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A ONE-PARAMETER FAMILY OF ANALYTIC MARKOV MAPS WITH AN INTERMITTENCY TRANSITION

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Abstract. In this paper we introduce and study a one-parameter family of piecewise analytic interval maps having the tent map and the Farey map as extrema. Among other things, we construct a Hilbert space of analytic functions left invariant by the Perron-Frobenius operator of all these maps and study the transition between discrete and continuous spectrum when approaching the intermittent situation.

1. **Introduction.** Expanding maps of the unit interval have been widely studied in the last decades in that several problems concerning their statistical behaviour can be treated by the powerful technique of transfer operators and thermodynamic formalism [Ba],[Co], [Ma2], [Rue1]. On the other hand, in recent years an increasing interest has been carried on maps which are expanding everywhere but on a marginally unstable fixed point in a neighbourhood of which trajectories are considerably slowed down leading to an interplay of chaotic and regular dynamics characteristic of intermittent systems [PM], [Sch]. Several approaches have been proposed to extend the above mentioned techniques to this situation, in particular to characterize the nature of possible phase transitions [P1], [PS] and that of the spectrum of the transfer operator [Rug], [Is]. In this paper we introduce a one-parameter family of piecewise analytic maps smoothly interpolating between the tent map and the Farey map and use it to investigate the passage between the uniformly expanding situation and the intermittent one in both perspectives: that of thermodynamics and that of spectral theory.The paper is organized as follows: Section 2 is devoted to introduce the model and derive some of its properties along with those of an induced version of it. In Section 3 we discuss large deviation properties and show in particular how the free energy gets non-analytic in the intermittent limit. Section 4 deals with the spectral analysis of the transfer, or Perron-Frobenius, operator. We construct a Hilbert space of analytic functions where this operator gets a particularly expressive integral representation (which becomes symmetric in the intermittent limit) and study the mechanism with which

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a continuous component in the spectrum (intermittent situation) comes out of a purely discrete spectrum (expanding situation). Finally we extend the above construction to study a family of operator-valued power series by which we obtain a characterization of the analytic properties of the dynamical zeta functions [Rue2] for both the original map and its induced version discussed in Section 2.

2. **Preliminaires.** Let $r \in [0,1]$ be a real parameter and consider the family of piecewise real-analytic maps F_r of the interval $[0,1]$ defined as

$$F_r(x) = \begin{cases} F_{r,0}(x), & \text{if } 0 \le x \le 1/2 \\ F_{r,1}(x), & \text{if } 1/2 < x \le 1 \end{cases} \tag{2.1}$$

where

$$F_{r,0}(x) = \frac{(2-r)x}{1-rx} \quad \text{and} \quad F_{r,1}(x) = F_{r,0}(1-x) = \frac{(2-r)(1-x)}{1-r+rx}\cdot \tag{2.2}$$

Some properties of this family are listed below:

1. For $r = 0$ we find the *tent map*

$$F_0(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1/2 \\ 2(1-x), & \text{if } 1/2 < x \le 1 \end{cases} \tag{2.3}$$

whereas for $r = 1$ one has the *Farey map*

$$F_1(x) = \begin{cases} \frac{x}{1-x}, & \text{if } 0 \le x \le 1/2 \\ \frac{1-x}{x}, & \text{if } 1/2 < x \le 1. \end{cases} \tag{2.4}$$

The latter provides an example of an *intermittent* map in that the fixed point at the origin is neutral (see below).

2. The left branch $F_{r,0}$ satisfies $F_{r,0}(0) = 0$, i.e. the origin is always a fixed point, and is conjugated to the map T_r defined as

$$T_r(x) = \frac{x-r}{2-r}\cdot \tag{2.5}$$

More precisely, we have, for all $n \ge 1$,

$$F^n_{r,0}(x) = J^{-1} \circ T^n_r \circ J(x) \tag{2.6}$$

with

$$J(x) = J^{-1}(x) = 1/x. \tag{2.7}$$

Notice that for $r = 1$ the map T_r becomes the left translation $x \to x - 1$. Moreover, for each $r \in [0,1]$ there is a unique point x_1 left fixed by $F_{r,1}$, i.e. $F_{r,1}(x_1) = x_1$ with

$$x_1 = \frac{\sqrt{9-4r} - (3-2r)}{2r}\,\cdot \tag{2.8}$$

3. The derivative is given by

$$F'_r(x) = \begin{cases} F'_{r,0}(x) = \frac{2-r}{(1-rx)^2}, & \text{if } 0 \le x \le 1/2 \\ F'_{r,1}(x) = -\frac{(2-r)}{(1-r+rx)^2}, & \text{if } 1/2 < x \le 1 \end{cases} \tag{2.9}$$

so that $F'_{r,0}(x) > 0$ and $F'_{r,1}(x) < 0$ for all $r \in [0,1]$. Therefore F_r is increasing on the interval $[0,1/2]$ and decreasing on $[1/2,1]$. In addition we have

$$\inf_{x\in[0,1]} |F'_r(x)| = F'_{r,0}(0) = -F'_{r,1}(1) = 2 - r =: \rho. \tag{2.10}$$

This means that for $r < 1$ the map F_r is uniformly expanding, i.e. $|F_r'| \geq \rho > 1$, thus providing an example of *analytic Markov map* [Ma2]. In particular we have

$$F'_{r,1}(x_1) = \frac{-4\rho}{\left(\sqrt{1+4\rho}-1\right)^2} = \frac{-\left(\sqrt{1+4\rho}+1\right)^2}{4\rho}\,. \tag{2.11}$$

On the contrary, for $r = 1$ one has $|F'(x)| > 1$ for $x > 0$ but $F'(0) = 1$.

2.1. **Inverse branches and renormalisation.** Set $\rho = 2 - r$ and

$$S_r(x) = T_r^{-1}(x) = \rho x + r \tag{2.12}$$

so that

$$S_0(x) = 2x \quad \text{and} \quad S_1(x) = x + 1. \tag{2.13}$$

The left inverse branch $\Phi_{r,0}$ of F_r is then given as

$$\Phi_{r,0}(x) = J^{-1} \circ S_r \circ J(x) = \frac{x}{\rho + rx} = \frac{1}{2} - \frac{1}{2}\left(\frac{\rho - \rho\, x}{\rho + rx}\right), \tag{2.14}$$

whereas the right inverse branch $\Phi_{r,1}$ is

$$\Phi_{r,1}(x) = 1 - \Phi_{r,0}(x) = 1 - \frac{x}{\rho + rx} = \frac{1}{2} + \frac{1}{2}\left(\frac{\rho - \rho\, x}{\rho + rx}\right)\cdot \tag{2.15}$$

Note that $\Phi_{r,0}(x)$ and $\Phi_{r,1}(x)$ are holomorphic in $H_{-\frac{\rho}{r}}$ and bounded in $H_{-\frac{\rho}{r}+\epsilon}$ for all $\epsilon > 0$, with

$$H_\alpha := \{x \in \mathbb{C} : \operatorname{Re} x > \alpha\}. \tag{2.16}$$

Eq. (2.14) allows us to write an explicit expression for the iterates $\Phi^n_{r,0}$ of $\Phi_{r,0}$.

Lemma 2.1.

$$\Phi^n_{r,0}(x) = \left(\frac{\rho^n}{x} + r\sum_{k=0}^{n-1}\rho^k\right)^{-1} \tag{2.17}$$

Proof. Reasoning inductively in n one obtains

$$S^n_r(x) = \rho^n x + r\sum_{k=0}^{n-1}\rho^k \tag{2.18}$$

and the claimed result follows upon applying (2.14). □

Note that (2.14) can be rewritten as

$$J(\Phi_{r,0}(x)) = S_r \circ J(x) = \rho\, J(x) + r \tag{2.19}$$

which can be viewed as a generalized Abel equation (see [deB]), to which it actually reduces when $r = 1$. This suggests that $\Phi_{r,0}$ satisfies some non-linear fixed point equation. Indeed, let $\mathcal{R}_r$ be the renormalisation operator acting as

$$\mathcal{R}_r\Psi(x) = \alpha\,\Psi\left(\Psi\left(\frac{x}{\beta}\right)\right), \quad \text{with} \quad \alpha = 3 - r \quad \text{and} \quad \beta = \frac{3-r}{2-r}\,\cdot \tag{2.20}$$

Proposition 2.1. *For all $r \in [0,1]$ the map $\Phi_{r,0}$ satisfies $\mathcal{R}_r\Phi_{r,0} = \Phi_{r,0}$ with boundary conditions $\Phi_{r,0}(0) = 0$ and $\Phi'_{r,0}(0) = (2-r)^{-1} = \rho^{-1}$.*

Proof. We have

$$
\begin{aligned}
\mathcal{R}_r\Phi_{r,0}(x) &= \alpha\,\Phi_{r,0}\left(\Phi_{r,0}\left(\frac{x}{\beta}\right)\right) = \alpha\,\Phi_{r,0}\left(\frac{\frac{x}{\beta}}{\rho+\frac{rx}{\beta}}\right)\\
&= \alpha\,\Phi_{r,0}\left(\frac{x}{\alpha+rx}\right) = \alpha\,\frac{\frac{x}{\alpha+rx}}{\rho+\frac{x}{\alpha+rx}}\\
&= \alpha\,\frac{x}{\rho(\alpha+rx)+rx} = \frac{x}{\rho+rx} = \Phi_{r,0}(x)\cdot
\end{aligned}
$$

□

Remark 1. *For $r=1$ the scaling factors α and β in (2.20) are both equal to* 2 *and the function $\Phi_{0,1}$ is but the fixed point of the Feigenbaum renormalisation equation with intermittency boundary conditions obtained in* [HR].

