Canonical Expansion of $\mathcal{PT}-$Symmetric Operators and Perturbation Theory

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Abstract

Let H be any $\mathcal{PT}$ symmetric Schrödinger operator of the type $-\hbar^2\Delta+(x_1^2+\ldots+x_d^2)+igW(x_1,\ldots,x_d)$ on $L^2(\mathbf{R}^d)$, where W is any odd homogeneous polynomial and $g\in\mathbf{R}$. It is proved that $\mathcal{P}H$ is self-adjoint and that its eigenvalues coincide (up to a sign) with the singular values of H, i.e. the eigenvalues of $\sqrt{H^*H}$. Moreover we explicitly construct the canonical expansion of H and determine the singular values μ_j of H through the Borel summability of their divergent perturbation theory. The singular values yield estimates of the location of the eigenvalues λ_j of H by Weyl's inequalities.

1 Introduction and statement of the results

A Schrödinger operator $H=-\Delta+V$ acting on $\mathcal{H}=L^2(\mathbf{R}^d)$ is called $\mathcal{PT}-$symmetric if it is left invariant by the $\mathcal{PT}$ operation. While generally speaking $\mathcal{P}$ could be the parity operator with respect to at least one variable, here for the sake of simplicity we consider only the case in which $\mathcal{P}$ is the parity operator with respect to all variables, $(\mathcal{P}u)(x_1,\ldots,x_d)=u(-x_1,\ldots,-x_d)$, and $\mathcal{T}$ the complex conjugation (equivalent to time-reversal symmetry) $(\mathcal{T}u)(x_1,\ldots,x_d):=\overline{u}(x_1,\ldots,x_d)$. The condition

$$\overline{V}(-x_1,\ldots,-x_d)=V(x_1,\ldots,x_d) \tag{1.1}$$

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defines the $\mathcal{PT}-$symmetry on the potential $V(x_1,\dots,x_d)$. The $\mathcal{PT}$-symmetric operators are currently the object of intense investigation because, while not self-adjoint, they admit in many circumstances a real spectrum. Hence the investigation is motivated (at least partially), by an attempt to remove the self-adjointess condition on the observables of standard quantum mechanics (see e.g.[1],[2],[3],[4],[5],[6], [7], [8],[9]).

The simplest and most studied class of $\mathcal{PT}$ symmetric operators is represented by the *odd anharmonic oscillators with purely imaginary coupling* in dimension one, namely the maximal differential operators in $L^2(\mathbf{R})$

$$Hu(x) := [-\frac{d^2}{dx^2} + x^2 + igx^{2m+1}], \quad g\in\mathbf{R}, \quad m=1,2,\dots \tag{1.2}$$

It has long been conjectured (Bessis Zinn-Justin), and recently proved [10], [11], that the spectrum $\sigma(H)$ is real for all g; there are however examples of one-dimensional $\mathcal{PT}$-symmetric operators with *complex* eigenvalues[5].

Now recall that there is a natural additional notion of spectrum associated with a non-normal operator T in a Hilbert space which is by construction real. Any closed operator T admits a *polar decomposition* ([12], Chapt. VI.7) $T=U|T|$, where $|T|$ is self-adjoint and U is unitary. The modulus of T is the self-adjoint operator $|T|=\sqrt{T^*T}$. The (obviously real and positive) eigenvalues of $|T|$ are called the *singular values* of T. In this paper we consider the self-adjoint operator $\sqrt{H^*H}$; its eigenvalues $\mu_j;\ j=0,1,\dots$, necessarily real and positive, are the by definition the *singular values* of H. A first immediate question arising in this context is to determine how these singular values are related to the $\mathcal{PT}$-symmetry of H. A related question is the explicit construction of the canonical expansion of H (see e.g.[12]) in terms of the spectral analysis of $\sqrt{H^*H}$, which entails the diagonalization of H with respect to a pair of dual bases (which do not form a biorthogonal pair); a further one is the actual computation of the singular values. The determination of the singular values reflects directly on the object of physical interest, namely the eigenvalues $\lambda_j;\ j=0,1,\dots$ of H. If the eigenvalues and the singular values are ordered according to increasing modulus, the Weyl inequalities (see e.g. [18]) indeed yield

$$\sum_{j=1}^{k}|\lambda_j| \le \sum_{j=1}^{k}\mu_j, \quad |\lambda_1\cdots\lambda_k| \le \mu_1\cdots\mu_k, \quad k=1,2,\dots \tag{1.3}$$

We intend in this paper to give a reply to these questions for the most general class of odd anharmonic oscillator in $\mathbf{R}^d$. Namely, we consider in $L^2(\mathbf{R}^d)$ the Schrödinger operator family

$$H(g)u(x) := H_0u(x) + igW(x)u(x), \quad x = (x_1, \ldots, x_d) \in \mathbf{R}^d \tag{1.4}$$

Here:

1. W is a real homogenous polynomial of odd order $2K+1$, $K = 1, 2\ldots$;

$$W(\lambda x) = \lambda^{2K+1}W(x)$$

2. H_0 is the Schrödinger operator of the harmonic oscillator in $\mathbf{R}^d$:

$$H_0u(x) = -\Delta u(x) + x^2u(x), \quad x^2 := x_1^2 + \ldots + x_d^2 \tag{1.5}$$

Under these conditions the operator family $H(g)$, which is obviously $\mathcal{PT}$-symmetric (see below for the mathematical definition), but non self-adjoint, enjoys the following properties (proved in [13] for $d = 1$ and in [14] for $d > 1$; see below for a more detailed statement):

1. The operator $H(g)$, defined as the closure of the minimal differential operator $\dot{H}(g)u = -\Delta u(x) + x^2u(x) + igW(x)$, $u \in C_0^\infty(\mathbf{R}^d)$, generates a holomorphic operator family with compact resolvents with respect to g in some domain $\mathcal{S} \subset \mathbf{C}$, with $H(g)^* = H(\overline{g})$. An operator family $T(g)$ depending on the complex variable $g \in \Omega$, where $\Omega \subset \mathbf{C}$ is open is holomorphic (see [12], VII.1) if the scalar products $\langle u, T(g)v\rangle$ are holomorphic functions of $g \in \Omega$ $\forall\,(u, v) \in T(g)$ and the resolvent $[T(g) - zI]^{-1}$ exist for at least one $g \in \Omega$.

