(1-\left|\frac{\mathrm{Re}\sigma}{\sigma}\right|\right)\left[(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})+|\sigma|^2x^4\right]\\
&+i\mathrm{Im}\sigma\cos\alpha[p^2,x^2]-\mathrm{Im}\sigma\sin\alpha(p^2x^2+x^2p^2)\\
&=\left(1-\left|\frac{\mathrm{Re}\sigma}{\sigma}\right|\right)\left[(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})+|\sigma|^2x^4\right]\\
&+2\mathrm{Im}\sigma\cos\alpha(px+xp)-\mathrm{Im}\sigma\sin\alpha\left([p,[p,x^2]]+2px^2p\right)\\
&=\left(1-\left|\frac{\mathrm{Re}\sigma}{\sigma}\right|\right)\left[(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})+|\sigma|^2x^4\right]\\
&-2\mathrm{Im}\sigma|\cos\alpha|(\mp px\mp xp)-\mathrm{Im}\sigma\sin\alpha(-2+2px^2p)\\
&(\text{since }\sin\alpha>0\text{ and }\mathrm{Im}\sigma<0)\\
&\ge\left(1-\left|\frac{\mathrm{Re}\sigma}{\sigma}\right|\right)\left[(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})+|\sigma|^2x^4\right]\\
&-2\mathrm{Im}\sigma|\cos\alpha|\left[(p\mp x)^2-p^2-x^2\right]+2\mathrm{Im}\sigma\sin\alpha\\
&\ge\left(1-\left|\frac{\mathrm{Re}\sigma}{\sigma}\right|\right)\left[(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})+|\sigma|^2x^4\right]\\
&+2\mathrm{Im}\sigma|\cos\alpha|(p^2+x^2)+2\mathrm{Im}\sigma\sin\alpha.
\end{aligned}$$

In [7] it was proved that there exist $a_1,b_1>0$, in general depending on $|\beta|$, such that

$$(e^{i\alpha}p^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+|\beta|x^{2k+1})\ge a_1(p^4+|\beta|^2x^{4k+2})-b_1\ ;$$

thus,

$$\begin{aligned}
&(e^{i\alpha}p^2+\overline{\sigma}x^2+|\beta|x^{2k+1})(e^{-i\alpha}p^2+\sigma x^2+|\beta|x^{2k+1})\\
&\ge A(p^4+|\beta|^2x^{4k+2})+B|\sigma|^2x^4+2\mathrm{Im}\sigma|\cos\alpha|(p^2+x^2)+2\mathrm{Im}\sigma\sin\alpha-b_1\\
&\ge[Aa'p^4+2\mathrm{Im}\sigma|\cos\alpha|p^2+2\mathrm{Im}\sigma\sin\alpha-b+b'/2]
\end{aligned}$$

$$+[Aa'|\beta|^2x^{4k+2}+2\mathrm{Im}\sigma|\cos\alpha|x^2+b'/2]$$
$$+A(1-a')p^4+A(1-a')|\beta|^2x^{4k+2}+B|\sigma|^2x^4-b'\ .$$

Now it suffices to choose $0<a'<1$ and $b'>0$ such that the two terms in square brackets are positive.

Lemma 2.4 *Let β and θ be fixed, satisfying conditions (2.7) and let $\alpha=\arg\beta+(2k+3)\mathrm{Im}\theta$, $\alpha\in]0,\pi[$. Then there exists $\xi>0$ such that*

$$\xi\mathrm{Re}\Big[e^{-i(\alpha-\frac{\pi}{2})}\left\langle u,K(\beta,\theta)u\right\rangle\Big]\geq\left\langle u,p^2u\right\rangle\ ,\quad\forall u\in C_0^\infty(\mathbf{R})\ . \tag{2.10}$$

Proof. We have

$$\mathrm{Re}\Big[e^{-i(\alpha-\frac{\pi}{2})}\Big\langle u,(p^2+e^{4\theta}x^2+|\beta|e^{(2k+3)\mathrm{Re}\theta+i\alpha}x^{2k+1})u\Big\rangle\Big]$$
$$=\cos(\alpha-\pi/2)\langle u,p^2u\rangle+e^{4\mathrm{Re}\theta}\cos(\pi/2-\alpha+4\mathrm{Im}\theta)\langle u,x^2u\rangle$$
$$+|\beta|e^{(2k+3)\mathrm{Re}\theta}\cos(\pi/2)\langle u,x^{2k+1}u\rangle$$
$$=\sin\alpha\langle u,p^2u\rangle+e^{4\mathrm{Re}\theta}\sin(\alpha-4\mathrm{Im}\theta)\langle u,x^2u\rangle$$
$$\geq\sin\alpha\langle u,p^2u\rangle\ ,$$

since $\sin(\arg\beta+(2k-1)\mathrm{Im}\theta)>0$ by the second of (2.7). Moreover, since $0<\alpha<\pi$, the lemma is proved with $\xi=(\sin\alpha)^{-1}$.

Theorem 2.5 *Let $s=\arg\beta$ and $t=\mathrm{Im}\theta$. Then $H(\beta,\theta)$ is a holomorphic family of type A of closed operators on $D(H(\beta,\theta)=D(p^2)\cap D(x^{2k+1})$ with compact resolvents for β and θ such that s and t vary in the parallelogram P of the (s,t)-plane defined by*

$$P=\{(s,t)\in\mathbf{R}^2:0<(2k-1)t+s<\pi,0<(2k+3)t+s<\pi\}\ . \tag{2.11}$$

Proof. Lemma 2.3 guarantees that $H(\beta,\theta)$ is closed on a domain independent of β and θ for $\arg\beta=s$ and $\mathrm{Im}\theta=t$ satisfying conditions (2.7):

$$\begin{cases}0<(2k+3)t+s<\pi\\0<(2k-1)t+s<\pi\end{cases}$$

which define the parallelogram P with vertices in the points of coordinates $(-(2k-1)\pi/4,\pi/4)$, $(0,0)$, $((2k+3)\pi/4,-\pi/4)$, $(\pi,0)$. From Lemma 2.4 it follows that, for β and θ in this region, $K(\beta,\theta)$ has numerical range in the half-plane $-\pi+\alpha\leq\arg z\leq\alpha$, with $\alpha=\arg\beta+(2k+3)\mathrm{Im}\theta$; thus $H(\beta,\theta)$ has numerical range contained in the half-plane

$$\Pi=\{z\in\mathbf{C}:-\pi+\arg\beta+(2k+1)\mathrm{Im}\theta\leq\arg z\leq\arg\beta+(2k+1)\mathrm{Im}\theta\}.$$

By standard arguments on the holomorphic families of type A (see [17] or [18] Vol.IV), taking into account the above mentioned results obtained in [7] for $-(2k-1)\pi/8 < \arg\beta < (2k+7)\pi/8$, we now obtain the analyticity of $H(\beta,\theta)$ in the region defined by P, which allows β to be extended to the sector $-(2k-1)\pi/4 < \arg\beta < (2k+3)\pi/4$, as well as the compactness of the resolvents. Finally, the (discrete) spectrum of $H(\beta,\theta)$ is contained in Π and $\forall z \notin \Pi$, $\|(z-H(\beta,\theta))^{-1}\| \le (\mathrm{dist}(z,\Pi))^{-1}$.