2.2. **Invariant measure, induced map, characteristic exponent.** We let $\mathcal{P}$ denote the Perron-Frobenius, or transfer operator associated to the map F_r (see [Ba]). It acts on a function $f:[0,1]\to\mathbb{C}$ as

$$
\mathcal{P}f(x) = \sum_{y\,:\,F_r(y)=x} \frac{f(y)}{|F_r'(y)|}, \tag{2.21}
$$

or, more explicitly,

$$
\mathcal{P}f(x) = \frac{\rho}{(\rho+rx)^2}\left[f\left(\frac{x}{\rho+rx}\right)+f\left(1-\frac{x}{\rho+rx}\right)\right]\cdot \tag{2.22}
$$

A fundamental property of this operator is that if there is a measurable function f which satisfies the fixed point equation

$$
\mathcal{P}f(x) = f(x) \tag{2.23}
$$

then f is the density of an absolutely continuous measure ν_r on $[0,1]$ which is F_r-invariant, that is $\nu_r(E)=\nu_r(F_r^{-1}E)$ for all measurable $E\subseteq[0,1]$.

Theorem 2.1. *The function $e_r(x)=K_r/(1-r+rx)$ where K_r is a given positive constant, is a solution of equation (2.23) and thus represents the density of an absolutely continuous F_r-invariant measure $d\nu_r(x)=e_r(x)dx$.*

Proof. By virtue of (2.22) we have

$$
\begin{aligned}
\mathcal{P}\,e_r(x) &= \frac{K_r\,\rho}{(\rho+rx)^2}\left[\frac{1}{1-r+\frac{rx}{\rho+rx}}+\frac{1}{1-\frac{rx}{\rho+rx}}\right]\\
&= \frac{K_r\,\rho}{(\rho+rx)^2}\left[\frac{\rho+rx}{\rho+rx-2r+r^2-r^2x+rx}+\frac{\rho+rx}{\rho}\right]\\
&= \frac{K_r}{(\rho+rx)}\left[\frac{\rho}{(1-r)\rho+r\rho x}+1\right] = \frac{K_r}{1-r+rx}\cdot
\end{aligned}
$$

□

Notice that

$$
\nu_r([0,1]) = \frac{K_r}{r}\log\left(\frac{1}{1-r}\right) \tag{2.24}
$$

and therefore $\nu_r([0,1])\to\infty$ when $r\nearrow 1$. In order to compare the F_r-invariant measure ν_r with a probability measure μ_r invariant w.r.t. to an induced map G_r

to be defined below, we shall choose the value of K_r so that $\nu_r([1/2,1]) = 1$. This renders

$$K_r = \frac{r}{\log 2 - \log \rho}\cdot \tag{2.25}$$

In particular we have

$$\lim_{r\searrow 0} K_r = 2 \qquad \text{and} \qquad \lim_{r\nearrow 1} K_r = \frac{1}{\log 2}\cdot \tag{2.26}$$

Let $\mathcal{A}_r = \{A_n\}_{n\in\mathbb{N}}$ be the countable partition of $[0,1]$ whose elements are the intervals $A_n = [c_n, c_{n-1}]$ with

$$c_0 = 1 \quad \text{and} \quad c_n = \Phi^n_{r,0}(1), \quad n \geq 1. \tag{2.27}$$

As a corollary of Lemma 2.1 we have the explicit expression

$$c_n = \frac{1-r}{\rho^n - r}\cdot \tag{2.28}$$

and in particular

$$\lim_{r\searrow 0} c_n = 2^{-n} \qquad \text{and} \qquad \lim_{r\nearrow 1} c_n = \frac{1}{n}\cdot \tag{2.29}$$

Set $X = (0,1]\setminus\{c_n\}_{n\in\mathbb{N}}$ and let $\tau : X \to \mathbb{N}$ be the *first passage time* in the interval A_1, that is

$$\tau(x) = 1 + \min\{n \geq 0 \ : \ F^n_r(x) \in A_1 \ \}, \tag{2.30}$$

so that A_n is the closure of the set $\{x \in X \ : \ \tau(x) = n\}$. On the other hand, the *return time* function $\ell : X \to \mathbb{N}$ in the interval A_1 is given by

$$\ell(x) = \min\{n \geq 1 \ : \ F^n_r(x) \in A_1 \ \} = \tau \circ F_r(x). \tag{2.31}$$

We now prove the following version of Kac's formula: the ν_r-measure of the whole interval $[0,1]$, that is (2.24), equals the conditional expectation of the function ℓ on the interval A_1 (recall that we have set $\nu_r(A_1) = 1$).

Lemma 2.2.

$$\int_{A_1} \ell(x)\nu_r(dx) = \nu_r([0,1]) = \frac{1}{\log 2 - \log \rho} \log\left(\frac{1}{1-r}\right) \tag{2.32}$$

Proof. Let $B_n = \overline{\{x \in A_1 \ : \ \ell(x) = n\}}$. Using (2.31) we have that $A_n = F_r(B_n)$. Let us show that $\nu_r(A_n) = \sum_{k\geq n} \nu_r(B_k)$. Indeed, for $n = 1$ we have $1 = \nu_r(A_1) = \sum_{k\geq 1} \nu_r(B_k)$. Moreover, since ν_r is F_r-invariant, $\nu_r(A_n) = \nu_r(F_r^{-1}A_n) = \nu_r(A_{n+1}) + \nu_r(B_{n+1})$ and the claim follows by induction. Therefore,

$$\nu_r([0,1]) = \sum_{n\geq 1} \nu_r(A_n) = \sum_{n\geq 1} n \cdot \nu_r(B_n) = \int_{A_1} \ell(x)\nu_r(dx),$$

and the last identity in (2.32) follows from (2.24). $\square$

Remark 2. *As already remarked, the one-parameter family F_r is well suited to study the transition from a strongly chaotic behaviour, corresponding to the uniformly expanding situation with $r < 1$, to an intermittent behaviour, corresponding to the the tangent bifurcation point at $r = 1$. One interesting item in this study is the divergence type of the average duration $< \ell >$ of the* laminar regime *as $r \nearrow 1$ (see, e.g.,* [Sch]*). In our situation this is nothing but the expectation of the return time function ℓ, and by formula (2.32) it diverges logarithmically:*

$$< \ell > \sim \frac{\log(1/\delta)}{\log 2} \quad \textit{as} \quad \delta \equiv 1 - r \searrow 0. \tag{2.33}$$

Now, using (2.31) we can express the expected return time $<\ell>$ as the expected first passage time w.r.t. an absolutely continuous probability measure μ_r obtained by pushing forward ν_r with $F_{r,1}$, i.e.

$$<\ell>=\int_0^1 \tau(x)\,\mu_r(dx) \tag{2.34}$$

where

$$\mu_r(E)=((F_{r,1})_*\,\nu_r)(E)=(\nu_r\circ\Phi_{r,1})(E). \tag{2.35}$$

Reasoning as in the proof of Lemma 2.2 one readily verifies that the converse relation is

$$\nu_r(E)=\sum_{n\geq 0}(\mu_r\circ\Phi^n_{r,0})(E). \tag{2.36}$$

In particular we have $\nu_r(A_n)=\sum_{l\geq n}\mu_r(A_l)$ and $\mu_r(A_n)=\mu_r(F_{r,1}(B_n))=\nu_r(B_n)$, where B_n is as in the proof of Lemma 2.2. The measure μ_r will play the role of reference probability measure in our construction. If we set $h_r(x)=\mu_r(dx)/dx$ then by the foregoing we have

$$h_r=|\Phi'_{r,1}|\cdot e_r\circ\Phi_{r,1},\qquad e_r=\sum_{k=0}^{\infty}(\Phi^k_{r,0})'\cdot h_r\circ\Phi^k_{r,0}\,. \tag{2.37}$$

Using the explicit expressions for e_r and $\Phi_{r,1}$ we get

$$h_r(x)=\frac{K_r}{2-r+rx}, \tag{2.38}$$

which is monotone non-increasing with $h_r(0)=K_r/\rho$ and $h_r(1)=K_r/2$.

The induced map. The measure μ_r is left invariant by a map G_r obtained from F_r by inducing w.r.t. the first passage time τ. Indeed, by the above we can write

$$\mu_r(E)=(\nu_r\circ\Phi_{r,1})(E)=\sum_{n\geq 0}(\mu_r\circ\Phi^n_{r,0}\circ\Phi_{r,1})(E)=\rho(G_r^{-1}E) \tag{2.39}$$

where $G_r:X\to X$ denotes the map:

$$x\to G_r(x)=F_r^{\tau(x)}(x), \tag{2.40}$$

which can be extended to all of $[0,1]$ as

$$G_r(x)=G_{r,n}(x)=F_r^n(x)=F_{r,1}\circ F^{n-1}_{r,0}(x)\quad\text{if}\quad x\in\overset{\circ}{A}_n\quad\text{for all}\quad n\geq 1, \tag{2.41}$$

$G_r(0)=G_r(1)=1$ and

$$\lim_{x\nearrow c_{r,n}}G_r(x)=1,\qquad \lim_{x\searrow c_{r,n}}G_r(x)=0,\quad n\geq 1. \tag{2.42}$$

The explicit expression for $G_{r,n}$ can be easily obtained from that of F_r and reads

$$G_{r,n}(x)=\frac{\rho}{1-r+rx}\left(1-\frac{x}{c_{n-1}}\right), \tag{2.43}$$

and

$$G'_{r,n}(x)=-\frac{\rho\,(1-r+rc_{n-1})}{c_{n-1}(1-r+rx)^2}, \tag{2.44}$$

with c_n as in (2.28) and $r\in[0,1]$.

Example. The induced maps corresponding to the maps F_0 and F_1 are the map $G_0(x)$ such that $G_{0,n}(x)=2(1-2^{n-1}x)$ and the Gauss map $G_1(x)=\frac{1}{x}\,(\mathrm{mod}\,1)$, respectively. Their invariant densities are $h_0(x)=1$ and $h_1(x)=\frac{1}{\log 2}\cdot\frac{1}{(1+x)}\,$.

We now list some properties of G_r which are relevant for our discussion.

Proposition 2.2.

