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Gevrey formal power series of Wannier–Stark ladders

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Abstract
We consider time-independent Schrödinger equations in one dimension with both periodic and Stark potentials. By means of an iterative procedure we obtain a formal power series for the Wannier–Stark ladders. In the case of strongly singular periodic potentials we prove that such a formal power series is of Gevrey type.

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1. Introduction

The nature of the spectrum of the Wannier–Stark operator, formally defined on $L^2(\mathbb{R}, \mathrm{d}x)$ as

$$H_\epsilon = p^2 + V + \epsilon x \qquad p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \qquad \epsilon \in \mathbb{R}$$

where $V(x)$ is a real valued periodic function with period d, i.e. $V(x) = V(x+d)$, is still an object of analysis. In fact, even if it is well known that the spectrum of H_ϵ is purely absolutely continuous when the periodic potential is a piecewise continuous function [5], it is not clear what happens when the periodic potential is singular enough. Avron *et al* [3] introduced Wannier–Stark operators with strongly singular periodic potential given by δ' interactions. In their seminal paper they discussed the nature of the spectrum of such Wannier–Stark operators suggesting that the absolutely continuous spectrum is empty; a few years later, this statement was rigorously proved by Exner himself [8] and, by means of different techniques, by Maioli and Sacchetti [9]. Thus, in such a case we expect to have eigenvalues

$$[H_\epsilon - \lambda]\psi = 0 \qquad \psi \in L^2(\mathbb{R}, \mathrm{d}x) \tag{1}$$

displaced on regular ladders, i.e. if λ is an eigenvalue then $\lambda + \epsilon d$ is an eigenvalue too, usually named Wannier–Stark ladders.

The problem of the computation of such Wannier–Stark ladders was posed by Wannier in the sixties [12] where he introduced an iterative procedure which gives the formal power series of the eigenvalues. This KAM-type method, later improved by [2, 7, 10, 11], consists in producing, at the mth step, an unitary transformation which makes the initial problem explicitly solvable up to a small correction of order ϵ^m. Since such an unitary transformation is obtained by means of the solution of an infinite-dimensional system of ordinary differential equations it turns out that the explicit construction of the formal power series is, from a computational point of view, very hard.

In this paper we propose a different iterative procedure which gives a formal power series for the eigenvalues and eigenvectors of H_ϵ where the mth term of the series is directly computed by means of the first $m-1$ terms (theorem 1). Hence, our method turns out to be much more computationally efficient than the previous one. Furthermore, we can prove that the formal power series we obtain is of Gevrey type (theorem 2).

The paper is organized as follows. In section 2 we give the main notation and briefly recall the crystal momentum representation (for more details see [11]). In section 3 we give the main results: that is we give the formal power series for the solutions of the eigenvalue equation (1) which is of Gevrey order γ for any $\gamma > 1$. In sections 4 and 5 we respectively prove theorems 1 and 2.

2. Notation and crystal momentum representation

Here, we recall some well-known facts about the Bloch operator, formally defined on $L^2(\mathbb{R}, \mathrm{d}x)$ as

$$H_0 = p^2 + V \qquad V(x) = \sum_{j\in\mathbb{Z}} \alpha\delta'(x - jd) \qquad \alpha \neq 0.$$

It is defined as the unique self-adjoint extension of p^2, on $C_0^\infty(\mathbb{R} - \{jd, j \in \mathbb{Z}\})$, with the conditions [1]

$$\varphi(jd + 0) = \varphi(jd - 0) + \alpha\varphi'(jd - 0) \qquad \varphi'(jd + 0) = \varphi'(jd - 0) \qquad j \in \mathbb{Z}.$$

Hereafter, we assume, for the sake of definiteness, that

$$d = 1 \quad \text{and} \quad \alpha = 1.$$

2.1. Band functions

H_0 is a self-adjoint operator and its spectrum is given by bands

$$\sigma(H_0) = \cup_{n=1}^{\infty}\left[E_n^b, E_n^t\right] \qquad E_n^b = \min_{k\in\mathsf{B}} E_n(k) < E_n^t = \max_{k\in\mathsf{B}} E_n(k) < E_{n+1}^b$$

where $E_n(k)$ denotes the nth band function and k is the crystal momentum variable which belongs to the Brillouin zone $\mathsf{B} = [0, 2\pi)$. The band functions are periodic functions, with period 2π, and

$$E_n^b = \begin{cases} E_n(\pi) & n \text{ even} \\ E_n(0) & n \text{ odd} \end{cases} \qquad E_n^t = \begin{cases} E_n(0) & n \text{ even} \\ E_n(\pi) & n \text{ odd.} \end{cases}$$

The associated wavefunctions, usually named Bloch functions, are denoted by $\varphi_n(k, x)$ and they are periodic functions, with respect to k, and such that

$$\varphi_n(x, k) = \mathrm{e}^{\mathrm{i}kx}\nu_n(x, k)$$

where $\nu_n(x, k)$ is a periodic function with respect to x:

$$\varphi_n(x, k + 2\pi) = \varphi_n(x, k) \qquad \nu_n(x + 1, k) = \nu_n(x, k).$$