Remark 2.6 Let us notice that, if we start from the operator $H(\beta)$ with $\mathrm{Im}\beta < 0$, analogous results can be obtained for the operator family $H(\beta,\theta)$ for β and θ such that $s = \arg\beta$, $t = \mathrm{Im}\theta$ vary in the parallelogram

$$P^1 = \{(s,t) \in \mathbf{R}^2 : -\pi < (2k-1)t + s < 0, -\pi < (2k+3)t + s < 0\}.$$

Furthermore the adjoint operator $H(\beta,\theta))^*$ of $H(\beta,\theta)$ is $H(\overline{\beta},\overline{\theta})$.

In order to complete the proof of Theorem 2.1 we need to extend to the wider sector $S(\delta)$ given by (2.4) the result obtained in [7] for $\beta \in S_1(\delta)$ (see (2.3)), on the existence of eigenvalues of $H(\beta,\theta)$ and on their convergence to the corresponding eigenvalues of the harmonic oscillator as $|\beta| \to 0$. To this end, since we cannot make use of the norm resolvent convergence which holds only for $\beta \in S_1(\delta)$, $|\beta| \to 0$, we will apply the more general criterion for the stability of the eigenvalues introduced in [16] and based on the strong convergence of the resolvents. More precisely, let us consider the operator

$$H(0,\theta) \equiv e^{-2\theta}p^2 + e^{2\theta}x^2\ , \quad D(H(0,\theta)) = D(p^2) \cap D(x^2)$$

corresponding to the dilated harmonic oscillator. We will prove that the eigenvalues of $H(0,\theta)$, independent of θ for $-\pi/4 < \mathrm{Im}\theta < \pi/4$, and represented by the sequence of the odd numbers $\{(2j+1) : j \in \mathbf{N}\}$, are stable in the sense of Kato with respect to the family $\{H(\beta,\theta) : |\beta| > 0\}$, β and θ in the region defined by P. For simplicity we will work with the operators $K(\beta,\theta) = e^{2\theta}H(\beta,\theta)$ and $K(0,\theta) = e^{2\theta}H(0,\theta)$; moreover, from now on we will assume θ purely imaginary, that is of the form $i\theta$, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, and (with slight abuse of notation) we will still denote $H(\beta,\theta)$ and $K(\beta,\theta)$ the operators $H(\beta,i\theta)$ and $K(\beta,i\theta)$ respectively. Notice that with this convention we should read θ in place of $\mathrm{Im}\theta$ wherever the notation $\mathrm{Im}\theta$ has been employed, in particular in the conditions (2.7). Finally, let $\sigma(K(\beta,\theta))$ denote the spectrum of $K(\beta,\theta)$. Then, in order to obtain the above mentioned stability result, we will prove the following

Theorem 2.7 *Let β and θ satisfy conditions (2.7). We have:*

(i) if $\lambda \notin \sigma(K(0,\theta))$*, then* $\lambda \in \Delta$*, where*

$$\Delta = \{z \in \mathbf{C} : z \notin \sigma(K(\beta,\theta)) \text{ and } (z - K(\beta,\theta))^{-1} \text{ is uniformly bounded as } |\beta| \to 0\}$$

(ii) if $\lambda \in \sigma(K(0,\theta)) = \{(2j+1)e^{2i\theta} : j \in \mathbf{N}\}$*, then* λ *is stable with respect to the family* $K(\beta,\theta)$*, i.e.: if* $r > 0$ *is sufficiently small, so that the only eigenvalue of* $K(0,\theta)$ *enclosed in* $\Gamma_r = \{z \in \mathbf{C} : |z - \lambda| = r\}$ *is* λ*, then there is* $B > 0$ *such that for* $|\beta| < B$*,* $dim P(\beta,\theta) = dim P(0,\theta)$*, where*

$$P(\beta,\theta) = (2\pi i)^{-1} \oint_{\Gamma_r} (z - K(\beta,\theta))^{-1} dz$$

is the spectral projection of $K(\beta,\theta)$ *corresponding to the part of the spectrum enclosed in* $\Gamma_r \subset \mathbf{C} - \sigma(K(\beta,\theta))$*. Similarly for* $P(0,\theta)$*.*

Proof. It is a straightforward application of Theorem 5.4 of [16] once we have proved the following

Theorem 2.8 *Let* $\arg\beta$ *and* θ *be fixed, satisfying conditions (2.7), and let* $K(\rho) = K(\beta,\theta)$ *with* $\rho = |\beta|$*. Then*

(a)

$$\lim_{\rho\to 0^+} K(\rho)u = K(0)u, \quad \lim_{\rho\to 0^+} K(\rho)^* u = K(0)^* u, \quad \forall u \in C_0^\infty(\mathbf{R}).$$

(b) $\Delta \neq \emptyset$*.*

(c) Let $\chi \in C_0^\infty(\mathbf{R})$ *be such that* $\chi(x) = 1$ *for* $|x| \le 1$*,* $0 \le \chi(x) \le 1$*,* $\forall x \in \mathbf{R}$*,* $\chi(x) = 0$ *for* $|x| \ge 2$*. For* $n \in \mathbf{N}$ *let* $\chi_n(x) = \chi(x/n)$ *and* $M_n(x) = 1 - \chi_n(x)$*. We have:*

(1) if $\rho_m \to 0^+$ *and* $u_m \in D(K(\rho_m))$ *are two sequences such that*

$$\|u_m\| \to 1, \; u_m \xrightarrow{w} 0, \quad \text{and} \quad \|K(\rho_m)u_m\| \le (\text{const.}), \; \forall m,$$

then there exists $a > 0$ *such that*

$$\limsup_{m\to\infty} \|M_n u_m\| \ge a > 0, \quad \forall n;$$

(2) for some $z \in \Delta$

$$\lim_{n\to\infty} \|[M_n, K(\rho)](z - K(\rho))^{-1}\| = 0,$$

uniformly as $\rho \to 0^+$*;*

(3) $\forall\lambda \in \mathbf{C}$, *there exists* $\delta > 0$ *such that*

$$d_n(\lambda, \rho) \equiv \inf\{\|(\lambda - K(\rho))M_n u\| : u \in D(K(\rho)), \|M_n u\| = 1\} > \delta\,,$$

$\forall n > n_0$ *and* $\rho \to 0^+$.