2. All eigenvalues of $H_0 := H(0)$ are stable with respect to the operator family $H(g)$. This means (see e.g.[12], VIII.1) that if λ_0 is any eigenvalue of $H(0)$ of multiplicity m, there is $B(\lambda_0) > 0$ such that $H(g)$ has exactly m (repeated) eigenvalues $\lambda_j(g) : j = 1, \ldots, m$ near λ_0 for $g \in \mathcal{S}$, $|g| < B(\lambda_0)$, and $\lim_{g\to 0, g\in\mathcal{S}} \lambda_j(g) = \lambda_0$.

3. The (Rayleigh-Schrödinger) perturbation series of each eigenvalue $\lambda(g)$ of $H(g)$ is Borel summable to $\lambda(g)$.

We denote $\mu_j(g) : j = 0, 1, \ldots$ the singular values of $H(g) : g \in \mathbf{R}$, i.e. the eigenvalues of $\sqrt{H(g)^*H(g)} = \sqrt{H(-g)H(g)}$.
Our first result concerns the identification of the singular values as the eigenvalues of a self-adjoint operator directly associated with $H(g)$ by the operator-theoretic implementation of the recently isolated *pseudohermiticity* notion ([1],[15],[16],[17]) in terms of the $\mathcal{P}$ symmetry itself.

Consider indeed the operator family $Q(g) := \mathcal{P}H(g)$. We will show that $D(Q(g)) = D(H(g))$. Since $[\mathcal{P}, H_0] = 0$, the explicit action of $Q(g)$ is

$$\begin{aligned} Q(g)u(x) &= H_0u(-x) + igW(-x)u(-x) \\ &= H_0u(-x) - igW(x)u(-x) = H(-g)\mathcal{P}u(x) \end{aligned}$$

Then we have:

Theorem 1.1 *Let $Q(g)$ be defined as above and $Q'(g) := H(g)\mathcal{P}$. Then:*

1. *If $g \in \mathbf{R}$ the operator families $Q(g)$ and $Q'(g)$ are self-adjoint.*

2. *The operator family $Q(g)$ defined on $D(H(g))$ is holomorphic with compact resolvents at least for g in a neighbourhood of $\mathbf{R}_+$.*

3. *If $g \in \mathbf{R}$ the eigenvalues of $Q(g)$ and of $\sqrt{H(g)^*H(g)}$ coincide (up to the sign);*

Remarks

1. $H(g)^* = H(-g)$ for $g \in \mathbf{R}$ by $\mathcal{PT}$-symmetry. Hence the relation $Q(g) = \mathcal{P}H(g) = H(-g)\mathcal{P} = H(g)^*\mathcal{P}$ can be equivalently written $\mathcal{P}H(g)\mathcal{P}^{-1} = H(g)^*$ which is the $\mathcal{P}-$ pseudohermiticity property of $H(g)$ [16].

2. The eigenvalues μ of the operator $Q(g)$ clearly solve the generalized spectral problem $H(g)u = \mu\mathcal{P}u$ (for this notion, see [12], SVII.6). Explicitly:

$$(H_0 + igW)u(x) = \mu(\mathcal{P}u)(x) \tag{1.6}$$

   By the above theorem the singular values coincide (up to a sign) with the generalized eigenvalues.

As a consequence of this, we obtain the explicit canonical expansion of $H(g)$ in terms of the eigenvectors ψ_k of Q and of the $\mathcal{P}$ operation:

Corollary 1.2 *Let $\{\psi_k(g)\} : k = 0, 1, \ldots$ be the eigenvectors of $Q(g)$, and μ_k the corresponding eigenvalues (counting multiplicy). Then $H(g)$ admits the following canonical expansion*

$$H(g)u = \sum_{k=0}^{\infty} \mu_k \langle u, \psi_k \rangle \mathcal{P}\psi_k, \quad u \in D(H(g)) \tag{1.7}$$

Remarks

1. Since $\mathcal{P}\psi_n(x) = \psi_n(-x)$, the canonical expansion (1.7) entails that $H(g)$ can be diagonalized in terms of the (repeated) real singular values μ_n and of the pair of orthonormal bases $\{\psi_n(x)\}$ and $\{\psi_n(-x)\}$.

2. For a general operator with compact resolvent the canonical expansion reads

$$Tu = \sum_{k=0}^{\infty} \mu_k \langle u, \psi_k \rangle \psi_k', \quad u \in D(T) \tag{1.8}$$

   Here $\{\mu_k\}$ is the sequence of singular values of T, ψ_k the corresponding eigenvectors, but the dual basis $\{\psi_k'\}$ is a priori unknown. In this case it is simply the $\mathcal{P}$-dual basis $\mathcal{P}\psi_k$. Remark that the orthogonal bases ψ_k and $\mathcal{P}\psi_k$ do not form a biorthogonal set.

3. The expansion (1.7) is useful even when all eigenvalues of $H(g)$ are real, because $H(g)$ is not normal and the spectral theorem does not hold.

4. Finally we note the following relation involving nonzero eigenvalues and eigenvectors on one side and nonzero singular values and corresponding eigenvectors on the other side: if $H(g)\psi_k = \mu_k(g)\mathcal{P}\psi_k$, and $H(g)\phi_l = \lambda_l(g)\phi_l$, then:

$$\lambda_l(g)\langle \phi_l, \mathcal{P}\psi_k(g) \rangle = \mu_k(g)\langle \phi_l, \psi_k(g) \rangle \tag{1.9}$$

   One has indeed (omitting the g-dependence):

$$\lambda_l \langle \phi_l, \mathcal{P}\psi_k \rangle = \langle H\phi_l, \mu_k^{-1} H\psi_k \rangle = \langle H\phi_l, \mu_k^{-1} H^* H\psi_k \rangle = \mu_k \langle \phi_l, \psi_k \rangle$$

Our third result deals with the actual computation of the singular values $\mu_j(g)$. To formulate the result, remark that the closed subspaces $\mathcal{P}\mathcal{H}$ and $(1-\mathcal{P})\mathcal{H}$ are invariant under H_0 because $[\mathcal{P}, H_0] = 0$. The operator $\mathcal{P}H_0$ has the same eigenvectors of H_0, but the eigenvalues $\lambda_l = 2l_1 + \ldots + 2l_d + d$, $l_k = 0, 1, \ldots,$ $k = 1, \ldots, d$, of H_0 split into *even* and *odd* eigenvalues. More precisely, introduce the usual *principal quantum number* $l := l_1 + \ldots + l_d : l = 0, 1, \ldots$. Then the eigenvalues of H_0 are $\lambda_l = 2l + d$, with multiplicity $m(l) = l^{d-1}$. The eigenvalues of $\mathcal{P}H_0$ are

$$\lambda_l = \begin{cases} 2l + d, & l \text{ even} \\ -(2l + d), & l \text{ odd} \end{cases} \tag{1.10}$$