1. *smoothness property:* $G_{r,n}$ *is a real analytic diffeomorphism of* A_n *onto* $[0,1]$*;*
2. *expanding property: for all* $r \in [0,1]$ $\inf_{x\in[0,1]} |(G_r^2)'(x)| = |(G_r^2)'(1/2)| = 4$;
3. *distortion property:* $\sup_{\substack{x,y,z\in\mathcal{A}_n \\ n\geq 1}} \left|\frac{G_r''(x)}{G_r'(y)G_r'(z)}\right| = L < \infty$.

Proof. Statement 1) is an immediate consequence of the definition. As for 2) notice that for $r < 1$ we have $\inf_{x\in[0,1]} |(G_r)'(x)| = |G_r'(1)| = \rho > 1$, whereas $\inf_{x\in[0,1]} |(G_1)'(x)| = |G_1'(1)| = 1$. Therefore, since G_r' is monotone decreasing we have $\inf_{x\in[0,1]} |(G_r^2)'(x)| = |(G_r^2)'(1/2)| = |G_r'(1/2)\cdot G_r'(1)| = \frac{4}{\rho}\cdot\rho = 4$. To show 3), we first observe that the chain rule yields

$$\frac{G_r''(x)}{(G_r')^2(x)} = \sum_{k=0}^{\tau(x)-1} \frac{F_r''(F_r^k(x))}{(F_r')^2(F_r^k(x))} \cdot \frac{1}{\prod_{j=k+1}^{\tau(x)-1} F_r'(F_r^j(x))} \cdot \tag{2.45}$$

On the other hand, one can easily find a positive constant C_1 such that

$$\sup_{x\in[0,1]} \frac{|F_r''(x)|}{|(F_r')^2(x)|} \leq C_1. \tag{2.46}$$

Moreover by an easy estimate using (2.44) one can find a constant $C_2 \geq 1$ so that $C_2^{-1}\, G_r'(y) \leq G_r'(x) \leq C_2\, G_r'(y)$ for any choice of $x, y \in A_n$ and any $n \geq 1$. Hence, by the mean value theorem $\prod_{j=0}^{n-1} |F_r'(F_r^j(x))| \equiv |G_{r,n}'(x)| \geq C_2^{-1}\, |A_n|^{-1}$ whenever $x \in A_n$. The assertion now follows putting together the above inequalities. $\square$

These properties yield a uniform bound for the buildup of non-linearity in the induction process.

Corollary 2.1. *Let* $x, y \in [0,1]$ *be such that* $G_r^j(x)$ *and* $G_r^j(y)$ *belong to the same atom* A_{k_j}*, for* $0 \leq j \leq n$ *and some* $n \geq 1$*. There is a constant* $C > 0$*, independent of* r*, such that*

$$\left|\log \frac{G_r'(x)}{G_r'(y)}\right| \leq C\, \left(\frac{1}{2}\right)^n \cdot$$

Proof. Taking $x, y \in A_{k_0}$, let $\eta \in A_{k_0}$ be such that $|G_r'(\eta)| = |A_{k_0}|^{-1}$. Then using the distortion property listed above we have

$$\begin{aligned} \left|\log \frac{G_r'(x)}{G_r'(y)}\right| &= \left|\frac{G_r''(\xi)}{G_r'(\xi)}\right| \cdot |x-y| \quad \text{for some} \quad \xi \in [x,y] \subseteq A_{k_0} \\ &= \left|\frac{G_r''(\xi)}{G_r'(\xi)G_r'(\eta)}\right| \cdot \frac{|x-y|}{|A_{k_0}|} \leq L\, \frac{|x-y|}{|A_{k_0}|} \cdot \end{aligned}$$

Now, by the expanding property we can find a constant $C > 0$ such that, under the above hypotheses, $|x-y| \leq C\, L^{-1} |A_{k_0}|\, \beta^n$ with $\beta = 4^{-\frac{1}{2}} = 1/2$. $\square$

Putting together the above and ([Wal], Theorem 22(3)) we have the following

Proposition 2.3. *The probability measure* μ_r *is the unique absolutely continuous invariant measure for the dynamical system* $([0,1], G_r)$*. Moreover* (G_r, μ_r) *is an exact endomorphism.*

Characteristic exponents. Finally we show that the measures ν_r and μ_r have the same characteristic exponent. Set

$$\chi_{\nu_r} = \int_0^1 \log|F_r'(x)|\,\nu_r(dx) \quad \text{and} \quad \chi_{\mu_r} = \int_0^1 \log|G_r'(x)|\,\mu_r(dx). \tag{2.47}$$

Then we have

Proposition 2.4. *For all $r \in (0,1)$ we have*

$$\begin{aligned}\chi_{\nu_r} = \chi_{\mu_r} = \frac{\log(2-r)\,\log(\frac{1}{1-r})}{\log(\frac{2}{2-r})} &- \log(4-2r)\\ &- \frac{1}{\log(\frac{2}{2-r})}\left[\frac{\pi^2}{6} - \log^2 2 - 2\,\mathrm{Li}_2\left(\frac{1}{2-r}\right)\right]\end{aligned}$$

which involves the dilogarithm function $\mathrm{Li}_2(q) = \sum_{k=1}^\infty \frac{q^n}{k^2}$. In particular,

$$\lim_{r\searrow 0} \chi_{\nu_r} = 2\log 2, \qquad \lim_{r\nearrow 1} \chi_{\nu_r} = \frac{\pi^2}{6\log 2}.$$

Proof. To prove the first identity we write, using (2.37),

$$\begin{aligned}&\int_0^1 \log|F_r'(x)|\,\nu_r(dx) = \int_0^1 \log|F_r'(x)|\,e_r(x)\,dx\\ =\ &\int_0^1 \log|F_r'(x)| \sum_{k=0}^\infty h(\Phi_{r,0}^k(x))\cdot(\Phi_{r,0}^k)'(x)\,dx = \sum_{k=0}^\infty \int_0^{c_k} \log|F_r'(F_{r,0}^k(x))|\,h_r(x)\,dx\\ =\ &\sum_{k=1}^\infty \int_{A_k} \prod_{j=0}^{k-1} \log|F_r'(F_{r,0}^j(x))|\,h_r(x)\,dx = \sum_{k=1}^\infty \int_{A_k} \log|G_{r,k}'(x)|\,h_r(x)\,dx\\ =\ &\int_0^1 \log|G_r'(x)|\,h_r(x)\,dx = \int_0^1 \log|G_r'(x)|\,\mu_r(dx).\end{aligned}$$

For the second identity we have, using (2.9):

$$\begin{aligned}\chi_{\nu_r} &= K_r\left[\int_0^{\frac12}\left(\log\frac{\rho}{(1-rx)^2}\right)\frac{dx}{1-r+rx} + \right.\\ &+ \left.\int_{\frac12}^1\left(\log\frac{\rho}{(1-r+rx)^2}\right)\frac{dx}{1-r+rx}\right] = \log\rho\cdot\nu_r([0,1]) +\\ &- 2K_r\left[\int_0^{\frac12}\frac{\log(1-rx)}{1-r+rx}\,dx + \int_{\frac12}^1 \frac{\log(1-r+rx)}{1-r+rx}\,dx\right].\end{aligned}$$

By (2.32) the first term in the r.h.s equals the first term in the r.h.s. of (2.48). Notice that this term has limits $2\log 2$ and 0 when $r \searrow 0$ and $r \nearrow 1$, respectively. Furthermore, we have

$$2K_r \int_{\frac12}^1 \frac{\log(1-r+rx)}{1-r+rx}\,dx = -\frac{K_r}{r}\log^2\left(\frac{2-r}{2}\right) = \log\left(\frac{2-r}{2}\right),$$

and

$$2K_r \int_0^{\frac{1}{2}} \frac{\log(1-rx)}{1-r+rx}\, dx =$$
$$= \frac{1}{\log\left(\frac{2}{2-r}\right)} \left[\frac{\pi^2}{6} - \log^2 2 - 2\log 2 \log\left(\frac{2-r}{2}\right) - 2\int_{\frac{1}{2-r}}^{0} \frac{\log(1-t)}{t} dt \right].$$

The claimed formula now follows by putting together the above expressions and noting that

$$\int_q^0 \frac{\log(1-t)}{t} dt = \mathrm{Li}_2(q)\,.$$

□

3. **Free energy and large deviations.** For $r \in [0,1)$ we shall consider the F_r-invariant probability measure p_r as well as its characteristic (or Lyapunov) exponent λ_r given by

$$p_r(\,\cdot\,) = \frac{\nu_r(\,\cdot\,)}{\nu_r([0,1])} \quad \text{and} \quad \lambda_r = \frac{\chi_{\nu_r}}{\nu_r([0,1])} \tag{3.48}$$

respectively. Set moreover

$$u(x) := \log|F_r'(x)| - \lambda_r, \tag{3.49}$$

and

$$S_n(x) = \sum_{i=0}^{n-1} u(F_r^i(x)) = \log|(F_r^n)'(x)| - n\,\lambda_r. \tag{3.50}$$

For $\beta \in I\!R$ and $n \geq 1$ we may then define the *partition function* $Z_n(\beta)$ as

$$Z_n(\beta) = \int_0^1 |(F_r^n)'(x)|^\beta\, p_r(dx) = e^{n\beta\lambda_r} \int_0^1 e^{\beta\, S_n(x)}\, p_r(dx) \tag{3.51}$$

and consider the sequence of functions

$$f_n(\beta) = \frac{1}{n}\log Z_n(\beta) = \lambda_r\,\beta + \frac{1}{n}\log\int_0^1 e^{\beta\, S_n(x)}\, p_r(dx). \tag{3.52}$$