Proof.

(a) It follows immediately from the convergence of the potential $V(\rho) = e^{4i\theta}x^2 + \rho e^{i(\arg\beta + (2k+3)\theta)}x^{2k+1}$ to $V(0) = e^{4i\theta}x^2$ as $\rho \to 0^+$, uniformly on the compact subsets of $\mathbf{R}$.

(b) As already observed in the proof of Theorem 2.5 $K(\rho)$ has numerical range contained in the half-plane

$$\Pi_\alpha = \{z \in \mathbf{C} : -\pi + \alpha \le argz \le \alpha\}, \quad \alpha = \arg\beta + (2k+3)\theta$$

indipendent of ρ, and $\forall z \notin \Pi_\alpha$, $\|(z - K(\rho))^{-1}\| \le (\mathrm{dist}(z, \Pi_\alpha))^{-1}$.

(c) Statement (1) follows from a standard argument based on an estimate which comes from Lemma 2.4: there exists $c > 0$ such that

$$\|(1 + p^2)^{\frac{1}{2}}u\| \le c(\|K(\rho)u\| + \|u\|)\,, \quad \forall u \in D(K(\rho))\,. \tag{2.12}$$

For the details see [16]. As for (2), following again [16], we have:

$$[M_n, K(\rho)] = [\chi_n, p^2] = 2in^{-1}\Phi_n p - n^{-2}\Psi_n\,,$$

where the functions Φ_n and Ψ_n, obtained by differentiating χ once and twice respectively, are uniformly bounded in n and ρ. Thus, the result follows applying again (2.12). Finally, given $\lambda \in \mathbf{C}$ we have

$$d_n(\lambda, \rho) = \inf\{\|(\lambda' - e^{i(\frac{\pi}{2} - \alpha)}K(\rho))M_n u\| : u \in D(K(\rho)), \|M_n u\| = 1\}$$

with $\lambda' = e^{i(\frac{\pi}{2} - \alpha)}\lambda$, $\alpha = \arg\beta + (2k+3)\theta$. Therefore $d_n(\lambda, \rho) \ge \mathrm{dist}(\lambda', G_n(\rho))$, where

$$G_n(\rho) = \left\{\left\langle M_n u, e^{i(\frac{\pi}{2} - \alpha)}K(\rho))M_n u\right\rangle : u \in D(K(\rho)), \|M_n u\| = 1\right\}\,,$$

whence

$$d_n(\lambda, \rho) \ge \inf\left\{\mathrm{Re}\left\langle M_n u, e^{i(\frac{\pi}{2} - \alpha)}K(\rho))M_n u\right\rangle - |\lambda'| : u \in D(K(\rho)), \|M_n u\| = 1\right\}\,.$$

Now the assertion follows from the proof of Lemma 2.4, which yields

$$\begin{aligned}&\mathrm{Re}\Big\langle M_n u, e^{i(\frac{\pi}{2}-\alpha)} K(\rho)) M_n u\Big\rangle \\ &\geq \sin\left(\arg\beta + (2k-1)\theta\right)\langle M_n u, x^2 M_n u\rangle \geq n^2 \sin\left(\arg\beta + (2k-1)\theta\right)\end{aligned}$$

and therefore

$$\lim_{\substack{n\to\infty \\ \rho\to 0^+}} d_n(\lambda,\rho) = +\infty\,.$$

Remark 2.9 It is immediate to check that all the results so far obtained, in particular the analyticity of the family $H(\beta,\theta)$ and the stability of the eigenvalues of the harmonic oscillator with respect to $H(\beta,\theta)$ as $\rho = |\beta| \to 0^+$, hold uniformly in β and θ such that $(\arg\beta,\theta)$ varies in any compact subset of P.

Proof of Theorem 2.1 It follows from Theorems 2.5 and 2.7 and from Remark 2.9. In particular if $(\arg\beta,\theta) \in P$, by the well-known Symanzik scaling properties (see [19]) the eigenvalues $E_j(\beta)$ of $H(\beta,\theta)$ do not depend on θ and represent the analytic continuation to the sector $S(\delta)$ of the eigenvalues of $H(\beta)$, $\mathrm{Im}\beta > 0$; in fact, as already observed, the condition $(\arg\beta,\theta) \in P$, allows us to extend $\arg\beta$ to the interval $]-(2k-1)\pi/4, (2k+3)\pi/4[$.

Remark 2.10 Let $E_j(\beta)$ denote the generic eigenvalue of $H(\beta)$ for $\mathrm{Im}\beta > 0$, which can be analytically continued to the sector $S(\delta)$, and $E^1_j(\beta)$ the generic eigenvalue of $H(\beta)$ for $\mathrm{Im}\beta < 0$, which can be analytically continued to the sector

$$\overline{S}(\delta) = \left\{\beta \in \mathbf{C} : 0 < |\beta| < B(\delta), -(2k+3)\frac{\pi}{4} + \delta < \arg\beta < (2k-1)\frac{\pi}{4} - \delta\right\}\,.$$

Then, from Remark 2.6 we have $E^1_j(\beta) = \overline{E_j(\overline{\beta})}$.

3 Analyticity of the eigenvalues in a Nevanlinna disk and distributional Borel summability

We begin this section by stating and proving the basic analyticity result needed to establish the distributional Borel summability.