The corresponding eigenvectors will be $\mathcal{P}$ even and $\mathcal{P}$ odd, respectively. We then have:

Theorem 1.3

1. *All eigenvalues λ_l of $\mathcal{P}H_0$ are stable as eigenvalues $\mu_j(g) : j = 1, \ldots, m(l)$ of $Q(g)$ as $|g| \to 0$, $g \in \mathcal{S}_1 \cup \mathcal{S}_2$ where:*

$$\mathcal{S}_1 \quad := \quad \{g \in \mathbf{C} \setminus \{0\} : -\frac{\pi}{2} < \arg g < \frac{\pi}{2}\} \tag{1.11}$$

$$\mathcal{S}_2 \quad := \quad \{g \in \mathbf{C} \setminus \{0\} : \frac{\pi}{2} < \arg g < \frac{3\pi}{2}\} \tag{1.12}$$

2. *All eigenvalues $\mu_j(g) : j = 1, \ldots, m(l)$ are holomorphic on the Riemann surface sector*

$$\mathcal{S}_{K,\delta} := \{g \in \mathbf{C} : 0 < |g| < B(\delta); -(2K+1)\frac{\pi}{4} + \delta < \arg(g) < (2K+1)\frac{\pi}{4} - \delta\}$$

where $\delta > 0$ is arbitrary.

3. *The Rayleigh-Schrödinger perturbation expansion for any eigenvalue $\mu_j(g) : j = 1, \ldots, m(l)$ of $Q(g)$ near the eigenvalue λ_l of $\mathcal{P}H_0$ for $|g|$ small is Borel summable to $\mu_j(g) : j = 1, \ldots, m(l)$.*

Remark

Let $\mu(g)$ be a singular value near an unperturbed eigenvalue λ. The Borel summability (see e.g.[19], Chapter XII.5) means that it can be uniquely reconstructed through its divergent perturbation expansion $\sum_{s=0}^{\infty} \mu_s g^s$, $\mu_o = \lambda$ in the following way:

$$\mu(g) = \frac{1}{q} \int_0^{\infty} \mu_B(gt) e^{-t^{1/q}} t^{-1+1/q} \, dt \tag{1.13}$$

Here $q = \dfrac{2K-1}{2}$ and $\mu_B(g)$, the *Borel transform of order* q of the perturbation series, is defined by the power series

$$\mu_B(g) = \sum_{s=0}^{\infty} \frac{\mu_s}{\Gamma[q(s+1)]} g^s$$

which has a positive radius of convergence. The proof of (1.13) consists precisely in showing that $\mu_B(g)$ has analytic continuation along the real positive axis and that the integral converges for some $0 \le g < B$, $B > 0$.

Example

The Hénon-Heiles potential, i.e. the third degree polynomial in $\mathbf{R}^2$

$$W(x) = x_1^2 x_2$$

2 Proof of the results

Let us begin by a more detailed quotation of Theorem 1.1 of [14]. The results are more conveniently formulated in the variable $\beta = ig$ instead of g.

Let $\beta \in \mathbf{C}$, $0 < |\arg \beta| < \pi$, and let $\dot{H}(\beta)$ denote the minimal differential operator in $L^2(\mathbf{R}^d)$ defined by $-\Delta + x^2 + \beta W(x)$ on $C_0^\infty(\mathbf{R}^d)$, with $x^2 = x_1^d + \ldots + x_d^2$. Then

(N1) $\dot{H}(\beta)$ is closable. Denote $H(\beta)$ its closure.

(N2) $H(\beta)$ represents a pair of type-A holomorphic families in the sense of Kato for $0 < \arg\beta < \pi$ and $-\pi < \arg\beta < 0$, respectively, with $H(\beta)^* = H(\overline{\beta})$. Recall that an operator family $T(g)$ depending on the complex variable g belonging to some open set $\Omega \subset \mathbf{C}$ is called type-A holomorphic if its domain D does not depend on g and the scalar products $\langle u, T(g)\rangle$ are holomorphic functions for $g \in D$ $\forall\ (u, v) \in D$. A general theorem of Kato ([12], VII.2) states that the isolated eigenvalues of a type-A holomorphic family are locally holomorphic functions of $g \in D$ with at most algebraic branch points.

(N3) $H(\beta)$ has compact resolvent $\forall\, \beta \in \mathbf{C}$, $0 < |\arg \beta| < \pi$.

(N4) All eigenvalues of $H_0 = H(0)$ are stable with respect to the operator family $H(\beta)$ for $\beta \to 0$, $0 < |\arg \beta| < \pi$;

(N5) Let $\beta \in \mathbf{C}$, $\sigma \in \mathbf{C}$, $0 < |\arg\beta| < \pi$, $-\pi + \arg\beta \le \arg\sigma \le \arg\beta$, and let $\dot{H}_\sigma(\beta)$ denote the minimal differential operator in $L^2(\mathbf{R}^d)$ defined by $-\Delta + \sigma x^2 + \beta W(x)$ on $C_0^\infty(\mathbf{R}^d)$. Then $\dot{H}_\sigma(\beta)$ is sectorial (and hence closable) because its numerical range is contained in the half-plane $\{z \in \mathbf{C} : -\pi + \arg\beta \le \arg\sigma \le \arg\beta\}$;

(N6) Let $H_\sigma(\beta)$ denote the closure of $\dot{H}_\sigma(\beta)$. Let $\sigma \in \mathbf{C}, \sigma \notin]-\infty, 0]$. Then the operator family $\beta \mapsto H_\sigma(\beta)$ is type-A holomorphic with compact resolvents for $\beta \in \mathcal{C}_\sigma := \{\beta \in \mathbf{C} : 0 < \arg\beta - \arg\sigma < \pi\}$. Moreover if $\beta \in \mathbf{C}, \mathrm{Im}\beta > 0$, the operator family $\sigma \mapsto H_\sigma(\beta)$ is type-A holomorphic with compact resolvents in the half-plane $\mathcal{D}_\beta = \{\sigma \in \mathbf{C} : 0 < \arg\beta - \arg\sigma < \pi\}$

Let us now introduce the operator

$$H(\beta, \theta) = e^{-2\theta}\Delta + e^{2\theta}x^2 + \beta e^{2(K+1)\theta}W(x) := e^{-2\theta}K(\beta, \theta) \tag{2.1}$$