The limit function

$$f(\beta) = \lim_{n\to\infty} f_n(\beta) \tag{3.53}$$

is called *free energy function* of the characteristic exponent (see [BR], [Co], [D]). Notice that $f(0) = 0$. If we set

$$\langle A\rangle_\beta := \int_0^1 A(x) \frac{e^{\beta\, S_n(x)}}{\int_0^1 e^{\beta\, S_n(x)}\, p_r(dx)}\, p_r(dx) \tag{3.54}$$

so that in particular

$$\langle A\rangle_0 = \int_0^1 A(x)\, p_r(dx), \tag{3.55}$$

then we get

$$f_n'(\beta) = \lambda_r + \frac{1}{n}\langle S_n\rangle_\beta \quad \text{and} \quad f_n''(\beta) = \frac{1}{n}\left[\,\langle S_n^2\rangle_\beta - \langle S_n\rangle_\beta^2\,\right]. \tag{3.56}$$

Now, for each $r<1$ the transformation F_r is a uniformly expanding Markov map and using standard arguments one sees that the sequence $\{f_n''(\beta)\,:\, n \geq 1\}$ is uniformly bounded on compact sets. This entails that the free energy function $f(\beta)$ is convex and C^1 with $f'(\beta)$ strictly increasing and given by

$$f'(\beta) = \lambda_r + \lim_{n\to\infty} \frac{1}{n}\langle S_n\rangle_\beta, \qquad \beta \in I\!R. \tag{3.57}$$

In particular we have

$$f'(\beta)|_{\beta=0} = \lambda_r \cdot \tag{3.58}$$

One finds moreover that the function $\beta \to \beta\, f'(\beta) - f(\beta)$ is decreasing for $\beta < 0$ and increasing for $\beta > 0$. This relates $f(\beta)$ to large deviation properties of the sequence of random variables $S_n(x)$. To see this, notice that $\int_0^1 S_n(x)\, p_r(dx) = 0$ for all $n \geq 1$. The ergodic theorem then yields

$$p_r \left(\left\{ x \in [0,1] \, : \, \lim_{n\to\infty} \frac{S_n(x)}{n} = 0 \right\} \right) = 1 \tag{3.59}$$

and therefore for each fixed $\alpha > 0$ we have

$$\lim_{n\to\infty} p_r\, (\{x \in [0,1] \, : \, S_n(x) \geq n\alpha\}) = 0. \tag{3.60}$$

For each $r < 1$ the dynamical system $([0,1], F_r, p_r)$ satisfies the assumptions of the large deviation theorem which says that (see [Co], [D])

$$\lim_{n\to\infty} \frac{1}{n} \log p_r\, (\{S_n \geq n f'(\beta) - n\lambda_r\}) = \beta\, f'(\beta) - f(\beta) \qquad (\beta \geq 0) \tag{3.61}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log p_r\, (\{S_n \leq n f'(\beta) - n\lambda_r\}) = \beta\, f'(\beta) - f(\beta) \qquad (\beta \leq 0). \tag{3.62}$$

In other words, the probability of finding a deviation of S_n/n from its average value 0 decays exponentially with n. Notice that if we set $\alpha = f'(\beta) - \lambda_r$ then the r.h.s. in (3.61) and (3.62) can be viewed as a Legendre transform

$$\phi(\alpha) = \beta\,\alpha - (f(\beta) - \beta\lambda_r), \qquad \alpha = f'(\beta) - \lambda_r. \tag{3.63}$$

We now derive upper and lower bounds for the free energy function. First, from $|F_r'| \geq \rho > 1$ it follows immediately that for $\beta \leq 0$ we have $f(\beta) \leq \beta \log \rho$, the opposite inequality being valid for positive values of β. In addition, using either the convexity of $f(\beta)$ or directly (3.52) we get the inequality $f(\beta) \geq \lambda_r\, \beta$, which is valid for all $\beta \in I\!R$. Whence, expressing λ_r by means of Lemma 2.2 and Proposition 2.4 we get

Lemma 3.3. *For all $\beta \leq 0$ and $r \in [0,1)$ we have*

$$\beta \log \rho \geq f(\beta) \geq \beta \log \rho + \beta\, \gamma_r$$

where $\gamma_r \geq 0$ is given by

$$\gamma_r = \frac{1}{\log(1-r)} \left[\frac{\pi^2}{6} - 2\,\mathrm{Li}_2 \left(\frac{1}{2-r} \right) + \log\left(\frac{2}{2-r} \right) \log(4-2r) - \log^2 2 \right].$$

Notice that $\lim_{r\searrow 0} \log \rho = \log 2$ whereas $\lim_{r\nearrow 1} \log \rho = 0$, moreover $\lim_{r\searrow 0} \gamma_r = 0$ and $\lim_{r\nearrow 1} \gamma_r = 0$.
Therefore from Lemma 3.3 we obtain the

Corollary 3.2. $\lim_{r\searrow 0} f(\beta) = \beta\, \log 2, \qquad \lim_{r\nearrow 1} f(\beta) = 0, \qquad (\beta \leq 0).$

Remark 3. *This result shows that for $r = 1$ the free energy has a discontinuity in its first derivative at $\beta = 0$. This can be interpreted in thermodynamic language as a second order phase-transition* [P1], [FKO].

4. **Transfer operators.** The transfer operator $\mathcal{P}$ associated to the map F_r has already been introduced in (2.22) and we write

$$\mathcal{P}f(x) = (\mathcal{P}_0 + \mathcal{P}_1)f(x) \tag{4.64}$$

with (recall that $\rho \equiv 2 - r$)

$$\mathcal{P}_0 f(x) = \frac{\rho}{(\rho + rx)^2} \cdot f\left(\frac{x}{\rho + rx}\right) \quad \text{and} \quad \mathcal{P}_1 f(x) = \frac{\rho}{(\rho + rx)^2} \cdot f\left(1 - \frac{x}{\rho + rx}\right). \tag{4.65}$$

Several interesting properties of the dynamics generated by F_r are intimately related to $\mathrm{sp}\,(\mathcal{P})$, the spectrum of $\mathcal{P}$ (see [Ba]). However, the latter depends crucially on the function space $\mathcal{P}$ is acting on, which is in general a Banach space. For smooth uniformly expanding maps and Banach spaces of sufficiently regular functions, e.g. the space $\mathcal{C}^k$ of k-times differentiable functions on $[0,1]$ with $k \geq 0$, the transfer operator is *quasi-compact*. This means that $\mathrm{sp}\,(\mathcal{P})$ is made out of a finite or at most countable set of isolated eigenvalues with finite multiplicity (the discrete spectrum) and its complementary, the essential spectrum. It has been proved in [CI] that for piecewise $\mathcal{C}^\infty$ expanding Markov maps of the unit interval the essential spectrum of $\mathcal{P}$ when acting on $\mathcal{C}^k$ is a disk of radius

$$r_{\mathrm{ess}}(\mathcal{P}) \;=\; \exp f(-k)\,, \tag{4.66}$$

where $f(\beta)$ is the free energy function discussed in the previous Section. Putting together this result and Lemma 3.3 we obtain the following

Theorem 4.2. *For $r \in [0,1)$ the essential spectrum of $\mathcal{P} : \mathcal{C}^k \to \mathcal{C}^k$ with $k \geq 0$ is a disk of radius*

$$e^{-k\,(\log\rho + \gamma_r)} \;\leq\; r_{\mathrm{ess}}(\mathcal{P}) \;\leq\; e^{-k\,\log\rho}\,.$$

The above bounds along with standard arguments [K] yield the following

Corollary 4.3. *For $r = 1$ and for each fixed $k \geq 0$ the essential spectrum of $\mathcal{P} : \mathcal{C}^k \to \mathcal{C}^k$ is the unit disk.*

4.1. **An invariant Hilbert space.** From the above discussion it follows that if we want to understand the nature of the spectrum lying under the 'essential spectrum rug' we have to let $\mathcal{P}$ acting on increasingly smooth test functions as r approaches 1. In particular, Corollary 4.3 suggests that for $r = 1$ one should consider suitable spaces of analytic functions. In the following definition we shall introduce a Hilbert space of analytic functions which will be shown to be invariant under $\mathcal{P}$ for each $r \in [0,1]$.

Definition 4.1. *We denote by $\mathcal{H}$ the Hilbert space of all complex-valued functions f which can be represented as a generalized Borel transform*

$$f(x) = (\mathcal{B}\,[\varphi])(x) := \frac{1}{x^2}\int_0^\infty e^{-\frac{t}{x}}\,e^t \varphi(t)\,dm(t), \quad \varphi \in L^2(m), \tag{4.67}$$

with inner product

$$(f_1, f_2) = \int_0^\infty \varphi_1(t)\,\overline{\varphi_2(t)}\,dm(t) \quad \text{if} \quad f_i = \mathcal{B}\,[\varphi_i], \tag{4.68}$$

and measure

$$dm(t) = t\,e^{-t}\,dt. \tag{4.69}$$

Remark 4. *An alternative representation can be obtained by a simple change of variable when x is real and positive:*

$$f(x) = \frac{1}{x}\int_0^\infty ds\, e^{-s}\,\psi(sx) \quad \textit{with} \quad \psi(t) = t\,\varphi(t)\,. \tag{4.70}$$

Note that a function $f \in \mathcal{H}$ is analytic in the disk (here $z = x + iy$)

$$D_1 = \{z \in \mathbb{C} : \mathrm{Re}\,\frac{1}{z} > \frac{1}{2}\} = \{z \in \mathbb{C} : |z-1| < 1\}. \tag{4.71}$$

A Hilbert space identical to $\mathcal{H}$ apart from a slightly different choice of the measure m was introduced in [Is] *to study the operator $\mathcal{P}$ for $r = 1$ (the Farey map), whereas a generalized version of $\mathcal{H}$ has been used by Prellberg in* [P2] *to study the spectrum of the operator $\mathcal{P}_\beta f(x) = \left(\frac{1}{1+x}\right)^{2\beta}\left[f\left(\frac{x}{1+x}\right) + f\left(\frac{1}{1+x}\right)\right]$ (thus again for the case $r = 1$).*

Remark 5. *The invariant densities $e_r(x) = K_r/(\delta + rx)$ and $h_r(x) = K_r/(\rho + rx)$ can be represented as*

$$e_r = \mathcal{B}[\phi_r] \quad \textit{and} \quad h_r = \mathcal{B}[\psi_r] \tag{4.72}$$

with

$$\phi_r(t) = \frac{K_r}{r}\left(\frac{1 - e^{-\frac{r}{\delta}t}}{t}\right) \quad \textit{and} \quad \psi_r(t) = \frac{K_r}{r}\left(\frac{1 - e^{-\frac{r}{\rho}t}}{t}\right), \tag{4.73}$$

respectively. For the limiting values $r = 1$ and $r = 0$ we get

$$\phi_1(t) \;=\; \frac{1}{t\,\log 2}, \qquad \psi_1(t) \;=\; \frac{1 - e^{-t}}{t\,\log 2}\,, \tag{4.74}$$

and

$$\phi_0(t) = \lim_{r\searrow 0}\phi_r(t) \;=\; 2, \qquad \psi_0(t) = \lim_{r\searrow 0}\psi_r(t) \;=\; 1\,. \tag{4.75}$$

We point out that ϕ_1 is not in $L^2(m)$.