Theorem 3.1 *Set $q = (2k-1)/2$. For each eigenvalue $E_j(\beta)$, $j \in \mathbf{N}$, of the odd anharmonic oscillator $H(\beta)$ there exists $R > 0$ such that $E_j(\beta)$ is analytic in the Nevanlinna disk $C_R = \{\beta \in \mathbf{C} : \mathrm{Re}\beta^{-1/q} \geq R^{-1}\}$ of the $\beta^{1/q}$-plane.*

Remark 3.2 (I) the sector $S(\delta)$ can be re-written in terms of the parameter q:

$$S(\delta) = \left\{\beta \in \mathbf{C} : |\beta| < B(\delta), -\frac{\pi}{2} + \frac{\delta}{q} < \arg \beta^{1/q} < \frac{\pi}{2} + \frac{\pi}{q} - \frac{\delta}{q}\right\} .$$

(II) The function $E_j(\beta)$, analytic in any sector $S(\delta)$ and for which we want to prove analyticity in a disk C_R, represents an eigenvalue of the operator $H(\beta, \theta)$ if the pair (β, θ) satisfies the condition $(\arg\beta, \theta) \in P$. In particular for $-\pi(2k-1)/4 < \arg\beta < 0$ we can choose the path inside P given by the straight line of equation

$$\theta = -\frac{1}{2k+1} \arg\beta + \frac{\pi}{2(2k+1)} ;$$

then, if we set

$$\arg\beta = -\frac{\pi}{4}(2k-1) + \frac{\epsilon}{2}(2k-1) = -\frac{\pi}{2}q + \epsilon q, \text{ i.e. } \arg\beta^{1/q} = -\frac{\pi}{2} + \epsilon, \epsilon \to 0^+$$

we obtain $\theta = \pi/4 - (2k-1)\epsilon/[2(2k+1)] = \pi/4 - \epsilon q/(2k+1)$, and the operator $H(\beta, \theta)$ takes the form

$$A(\rho) = e^{-i(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon)} p^2 + e^{i(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon)} x^2 + i\rho x^{2k+1} , \quad \text{with } \rho = |\beta| .$$

(III) For $\beta = \rho e^{i\arg\beta}$ and $\arg\beta = (-\frac{\pi}{2} + \epsilon)q$, the boundary of C_R has equation

$$\sin\epsilon = \frac{\rho^{1/q}}{R} . \tag{3.1}$$

Since the disk C_R can be regarded as the union of the boundaries of disks of smaller radius, the proof of Theorem 3.1 reduces to a stability argument with respect to the family $A(\rho)$, as $\rho \to 0^+$, under condition (3.1), for the eigenvalues of a suitable limiting operator, which we proceed to define.

The argument is similar to the one already developed in [9] and [10] to obtain analyticity of the eigenvalues for the operators associated with the Stark effect and the double well oscillators respectively. More precisely, let D denote the dense subset of $L^2(\mathbf{R})$ of the functions which are translation analytic in a suitable strip $|\mathrm{Im} x| < \eta_0$, for some $0 < \eta_0 < 1$ (recall that $u \in L^2(\mathbf{R})$ is translation analytic for $|\mathrm{Im} x| < r$ if $(T_a u)(x) = u(x+a)$ admits an L^2-valued analytic continuation to $|\mathrm{Im} a| < r$); D represents a core for $A(\rho)$.

Definition 3.3 *Let $\eta > 0$ be fixed and small. For fixed $a_k > 0$, set $x_0 = -\frac{a_k}{\rho^{1/(2k-1)}}$ and let $\mathcal{U}$ denote the unitary operator in $L^2(\mathbf{R})$ defined by*

$$(\mathcal{U}\psi)(x) = (\xi'_\rho(x))^{\frac{1}{2}} \psi(\xi_\rho(x)) , \quad \forall \psi \in D ,$$

where, for any given $\rho > 0$, $\xi_\rho \in C^\infty(\mathbf{R})$ *satisfies the conditions:*

$$\begin{array}{ll} \xi_\rho(x) = x - i\eta \arctan\left[x/(1+x^2)^{1/4}\right] , & -x_0 \le x < +\infty \\ \xi_\rho(x) = x , & x \le x_0 - \eta \end{array} \tag{3.2}$$

and $\mathrm{Im}\xi_\rho(x)$ *is monotone in the remaining region.*
Then the closed operator $H_\rho \equiv \mathcal{U}A(\rho)\mathcal{U}^{-1}$, *unitarily equivalent to* $A(\rho)$ *and with the same (discrete) spectrum, has* $D_1 \equiv \mathcal{U}(D)$ *as a core, and its action on* D_1 *is given by*

$$H_\rho u = e^{-i(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon)} \left\{ p f_\rho^2 p + 4^{-1}(f_\rho^2)'' \right\} u + e^{i(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon)} \xi_\rho^2 u + i\rho \xi_\rho^{2k+1} u , \quad \forall u \in D_1 , \tag{3.3}$$

where $f_\rho(x) = (\xi_\rho'(x))^{-1}$, $\forall x \in \mathbf{R}$.

Remark 3.4 In a similar way we can define the dilated harmonic oscillator, having D_1 as a core:

$$H_0 u = -i \left\{ p f_0^2 p + 4^{-1}(f_0^2)'' \right\} u + i\xi_0^2 u , \quad \forall u \in D_1 ,$$

where $f_0(x) = (\xi_0'(x))^{-1}$ and $\xi_0'(x) = x - i\eta \arctan\left[x/(1+x^2)^{1/4}\right]$, $\forall \in \mathbf{R}$. In Corollary 3.9 we will prove that H_0 is the limit in the strong resolvent sense of H_ρ as $\rho \to 0^+$. Therefore, as anticipated after Remark 3.2, the proof of Theorem 3.1 consists in obtaining a stability result for the eigenvalues $E_j = (2j+1)$, $j \in \mathbf{N}$, of H_0, which coincide with those of the harmonic oscillator, with respect to the family H_ρ as $\rho \to 0^+$.

Proceeding in analogy with [9] and [10], this result will be obtained by proving some preliminary lemmas aimed to verify the hypotheses of Theorem A.1 of [10]. This theorem represents a simpler tool for applications, in the context of the more general stability theory developed by Hunziker and Vock in [16]. In particular in the subsequent Lemmas 3.5, 3.6, 3.9, 3.10 and Corollaries 3.7, 3.8, we follow the corresponding steps used in [9] and [10] to obtain similar results, each one adapted to the specific characteristics of the present problem; we will describe here the relevant details.