The band functions, implicitly defined by the equation [1]

$$\cos k = \cos K_n(k) - \tfrac{1}{2}K_n(k)\sin K_n(k) \qquad E_n(k) = K_n^2(k)$$

are analytic functions on a strip containing the real axis with amplitude κ which does not depend on the band index

$$S_\kappa = \{k \in \mathbb{Z} : \operatorname{Re} k \in \mathsf{B}, |\operatorname{Im} k| \leqslant \kappa\}.$$

In particular, it follows that

$$\inf_{n\in\mathbb{N}} \inf_{k\in S_\kappa} |E_n(k) - E_{n+1}(k)| > 0 \tag{2}$$

and they satisfy the asymptotic behaviour

$$E_n(k) = [n\pi + O(n^{-1})]^2 \qquad \text{as} \quad n \to \infty \tag{3}$$

uniformly for any k belonging to S_κ.

2.2. *Crystal momentum representation*

Here, we define a unitary transformation U_1 such as

$$\psi \in L^2(\mathbb{R}, \mathrm{d}x) \to a = U_1\psi \in \mathsf{H} = \oplus_{n=1}^{\infty} L^2(\mathsf{B})$$

where [6]

$$a = (a_n(k))_{n\in\mathbb{N}} \qquad k \in \mathsf{B} \qquad a_n \in L^2(\mathsf{B})$$

are periodic functions defined as

$$a_n(k) = \int_R \bar{\varphi}_n(x,k)\psi(x)\,\mathrm{d}x$$

such that

$$\|\psi\|^2_{L^2(\mathbb{R},\mathrm{d}x)} = \frac{1}{2\pi}\sum_{n=1}^{\infty} \|a_n(k)\|^2_{L^2(\mathsf{B})}.$$

In such a representation the operator H_ϵ takes the form

$$H^1_\epsilon = U_1 H_\epsilon U_1^{-1} = \mathrm{i}\epsilon\frac{\mathrm{d}}{\mathrm{d}k} + E + \epsilon X$$

where $E = \operatorname{diag}(E_n(k))$ is a diagonal matrix, whose diagonal elements are the band functions $E_n(k)$, and where $X = X(k)$ is the coupling term

$$(Ea)_n(k) = E_n(k)a_n(k) \qquad \text{and} \qquad (Xa)_n(k) = \sum_{p=1}^{\infty} X_{n,p}(k)a_p(k).$$

The functions $X_{n,p}(k)$ are defined by means of the Bloch functions as

$$X_{n,p}(k) = \mathrm{i}\int_{-1/2}^{1/2} \bar{\nu}_n(x,k)\frac{\partial \nu_p(x,k)}{\partial k}\,\mathrm{d}x.$$

In such a representation the eigenvalue equation (1) takes the form

$$\left[H^1_\epsilon - \lambda\right]a = 0 \qquad a \in \mathsf{H}. \tag{4}$$

We collect now some useful results.

Lemma 1. *The coupling term X is such that*

(i) $X_{n,p}(k) = \bar{X}_{p,n}(k)$*;*
(ii) $X_{n,n}(k)$ *is constant;*

(iii) $X_{n,p}(k) = x_{n,p} + R^1_{n,p}(k) + R^2_{n,p}(k)$ *where*

$$x_{n,p} = \begin{cases} \frac{1}{2} & n = p \\ 0 & n+p \text{ even} \qquad n \neq p \\ -\frac{2}{\pi}\left[\frac{\sin[(n+p)\pi/2]}{n+p} + \frac{\sin[(n-p)\pi/2]}{n-p}\right] & n+p \text{ odd} \end{cases}$$

and where

$$R^1_{n,p}(k) = O[(n-p)^{-2}] \qquad \text{and} \qquad R^2_{n,p}(k) = O[q^{-1}(n-p)^{-1}] \qquad q = \min(n,p) \tag{5}$$

uniformly with respect to $k \in S_\kappa$.

Proof. Property (i) is an immediate consequence of the derivation, with respect to k, of the normalization condition

$$\int_{-1/2}^{1/2} \bar{v}_n(x,k) v_p(x,k)\, \mathrm{d}x = \delta^p_n.$$

In order to prove the second statement we observe that the Bloch functions are defined up to a gauge choice $v_n(x,k) \to v_n(x,k)\,\mathrm{e}^{\mathrm{i}\omega_n(k)}$ where $\omega_n(k)$ is a periodic real valued function. Hence, it follows that

$$X_{n,n}(k) \to X_{n,n}(k) - \omega'_n(k) = \langle X_{n,n}\rangle$$

is constant by choosing $\omega_n(k) = \int_0^k X_{n,n}(q)\,\mathrm{d}q - \langle X_{n,n}\rangle k$, here $\langle X_{n,n}\rangle$ denotes the mean value of the function $X_{n,n}(k)$ (see equation (7)). Finally, property (iii) has already been proved in [9] for any $k \in \mathsf{B}$. The same arguments give the extension of estimate (5) to the complex strip S_κ. □