Lemma 4.4. *The space $\mathcal{H}$ is invariant for $\mathcal{P}_0$ and we have*

$$\mathcal{P}_0\mathcal{B}\,[\varphi] = \mathcal{B}\,[M_r\varphi]\,, \tag{4.76}$$

where $M_r : L^2(m) \to L^2(m)$ is defined as

$$M_r\varphi(t) = \frac{1}{\rho}\,e^{-\frac{r}{\rho}t}\,\varphi\left(\frac{t}{\rho}\right). \tag{4.77}$$

Proof.

$$\begin{aligned}
(\mathcal{P}_0\mathcal{B}\,[\varphi])(x) \quad &= \quad \frac{\rho}{(\rho+rx)^2}\mathcal{B}\,[\varphi]\left(\frac{x}{\rho+rx}\right) = \frac{\rho}{x^2}\int_0^\infty e^{-\frac{\rho+rx}{x}t}\,\varphi(t)\,t\,dt \\
&= \quad \frac{1}{x^2}\int_0^\infty e^{-\frac{s}{x}}\,\frac{1}{\rho}\,e^{-\frac{r}{\rho}s}\,\varphi\left(\frac{s}{\rho}\right)\,s\,ds \\
&= \quad \frac{1}{x^2}\int_0^\infty e^{-\frac{s}{x}}\,e^{s}\,(M_r\varphi)\,(s)\,dm(s) = (\mathcal{B}\,[M_r\varphi])(x)\,.
\end{aligned}$$

□

The following lemma will instead specify the action of $\mathcal{P}_1$ on $\mathcal{H}$.

Lemma 4.5. *We have*

$$\mathcal{P}_1\mathcal{B}\,[\varphi] = \mathcal{B}\,[N_r\varphi]\,, \tag{4.78}$$

where $N_r : L^2(m) \to L^2(m)$ *is the operator acting as*

$$N_r\varphi(t) = \frac{1}{\rho}\, e^{\frac{\delta}{\rho}t} \int_0^\infty \frac{J_1\left(2\sqrt{st/\rho}\right)}{\sqrt{st/\rho}}\, \varphi(s)\, dm(s) \tag{4.79}$$

where J_p *denotes the Bessel function of order* p.

Proof.

$$\begin{aligned}
(\mathcal{P}_1\mathcal{B}\,[\varphi])(x) &= \frac{\rho}{(\rho+rx)^2}\,\mathcal{B}\,[\varphi]\left(1-\frac{x}{\rho+rx}\right) = \frac{\rho}{(\rho-\delta\, x)^2}\int_0^\infty e^{-\frac{\rho+rx}{\rho-\delta x}t}\,\varphi(t)\,t\,dt \\
&= \frac{\rho}{(\rho-\delta\, x)^2}\int_0^\infty e^{-\frac{\rho-\delta x+\delta x+rx}{\rho-\delta x}t}\varphi(t)\,t\,dt \\
&= \frac{\rho}{x^2}\frac{1}{\left(\frac{\rho}{x}-\delta\right)^2}\int_0^\infty e^{-\frac{t}{\frac{\rho}{x}-\delta}}\,e^{-t}\varphi(t)\,t\,dt \\
&= \frac{\rho}{x^2}\int_0^\infty e^{-\left(\frac{\rho}{x}-\delta\right)t}\left(\int_0^\infty J_1(2\sqrt{st})\,\sqrt{\frac{t}{s}}\,\varphi(s)\,s\,e^{-s}\,ds\right)dt \\
&= \frac{1}{x^2}\int_0^\infty e^{-\frac{t}{x}}\,e^{\frac{\delta}{\rho}t}\left(\int_0^\infty J_1\left(2\sqrt{\frac{st}{\rho}}\right)\sqrt{\frac{t}{\rho s}}\,\varphi(s)\,dm(s)\right)dt \\
&= \frac{1}{x^2}\int_0^\infty e^{-\frac{t}{x}}\,e^{t}\,\frac{1}{\rho}\,e^{\frac{\delta}{\rho}t}\left(\int_0^\infty \frac{J_1\left(2\sqrt{st/\rho}\right)}{\sqrt{st/\rho}}\,\varphi(s)\,dm(s)\right)dm(t) \\
&= (\mathcal{B}\,[N_r\varphi])(x)\,,
\end{aligned}$$

where we have used the identity [GR]

$$\frac{1}{u^{p+1}}\int_0^\infty e^{-t/u}\psi(t)\,dt = \int_0^\infty e^{-tu}\left(\int_0^\infty \left(\frac{t}{s}\right)^{\frac{p}{2}} J_p(2\sqrt{st})\,\psi(s)\,ds\right)dt \tag{4.80}$$

with $p=1$, $u=\frac{\rho}{x}-\delta$ and $\psi(s)=s\,e^{-s}\varphi(s)$. Finally, by the estimates (see [Er]) $J_1(x)\sim x/2$ as $x\searrow 0$ and $J_1(x)=\mathcal{O}(x^{-\frac{1}{2}})$ as $x\to\infty$, along with the inequality $2\delta/\rho<1$, which holds for all $r\in[0,1]$, one easily checks that $N_r : L^2(m)\to L^2(m)$. □

We can summarize the above in the following

Theorem 4.3. *The space* $\mathcal{H}$ *is invariant for* $\mathcal{P}$ *and we have*

$$\mathcal{P}\mathcal{B}\,[\varphi] = \mathcal{B}\,[(M_r+N_r)\varphi]\,. \tag{4.81}$$

4.2. **The spectrum of M_r and $\mathcal{P}_0$.** We are now going to study the operator M_r. First, note that it reduces to $M_0\,\varphi(t)=(1/2)\varphi(t/2)$ when $r=0$, whereas for $r=1$ yields the multiplication operator $M_1\varphi(t)=e^{-t}\varphi(t)$. Moreover, for $r\in[0,1)$ its iterates are given by

$$M_r^n\varphi(t) = \left(\prod_{k=1}^n e^{-\frac{r}{\rho^k}t}\right)\varphi\left(\frac{t}{\rho^n}\right) = \frac{1}{\rho^n}\,e^{-\frac{r}{\delta}t}\,e^{\frac{r}{\delta}\frac{t}{\rho^n}}\,\varphi\left(\frac{t}{\rho^n}\right). \tag{4.82}$$

Assume that φ is analytic in some open neighbourhood of $t = 0$ and satisfies the eigenvalue equation

$$M_r\varphi(t) = \frac{1}{\rho}\, e^{-\frac{r}{\rho}t}\varphi\left(\frac{t}{\rho}\right) = \lambda\varphi(t). \tag{4.83}$$

The above equation at $t = 0$ writes $\varphi(0) = \lambda\,\rho\,\varphi(0)$, so that if $\varphi(0) \neq 0$ then $\lambda = 1/\rho$. In this case, by (4.82) we get

$$\varphi(t) = \rho^n\, M_r^n\varphi(t) \to e^{-\frac{r}{\delta}t}\,\varphi(0) \quad \text{as} \quad n \to \infty, \tag{4.84}$$

and therefore $\varphi(t) = e^{-\frac{r}{\delta}t}\,\varphi(0)$. If instead $\varphi(0) = 0$ we differentiate (4.83) to get

$$-\frac{r}{\rho^2}e^{-\frac{r}{\rho}t}\varphi(t/\rho) + \frac{e^{-\frac{r}{\rho}t}\varphi'(t/\rho)}{\rho^2} = \lambda\varphi'(t) \tag{4.85}$$

which at $t = 0$ writes $\varphi'(0)/\rho^2 = \lambda\varphi'(0)$. Therefore if $\varphi'(0) \neq 0$ then $\lambda = 1/\rho^2$. In this case, differentiating (4.82)-(4.83) we get

$$\frac{1}{\rho^{2n}}\varphi'(t) = \frac{r}{\delta\,\rho^n}\left(\frac{1}{\rho^n} - 1\right) e^{-\frac{r}{\delta}t}\, e^{\frac{r}{\delta}\frac{t}{\rho^n}}\,\varphi\left(\frac{t}{\rho^n}\right) + \frac{e^{-\frac{r}{\delta}t}\, e^{\frac{r}{\delta}\frac{t}{\rho^n}}}{\rho^{2n}}\varphi'\left(\frac{t}{\rho^n}\right). \tag{4.86}$$