Lemma 3.5 *Let* $V_\rho(x) = e^{i(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon)} \xi_\rho^2(x) + i\rho\xi_\rho^{2k+1}(x)$. *Then for a suitable choice of the constant* $a_k > 0$ *in Definition 3.3 there exist constants* $c_1 > 0$ *and* $c_2 \in \mathbf{R}$ *such that*

$$\mathrm{Re} V_\rho(x) \ge \frac{c_1}{R} + c_2 , \quad \forall x \notin (-n, n) \tag{3.4}$$

$\forall n \ge n_0$, $0 < \rho < \rho_0$.

Proof. Set $\eta(x) = \mathrm{Im}\xi_\rho(x)$; then $\eta(x) \le 0$ for $x > 0$, $\eta(x) \ge 0$ for $x \le 0$, and $-\eta\pi/2 \le \eta(x) \le \eta\pi/2$, $\forall x \in \mathbf{R}$. Now a simple calculation gives

$$\begin{aligned}
&\mathrm{Re}V_\rho(x) = \sin\{\epsilon(2k-1)/(2k+1)\}\left(x^2-\eta(x)^2\right) - \cos\{\epsilon(2k-1)/(2k+1)\}(2x\eta(x))\\
&-\rho\eta(x)\left[(2k+1)x^{2k} - \begin{pmatrix} 2k+1 \\ 3 \end{pmatrix} x^{2k-2}\eta(x)^2 + \begin{pmatrix} 2k+1 \\ 5 \end{pmatrix} x^{2k-4}\eta(x)^4\right.\\
&\left.+ ... + (-1)^{k-1}\begin{pmatrix} 2k+1 \\ 2k-1 \end{pmatrix} x^2\eta(x)^{2k-2} + (-1)^k\eta(x)^{2k}\right],
\end{aligned} \tag{3.5}$$

Next we notice that the term inside the square brackets can be bounded from below by a constant (independent of ρ), and for $x \ge n \ge n_0$, $0 < \rho < \rho_0$ we have $x^2 > \eta(x)^2$, whence

$$\mathrm{Re}V_\rho(x) \ge cn + c' \ge \frac{c_1}{R} + c_2\,. \tag{3.6}$$

For $x \le -n$ we still have $x^2 > \eta(x)^2$, and the term inside square brackets in (3.5) can be bounded from above by

$$Ax^{2k} + B\,,$$

for suitable constants $A > 0$ and $B \in \mathbf{R}$, independent of ρ and n. Thus,

$$\begin{aligned}
&\mathrm{Re}V_\rho(x) \ge \sin[\epsilon(2k-1)/(2k+1)]\left(x^2-\eta(x)^2\right)\\
&-\cos[\epsilon(2k-1)/(2k+1)](2x\eta(x))\\
&-\rho\eta(x)(Ax^{2k}+B)
\end{aligned} \tag{3.7}$$

Now, if the number $a_k > 0$ in Definition 3.3 is chosen so that the polynomial term

$$-2x\cos[\epsilon(2k-1)/(2k+1)] - \rho(Ax^{2k}+B) \tag{3.8}$$

attains its (positive) maximum at $x_0 = -\dfrac{a_k}{\rho^{1/(2k-1)}}$, estimate (3.6) still holds in the interval $x_0 \le x \le -n$, if we make the assumption, not restrictive in this context, that $n \ll \rho^{-2k}$. Finally, notice that at some point smaller than x_0 the term (3.8) becomes negative and tends to $-\infty$ as $\rho \to 0^+$, without being compensated by the term

$$\sin[\epsilon(2k-1)/(2k+1)]\left(x^2-\eta(x)^2\right)$$

which behaves as $\dfrac{\rho^{1/q}}{R}x^2$, if we recall that $\sin\epsilon = \dfrac{\rho^{1/q}}{R}$. This is the reason why it was necessary to set $\eta(x) = 0$ for $x \le x_0 - \eta$. In particular in this region we have

$$\mathrm{Re}V_\rho(x) = (\sin[\epsilon(2k-1)/(2k+1)])\,x^2 \ge c\left(\frac{\rho^{1/q}}{R}\right)\left(-\frac{a_k}{\rho^{1/(2k-1)}} - \eta\right)^2 \ge \frac{c_1}{R} + c_2\,,$$

whence the assertion.

¿From now on the constant $a_k > 0$ in Definition 3.3 will be chosen so as to satisfy Lemma 3.5.

Lemma 3.6 *There exist constants c_3,$c_4 > 0$ such that*

$$\mathrm{Re}\langle u, H_\rho u\rangle \geq c_3 \int_{x_0}^{+\infty} \frac{(1+x^2)^{\frac{1}{4}}}{x^2+(1+x^2)^{\frac{1}{2}}}|pu|^2 dx - c_4\|u\|^2 \,, \tag{3.9}$$

$\forall u \in D(H_\rho)$, $0 < \rho < \rho_0$.

Proof. Set $\omega = e^{-i(\frac{\pi}{2}-\frac{2k-1}{2k+1}\epsilon)}$. Then we have

$$\mathrm{Re}\langle u, H_\rho u\rangle = \mathrm{Re}\int_{-\infty}^{+\infty} \left\{\omega f_\rho^2|pu|^2 + \frac{\omega}{4}(f_\rho^2)''|u|^2 + V_\rho(x)|u|^2\right\} dx \,. \tag{3.10}$$

As for the first term in the right hand side of (3.10) we have

$$\mathrm{Re}(\omega f_\rho^2) = \sin\left[\epsilon(2k-1)/(2k+1)\right]\mathrm{Re}f_\rho^2 + \cos\left[\epsilon(2k-1)/(2k+1)\right]\mathrm{Im}f_\rho^2 \,. \tag{3.11}$$

For $x \geq x_0$ it is easy to check that

$$\mathrm{Re}f_\rho^2 \geq \frac{1}{4}\left(1-\eta^2\frac{(1+x^2)^{\frac{1}{2}}}{[x^2+(1+x^2)^{\frac{1}{2}}]^2}\right) \tag{3.12}$$

and

$$\mathrm{Im}f_\rho^2 \geq \eta\left[\frac{(1+x^2)^{\frac{1}{4}}}{x^2+(1+x^2)^{\frac{1}{2}}}\right] \tag{3.13}$$

whence

$$\mathrm{Re}(\omega f_\rho^2) \geq \eta\left(\cos\left[\epsilon(2k-1)/(2k+1)\right]\frac{(1+x^2)^{\frac{1}{4}}}{x^2+(1+x^2)^{\frac{1}{2}}}\right) \,. \tag{3.14}$$

In the region $x \leq x_0 - \eta$ we have $f_\rho(x) = 1$, so that

$$\mathrm{Re}(\omega f_\rho^2) = \sin\left[\epsilon(2k-1)/(2k+1)\right]. \tag{3.15}$$

Now simple calculations allow us to verify that $|(f_\rho^2)''|$ is bounded. Moreover from (3.5) it follows that $\mathrm{Re}V_\rho(x)$ is bounded from below in the interval $(-n_0, n_0)$, and therefore in $\mathbf{R}$ by Lemma 3.5. Now the assertion follows combining this result with (3.14) and (3.15).