For $\theta \in \mathbf{R}$ $H(\beta, \theta)$ is unitarily equivalent to $H(\beta)$, $\mathrm{Im}\beta > 0$, via the dilation operator defined by

$$(U(\theta)\psi)(x) = e^{d\theta/2}\psi(e^\theta x), \qquad \forall\, \psi \in L^2(\mathbf{R}^d) \tag{2.2}$$

As a consequence of (N6) (see again [14], or also [20], where all details are worked out for $d = 1$, and where the reader is referred also for the proof of statement(N8) below) we have:

(N7) $H(\beta, \theta)$ defined on $D(H(\beta))$ represents a type-A holomorphic family with compact resolvents for β and θ such that $s = \arg\beta$, $t = \mathrm{Im}\theta$ are variable in the parallelogram $\mathcal{R}$ defined as

$$\mathcal{R} = \{(s,t) \in \mathbf{R}^2 : 0 < (2K-1)t + s < \pi, 0 < (2K+3)t + s < \pi\}, \tag{2.3}$$

Moreover C_0^∞ is a core of $H(\beta, \theta)$. The spectrum of $H(\beta, \theta)$ does not depend on θ. Note that $(s,t) \in \mathcal{R}$ entails that the maximal range of β is $-(2K-1)\pi/4 < \arg\beta < (2K+3)\pi/4$ and that the maximal range of θ is $-\pi/4 < \mathrm{Im}\theta < \pi/4$;

N8) Let β and θ be such that $(s,t) \in \mathcal{R}$. Then:

(i) If $\lambda \notin \sigma(K(0,\theta))$, then $\lambda \in \tilde{\Delta}$, where:

$$\begin{aligned}\tilde{\Delta} := \{z \in \mathbf{C} : z \notin \sigma(K(\beta,\theta)); \|[z-(K(\beta,\theta)]^{-1}\| \\ \text{is uniformly bounded for } |\beta| \to 0\};\end{aligned} \tag{2.4}$$

(ii) If $\lambda \in \sigma(K(0,\theta))$, then λ is stable with respect to the operator family $K(\beta,\theta)$.

(N7) and (N8) entail:

(N9) Let $\beta \in \mathbf{C}$ with $0 < \arg(\beta) < \pi$. Then for any $\delta > 0$ and any eigenvalue $\lambda(g)$ of $H(\beta)$ there exists $\rho > 0$ such that the function $\lambda(\beta)$, a priori holomorphic for $0 < |g| < \rho$, $\delta < \arg(\beta) < \pi - \delta$, has an analytic continuation to the Riemann surface sector $\tilde{\mathcal{S}}_{K,\delta} := \{\beta \in \mathbf{C} : 0 < |\beta| < \rho; -(2K-1)\frac{\pi}{4} + \delta < \arg(\beta) < (2K+3)\frac{\pi}{4} - \delta\}$.

Remarks

1. The stability statement means the following: if $r > 0$ is sufficiently small, so that the only eigenvalue of $K(0,\theta)$ enclosed in $\Gamma_r := \{z \in \mathbf{C} : |z-\lambda| = r\}$ is λ, then there is $B > 0$ such that for $|\beta| < B$ $\dim P(\beta,\theta) = \dim P(0,\theta)$, where

$$P(\beta,\theta) = \frac{1}{2\pi i}\oint_{\Gamma_r} [z-(K(\beta,\theta)]^{-1}\, dz \tag{2.5}$$

is the spectral projection of $K(\beta,\theta)$ corresponding to the points of the spectrum enclosed in $\Gamma_r \subset \mathbf{C} \setminus \sigma(K(\beta,\theta))$. Similarly for $P(0,\theta)$.

2. Starting from the operator $H(\beta)$, $\mathrm{Im}\beta < 0$, analogous results hold for the operator family $H(\beta,\theta)$ where this time β and θ are such that $s = \arg\beta$ and $t = \mathrm{Im}\theta$ describe the parallelogram

$$\mathcal{R}^1 = \{(s,t) \in \mathbf{R}^2 : -\pi < (2K-1)t + s < 0, -\pi < (2K+3)t + s < 0\}. \tag{2.6}$$

Moreover, $H(\beta,\theta)^* = H(\overline{\beta},\overline{\theta})$.

We now set $\beta = ig$ and with slight abuse of notation the operator $H(\beta) = H(ig)$ will be denoted $H(g)$.

Let once again $\mathcal{P}$ denote the parity operator in $\mathcal{H}$

$$\mathcal{P}\psi(x) = \psi(-x), \qquad \forall\, \psi \in \mathcal{H}$$

P is a self-adjoint, unitary involution, i.e. $\mathcal{P}^2 = I$, and $\mathcal{P}W(x) = -W(x)\,\forall\, x \in \mathbf{R}^d$.

To show Theorem 1.1, let us first state and prove the following preliminary result:

Proposition 2.1 *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be the complex sectors defined by (1.11,1.12). Then:*

(1) $D(\mathcal{P}H(g)) = D(H(g)\mathcal{P}) = D(H(g)^*\mathcal{P}) = D(\mathcal{P}H(g)^*) = D(H(g))$ for all $g \in \mathcal{S}_1 \cup \mathcal{S}_2$;

(2) $\mathcal{P}H(g)) = H(-g)\mathcal{P}$ for all $g \in \mathcal{S}_1 \cup \mathcal{S}_2$. In particular, for $g \in \mathbf{R}$, $\mathcal{P}H(g) = H(g)^*\mathcal{P}$ whence $\mathcal{P}H(g)\mathcal{P} = H(g)^*$, i.e. $H(g)$ and $H(g)^*$ are unitarily equivalent;

(3) $\overline{\mathcal{P}H(g)\psi} = H(g)\mathcal{P}\overline{\psi}$, $\forall\, \psi \in D(H(g))$, $\forall\, g \in \mathbf{R}$.

Proof

(1) Since $H(g)^* = H(-\overline{g})$, and $D(H(g))$ is independent of $g \in \mathcal{S}_1 \cup \mathcal{S}_2$, it is enough to prove that, for all $g \in \mathcal{S}_1 \cup \mathcal{S}_2$:

$$(a) \qquad D(\mathcal{P}H(g)) = D(H(g)); \qquad\qquad (b) \qquad D(H(g)\mathcal{P}) = D(H(g))\,\forall\, g \in S_1 \cup S_2$$

(a) follows from $D(\mathcal{P}) = \mathcal{H}$. As for (b) notice that $u \in D(H(g))$ if and only if $\exists\{u_n\} \in C_0^\infty(\mathbf{R}^d)$ such that $u_n \to u$ and $H(g)u_n \to v = H(g)u$. Then $u_n(-x) \in C_0^\infty(\mathbf{R}^d) \to u(-x)$ and $H(-g)u_n(-x) \to v(-x)$. Thus $\mathcal{P}u = u(-x) \in D(H(-g)) = D(H(g))$, i.e. $u \in D(H(g)\mathcal{P})$. Conversely, if $u \in D(H(g)\mathcal{P})$ then $u(-x) \in D(H(g))$ and $u \in D(H(-g)) = D(H(g))$, whence $D(H(g)) = D(H(g)\mathcal{P})$.