Lemma 2. *For any positive integer m there exists a positive constant C independent of n, p and* $k \in \mathsf{B}$ *such that*

$$\left|\frac{\mathrm{d}^m E_n(k)}{\mathrm{d}k^m}\right| \leqslant Cn^2$$

and

$$\left|\frac{\mathrm{d}^m X_{n,p}(k)}{\mathrm{d}k^m}\right| \leqslant C\left[\frac{1}{|n-p|^2+1} + \frac{1}{q[|n-p|+1|]}\right] \qquad q = \min(n,p). \tag{6}$$

Proof. These estimates are a consequence of the behaviour (3) and (5), the analyticity of the band and Bloch functions on S_κ, and the Cauchy theorem. □

Furthermore:

Lemma 3. *Let* $X^{(m)}$ *be the operator formally defined as*

$$(X^{(m)}a)_n = \sum_{p=1}^{\infty} X^m_{n,p} a_p \qquad X^m_{n,p} = \frac{\mathrm{d}^m X_{n,p}(k)}{\mathrm{d}k^m}$$

then $X^{(m)}$ *is a bounded operator for any* $m \geqslant 0$.

Proof. The boundedness of X has already been proved in [9]; the same arguments apply to $X^{(m)}$ by (6). □

2.3. Notation

We denote by P_1 the projection operator on the first band defined on H as

$$(P_1 u)_n = \delta_1^n u_1 \qquad u = (u_n)_n \in \mathsf{H}.$$

In the following we denote

$$u^{\parallel} = P_1 u \qquad \text{and} \qquad u^{\perp} = (1 - P_1)u.$$

The scalar product and the norm on the Hilbert space H are denoted as

$$\langle u, v\rangle = \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \bar{u}_n(k) v_n(k)\,\mathrm{d}k \qquad \|u\| = \sqrt{\langle u, u\rangle}.$$

Given a function $f \in L^2(\mathsf{B}, \mathrm{d}k)$ we denote by $\langle f\rangle$ the mean value

$$\langle f\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(k)\,\mathrm{d}k. \tag{7}$$

3. Gevrey power series for the Wannier–Stark ladder

3.1. Preliminary result

In order to obtain the formal power series let us give the following preliminary results. Following Wannier [12] we expect that the eigenvalue equation (4) has solutions displaced on infinitely many regular ladders and that any eigenvector of a given ladder, say the ladder with index r, has the same phase factor $\theta_r(k)$ depending on ϵ given by

$$\theta_r(k) = -\frac{1}{\epsilon} \int_0^k [E_r(\tau) - \langle E_r\rangle]\,\mathrm{d}\tau$$

where

$$\langle E_r\rangle = \frac{1}{2\pi} \int_0^{2\pi} E_r(k)\,\mathrm{d}k$$

is the mean value of the band function $E_r(k)$. In order to look for the formal expansion of the rth ladder of solutions of equation (4) we have to factorize such a phase factor by means of a unitary transformation defined as

$$a \in \mathsf{H} \to u = U_2 a \in \mathsf{H}$$

where U_2 is defined as

$$\begin{aligned} U_2 = U_2 &= \exp(\mathrm{i}\theta_r(k))1 \\ &= \mathrm{diag}\left(\exp\left(-\frac{\mathrm{i}}{\epsilon}\int_0^k [E_r(\tau) - \langle E_r\rangle]\,\mathrm{d}\tau\right)\right) \end{aligned}$$

with fixed r; in the following we choose

$$r = 1$$

and we consider the formal power series for the first Wannier–Stark ladder.

Hence, the operator H_ϵ^1 takes the form

$$H_\epsilon^2 = U_2 H_\epsilon^1 U_2^{-1} = \mathrm{i}\epsilon \frac{\mathrm{d}}{\mathrm{d}k} + F + \epsilon X$$

where F is the diagonal matrix given by

$$F = E - (E_1 - \langle E_1\rangle)1 = \mathrm{diag}(E_n - E_1 + \langle E_1\rangle).$$

In such a representation the eigenvalue equation takes the form

$$\left[H_\epsilon^2 - \lambda\right]u = 0. \tag{8}$$

Lemma 4. *Let P_1 be the projection operator on the first band and let $\lambda_0 = \langle E_1\rangle$, then*

$$\langle u, Fu\rangle \geqslant [\lambda_0 + G_1]\|u\|^2 \qquad \forall u \in [1 - P_1]\mathsf{H}$$

and

$$\|[F - \lambda_0]^{-1}[1 - P_1]\| \leqslant [G_1]^{-1}$$

where

$$G_1 = \min_{k\in\mathsf{B}}[E_2(k) - E_1(k)]$$

is the width of the first gap.