Taking the limit $n \to \infty$ and noting that $\lim_{n\to\infty}\rho^n\varphi(t/\rho^n) = t\,\varphi'(0)$ we obtain

$$\varphi'(t) = \varphi'(0)\, e^{-\frac{r}{\delta}t}\left(1 - \frac{r}{\delta}\,t\right), \tag{4.87}$$

which upon integration renders

$$\varphi(t) = \varphi'(0)\, t\, e^{-\frac{r}{\delta}t}. \tag{4.88}$$

Iterating this argument we have that if φ satisfies (4.83) with $\varphi^{(l)}(0) = 0$ for $0 \le l < k-1$ but $\varphi^{(k-1)}(0) \neq 0$ for some $k \ge 1$ then $\lambda = \rho^{-k}$ and $\varphi(t) = t^{k-1}\, e^{-\frac{r}{\delta}t}$ (up to a non-zero but otherwise arbitrary constant multiplicative factor). Denoting by $\|\ \|_2$ the norm in $L^2(m)$ we also have

$$\|t^{k-1}\, e^{-\frac{r}{\delta}t}\|_2^2 = \left(\frac{\delta}{1+r}\right)^k (2k-1)! \tag{4.89}$$

It is not hard to see that for all $r \in [0, 1)$ the sequence $\{t^{k-1}\, e^{-\frac{r}{\delta}t}\}_{k=1}^{\infty}$ is a linearly independent family in $L^2(m)$, and by adapting ([He], p.62, Thm.8) we have that the linear span of this family is dense in $L^2(m)$. Putting together the above along with standard arguments we have proved the following

Proposition 4.5. *For all $r \in [0, 1)$ the operator $M_r : L^2(m) \to L^2(m)$ is compact and its spectrum is given by* $\mathrm{sp}\,(M_r) = \{\mu_k\}_{k\ge1} \cup \{0\}$ *with*

$$\mu_k = \frac{1}{\rho^k} \equiv \left(\Phi'_{r,0}(0)\right)^k. \tag{4.90}$$

Each eigenvalue μ_k is simple

and the corresponding (normalized) eigenfunction φ_k is given by

$$\varphi_k(t) = A_k\, t^{k-1}\, e^{-\frac{r}{\delta}t} \tag{4.91}$$

with

$$A_k = \left(\frac{1+r}{\delta}\right)^k \frac{1}{\sqrt{(2k-1)!}}. \tag{4.92}$$

Remark 6. *For each fixed $r \in [0,1)$ the operator $M_r : L^2(m) \to L^2(m)$ is actually trace-class. Its trace is easily computed:*

$$\operatorname{tr} M_r = \sum_{k=1}^{\infty} \rho^{-k} = \frac{1}{\delta}\,, \tag{4.93}$$

and satisfies

$$\lim_{r\searrow 0} \operatorname{tr} M_r \;=\; 1\,, \qquad \lim_{r\nearrow 1} \operatorname{tr} M_r \;=\; \infty\,. \tag{4.94}$$

We recall that for $r = 1$ we get the multiplication operator M_1 which is self-adjoint in $L^2(m)$, its spectrum is continuous and given by the closure of the range of the multiplying function, that is the interval $[0,1]$ (see, e.g., [DeV]). By the above Proposition we see how the continuous spectrum is approached as $r \nearrow 1$: having fixed an interval $[a,b] \subseteq (0,1]$ an easy computation yields

$$\#\{\mu_k \in [a,b]\} \sim \frac{\log\left(\frac{b-a}{ab}\right)}{1-r} \quad \text{as} \quad r \nearrow 1. \tag{4.95}$$

Moreover a simple calculation gives

$$\mathcal{B}\,[t^{k-1}\, e^{-\frac{r}{\delta}t}] = \frac{k!\,\delta^{k+1}\,x^{k-1}}{(\delta+rx)^{k+1}}. \tag{4.96}$$

Therefore by the above and Lemma 4.4 we have the following

Corollary 4.4. *The spectrum of $\mathcal{P}_0$ when acting upon $\mathcal{H}$ is given by $\operatorname{sp}(\mathcal{P}_0) = \operatorname{sp}(M_r)$. For $r \in [0,1)$ each eigenvalue μ_k, with $k \geq 1$, is simple and the corresponding (normalized) eigenfunction χ_k is given by*

$$\chi_k(x) = (\mathcal{B}_r\,[\varphi_k])(x) = \frac{(1+r)^k\,k!\,\delta}{\sqrt{(2k-1)!}}\,\frac{x^{k-1}}{(\delta+rx)^{k+1}}\,. \tag{4.97}$$

4.3. **The spectrum of N_r and $\mathcal{P}_1$.** We now turn to study the action of the operator N_r on $L^2(m)$. We first note that N_r can be viewed as the composition of the multiplication operator

$$\varphi(t) \to \frac{1}{\rho}\,e^{\frac{\delta}{\rho}t}\,\varphi(t), \tag{4.98}$$

(which reduces to the identity for $r = 1$) and the symmetric integral operator

$$\varphi(t) \to \int_0^\infty \frac{J_1\left(2\sqrt{st/\rho}\right)}{\sqrt{st/\rho}}\,\varphi(s)\,dm(s). \tag{4.99}$$

Observing that the (associated) Laguerre polynomials L_k^1 given by [GR]

$$L_k^1(s) = \frac{e^s\,s^{-1}}{k!}\,\frac{d^k}{ds^k}\left(e^{-s}\,s^{k+1}\right) = \sum_{l=0}^{k} \binom{k+1}{k-l}\frac{(-s)^l}{l!} \tag{4.100}$$

form a complete orthogonal basis in $L^2(m)$ and expanding the kernel of the integral operator defined above on this basis we get

$$\frac{J_1\left(2\sqrt{st/\rho}\right)}{\sqrt{st/\rho}} = \sum_{k=0}^{\infty} L_k^1(s)\;\frac{t^k\,e^{-\frac{t}{\rho}}}{\rho^k\,(k+1)!} \tag{4.101}$$

from which we see that N_r has the representation (we keep using the symbol (φ_1, φ_2) to denote the inner product in $L^2(m)$ as well)

$$N_r\varphi = \sum_{k=1}^{\infty}(\varphi, e_k)\, f_k \tag{4.102}$$

where $e_k, f_k \in L^2(m)$ are given by

$$e_k(t) = L^1_{k-1}(t) \quad \text{and} \quad f_k(t) = N_r e_k(t) = \frac{t^{k-1}\, e^{-\frac{r}{\rho}t}}{\rho^k\, k!}\,\cdot \tag{4.103}$$

We find

$$\|e_k\|_2 = \sqrt{k} \quad \text{and} \quad \|f_k\|_2 = \frac{\sqrt{(2k-1)!}}{(2+r)^k\, k!}, \tag{4.104}$$

and therefore, for all $r \in (0, 1]$,

$$\sum ?\|e_k\|_2\, \|f_k\|_2 < \infty \tag{4.105}$$

showing that the operator N_r is nuclear in $L^2(m)$. Its trace can be computed as

$$\operatorname{tr} N_r = \int_0^\infty \frac{e^{-\frac{s}{\rho}}}{\sqrt{\rho}}\, J_1\left(\frac{2s}{\sqrt{\rho}}\right) ds = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+4\rho}}\right), \tag{4.106}$$

with limit values

$$\lim_{r\searrow 0} \operatorname{tr} N_r \;=\; \frac{1}{3} \quad \text{and} \quad \lim_{r\nearrow 1} \operatorname{tr} N_r \;=\; \frac{\sqrt{5}-1}{2\sqrt{5}}\,\cdot \tag{4.107}$$

Moreover we find

$$\operatorname{tr} N_r^2 = \frac{1}{\rho}\int_0^\infty\int_0^\infty e^{-\frac{t+s}{\rho}}\left[J_1\left(2\sqrt{st/\rho}\right)\right]^2 ds\, dt = \frac{1}{2}\left(\frac{1+2\rho}{\sqrt{1+4\rho}} - 1\right). \tag{4.108}$$

Note that $\operatorname{tr} N_r^2 \le \operatorname{tr} N_r$, with strict inequality unless $r = 0$ where $\operatorname{tr} N_0 = \operatorname{tr} N_0^2 = 1/3$. This suggests that the eigenvalues of N_r are alternately positive and negative. Indeed we have the following

Proposition 4.6. *For all $r \in [0, 1]$ the spectrum of the operator $N_r : L^2(m) \to L^2(m)$ is given by $\operatorname{sp}(N_r) = \{\nu_k\}_{k=1}^\infty \cup \{0\}$ with*

$$\nu_k = (-1)^{k-1}\left(\frac{4\rho}{(1+\sqrt{1+4\rho})^2}\right)^k \equiv -\left(\Phi'_{r,1}(x_1)\right)^k. \tag{4.109}$$

Each eigenvalue is simple and the corresponding (normalized) eigenfunction ψ_k is given by

$$\psi_k(t) = B_k\, L^1_{k-1}(\alpha_r t)\; e^{-\beta_r t}, \tag{4.110}$$

where $\alpha_r = \frac{\sqrt{1+4\rho}}{\rho}$, $\beta_r = \frac{1+\sqrt{1+4\rho}}{2\rho} - 1$ and

$$B_k = \frac{\sqrt{1+4\rho}}{\rho\sqrt{k}}\left(1 - \frac{\delta}{\sqrt{1+4\rho}}\right)^k \left[\sum_{j=0}^{k-1}\binom{k}{j}\binom{k-1}{j}\left(\frac{\delta^2}{1+4\rho}\right)^j\right]^{-\frac{1}{2}}.$$

Remark 7. *Note that the eigenvalues of N_r can be written in terms of the function β_r as*

$$\nu_k = \frac{(-1)^{k-1}}{\rho^k}\left(\frac{1}{1+\beta_r}\right)^{2k}\cdot \tag{4.111}$$

In particular, $\lim_{r\searrow 0}\beta_r = 0$ so that for $r = 0$ we have $\nu_k = (-1)^{k-1}(2)^{-k}$, moreover $\lim_{r\nearrow 1}\beta_r = \frac{\sqrt{5}-1}{2}$ and thus for $r = 1$ we see that $\nu_k = (-1)^{k-1}\left(\frac{\sqrt{5}-1}{2}\right)^{2k}$.