(2) From (1) we have $D(\mathcal{P}H(g)) = D(H(-g)\mathcal{P}) = D(H(g))$; moreover $C_0^\infty(\mathbf{R}^d)$ is a core for both operators $\mathcal{P}H(g)$ and $H(-g)\mathcal{P}$. Therefore it is enough to prove that $\mathcal{P}H(g)u = H(-g)\mathcal{P}u\,\forall\, u \in C_0^\infty(\mathbf{R}^d)$. Indeed, if $u \in C_0^\infty(\mathbf{R}^d)$ then $\mathcal{P}u \in C_0^\infty(\mathbf{R}^d)$ and

$$\mathcal{P}H(g)u = \mathcal{P}(-\Delta u + x^2\psi + igWu) = -\Delta\mathcal{P}u + x^2\mathcal{P}u - igW\mathcal{P}u = H(-g)\mathcal{P}u$$

(3) Again it is enough to prove the identity for $u \in C_0^\infty(\mathbf{R}^d)$. By direct inspection:

$$\overline{\mathcal{P}H(g)\psi} = \overline{H(-g)\mathcal{P}\psi} = \overline{-\Delta\mathcal{P}\psi + x^2\mathcal{P}\psi - igW\mathcal{P}\psi} = H(g)\mathcal{P}\overline{\psi}$$

because $\overline{\mathcal{P}\psi} = \mathcal{P}\overline{\psi}$. This proves the Proposition.

Proof of Theorem 1.1.

1. Since $\mathcal{P}$ is continuous in $\mathcal{H}$ we have ([12], Problem 5.26) $Q(g)^* = (\mathcal{P}H(g))^* = H(g)^*\mathcal{P} = \mathcal{P}H(g)$, where the last equality follows from Assertion (2) of Proposition 2.1. Since $\mathcal{P}H(g) = Q(g)$, $Q(g)^* = Q(g)$. Same argument for $Q'(g)$.

2. The domain of $H(g)$ does not depend on g by (N2) for $g \in \mathcal{S}_1 \cup \mathcal{S}_2$. Hence also the domain of $Q(g)$ is $g-$independent. Moreover the scalar products $\langle Q(g)u, u\rangle$ are obviously entire holomorphic functions of $g\ \forall\, u \in D(Q(g))$. Thus $Q(g)$ is by definition a type-A holomorphic family in the sense of [12] (Section VII.1.3) in the stated domain. We now verify that $\rho(Q(g)) \neq \emptyset$ for g belonging to a neighbourhood of $\mathbf{R}_+$. Since $0 \notin \sigma(H_0)$, by (2.4) with $\theta = 0$ there is $B > 0$ such $H(g)^{-1}$ is uniformly bounded in $\tilde{S} := \{g \in \mathcal{S}_1 \cup \mathcal{S}_2, |g| < B\}$. Hence $\mu = 0$ is not an eigenvalue of $Q(g) = \mathcal{P}H(g)$ because $\mathcal{P}$ is invertible. Therefore $Q(g)$ is invertible for $g \in \tilde{S}$. Now $Ran(Q(g) = L^2$: if indeed $v \in L^2$, then $\mathcal{P}v \in R(H(g)) = L^2$, i.e. there exists $u \in D(H(g))$ such that $H(g)u = \mathcal{P}v$. Hence $\mathcal{P}H(g))u = v$ and $v \in Ran(\mathcal{P}H(g))$. The inverse $Q(g)^{-1} = H(g)^{-1}\mathcal{P}$ is compact as the product of the compact operator $H(g)^{-1}$ times the continuous operator $\mathcal{P}$. Since $Q(g)$ is self-adjoint for $g \in \mathbf{R}$, the compactness of the resolvent $[Q(g) - z]^{-1}$ extends to all g in a neighbourhod of the real axis (see [12], Thm VII.2.8).

3. Let us first prove the coincidence between the eigenvalues of $Q(g)$ and those of $Q'(g)$. We have:

$$\mathcal{P}H(g)\psi = \lambda\psi \Longleftrightarrow H(g)\psi = \lambda\mathcal{P}\psi \Longleftrightarrow (H(g)\mathcal{P})\mathcal{P}\psi = Q'(g)\mathcal{P}\psi = \lambda\mathcal{P}\psi$$

Hence λ is eigenvalue of $Q(g)$ with eigenvector ψ if and only if λ is eigenvalue of $Q'(g)$ with eigenvector $\mathcal{P}\psi$.

Let now μ be any eigenvalue of $Q = \mathcal{P}H$, and let ψ be any corresponding eigenvector. Then, by the self-adjointness of $\mathcal{P}H$:

$$Q\psi = \mathcal{P}H\psi = \mu\psi \Longrightarrow H^*H\psi = H^*\mathcal{P}\mathcal{P}H\psi = Q^2\psi = \mu^2\psi.$$

Thus μ^2 is an eigenvalue of H^*H with the same eigenvector of Q. On the other hand, since, as we have seen, Q^{-1} exists and is compact the eigenvectors of Q form a complete set. Therefore μ^2 is an eigenvalue of H^*H if and only if μ or $-\mu$ is an eigenvalue of Q. This completes the proof of Theorem 1.1.

Proof of Corollary 1.2.

By the spectral theorem we have, if $u \in D(Q)$:

$$Qu = \mathcal{P}Hu = \sum_{n=0}^{\infty} \mu_n \langle u, \psi_n \rangle \psi_n$$

(counting multiplicities). Since $\mathcal{P}Q = H$, and $\mathcal{P}$ is continuous:

$$Hu = \sum_{n=0}^{\infty} \mu_n \langle u, \psi_n \rangle \mathcal{P}\psi_n, \quad \forall\, u \in D(H(g))$$

Now $(\mathcal{P}\psi_n)(x) = \psi_n(-x)$

Define now $Q(\beta, \theta) := \mathcal{P}H(\beta, \theta)$ and let us prove that this operator family enjoys the same properties of $H(\beta, \theta)$. We have:

Proposition 2.2 *$Q(\beta, \theta)$ defined on $D(Q(\beta)) = D(H(\beta))$ is a type-A holomorpic family with compact resolvents in a neighbourhood of $\mathbf{R}_+$ for β and θ such that $(s,t) \in \mathcal{R}$, $s = \arg\beta$, $t = \mathrm{Im}\theta$. Moreover $C_0^\infty(\mathbf{R}^d)$ is a core of $Q(\beta, \theta)$. Analogous results hold for the operator family $Q(\beta, \theta)$ for β and θ such that $(s,t) \in \mathcal{R}'$, and $Q(\beta, \theta)^* = Q(\overline{\beta}, \overline{\theta})$.*

Proof.