Corollary 3.7 *(1)* $\lim_{\rho\to 0^+} H_\rho u = H_0 u \,, \quad \forall u \in D_1 \,.$

(2) $\Delta' \neq \emptyset$, *where*

$$\Delta' = \{z \in \mathbf{C} : z \notin \sigma(H_\rho) \text{ and } (z-H_\rho)^{-1} \text{ is uniformly bounded as } \rho \to 0^+\}\,.$$

(3) H_ρ *converges strongly to* H_0 *in the generalized sense.*

Proof. Statement (1) follows from the fact that $\xi_\rho(x) \to \xi_0(x)$ as $\rho \to 0^+$, uniformly on compacts. By Lemma 3.6 we have that the numerical range of H_ρ is contained in a right half-plane Π, and since H_ρ has discrete spectrum, $\|(z-H_\rho)^{-1}\| \le (\mathrm{dist}(z,\Pi))^{-1}$, $\forall z \notin \Pi$. Finally (3) follows from (1) and (2), since D_1 is a core for H_ρ, $\rho \ge 0$ (see [17], Theorem VIII.1.5).

Corollary 3.8 *Let $\chi \in C_0^\infty(\mathbf{R})$ be the function defined in Theorem 2.8(c), and again let $\chi_n(x) = \chi(x/n)$, $M_n(x) = 1 - \chi_n(x)$, $\forall n \in \mathbf{N}$. Then there exists $c_5 > 0$ such that*

$$\|[H_\rho, \chi_n]u\| \le \frac{c_5}{n^{\frac{1}{4}}}(\|H_\rho u\| + \|u\|) \tag{3.16}$$

$\forall u \in D(H_\rho)$, $0 \le \rho < \rho_0$.

Proof. Let $u \in D(H_\rho)$, $\|u\| = 1$, and γ_{2n} be the characteristic function of the interval $[-2n, 2n]$. We have

$$[H_\rho, \chi_n] = \omega[pf_\rho^2 p, \chi_n] = \omega\gamma_{2n}\{2in^{-1}f_\rho^2\chi'(x/n)p + 2n^{-1}f_\rho f_\rho'\chi'(x/n) + n^{-2}f_\rho^2\chi''(x/n)\}\ . \tag{3.17}$$

Now, since χ', χ'', f_ρ, f_ρ', f_ρ^2 are all bounded functions, we have the pointwise estimate

$$|[H_\rho, \chi_n]u(x)| \le \frac{c}{n}(|u(x)| + |(pu)(x)|)\,. \tag{3.18}$$

Thus, for $\|u\| = 1$,

$$\begin{aligned}
&\|[H_\rho, \chi_n]u\| \\
&\le \frac{c'}{n}\left\{\left(\int_{-2n}^{2n} |pu|^2 \frac{(1+x^2)^{\frac{1}{4}}}{x^2+(1+x^2)^{\frac{1}{2}}}\frac{x^2+(1+x^2)^{\frac{1}{2}}}{(1+x^2)^{\frac{1}{4}}}dx\right)^{\frac{1}{2}} + 1\right\} \\
&\le \frac{c''}{n}\left\{n^{\frac{3}{4}}\left(\int_{x_0}^{+\infty} |pu|^2 \frac{(1+x^2)^{\frac{1}{4}}}{x^2+(1+x^2)^{\frac{1}{2}}}dx\right)^{\frac{1}{2}} + 1\right\} \\
&\le \frac{c_5}{n^{\frac{1}{4}}}\left\{\mathrm{Re}\langle u, H_\rho u\rangle + 1\right\}\ ,
\end{aligned}$$

whence the assertion. Notice that to obtain the second inequality we assumed again, without loss, $n \ll |x_0|$, while for the last inequality we have used Lemma 3.6.

Lemma 3.9 *Let the sequences $\rho_m \to 0^+$ and $u_m \in D(H_{\rho_m})$ be given such that $\|H_{\rho_m}u_m\|$ is bounded, $\|u_m\| = 1$, $u_m \xrightarrow{w} 0$. Then $\forall n$*

$$\lim_{m\to\infty} \|\chi_n u_m\| = 0\ .$$

Proof. Set $H'_\rho = \omega^{-1} H_\rho$ and let $\lambda \in \mathbf{C} - \sigma(H'_0)$ be fixed. Then we have

$$\|\chi_n u_m\|^2 \leq c \left(\|\chi_n R'_0 (H'_0 - H'_{\rho m}) u_m\|^2 + \|\chi_n R'_0 (H'_{\rho m} - \lambda) u_m\|^2 \right) ,$$

where $R'_0 = (\lambda - H'_0)^{-1}$. Now we can proceed as in the proof of Lemma 5 of [9].

Lemma 3.10 *For any $\lambda \in \mathbf{C}$ there exist R, n_0, $\delta > 0$ such that*

$$d_{n,\rho}(\lambda) \equiv \inf \{ \|(\lambda - H_\rho) M_n u\| : u \in D(H_\rho), \|M_n u\| = 1 \} \geq \delta ,$$

$\forall n > n_0$,$\forall \rho \leq \rho_0$.

Proof. By Lemma 3.5

$$\mathrm{Re}\langle M_n u, V_{\rho m} M_n u \rangle \geq \frac{c_1}{R} + c_2 > \delta > 0$$

if $\|M_n u\| = 1$ and R is chosen sufficiently small. Finally, from the proof of Lemma 3.6 the kinetic part of H_ρ is bounded from below and this proves the lemma.

Proof of Theorem 3.1. From Corollary 3.8 and Lemmas 3.9, 3.10 the proof of a theorem analogous to Theorem 2.8 immediately follows, with the operator $K(\rho)$ replaced by H_ρ, $\rho \geq 0$. Thus, we can apply Theorem A1 of [10], in order to obtain the following stability result:

(i') if $\lambda \notin \sigma(H_0)$ then $(\lambda - H_\rho)^{-1}$ is uniformly bounded as $\rho \to 0^+$;

(ii') if $\lambda \in \sigma(H_0)$ then λ is a stable eigenvalue with respect to the family $\{H_\rho\}_{\rho > 0}$.