Proof. By definition it follows that for any $u \in [1 - P_1]\mathsf{H}$ then

$$\begin{aligned}\langle u, Fu\rangle &= \sum_{n=2}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}[E_n(k) - E_1(k) + \langle E_1\rangle]|u_n(k)|^2\,\mathrm{d}k\\ &\geqslant [G_1 + \langle E_1\rangle]\left[\sum_{n=2}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}|u_n(k)|^2\,\mathrm{d}k\right]\\ &= [G_1 + \langle E_1\rangle]\|u\|^2\end{aligned}$$

proving the first estimate. Furthermore, the second estimate will follow from the previous estimate and the self-adjointness of F. □

3.2. Main results

We give now a formal power series for the solution of the eigenvalue equation (8).

Theorem 1. *Let*

$$\lambda^N = \lambda^N(\epsilon) = \sum_{m=0}^{N}\epsilon^m\lambda_m \tag{9}$$

and

$$u^N = u^N(\epsilon) = \sum_{m=0}^{N}\epsilon^m u_m \tag{10}$$

be two power series with terms recursively defined for any $m \geqslant 1$ as

$$u_m = u_m^\perp + u_m^\parallel \qquad u_m^\perp = (1 - P_1)u_m \qquad u_m^\parallel = P_1 u_m \tag{11}$$

$$u_m^\perp(k) = [\lambda_0 - F]^{-1}\left(\mathrm{i}\frac{\mathrm{d}u_{m-1}^\perp}{\mathrm{d}k} + (1 - P_1)Xu_{m-1} - \sum_{r=1}^{m}\lambda_r u_{m-r}^\perp\right) \tag{12}$$

$$\lambda_1 = \langle X_{1,1}\rangle \qquad \lambda_{m+1} = \left[\left\langle P_1X(1 - P_1)u_m^\perp\right\rangle - \left\langle\sum_{r=2}^{m}\lambda_r u_{m+1-r}^\parallel\right\rangle\right] \tag{13}$$

and

$$u_m^\parallel(k) = \mathrm{i}\int_0^k\left[P_1X(1 - P_1)u_m^\perp - \sum_{r=2}^{m+1}\lambda_r u_{m+1-r}^\parallel\right]\mathrm{d}q \tag{14}$$

where

$$\lambda_0 = \langle E_1 \rangle \qquad \textit{and} \qquad u_0 = \left(\delta_n^1\right)_n. \tag{15}$$

Then $\lambda^N(\epsilon)$ *and* $v^N(\epsilon)$ *satisfy the eigenvalue equation* $\left[H_\epsilon^2 - \lambda\right]u = 0$ *at any order. That is, for any* $N \geqslant 1$ *there exists a positive constant* C_N*, independent of* ϵ*, such that*

$$\left\| \left[H_\epsilon^2 - \lambda^N\right]u^N \right\| \leqslant C_N \epsilon^{N+1}. \tag{16}$$

Furthermore, such a formal power series is of Gevrey type [4]; that is

Theorem 2. *The formal power series (9) and (10) are of Gevrey order* γ *for any* $\gamma > 1$*; that is*

$$|\lambda_m| \leqslant f(m) \qquad \textit{and} \qquad \|u_m\| \leqslant 4f(m) \qquad \forall m \geqslant 1 \tag{17}$$

where

$$f(m) = CK^m\Gamma(1+m\gamma)$$

for some positive constants C and K independent of m, here $\Gamma(\cdot)$ *denotes the Gamma function.*

Remark 1. As appears in the proof of theorem 2 we are not able to obtain estimate (17) with $\gamma = 1$, however (17) could be improved in order to have

$$f(m) = CK^m m!\, \mathrm{e}^{2m\ln(\ln(m))}.$$

3.3. Explicit computation of the first two orders

Here, we explicitly compute the first two terms of the formal power series (13). It turns out that this result agrees with that previously given in [11] by means of a different method. Indeed, from (13) it follows that

$$\begin{aligned}\lambda_2 &= \left\langle P_1 X(1-P_1)u_1^\perp \right\rangle \\ &= \langle P_1 X(1-P_1)[\lambda_0 - F]^{-1}(1-P_1)Xu_0\rangle \\ &= -\sum_{n=2}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{X_{1,n}(k)X_{n,1}(k)}{E_n(k)-E_1(k)}\,\mathrm{d}k.\end{aligned}$$

4. Proof of theorem 1

We split the proof in several steps. Initially we obtain the formal power series, then we show that the vectors u_m defined by (11) belongs to H. Finally, we prove the estimate (16).