The fact that $Q(\beta, \theta)$ is closed on $D(H(\beta)) = D(H(\beta, \theta))$ can be proved by the same argument of Proposition 2.1, (1). To complete the proof we then proceed as in Theorem 1.1, (2). This proves the proposition.

Proof of Theorem 1.3

Set $T(\beta, \theta) := e^{2\theta}Q(\beta, \theta) = \mathcal{P}K(\beta, \theta)$. Given the analyticity property of the operator family $Q(\beta, \theta)$, we have only to verify the analogous of N5); namely that, for all (β, θ) such that $(s,t) \in \mathcal{R}$, the following two properties hold:

(i) If $\lambda \notin \sigma(T(0, \theta))$, then $\lambda \in \tilde{\Delta}_1$ where:

$$\tilde{\Delta}_1 := \{z \in \mathbf{C} : z \notin \sigma(T(\beta, \theta)); \|[z - (T(\beta, \theta)]^{-1}\| \quad \text{is uniformly bounded for } |\beta| \to 0\}; \tag{2.7}$$

(ii) If $\lambda \in \sigma(T(0,\theta))$, then λ is stable with respect to the operator family $T(\beta,\theta)$.

To prove these assertions, we generalize the argument of [20] valid for $d = 1$. First set $\rho := |\beta|$, $K(\rho) := K(\beta,\theta)$, $T(\rho) := T(\beta,\theta)$. The proof of N10) relies on the following results (see [21]):

(a) $\lim_{\rho\downarrow 0} K(\rho)u = K(0)u$, $\lim_{\rho\downarrow 0} K(\rho)^* u = K(0)^* u$, $\forall\, u \in C_0^\infty(\mathbf{R}^d)$;

(b) $\tilde{\Delta}_1 \neq \emptyset$;

(c) Let $\chi \in C_0^\infty(\mathbf{R}^d)$ be such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ for$|x| \leq 1$, $\chi(x) = 0$ for$|x| \geq 2$. For $h \in \mathbf{N}$ let $\chi_h(x) := \chi(x/h)$, and $M_h(x) = 1 - \chi_h(x)$. Then:

  (1) If $\rho_m \downarrow 0$ and $u_m \in D(K(\rho_m))$ are two sequences such that $\|u_m\| \to 1$, $u_m \to 0$ weakly, and $\|(K(\rho_m))u_m\|$ is bounded in m, then there exists $a > 0$ such that
$$\liminf_{m\to\infty} \|M_h u_m\| \geq a > 0, \quad \forall\, h$$

  (2) For some $z \in \tilde{\Delta}_1$:
$$\lim_{h\to\infty} \|[M_h, K(\rho)][z - K(\rho)]^{-1}\| = 0$$

  (3) $\lim_{\substack{h\to\infty \\ \rho\downarrow 0}} d_h(\lambda,\rho) = +\infty\ \forall\, \lambda \in \mathbf{C}$, where:
$$d_h(\lambda,\rho) := \inf\{\|[\lambda - K(\rho)]M_h u\| : u \in D(K(\rho)), \|M_h u\| = 1\}$$

Hence we must verify the analogous properties, denoted (a'), (b'), (c'), for the operator family $T(\rho)$. Remark that, as in [20], the verification of (b') requires an argument completely independent of [21] because the operator family $T(\rho)$ is not sectorial. We have:

(a') From (a) and the continuity of $\mathcal{P}$ we can write
$$\lim_{\rho\downarrow 0} T(\rho)u = T(0)u, \quad \lim_{\rho\downarrow 0} T(\rho)^* u = T(0)^* u, \quad \forall\, u C_0^\infty(\mathbf{R}^d)$$

(b') First remark that $0 \in \tilde{\Delta}$ by N9) (i) since $0 \notin \sigma(K(0,\theta))$. Then there is $B > 0$ such that
$$\sup_{0\leq|\beta|<B} \|K(\beta,\theta)^{-1}\| < +\infty.$$

To prove the analogous bound with $T(\beta,\theta)$ in place of $K(\beta,\theta)$, note that $\mathcal{P}K(\beta,\theta)\psi = 0$ if and only if $\psi = 0$. Hence there exists $B > 0$ such that $\mu = 0$ is not an eigenvalue of $T(\beta,\theta)$ for $|\beta| < B$. Thus $T(\beta,\theta)$ is invertible. Its range is $\mathcal{H}$: if $v \in \mathcal{H}$, then $\mathcal{P}v \in Ran(K(\beta,\theta)) = \mathcal{H}$, i.e. there exists $u \in D(K(\beta,\theta))$ such that $K(\beta,\theta)u = \mathcal{P}v$. Thus $\mathcal{P}K(\beta,\theta)u = v$ and $v \in Ran(\mathcal{P}K(\beta,\theta))$. Finally $T(\beta,\rho)^{-1} = (\mathcal{P}K(\beta,\theta))^{-1} = K(\beta,\theta)^{-1}\mathcal{P}$ is uniformly bounded for $|\beta| < B$ because $\mathcal{P}$ is bounded and $K(\beta,\theta)^{-1}$ is uniformly bounded.

(c') Let $\chi \in C_0^\infty(\mathbf{R}^d)$ be as in (c) with the additional condition $\chi(x) = \chi(-x)$, i.e. $\mathcal{P}\chi = \chi$. Then $\mathcal{P}\chi_h = \chi_h$, and $\mathcal{P}M_h = M_h$. We have:

(1') Let $\rho_m \downarrow 0$ and $u_m \in D(T(\rho_m))$ be such that $\|u_m\| \to 1$, $u_m \to 0$ weakly and $\|T(\rho_m)u_m\| \le$ const. $\forall\, m$. Then $\|K(\rho_m)u_m\| = \|T(\rho_m)u_m\| \le$ const.; hence by (c1) there exists $a > 0$ such that

$$\liminf_{m\to\infty} \|M_h u_m\| \ge a > 0, \quad \forall\, h$$

(2') As proved in [21], if (c2) holds for some $z \in \tilde{\Delta}_1$ then it holds for all $z \in \tilde{\Delta}_1$. Thus we can take $z = 0 \in \tilde{\Delta} \cap \tilde{\Delta}_1$ and we have:

$$\begin{aligned}\lim_{h\to\infty} \|[M_h, T(\rho)](\mathcal{P}K(\rho))^{-1}\| &= \lim_{h\to\infty} \|(M_h\mathcal{P}K(\rho) - \mathcal{P}K(\rho)M_h)(\mathcal{P}K(\rho))^{-1}\| \\ &= \lim_{h\to\infty} \|\mathcal{P}[M_h, K(\rho)]K(\rho))^{-1}\mathcal{P}\| = 0\end{aligned}$$

where the last equality follows from the unitarity of $\mathcal{P}$ and (c2).