With an argument analogous to the one used to prove Theorem 3.1 we now obtain the following

Theorem 3.11 *Let $q = (2k-1)/2$. Then for each eigenvalue $E_j(\beta)$, $j \in \mathbf{N}$, of $H(\beta)$, $\mathrm{Im}\beta > 0$, there exists $R' > 0$ such that $E_j(\beta)$ is analytic in the Nevanlinna disk of the $\beta^{1/q}$-plane*

$$D_{R'} = \{\beta \in \mathbf{C} : |\beta^{1/q} - (R/2)e^{i\pi/q}| \leq R/2\}$$

contained in the half-plane $-\frac{\pi}{2} + \frac{\pi}{q} < \arg \beta^{1/q} < \frac{\pi}{2} + \frac{\pi}{q}$, with radius $R/2$ and center at $C = (R/2)e^{i\pi/q}$.

Remark 3.12 Set $\beta' = \beta e^{-i\pi}$; then, by Theorem 3.11, $E_j(\beta)$ is analytic in the Nevanlinna disk

$$C_{R'} = \left\{ \beta \in \mathbf{C} : \mathrm{Re}(\beta')^{-1/q} \geq (R')^{-1} \right\}$$

of the $(\beta')^{1/q}$-plane.

Theorem 3.13 *For any $j \in \mathbf{N}$, the eigenvalue $E_j(\beta)$ of $H(\beta)$ is Borel summable in the ordinary sense for $0 < \arg\beta < \pi$ and in the distributional sense for $\arg\beta = 0$ and $\arg\beta = \pi$.*

Proof. We will examine only the "singular" cases $\arg\beta = 0, \pi$; the others can be treated in the standard way (see also [7] for $\pi/8 < \arg\beta < 7\pi/8$). Let us consider first the case $\arg\beta = 0$. Then Theorem 3.1 allows us to apply the criterion for the distributional Borel-Leroy sum of order q given in [8]. More precisely, the criterion requires the analyticity of $E_j(\beta)$ in a disk $C_R = \{\beta : \mathrm{Re}\beta^{-1/q} \geq R^{-1}\}$, as obtained in Theorem 3.1, and the well-known estimates for the remainders:

$$\left| E_j(\beta) - \sum_{s=0}^{N-1} a_s\beta^s \right| \leq A\sigma^N \Gamma(qN+1)|\beta|^N \,, \quad \forall N = 1, 2, \ldots \tag{3.19}$$

uniformly in $C_{R,\epsilon} = \{\beta \in C_R : \arg\beta^{1/q} \geq -\pi/2+\epsilon\}$, $\forall \epsilon > 0$, where the constants A and σ may depend on ϵ, and $\sum_{s=0}^{\infty} a_s\beta^s$ is the Rayleigh-Schrödinger perturbation expansion corresponding to $E_j(\beta)$ (see also [18],Vol.IV, for the standard proof of such estimates). As for the case $\arg\beta = \pi$, we first notice that (3.19) is known to hold uniformly in β in any sector

$$S(\delta) = \left\{ \beta \in \mathbf{C} : |\beta| < B(\delta), -\frac{\pi}{2} + \frac{\delta}{q} < \arg\beta^{1/q} < \frac{\pi}{2} + \frac{\pi}{q} - \frac{\delta}{q} \right\} .$$

Next observe that the direction $\arg\beta = \pi$ in the β-plane corresponds to the direction $\arg\beta' = 0$ in the β'-plane, $\beta' = \beta e^{-i\pi}$. Now, in analogy with [8] (Theorems 3 and 4), the criterion for the distributional Borel-Leroy summability of order q of $E_j(\beta)$ in the direction $\arg\beta = \pi$ can be stated in terms of the "adapted" variable β', in the sense that it relies on the following two conditions:

(1) $E_j(\beta)$ is analytic in

$$C_{R'} = \left\{ \beta \in \mathbf{C} : \mathrm{Re}(\beta')^{-1/q} \geq (R')^{-1} \right\} ;$$

(2) $\forall \epsilon > 0$, there exist A,$\sigma > 0$ such that

$$\left| F_j(\beta') - \sum_{s=0}^{N-1} (-1)^s a_s(\beta')^s \right| \leq A\sigma^N \Gamma(qN+1)|\beta'|^N \,, \quad \forall N = 1, 2, \ldots \tag{3.20}$$

uniformly in $C_{R',\epsilon} = \{\beta \in C_{R'} : \arg(\beta')^{1/q} \geq -\pi/2 + \epsilon\}$, where

$$F_j(\beta') \equiv \overline{E_j\left(\overline{\beta'} e^{-i\pi}\right)} = \overline{E_j(\overline{\beta})} \,.$$

Now, (1) is given in Remark 3.12 and (2) follows from the fact that the sector $S(\delta)$, where (3.19) holds uniformly, can be rewritten in terms of $(\beta')^{1/q}$ as

$$S(\delta)=\left\{\beta\in\mathbf{C}:|\beta|<B(\delta),-\frac{\pi}{2}-\frac{\pi}{q}+\frac{\delta}{q}<\arg\,(\beta')^{1/q}<\frac{\pi}{2}-\frac{\delta}{q}\right\}\,.$$

Indeed, since the coefficients a_s of the power series are real, (2) is equivalent to

$$\left|\overline{F_j(\overline{\beta'})}-\sum_{s=0}^{N-1}(-1)^s a_s(\beta')^s\right|\leq A\sigma^N\Gamma(qN+1)|\beta'|^N\ ,\quad \forall N=1,2,... \tag{3.21}$$

uniformly in $\overline{C}_{R',\epsilon}=\{\beta\in C_{R'}:\arg(\beta')^{1/q}\leq\pi/2-\epsilon\}$, where $\overline{F_j(\overline{\beta'})}=E_j(\beta)$.