Proof of Proposition 4.6. The proof is based on the following Hankel transform (see [Er], vol 2)

$$\int_0^\infty x^{p+\frac{1}{2}}\, e^{-bx^2}\, L_k^p(ax^2)\, J_p(xy)\, \sqrt{xy}\, dx = \frac{(b-a)^k\, y^{p+\frac{1}{2}}}{2^{p+1}\, b^{p+k+1}}\, e^{-\frac{y^2}{4b}}\, L_k^p\left[\frac{ay^2}{4b(a-b)}\right], \tag{4.112}$$

which for $p = 1$ can be rewritten in terms of the operator N_r as

$$N_r\left[e^{-(\frac{2b}{\sqrt{\rho}}-1)t}\, L_k^1\left(\frac{2at}{\sqrt{\rho}}\right)\right] = \frac{(b-a)^k}{4\, b^{k+2}}\, e^{-(\frac{1}{2b\sqrt{\rho}}-\frac{\delta}{\rho})t}\, L_k^1\left[\frac{at}{2\sqrt{\rho}\, b(a-b)}\right]. \tag{4.113}$$

To make the above identity an eigenvalue equation the following relations have to be satisfied

$$\frac{2b}{\sqrt{\rho}} - 1 = \frac{1}{2b\sqrt{\rho}} - \frac{\delta}{\rho} \quad \text{and} \quad \frac{2a}{\sqrt{\rho}} = \frac{a}{2\sqrt{\rho}\, b(a-b)}. \tag{4.114}$$

This renders

$$a = b + \frac{1}{4b} \quad \text{and} \quad b = \frac{1 \pm \sqrt{1+4\rho}}{4\sqrt{\rho}}. \tag{4.115}$$

It is now easy to check that the only solution giving a function in $L^2(m)$ is that with $b = (1+\sqrt{1+4\rho})/4\sqrt{\rho}$ and this choice yields the eigenvalues and the eigenfunctions given in the Proposition. Moreover, a standard evaluation of

$$\|L_{k-1}^1\,(\alpha(r)t)\; e^{-\beta_r\, t}\|_2^2 = \frac{\rho^2}{1+4\rho}\int_0^\infty \left[L_{k-1}^1(s)\right]^2\, s\, e^{-(1-\frac{\delta}{\sqrt{1+4\rho}})s}\, ds \tag{4.116}$$

yields the claimed expression for the normalizing factor B_k (see [Er], vol 1). □

Direct application of the Lemma 4.5 now yields

Corollary 4.5. *The spectrum of $\mathcal{P}_1$ when acting upon $\mathcal{H}$ is given for each $r \in [0,1]$ by $\mathrm{sp}\,(\mathcal{P}_1) = \mathrm{sp}\,(N_r)$. Each eigenvalue ν_k, with $k \geq 1$, is simple and the corresponding (normalized) eigenfunction ξ_k is given by*

$$\xi_k(x) = (\mathcal{B}_r\,[\psi_k])(x) = k\, B_k\, \frac{(1+(\beta_r-\alpha_r)x)^{k-1}}{(1+\beta_r x)^{k+1}}. \tag{4.117}$$

4.4. **The spectrum of $\mathcal{P}$.** First, putting together (4.93), (4.106) and Theorem 4.3 we deduce the following result,

Theorem 4.4. *For all $r \in [0,1)$ the transfer operator $\mathcal{P}$ when acting upon $\mathcal{H}$ is of the trace-class, with*

$$\mathrm{tr}\,\mathcal{P} = \frac{1}{\delta} + \frac{\sqrt{1+4\rho}-1}{2\sqrt{1+4\rho}} \tag{4.118}$$

with $\delta = 1-r$ and $\rho = 2-r$.

Remark 8. *Using (2.10) and setting set $x_0 \equiv 0$ one easily verifies that*

$$\frac{1}{\delta} = \frac{|\Phi'_{r,0}(x_0)|}{1-\Phi'_{r,0}(x_0)}\,. \tag{4.119}$$

Moreover, a straightforward computation based on the integral

$$\int_0^\infty e^{-ax}\, J_1(bx)\, dx = \frac{\sqrt{a^2+b^2}-a}{b\sqrt{a^2+b^2}} \tag{4.120}$$

and using (2.10) shows that

$$\frac{\sqrt{1+4\rho}-1}{2\sqrt{1+4\rho}} = \frac{|\Phi'_{r,1}(x_1)|}{1-\Phi'_{r,1}(x_1)}\,, \tag{4.121}$$

where x_1, defined in (2.8), is the unique fixed point of F_r besides x_0 (obviously one would arrive at the same conclusion by directly summing the geometric series $\sum_{k\geq 1}\mu_k + \sum_{k\geq 1}\nu_k$ with μ_k and ν_k given by (4.90) and (4.111), respectively). As a result we can rewrite (4.118) as

$$\operatorname{tr}\mathcal{P} = \sum_{i=0,1} \frac{|\Phi'_{r,i}(x_i)|}{1-\Phi'_{r,i}(x_i)}. \tag{4.122}$$

This expression does'nt come unexpectedly: it is an instance of a trace formula valid for more general analytic Markov maps (see [Ma2], *Sec. 7.3.1).*

We are now ready to investigate the spectrum $\operatorname{sp}(\mathcal{P})$ on the space $\mathcal{H}$. We first notice that from (4.113) with $a = \sqrt{\rho}$ and $b = \sqrt{\rho}/2$ it follows that

$$N_r\left[L_k^1(2t)\right] = \frac{(-1)^k}{\rho}\, e^{-\frac{r}{\rho}t}\, L_k^1\left(\frac{2t}{\rho}\right) = (-1)^k\, M_r\left[L_k^1(2t)\right] \tag{4.123}$$

Therefore for all odd k the function $L_k^1(2t)$ lies in the kernel of $M_r + N_r$. This is in agreement with (2.14) and (2.15) since we have

$$\mathcal{B}\left[L_k^1(2t)\right] = (k+1)(1-2x)^k \tag{4.124}$$

which for k odd is an odd function w.r.t. $x = 1/2$. This implies that 0 is an eigenvalue of infinite multiplicity for $\mathcal{P}$ for all $r \in [0,1]$.

Let us first consider the two extremal cases $r = 0$ and $r = 1$. For $r = 0$ we the above theorem gives $\operatorname{tr}\mathcal{P} = 4/3$, suggesting the eigenvalues 2^{-2n}, $n \geq 0$. To check, we first note that

$$M_0\, t^k = \frac{t^k}{2^{k+1}} \quad \text{and} \quad N_0\, t^k = \frac{k!}{2}\, L_k^1(t/2) = \frac{(-t)^k}{2^{k+1}} + \sum_{j=0}^{k-1} d_j\, t^j$$

with $d_j = \frac{(-1)^j\,(k+1)!\,k!}{2^{j+1}\,(j+1)!\,j!\,(k-j)!}$. Therefore the set of polynomials with even degree is a subset of $L^2(m)$ which is invariant for M_0+N_0, and any polynomial with odd degree is mapped into this subset. Given $k = 2n$ it is now a simple task to construct a polynomial eigenfunction ϕ_{2n} (of degree $2n$) to the eigenvalue 2^{-2n}. The first three are

$$\phi_0(t) = 1, \quad \phi_2(t) = \frac{t^2}{2} - 3t + 2, \quad \phi_4(t) = \frac{t^4}{24} - \frac{5t^3}{6} + \frac{10t^2}{3} - \frac{32}{15}\,.$$

Note moreover that for $\phi_k(t) = \sum_{j=0}^k a_j\, t^j$ we have $(\mathcal{B}\,[\phi_k])(x) = \sum_{j=0}^k a_j\,(j+1)!\,x^j$. These observations along with standard arguments yield the following

Proposition 4.7. *For $r = 0$ the spectrum of $\mathcal{P}$ when acting upon $\mathcal{H}$ is the set $\operatorname{sp}(\mathcal{P}) = \{2^{-2n}\}_{n\geq 0} \cup \{0\}$. Each eigenvalue 2^{-2n} is simple and the corresponding eigenfunction is a polynomial of degree $2n$.*

At the opposite extremum we have the following result.

Proposition 4.8. *For $r = 1$ the spectrum of $\mathcal{P}$ when acting upon $\mathcal{H}$ is the union of $[0,1]$ and a (possibility empty) countable set of real eigenvalues of finite multiplicity.*

Proof. The assertion follows by noting that for $r = 1$ the operator $\mathcal{P}$ when acting on $\mathcal{H}$ is isomorphic to $M_1 + N_1$, which is a self-adjoint compact perturbation of the (self-adjoint) multiplication operator M_1. The assertion is now a consequence of Theorem 5.2 in [GK]. Note that although the function $e_1(x) = (\log 2)^{-1}/x$ satisfies $\mathcal{P}e_1 = e_1$ it does not belong to $\mathcal{H}$ and therefore $1 \notin \operatorname{sp}(\mathcal{P} : \mathcal{H} \to \mathcal{H})$. $\square$

The above construction, in particular Propositions 4.7 and 4.8, along with the arguments outlined in [Is] (where the $r = 1$ case has been studied in detail) lead us to state the following

Conjecture 1. *For all $0 \le r < 1$ the spectrum of $\mathcal{P}$ when acting upon $\mathcal{H}$ is a countable subset of $[0,1]$ densely filling $[0,1]$ as $r \nearrow 1$. When $r = 1$ the spectrum is purely continuous (i.e. there are no eigenvalues).*

4.5. **Operator-valued functions and zeta functions.** We may consider the operator-valued function $\mathcal{Q}_z$ defined as

$$\mathcal{Q}_z := z\,\mathcal{P}_1(1 - z\mathcal{P}_0)^{-1}. \tag{4.125}$$

Its relevance is twofold: first, expanding formally in powers of z we get

$$\mathcal{Q}_z = \sum_{n=1}^{\infty} z^n \mathcal{P}_1 \mathcal{P}_0^{n-1} \tag{4.126}$$

and using (2.41) we see that for $z = 1$ the operator $\mathcal{Q} \equiv \mathcal{Q}_1$ is the transfer operator associated to the induced map G_r. Second, it is related to $\mathcal{P}$ by the identity

$$(1 - \mathcal{Q}_z)(1 - z\,\mathcal{P}_0) = 1 - z\,\mathcal{P}. \tag{4.127}$$

Now, as a consequence of (4.125) and Lemmas 4.4 - 4.5, we have the following expression for the operator-valued function $\mathcal{Q}_z$ when acting on $\mathcal{H}$:

$$\mathcal{Q}_z\mathcal{B}\,[\varphi] = \mathcal{B}\left[N_r \left(\frac{1}{z} - M_r \right)^{-1} \varphi \right] \cdot \tag{4.128}$$