(3') Let $\lambda \in \mathbf{C}$ and

$$d'_h(\lambda,\rho) := \inf\{\|(\lambda - T(\rho))M_h u\| : u \in D(T(\rho)), \|M_h u\| = 1\}$$

Then:

$$\begin{gathered}\|[\lambda - T(\rho)]M_h u\| = \|[\lambda(1-\mathcal{P}) + \mathcal{P}(\lambda - K(\rho))]M_h u\| \ge \\ \|[\lambda - K(\rho)]M_h u\| - |\lambda|\|(1-\mathcal{P})M_h u\| \ge \|[\lambda - K(\rho)]M_h u\| - |\lambda|\end{gathered}$$

Hence $d'_h(\lambda,\rho) \ge d_h(\lambda,\rho) - |\lambda|$ and by (3) $\lim_{h\to\infty} d'_h(\lambda,\rho) = +\infty$. The assertion is now a direct application of [21], Theorem 5.4. This concludes the proof of Assertions 1 and 2 of Theorem 1.3.

Let us now turn to the proof of Assertion 3, i.e. the Borel summability of the eigenvalues of the operator family $Q(g,\theta) := Q(\beta,\theta)$ for $\beta = ig$, $-\pi/4 < \arg g < \pi/4$, $|g|$ suitably small (depending on the unperturbed eigenvalue).

To this end, we adapt to the present situation the proof [14] valid for the operator family $H(g,\theta) := H(\beta,\theta), \beta = ig$, in turn based on the general argument of [22].

First remark that if (β,θ) generates the parallelogram $\mathcal{R}$ defined in (2.3) then (g,θ) generates the parallelogram

$$\widehat{\mathcal{R}} = \{(s,t) \in \mathbf{R}^2 : -\pi/2 < (2K-1)t + s < \pi/2, -\pi/2 < (2K+3)t + s < \pi/2\}, \quad (2.8)$$

where now $s = \arg g = \arg\beta - \pi/2$. From now on, with abuse of notation, we write $(g,\theta) \in \widehat{\mathcal{R}}$ whenever $(s,t) \in \mathcal{R}$.

Let λ be an eigenvalue of $H_0(\theta) := H(0,\theta)$ of multiplicity $m(\lambda) := m$. Denote $P(0,\theta)$ the corresponding projection. By the above stability result, this means that if Γ is a circumference of radius ϵ centered at λ there is $C > 0$ independent of $(g,\theta) \in \widehat{\mathcal{R}}$ such that, denoting $R_Q(z,g,\theta) := [Q(g,\theta) - z]^{-1}$ the resolvent of $Q(g,\theta)$:

$$\sup_{z\in\Gamma_0} \|[Q(g,\theta) - z]^{-1}\| \le C, \quad |g| \to 0$$

and that $\dim \widehat{P}(g,\theta) = \dim \widehat{P}$ as $|g| \to 0$, $(g,\theta) \in \widehat{\mathcal{R}}$, $\arg g$ fixed. This time:

$$\widehat{P}(g,\theta) := \frac{1}{2\pi i}\int_\Gamma R_Q(z,g,\theta)\,dz, \quad \widehat{P} \equiv \widehat{P}(0,\theta) := \frac{1}{2\pi i}\int_\Gamma R_Q(z,0,\theta)\,dz \quad (2.9)$$

are the projections on the parts of $\sigma(Q(g,\theta))$, $\sigma(\mathcal{P}H(0,\theta))$ enclosed in Γ. We recall that $\sigma(Q(g,\theta))$ is independent of θ for all (g,θ) in the stated analyticity region, and that $\widehat{P}(0,\theta) = P(0,\theta)$. It follows that $Q(g,\theta)$ has exactly m eigenvalues (counting multiplicities) in Γ, denoted once again $\mu_1(g),\dots,\mu_m(g)$. We explicitly note that, unlike the $m = 1$ case, when the unperturbed eigenvalue is degenerate, the analyticity of the operator family does not a priori entail the same property of the eigenvalues $\mu_1(g),\dots,\mu_m(g)$, so that the analysis of [14],[22] is necessary. Following [[22], Sect.5] set:

$$\mathcal{M}(g,\theta) := Ran(\widehat{P}_Q(g,\theta)); \qquad \widehat{D}(g,\theta) := \widehat{P}(0,\theta)\widehat{P}(g,\theta)\widehat{P}(0,\theta)$$

Under the present conditions $\widehat{D}(g,\theta)$ is invertible on $\mathcal{M}(0) := Ran(\widehat{P}(0,\theta))$. Hence the present problem can be reduced to a finite-dimensional one in $\mathcal{M}(0,\theta)$ by setting

$$E(g,\theta) \;:=\; \widehat{D}(g,\theta)^{-1/2} N(g,\theta) \widehat{D}(g,\theta)^{-1/2};$$

$$N(g,\theta) \;:=\; \widehat{P}(0,\theta)\widehat{P}(g,\theta)[Q(g,\theta)-\lambda]\widehat{P}(g,\theta)\widehat{P}(0,\theta)$$

As in [[22], Thms 4.1, 4.2] the Rayleigh-Schrödinger series for each eigenvalue $\mu_s(g)$: $s=1,\ldots,m$ near λ is Borel summable upon verification of the two following assertions: there exist $\eta(\delta)>0$ and a sequence of linear operators $\{E_i(0,\theta)\}$ in $\mathcal{M}(0,\theta)$ such that

(i) $E(g,\theta)$ is an operator-valued analytic function for $(g,\theta)\in\widehat{\mathcal{R}}$; As we know, this entails that $E(g)$ is is an operator-valued analytic function in the sector

$$\mathcal{S}_{K,\delta} := \{g\in\mathbf{C} : 0<|g|<\eta(\delta); -(2K-1)\frac{\pi}{2}+\delta<\arg(g)<(2K+3)\frac{\pi}{2}-\delta\}$$