Proof of Theorem 1.1 According to the terminology introduced in [8] about the distributional Borel summability, by (3.19) $E_j(\beta)$ represents the so-called "upper sum" and $\overline{E_j(\overline{\beta})}$ the "lower sum" for $\beta\in C_R$; conversely, by (3.20), $E_j(\beta)$ is the lower sum and $\overline{E_j(\overline{\beta})}$ the upper sum for $\beta\in C_{R'}$. More precisely, $E_j(\beta)$ admits for $\beta\in C_R$ the integral representation

$$E_j(\beta)=\frac{1}{q\beta}\int_0^\infty B_j(t+i0)e^{-(t/\beta)^{1/q}}\left(\frac{t}{\beta}\right)^{-1+1/q}dt \tag{3.22}$$

and the analogous representation holds for $\overline{E_j(\overline{\beta})}$ with $\overline{B(t+i0)}$ in place of $B(t+i0)$. For $\beta\in C_{R'}$ the representation analogous to (3.22) holds in terms of the adapted variable β', i.e.:

$$\overline{E_j(\overline{\beta})}=F_j(\beta')=\frac{1}{q\beta'}\int_0^\infty B_j(t+i0)e^{-(t/\beta')^{1/q}}\left(\frac{t}{\beta'}\right)^{-1+1/q}dt \tag{3.23}$$

because the odd terms in the power series are identically zero. The distributional Borel sum, which must be real for $\beta\in\mathbf{R}$ since the Rayleigh-Schrödinger perturbation series $\sum_{s=0}^{\infty}a_s\beta^s$ has real coefficients, is given by

$$f_j(\beta)=\frac{E_j(\beta)+\overline{E_j(\overline{\beta})}}{2}\,, \tag{3.24}$$

while the difference

$$d_j(\beta)\equiv 2ig_j(\beta)=\begin{cases}E_j(\beta)-\overline{E_j(\overline{\beta})}, & \beta\in C_R\\[2ex] \overline{E_j(\overline{\beta})}-E_j(\beta), & \beta\in C_{R'}\end{cases} \tag{3.25}$$

represents the so-called "discontinuity", which has zero asymptotic expansion. Now, if $\beta\in\mathbf{R}$, by (3.22) and (3.23) we have

$$E_j(-\beta)=\overline{E_j(\beta)}$$

since, once again, the perturbation series $\sum_{s=0}^{\infty} a_s\beta^s$ is such that $a_s = 0$ if s is odd, and therefore it can be written in the form $\sum_{l=0}^{\infty} a_{2l}\beta^{2l}$. It follows that $f_j(\beta) = f_j(-\beta)$ and $g_j(-\beta) = -g_j(\beta)$, i.e. $E_j(\beta)$ and $E_j(-\beta)$ have the same real part and opposite imaginary one. This concludes the proof of the theorem.

Remark 3.14 *1.* For $\beta \in \mathbf{R}$, it follows from (3.24) and (3.25) that

$$f_j(\beta) = \mathrm{Re}E_j(\beta), \quad d_j(\pm|\beta|) = \pm 2i\mathrm{Im}E_j(\pm|\beta|). \tag{3.26}$$

Since $E_j(\beta)$ can be interpreted as a resonance of the problem ([11]), $f_j(\beta)$ represents the position of the resonance and $|d_j(\beta)|/2$ its width. As in the Stark effect, the distributional Borel summability completely determines the resonance.

2. In the present case $f_j(\beta)$ and $d_j(\beta)$ admit a further interpretation, since by Remark 2.10, $\overline{E_j(\overline{\beta})} = E_j^1(\beta)$, where $E_j^1(\beta)$ represents the j-th eigenvalue of $H(\beta)$ for $\mathrm{Im}\beta < 0$. As proved for $E_j(\beta)$, $E_j^1(\beta)$ can be analytically continued to Nevanlinna disks analogous to C_R and $C_{R'}$ across the positive and negative real axis respectively. Thus,

$$f_j(\beta) = \frac{E_j(\beta) + E_j^1(\beta)}{2} \quad \text{and} \quad d_j(\beta) = \pm[E_j(\beta) - E_j^1(\beta)]\,,$$

where the $+$ holds for $\beta \in C_R$, and the $-$ for $\beta \in C_{R'}$.

3. As already recalled, the eigenvalues admit the classical Borel integral representation for $\pi/8 + \eta < \arg\beta < 7\pi/8 - \eta, \eta > 0$ ([7]). Formulas (3.22), (3.23) yield their explicit analytic continuation to the regions C_R and $C_{R'}$ across the real axis.

References

[1] G.Alvarez, Phys.Rev.A **37** (1988) 4079

[2] G.Alvarez, J.Phys.A: Math.Gen. **27** (1995) 4589

[3] C.M.Bender *et al*, Phys.Rev.Lett. **24** (1998) 5243

[4] C.M.Bender and G.V.Dunne, *Large-order Perturbation Theory for a Non-Hermitian PT-symmetric Hamiltonian*, quant-ph/9812039

[5] M.P.Blencowe, H.Jones and A.P.Korte, Phys.Rev.D **57** (1998) 5092

[6] M.Born: *Mechanics of the Atom*, Mac Millan (1960)

[7] E.Caliceti, S.Graffi and M.Maioli, Commun.Math.Phys. **75** (1980) 51

[8] E.Caliceti, V.Grecchi and M.Maioli, Commun.Math.Phys. **104** (1986) 163

[9] E.Caliceti, V.Grecchi and M.Maioli, Commun.Math.Phys. **157** (1993) 347

[10] E.Caliceti, V.Grecchi and M.Maioli, Commun.Math.Phys. **176** (1996) 1

[11] E.Caliceti and M.Maioli, Ann.Inst.H.Poincaré Sect.A **38** (1983) 175

[12] F.Cannata, G.Junker and J.Trost, Phys.Lett.A **246** (1998) 219

[13] E.Delabaere and F.Pham, Phys.Lett.A **250** (1998) 25

[14] E.Delabaere and F.Pham, Phys.Lett.A **250** (1998) 29

[15] A.Galindo and P.Pascual: *Quantum Mechanics.* Texts and Monographs in Physics. Berlin, Heidelberg, New York: Springer (1991)

[16] W.Hunziker and E.Vock, Commun.Math.Phys. **83** (1982) 281

[17] T.Kato: *Perturbation theory for linear operators*, Berlin, Heidelberg, New York: Springer (1966)

[18] M.Reed and B.Simon: *Methods of modern mathematical physics., II, IV*, New York: Academic Press (1978)

[19] B.Simon, Ann.Phys. **58** (1970) 76

[20] M.Znojil, *PT-symmetric harmonic oscillators*, quant-ph/9905020