Putting together the above we obtain

Theorem 4.5. *For all $r \in [0,1]$ the operator-valued function $\mathcal{Q}_z$ when acting on the Hilbert space $\mathcal{H}$ is analytic for $z \in \mathbb{C} \setminus \Lambda_r$, where $\Lambda_r = \{\rho^k\}_{k=1}^{\infty}$ for $r < 1$ and $\Lambda_1 = (1, \infty)$. For each $z \in \mathbb{C} \setminus \Lambda_r$ it defines a trace-class operator.*

Remark 9. *Introducing the operator*

$$L_r := (1 - M_r)^{-1}\, N_r\,, \tag{4.129}$$

we can rewrite (4.81) and (4.128) with $z = 1$ as

$$\mathcal{P}\mathcal{B}\,[\varphi] = \mathcal{B}\,[\,(M_r + (1 - M_r)L_r\,)\,\varphi]\,, \tag{4.130}$$

and (recall that $\mathcal{Q} \equiv \mathcal{Q}_1$)

$$\mathcal{Q}\,\mathcal{B}\,[\varphi] = \mathcal{B}\,\left[(1 - M_r)L_r(1 - M_r)^{-1}\,\varphi\right], \tag{4.131}$$

respectively. We thus see that the functions ϕ_r and ψ_r defined in (4.73) satisfy

$$L_r\phi_r = \phi_r \quad \textit{and} \quad (1 - M_r)L_r(1 - M_r)^{-1}\psi_r = \psi_r, \tag{4.132}$$

so that

$$\phi_r = (1 - M_r)^{-1}\psi_r \quad \textit{and} \quad \psi_r = N_r\phi_r. \tag{4.133}$$

We now consider the dynamical zeta functions ζ_{F_r} and ζ_{G_r} associated to the maps F_r and G_r, respectively, and defined by the following formal series [Rue2]:

$$\zeta_{F_r}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(F_r) \quad \text{and} \quad \zeta_{G_r}(s) = \exp \sum_{n=1}^{\infty} \frac{s^n}{n} Z_n(G_r), \tag{4.134}$$

where the 'partition functions' $Z_n(F_r)$ and $Z_n(G_r)$ are given by

$$Z_n(F_r) = \sum_{x=F_r^n(x)} \prod_{k=0}^{n-1} \frac{1}{|F_r'(F_r^k(x))|} \quad \text{and} \quad Z_n(G_r) = \sum_{x=G_r^n(x)} \prod_{k=0}^{n-1} \frac{1}{|G_r'(G_r^k(x))|}. \tag{4.135}$$

Moreover, let us define the 'grand partition function'

$$\Xi_n(z) := \sum_{\ell=0}^{\infty} z^{\ell+n} \sum_{x=G_r^n(x)=F_r^{\ell+n}(x)} \prod_{k=0}^{n-1} \frac{1}{|G_r'(G_r^k(x))|}, \tag{4.136}$$

and the two-variable zeta function

$$\zeta_2(s,z) := \exp \sum_{n=1}^{\infty} \frac{s^n}{n}\, \Xi_n(z). \tag{4.137}$$

A straightforward extension of ([Is], Proposition 4.3) to the present situation yields the identities

$$\zeta_2(1,z) = (1-z)\,\zeta_{F_r}(z) \quad \text{and} \quad \zeta_2(s,1) = \zeta_{G_r}(s) \tag{4.138}$$

which are valid for all $r \in [0,1]$ and wherever the series expansions converge absolutely. Therefore the analytic properties of the dynamical zeta functions ζ_{F_r} and ζ_{G_r} can be deduced from those of $\zeta_2(s,z)$. In turn, the latter can be studied as follows. For $q = 0, 1, \dots$ define

$$\begin{aligned} \mathcal{P}_{0,q} f(x) &:= \left[\frac{\rho}{(\rho+rx)^2}\right]^{1+q} \cdot f\left(\frac{x}{\rho+rx}\right) \\ \mathcal{P}_{1,q} f(x) &:= \left[\frac{\rho}{(\rho+rx)^2}\right]^{1+q} \cdot f\left(1-\frac{x}{\rho+rx}\right) \end{aligned}$$

so that $\mathcal{P}_{0,0} \equiv \mathcal{P}_0$ and $\mathcal{P}_{1,0} \equiv \mathcal{P}_1$. These operators are supposed to act upon the Hilbert space $\mathcal{H}_q \subseteq \mathcal{H}$ such that a function $f \in \mathcal{H}_q$ can be represented as

$$f(x) = (\mathcal{B}_q[\varphi])(x) := \frac{1}{x^{2(1+q)}} \int_0^{\infty} e^{-\frac{t}{x}}\, e^t\, \varphi(t)\, dm_q(t), \quad \varphi \in L^2(m_q), \tag{4.139}$$

with

$$dm_q(t) = t^{2q+1}\, e^{-t}\, dt \tag{4.140}$$

A straightforward computation extending to non zero q values those performed in the previous Section yields

$$\mathcal{P}_{0,q}\, \mathcal{B}_q[\varphi] = \mathcal{B}_q[M_{r,q}\varphi] \quad \text{and} \quad \mathcal{P}_{1,q}\, \mathcal{B}_q[\varphi] = \mathcal{B}_q[N_{r,q}\varphi] \tag{4.141}$$

where $M_{r,q} : L^2(m_q) \to L^2(m_q)$ is defined as

$$(M_{r,q}\varphi)(t) := \frac{1}{\rho^{1+q}}\, e^{-\frac{r}{\rho}t}\, \varphi\left(\frac{t}{\rho}\right). \tag{4.142}$$

and $N_{r,q} : L_2(m_q) \to L_2(m_q)$ is given by

$$(N_{r,q}\varphi)(t) := \frac{1}{\rho^{1+q}}\, e^{\frac{\delta}{\rho}t} \int_0^{\infty} \frac{J_{2q+1}\left(2\sqrt{st/\rho}\right)}{(st/\rho)^{q+\frac{1}{2}}}\, \varphi(s)\, dm_q(s). \tag{4.143}$$

We can now generalize (4.126) defining

$$\mathcal{Q}_{z,q} := \sum_{n=1}^{\infty} z^n \mathcal{P}_{1,q} \mathcal{P}_{0,q}^{n-1}. \tag{4.144}$$

In particular $\mathcal{Q}_{z,0} \equiv \mathcal{Q}_z$. Now set $\Lambda_{r,q} = \{\rho^{k+q}\}_{k=1}^{\infty}$ and note that $\Lambda_{r,q} \subseteq \Lambda_r$ for all $q \geq 0$. Reasoning as above and using (4.141) we have that for any given $q = 0, 1 \ldots$ the operator valued function $z \to \mathcal{Q}_{z,q}$ when acting on $\mathcal{H}_q$ is analytic for $z \in \mathbb{C} \setminus \Lambda_{r,q}$ and for each z in this domain $\mathcal{Q}_{z,q}$ defines a trace-class operator with

$$\mathcal{Q}_{z,q}\,\mathcal{B}_q\,[\varphi] = \mathcal{B}_q\,[(-1)^q\,N_{r,q}\left(\frac{1}{z} - M_{r,q}\right)^{-1}\varphi]. \tag{4.145}$$

Furthermore, a straightforward adaptation of ([Ma1], Corollaries 4 and 5) to our z-dependent situation leads to the following expression for the grand partition function:

$$\Xi_n(z) = \mathrm{tr}\ \mathcal{Q}_{z,0}^n - \mathrm{tr}\ \mathcal{Q}_{z,1}^n. \tag{4.146}$$

This trace formula along with standard arguments (see [Ma1]) allow us to write the two-variables zeta function (4.137) as a ratio of Fredholm determinants,

$$\zeta_2(s,z) = \frac{\det(1 - s\,\mathcal{Q}_{z,1})}{\det(1 - s\,\mathcal{Q}_{z,0})}, \tag{4.147}$$

where by definition

$$\det(1 - s\,\mathcal{Q}_{z,q}) = \exp\left(-\sum_{n=1}^{\infty}\frac{s^n}{n}\,\mathrm{tr}\ \mathcal{Q}_{z,q}^n\right) \tag{4.148}$$

is in the sense of Grothendieck [G]. Putting together the above we obtain the following result from which the analytic properties of the dynamical zeta functions associated to the maps F_r and G_r can be readily deduced via (4.138).

Theorem 4.6.

1. *for each $s \in \mathbb{C}$, the function $\zeta_2(s,z)$, considered as a function of the variable z, extends to a meromorphic function in $\{z \in \mathbb{C} : z \notin \Lambda_r\}$. Its poles are located among those z-values such that $\mathcal{Q}_z : \mathcal{H} \to \mathcal{H}$ has $1/s$ as an eigenvalue;*
2. *for each $z \in \mathbb{C} \setminus \Lambda_r$, the function $\zeta_2(s,z)$, considered as a function of the variable s, extends to a meromorphic function in $\mathbb{C}$. Its poles are located among the inverses of the eigenvalues of $\mathcal{Q}_z : \mathcal{H} \to \mathcal{H}$.*

Remark 10. *The first statement with $s = 1$ shows that for $r = 1$ the function $\zeta_{F_r}(z)$ has a non-polar singularity at $z = 1$. This can be related to the non-analytic behaviour of the free energy at $\beta = 0$ discussed in Section 3.*

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