4.1. Formal power series

By inserting the formal power series

$$u = \sum_{m=0}^{\infty} \epsilon^m u_m \qquad \text{and} \qquad \lambda = \sum_{m=0}^{\infty} \epsilon^m \lambda_m \tag{18}$$

into the eigenvalue equation $\left[H_\epsilon^2 - \lambda\right]u = 0$ then the eigenvalue equation takes the form

$$\sum_{m=0}^{\infty} \mathrm{i}\epsilon^{m+1}\frac{\mathrm{d}u_m}{\mathrm{d}k} + \sum_{m=0}^{\infty}\epsilon^m F u_m + \sum_{m=0}^{\infty}\epsilon^{m+1}Xu_m - \sum_{r,m=0}^{\infty}\epsilon^{r+m}\lambda_m u_r = 0. \tag{19}$$

For a moment we assume that u_m is such that

$$u_m \in \mathsf{H} \qquad \frac{\mathrm{d}u_m}{\mathrm{d}k} \in \mathsf{H} \qquad Xu_m \in \mathsf{H} \qquad \text{and} \qquad Fu_m \in \mathsf{H} \ \text{ for any } \ m \geqslant 1. \tag{20}$$

By comparing the terms of the same order, with respect to ϵ, it follows that

Order 0. The zeroth order term in equation (19) is given by

$$Fu_0 - \lambda_0 u_0 = 0$$

which has solution given by

$$\lambda_0 = \langle E_1 \rangle$$

and

$$u_0 = \left(\delta_1^n u_0^{\parallel}\right)_n$$

where $u_0^{\parallel}(k)$ is a function belonging to $L^2(\mathsf{B}, \mathrm{d}k)$ that will be defined later.

Order 1. The first order term in equation (19) is given by

$$\mathrm{i}\frac{\mathrm{d}u_0}{\mathrm{d}k} + Fu_1 + Xu_0 - (\lambda_0 u_1 + \lambda_1 u_0) = 0.$$

Let $u_m = u_m^{\parallel} + u_m^{\perp}$, where $u_m^{\parallel} = P_1 u_m$, $u_m^{\perp} = (1 - P_1)u_m$, and let us project the above equation onto the spaces $P_1\mathsf{H}$ and $(1 - P_1)\mathsf{H}$. We obtain the two equations

$$\mathrm{i}\frac{\mathrm{d}u_0^{\parallel}}{\mathrm{d}k} + Fu_1^{\parallel} + P_1 X u_0 - \left(\lambda_0 u_1^{\parallel} + \lambda_1 u_0^{\parallel}\right) = 0 \tag{21}$$

and

$$\mathrm{i}\frac{\mathrm{d}u_0^{\perp}}{\mathrm{d}k} + Fu_1^{\perp} + (1 - P_1) X u_0 - \left(\lambda_1 u_0^{\perp} + \lambda_0 u_1^{\perp}\right) = 0. \tag{22}$$

Recalling that $FP_1 = \langle E_1 \rangle P_1 = \lambda_0 P_1$ and $u_0^{\perp} = 0$ then the first equation takes the form

$$\mathrm{i}\frac{\mathrm{d}u_0^{\parallel}}{\mathrm{d}k} + X_{1,1} u_0^{\parallel} - \lambda_1 u_0^{\parallel} = 0$$

whose solution is given by

$$u_0^{\parallel}(k) = c_0 \exp\left[\mathrm{i}\int_0^k (X_{1,1} - \lambda_1)\,\mathrm{d}q\right]$$

where c_0 is a multiplicative factor and where λ_1 is such that $u_0^{\parallel}(k)$ is a periodic function with period 2π, i.e.

$$\lambda_1 = \langle X_{1,1} \rangle + j \qquad j \in \mathbb{Z}. \tag{23}$$

In particular, choosing $j = 0$ for the sake of definiteness, we have that

$$\lambda_1 = \langle X_{1,1} \rangle = X_{1,1}$$

since $X_{1,1}(k) = \text{const}$ and $u_0^{\parallel}(k) = c_0$. In the following let us choose $c_0 = 1$, hence $u_0^{\parallel}(k) = 1$. The second equation (22) takes the form

$$Fu_1^{\perp} + (1 - P_1) X u_0 - \lambda_0 u_1^{\perp} = 0$$

which has solution given by

$$u_1^{\perp} = [\lambda_0 - F]^{-1}(1 - P_1) X u_0.$$

At present the function $u_1^{\parallel}$ is not defined, it will be given in the following step.

Order 2. The second order term in equation (19) is given by

$$\mathrm{i}\frac{\mathrm{d}u_1}{\mathrm{d}k} + Fu_2 + Xu_1 - (\lambda_0 u_2 + \lambda_1 u_1 + \lambda_2 u_0) = 0.$$

Let us project the above equation onto the spaces $P_1\mathsf{H}$ and $(1 - P_1)\mathsf{H}$. We obtain the two equations

$$\mathrm{i}\frac{\mathrm{d}u_1^\parallel}{\mathrm{d}k} + Fu_2^\parallel + P_1 X u_1 - \left(\lambda_0 u_2^\parallel + \lambda_1 u_1^\parallel + \lambda_2 u_0^\parallel\right) = 0 \tag{24}$$

and

$$\mathrm{i}\frac{\mathrm{d}u_1^\perp}{\mathrm{d}k} + Fu_2^\perp + (1 - P_1) X u_1 - \left(\lambda_0 u_2^\perp + \lambda_1 u_1^\perp + \lambda_2 u_0^\perp\right) = 0. \tag{25}$$