(ii) $E(g,\theta)$ fulfills a strong asymptotic condition in $\widehat{\mathcal{R}}$ (and thus, in particular, for $g\in\mathcal{S}_{K,\delta}$) and admits $\sum_{i=0}^{\infty}E_i(0,\theta)g^i$ as asymptotic series; namely, there exist $A(\delta)>0$, $C(\delta)>0$ such that

$$||R_N(g)\| := \|E(g,\theta)-\sum_{i=0}^{N-1}E_i(0,\theta)g^i\| \le AC^N\Gamma((2K-1)N/2)|g|^N \tag{2.10}$$

as $|g|\to 0$, $(g,\theta)\in\widehat{\mathcal{R}}$, $g\in\mathcal{S}_{K,\delta}$;

(iii) $\qquad E_i(0,\theta)=E_i^*(0,\theta), \quad i=0,1,\ldots, \quad \theta\in\mathbf{R}.$

Given the stability result (Assertion 2 of the present Theorem 1.3) the proof of (i) and (iii) is identical to that of [14], Lemma 2.5 (i) and is therefore omitted. We prove assertion (ii). Under the present conditions the Rayleigh-Schrödinger perturbation expansion is generated by inserting in (2.9) the (formal) expansion of the resolvent $R_Q(z,g,\theta):=[Q(g,\theta)-z]^{-1}$:

$$R_Q(z,g,\theta)=R_Q(z,g,\theta)\sum_{p=0}^{N-1}[igWR_{\mathcal{P}}(z,0,\theta)]^p+R_Q(z)[igWR_{\mathcal{P}}(z,0,\theta)]^N \tag{2.11}$$

and performing the contour integration. Moreover (see once more [22], Section 5.7), to prove (2.10) it is enough to prove the analogous bound on $\widehat{D}(g,\theta)$ and $N(g,\theta)$. Since $\widehat{D}(g)=\widehat{P}(0,\theta)\widehat{P}_Q(g,\theta)\widehat{P}(0,t)$, we have, inserting (2.11)

$$\begin{aligned} D_N(g,\theta) \;&:=\; D(g,\theta)-\sum_{i=0}^{N-1}D_i(0,\theta)g^i \\ &=\; \widehat{P}(0,\theta)\frac{1}{2\pi i}\int_{\Gamma_0}R_Q(z,g,\theta)[W(x)R_{\mathcal{P}}(z,0,\theta)]^N\widehat{P}(0,\theta) \end{aligned}$$

By the analyticity and uniform boundedness of the resolvent $R_Q(z,g,\theta)$ in $\widehat{\mathcal{R}}$ (and hence in particular for $g\in\mathcal{S}_{K,\delta}$), it is enough to prove the estimate

$$\sup_{z\in\Gamma_j}\|[igWR_{\mathcal{P}}(z,0,\theta)]^N\widehat{P}_0\|\le AC^N\Gamma((2K-1)N/2)|g|^N \tag{2.12}$$

In turn, since $\widehat{P}(0,\theta)=P(0,\theta)$, by the Combes-Thomas argument (see [22], Sect. 5 for details) to prove (2.12) it is enough to to find a function $f:\mathbf{R}^d\to\mathbf{R}$ such that

$$\|e^fP(0,\theta)\|<+\infty;\qquad \sup_{x\in\mathbf{R}^d}|W(x)e^{-f/N}|\le N^{\frac{2K-1}{2}} \tag{2.13}$$

Now a basis in $Ran(P_j)$ is given by m functions of the type

$$\mathcal{Q}(e^{\theta/2}x_1,\dots,e^{\theta/2}x_d)e^{-e^{\theta/2}|x|^2}$$

where $\mathcal{Q}$ is a polynomial of degree at most m. Therefore both estimates are fulfilled by choosing $f=\alpha|x|^2$ with $\alpha=\alpha(\theta)<1/2$. This condition is always satsfied if $(g,\theta)\in\widehat{\mathcal{R}}$ because $|\mathrm{Im}\,\theta|<\pi/4$. This concludes the proof of the Theorem.

Remark

The summability statement just proved, called Borel summability for the sake of simplicity, is more precisely the Borel-Leroy summability of order $q:=(K-1)/2$.

3 Conclusion

Even though the object of main physical interest are the eigenvalues of $H(g)$ rather than its singular values $\mu_k(g)$ determined in this paper, the singular values yield a property that the eigenvalues cannot in general yield since the operator $H(g)$ is not normal: namely, a diagonal form. If an operator is physically interesting a diagonalization of it is clearly useful. To examine this point in more detail, consider once again the canonical expansion (1.7) of Corollary 1.2:

$$H(g)u=\sum_{k=0}^{\infty}\mu_k(g)\langle u,\psi_k\rangle\mathcal{P}\psi_k,\quad u\in D(H(g))$$

Since both vector sequences $\{\psi_k\}$ and $\{\mathcal{P}\psi_k\}$ are orthonormal we have

$$\langle\mathcal{P}\psi_k,H(g)\psi_l\rangle=\mu_k(g)\delta_{k,l} \tag{3.14}$$

Moreover the orthonormal sequences $\{\psi_k\}$ and $\{\mathcal{P}\psi_k\}$ are complete in the Hilbert space. Hence formula (3.14) is an actual diagonalization of $H(g)$. The basis $\{\psi_k\}$ acts in the domain, and the basis $\{\mathcal{P}\psi_k\}$ in the range. A complete diagonalization of the $\mathcal{PT}$-symmetric but non-normal operator $H(g)$ has been therefore obtained: the singular values $\mu_k(g)$ and the eigenvectors ψ_k (and thus also the vectors $\mathcal{P}\psi_k$) are indeed uniquely defined by perturbation theory through the Borel summability.

More precisely, the general formula (1.8)

$$Hu = \sum_{k=0}^{\infty} \mu_k \langle u, \psi_k \rangle \psi'_k, \quad u \in D(H)$$

which provides a diagonalization for an operator H with compact resolvent with respect to the pair of orthonormal bases $\{\psi_k\}$ and $\{\psi'_k\}$, requires a priori the computation of μ_k and ψ_k as solutions of the spectral problem

$$H^*(g)H(g)\psi = \mu^2\psi \tag{3.15}$$

which represents an eigenvalue problem more complicated than $H(g)\phi = \lambda\phi$. The result of this paper means that the eigenvalue problem (3.15) can be replaced by the more tractable one

$$H(g)\psi = \mu\mathcal{P}\psi$$

which can be solved by perturbation theory and Borel summability.

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