As above, and recalling also that $P_1 X P_1 = \lambda_1$, then equation (24) takes the form

$$\mathrm{i}\frac{\mathrm{d}u_1^\parallel}{\mathrm{d}k} + P_1 X (1 - P_1) u_1^\perp - \lambda_2 u_0^\parallel = 0.$$

By multiplying this equation by $u_0^\parallel$ then it follows that

$$\lambda_2 = \frac{\left\langle u_0^\parallel, P_1 X (1 - P_1) u_1^\perp \right\rangle}{\left\| u_0^\parallel \right\|^2} = \left\langle P_1 X (1 - P_1) u_1^\perp \right\rangle$$

since $u_0^\parallel = c_0 = 1$. Furthermore, the above equation has a periodic solution

$$u_1^\parallel(k) = \left[\mathrm{i}\int_0^k \left(P_1 X (1 - P_1) u_1^\perp - \lambda_2\right) \mathrm{d}q\right].$$

Equation (25) takes the form

$$\mathrm{i}\frac{\mathrm{d}u_1^\perp}{\mathrm{d}k} + Fu_2^\perp + (1 - P_1) X u_1 - \left(\lambda_0 u_2^\perp + \lambda_1 u_1^\perp\right) = 0$$

which has solution given by

$$u_2^\perp = [\lambda_0 - F]^{-1}\left(\mathrm{i}\frac{\mathrm{d}u_1^\perp}{\mathrm{d}k} + (1 - P_1) X u_1 - \lambda_1 u_1^\perp\right).$$

Order m. The mth order term, $m \geqslant 1$, in equation (19) is given by

$$\mathrm{i}\frac{\mathrm{d}u_{m-1}}{\mathrm{d}k} + Fu_m + Xu_{m-1} - \sum_{r=0}^{m} \lambda_r u_{m-r} = 0$$

where all the terms λ_r and u_r are recursively defined for $r = 0, \ldots, m - 1$, but $u_{m-1}^\parallel$. Let us project the above equation onto the spaces $P_1\mathsf{H}$ and $(1 - P_1)\mathsf{H}$. We obtain the two equations

$$\mathrm{i}\frac{\mathrm{d}u_{m-1}^\parallel}{\mathrm{d}k} + Fu_m^\parallel + P_1 X u_{m-1} - \sum_{r=0}^{m} \lambda_r u_{m-r}^\parallel = 0 \tag{26}$$

and

$$\mathrm{i}\frac{\mathrm{d}u_{m-1}^\perp}{\mathrm{d}k} + Fu_m^\perp + (1 - P_1) X u_{m-1} - \sum_{r=0}^{m} \lambda_r u_{m-r}^\perp = 0.$$

As above, and recalling that $FP_1 = \lambda_0 P_1$ and $P_1 X P_1 = \lambda_1$, then equation (26) takes the form

$$\mathrm{i}\frac{\mathrm{d}u_{m-1}^\parallel}{\mathrm{d}k} + P_1 X (1 - P_1) u_{m-1}^\perp - \sum_{r=2}^{m} \lambda_r u_{m-r}^\parallel = 0$$

from which it follows that λ_m is given by

$$\lambda_m = \frac{\left\langle u_0^{\|}, P_1 X(1-P_1)u_{m-1}^{\perp}\right\rangle - \left\langle u_0^{\|} \sum_{r=2}^{m-1} \lambda_r u_{m-r}^{\|}\right\rangle}{\left\| u_0^{\|}\right\|^2}$$
$$= \left\langle P_1 X(1-P_1)u_{m-1}^{\perp}\right\rangle - \left\langle \sum_{r=2}^{m-1} \lambda_r u_{m-r}^{\|}\right\rangle$$

and that

$$u_{m-1}^{\|}(k) = \left[\mathrm{i}\int_0^k \left(P_1 X(1-P_1)u_{m-1}^{\perp} - \sum_{r=2}^{m} \lambda_r u_{m-r}^{\|}\right)\mathrm{d}q\right]$$

is a periodic solution. Equation (26) has solution given by

$$u_m^{\perp} = [\lambda_0 - F]^{-1}\left(\mathrm{i}\frac{\mathrm{d}u_{m-1}^{\perp}}{\mathrm{d}k} + (1-P_1)Xu_{m-1} - \sum_{r=1}^{m} \lambda_r u_{m-r}^{\perp}\right).$$

4.2. Regularity of the functions u_m

Now we have to prove that u_m satisfies conditions (20) for any $m \geqslant 0$. By induction let us assume that (20) hold for some m (they are certainly true for $m = 0$) and then we prove that they hold for $m + 1$. Indeed $u_{m+1} \in \mathsf{H}$ and $Xu_{m+1} \in \mathsf{H}$ since X and $[\lambda_0 - F]^{-1}(1 - P_1)$ are bounded operators (lemmas 3 and 4). Furthermore $\frac{\mathrm{d}u_{m+1}}{\mathrm{d}k} \in \mathsf{H}$ since $X^{(1)}$ is a bounded operator (lemma 3) and

$$\frac{\mathrm{d}[\lambda_0 - F]^{-1}}{\mathrm{d}k} = \mathrm{diag}\left(\frac{E_n' - E_1'}{(E_n - E_1)^2}\right)$$

is bounded too since $\frac{E_n' - E_1'}{(E_n - E_1)^2} \sim 1$ as $n \to \infty$ (lemma 3).

4.3. Estimate (16)

Now, we have to prove the estimate (16). Indeed, by definition, it follows that

$$\left[H_\epsilon^2 - \lambda^N\right]u_N = \epsilon^{N+1} v_N$$

where

$$v_N = \mathrm{i}\frac{\mathrm{d}u_N}{\mathrm{d}k} + Xu_N - \sum_{m=N+1}^{2N} \epsilon^{m-(N+1)} \sum_{r=0}^{m} \lambda_r u_{m-r}$$

is such that $\|v_N\| \leqslant C_N$ for some positive constant C_N independent of ϵ since $u_m \in \mathsf{H}$, $m = 0, 1, \ldots, N$, does not depend on ϵ, $\frac{\mathrm{d}u_N}{\mathrm{d}k} \in \mathsf{H}$ and $Xu_N \in \mathsf{H}$.

5. Proof of theorem 2

In order to prove theorem 2 let us introduce the following notation. Let $\rho \in [0, \kappa]$ be given and let $u \in \mathsf{H}$ be analytic on the strip S_ρ, that is $u = (u_n)_n$ is such that the functions $u_n(k)$ are analytic on the same strip S_ρ. We denote

$$\lceil u \rceil_\rho = \left\{\sum_{n=1}^{\infty} \sup_{k \in S_\rho} |u_n(k)|^2\right\}^{1/2}.$$

It immediately follows that

$$\|u\| \leqslant \lceil u \rceil_\rho \qquad \forall \rho \geqslant 0 \qquad \text{and} \qquad \lceil u \rceil_{\rho_1} \leqslant \lceil u \rceil_{\rho_2} \qquad \rho_1 \leqslant \rho_2.$$

From (15) it follows that u_0 is analytic on S_ρ for any $\rho \geqslant 0$ and $\lceil u_0 \rceil_\rho = 1$.

We now prove theorem 2 by induction; we assume that u_m is analytic on a strip S_{ρ_m} and that for any $\gamma > 1$ fixed there exist C and K such that

$$|\lambda_m| \leqslant C(mK)^{m\gamma} \qquad \text{and} \qquad \lceil u_m \rceil_{\rho_m} \leqslant 4C(mK)^{m\gamma} \tag{27}$$

where

$$\rho_0 = \kappa \qquad \rho_{m+1} = \rho_m - \frac{\kappa}{2(m+2)[\ln(e+m)]^2}.$$

The estimates (27) are clearly true for $m = 0$ and $m = 1$.

In particular we assume that K is large enough and C is small enough; that is

$$C \leqslant \frac{1}{4(1+2^{2\gamma})} \leqslant 1 \qquad \frac{8c_1}{c_2} \leqslant K \qquad \text{and} \qquad \frac{16 m^{1-\gamma}\ln^2(m)}{\kappa c_2} \leqslant K \qquad \forall m \geqslant 1 \tag{28}$$

where $c_1 \geqslant \|X\|$ will be defined later (see equation (30)) and

$$c_2 = \inf\left\{1, \inf_{k\in S_\kappa} |E_2(k) - E_1(k)|\right\} > 0.$$

Now, we prove that u_{m+1} is analytic on the strip $S_{\rho_{m+1}}$ and the estimates (27) hold when we replace m by $m+1$. Indeed, from (13) we have that

$$\begin{aligned}
|\lambda_{m+1}| &\leqslant \left|\left\langle P_1 X(1-P_1)u_m^\perp\right\rangle\right| + \sum_{r=2}^{m} |\lambda_r| \cdot \left\|u_{m+1-r}^\parallel\right\| \\
&\leqslant \|X\| \cdot \|u_m\| + \sum_{r=2}^{m} |\lambda_r| \cdot \|u_{m+1-r}\| \\
&\leqslant \|X\| \cdot \lceil u_m \rceil_{\rho_m} + \sum_{r=2}^{m} |\lambda_r| \cdot \lceil u_{m+1-r} \rceil_{\rho_{m+1-r}} \\
&\leqslant \|X\| 4C(Km)^{m\gamma} + \sum_{r=2}^{m} C(Kr)^{r\gamma} 4C[K(m+1-r)]^{(m+1-r)\gamma} \\
&\leqslant 4\|X\| C(Km)^{m\gamma} + 4C^2 K^{(m+1)\gamma} \left\{ m^{m\gamma} + \sum_{r=2}^{m-1} r^{r\gamma}(m+1-r)^{(m+1-r)\gamma} \right\} \\
&\leqslant CK^{(m+1)\gamma} m^{m\gamma} + 4C^2 K^{(m+1)\gamma}(1+2^{2\gamma}) m^{m\gamma} \\
&\leqslant CK^{(m+1)\gamma}(m+1)^{(m+1)\gamma}
\end{aligned}$$

where $m \geqslant 2$ and since K and C satisfy (28).

Concerning the estimate of u_{m+1} from (14) it immediately follows that u_{m+1} is analytic on the same strip S_{ρ_m} and the same arguments as above give that

$$\begin{aligned}
\left\lceil u_{m+1}^\parallel \right\rceil_{\rho_{m+1}} &\leqslant CK(Km)^{m\gamma} + 4C^2(1+2^{2\gamma}) m^{m\gamma} K^{(m+1)\gamma} + |\lambda_{m+1}| \\
&\leqslant 2C[K(m+1)]^{(m+1)\gamma}.
\end{aligned}$$

In order to treat the last term $u_{m+1}^\perp$ we introduce the following lemma:

Lemma 5. *Let u be analytic on a given strip S_ρ for some $\rho \leqslant k$. Then the vectors $[\lambda_0 - F]^{-1}u$, Xu and $\frac{\mathrm{d}u}{\mathrm{d}k}$ are analytic on the same strip. Furthermore the following estimates hold:*

$$\lceil[\lambda_0 - F]^{-1}u\rceil_\rho \leqslant c_2^{-1}\lceil u\rceil_\rho \tag{29}$$

$$\lceil Xu\rceil_\rho \leqslant c_1\lceil u\rceil_\rho \tag{30}$$

for some positive constant $c_1 \geqslant \|X\|$, and

$$\left\lceil\frac{\mathrm{d}u}{\mathrm{d}k}\right\rceil_\rho \leqslant \frac{1}{\mu-\rho}\lceil u\rceil_\mu \qquad \mu > \rho. \tag{31}$$

Proof. The above estimate immediately follows from the analyticity of the band and Bloch functions on S_κ. Indeed, by definition it follows that

$$([\lambda_0 - F]^{-1}u)_n = \frac{u_n(k)}{E_1(k) - E_n(k)}$$

and so

$$\lceil[\lambda_0 - F]^{-1}u\rceil_\rho^2 = \sum_{n=1}^{\infty}\sup_{k\in S_\rho}\left|\frac{u_n(k)}{E_1(k)-E_n(k)}\right|^2 \leqslant \frac{\lceil u\rceil_\rho^2}{c_2^2}$$

since

$$\inf_{k\in S_\rho}|E_1(k) - E_n(k)| \geqslant \inf_{k\in S_\rho}|E_1(k) - E_2(k)| \geqslant c_2$$

for any n. In order to prove estimate (30) let

$$(Xu)_n = \sum_{p=1}^{\infty} X_{n,p}(k)u_p(k)$$

where $X_{n,p}(k)$ are given by lemma 1. Then, the same arguments as [9] prove that

$$\begin{aligned}\lceil Xu\rceil_\rho^2 &\leqslant \sum_{n,p=1}^{\infty}\sup_{k\in S_\rho}[|X_{n,p}(k)|^2|u_p(k)|^2]\\ &\leqslant \sum_{n,p=1}^{\infty}\Big[\sup_{k\in S_\rho}|X_{n,p}(k)|^2\Big]\cdot\Big[\sup_{k\in S_\rho}|u_p(k)|^2\Big]\\ &\leqslant c_1\lceil u\rceil_\rho\end{aligned}$$

for some positive constant c_1 independent of $\rho \leqslant \kappa$. Finally, estimate (31) follows from the analyticity of the function u and from the Cauchy theorem. □

Finally, from (12) we have that

$$\begin{aligned}\lceil u_{m+1}^\perp\rceil_{\rho_{m+1}} &\leqslant c_2^{-1}\left[\left\lceil\frac{\mathrm{d}u_m^\perp}{\mathrm{d}k}\right\rceil_{\rho_{m+1}} + \lceil Xu_m\rceil_{\rho_m} + \sum_{r=1}^{m+1}|\lambda_r|\lceil u_{m+1-r}\rceil_{\rho_{m+1}}\right]\\ &\leqslant c_2^{-1}\left[\frac{1}{\rho_m-\rho_{m+1}}\lceil u_m\rceil_{\rho_m} + c_1\lceil u_m\rceil_{\rho_{m+1}} + \sum_{r=1}^{m+1}|\lambda_r|\lceil u_{m+1-r}\rceil_{\rho_{m+1-r}}\right]\\ &\leqslant c_2^{-1}\left\{\left[\frac{1}{\rho_m-\rho_{m+1}} + c_1\right]\lceil u\rceil_{\rho_m} + \sum_{r=1}^{m} C(rK)^{r\gamma}4C[K(m-r+1)]^{(m-r+1)\gamma}\right\}\\ &\leqslant c_2^{-1}\left\{\left[2\frac{(m+2)\ln^2(m+e)}{k} + c_1\right]4C(Km)^{m\gamma} + 4C^2K^{(m+1)\gamma}(1+2^{2\gamma})m^{m\gamma}\right\}\\ &\leqslant 2C[K(m+1)]^{\gamma(m+1)}\end{aligned}$$

which completes the proof of theorem 2.

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