TIME QUASI-PERIODIC UNBOUNDED PERTURBATIONS OF SCHRÖDINGER OPERATORS AND KAM METHODS

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ABSTRACT. We eliminate by KAM methods the time dependence in a class of linear differential equations in ℓ^2 subject to an unbounded, quasi-periodic forcing. This entails the pure-point nature of the Floquet spectrum of the operator $H_0 + \epsilon P(\omega t)$ for ϵ small. Here H_0 is the one-dimensional Schrödinger operator $p^2 + V$, $V(x) \sim |x|^\alpha, \alpha > 2$ for $|x| \to \infty$, the time quasi–periodic perturbation P may grow as $|x|^\beta, \beta < (\alpha - 2)/2$, and the frequency vector ω is non resonant. The proof extends to infinite dimensional spaces the result valid for quasiperiodically forced linear differential equations and is based on Kuksin's estimate of solutions of homological equations with non constant coefficients.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Consider the non-autonomous, linear differential equation in a separable Hilbert space $\mathcal{H}$

$$\mathrm{i}\dot{\psi}(t) = (A + \epsilon P(\omega_1 t, \omega_2 t, ..., \omega_n t))\psi(t), \quad \psi(t) \in \mathcal{H},\ \epsilon \in \mathbb{R} \tag{1.1}$$

under the following conditions:

A1 The operator A is positive self-adjoint. Spec(A) is discrete, and all eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3, ...$ are simple. There is $d > 1$ such that

$$\lambda_i \sim i^d\ , \quad i \to \infty. \tag{1.2}$$

A2 $P(\phi_1, ..., \phi_n) \equiv P(\phi)$ is a function from the n-dimensional torus $\mathbb{T}^n \equiv \mathbb{R}^n/2\pi Z^n$ into the symmetric operators in $\mathcal{H}$, $\omega := (\omega_1, \dots, \omega_n) \in [0,1]^n$ is a frequency vector.

A3 For $\delta \geq 0$, denote $\mathcal{B}^\delta$ the Banach space of all closed operators T in $\mathcal{H}$ such that $A^{-\delta/d}T$ is bounded (remark that $\mathcal{B}^0 = \mathcal{L}(\mathcal{H})$), with norm

$$\|T\|_\delta := \sup_{\|x\|_{\mathcal{H}}=1} \|A^{-\delta/d}Tx\|_{\mathcal{H}} \tag{1.3}$$

Then the map $\mathbb{T}^n \ni \phi \to P(\phi) \in \mathcal{B}^\delta$ is analytic for some $\delta < d - 1$.

Our purpose is to prove the following

Theorem 1.1. *There exist $\epsilon_* > 0$, a subset $\Pi^\epsilon \subset \Pi := [0,1]^n$ and, if $|\epsilon| < \epsilon_*$ and $\omega \in \Pi^\epsilon$, a unitary operator $U_\epsilon(\omega t) \equiv U_\epsilon(\omega_1 t, \omega_2 t, ..., \omega_n t)$ in $\mathcal{H}$ with the following properties:*

T1 *$U_\epsilon(\omega t)$ is analytic in t and quasiperiodic with frequencies ω;*

T2 *$U_\epsilon(\omega t)$ transforms equation (1.1) into a system of the form*

$$\mathrm{i}\dot{\chi}(t) = A_\infty(\omega t)\chi(t) \tag{1.4}$$

$$A_\infty := \mathrm{diag}(\lambda_1^\infty + \mu_1^\infty(\omega t), \lambda_2^\infty + \mu_2^\infty(\omega t), \lambda_3^\infty + \mu_3^\infty(\omega t), ...) \tag{1.5}$$

Here $\{\lambda_i^\infty\}_{i=1}^\infty \in \mathbb{R}$ *and any function* $\mu_i^\infty(\phi) : \mathbb{T}^n \to \mathbb{R}$ *is analytic with zero average;*

T3 *There exists* $C > 0$ *such that:*

$$\|1 - U_\epsilon(\omega t)\|_0 \le C\epsilon \ , \quad |\lambda_i^\infty - \lambda_i| \le C i^\delta \epsilon \ , \quad |\mu_i(\omega t)| \le C i^\delta \epsilon \ , \quad |\Pi - \Pi^\epsilon| \overset{\epsilon \to 0}{\to} 0 \ .$$

Straightforward integration of (1.4) reduces (1.1) to an autonomous system which makes the almost-periodic nature of all its solutions evident.

Corollary 1.1. 1. *If* $|\epsilon| < \epsilon_*$, $\omega \in \Pi^\epsilon$ *there exists a unitary transformation* $U_F(\omega t)$, *quasiperiodic with frequency* ω *and such that* $\|1 - U_F(\omega t)\|_\delta \le C\epsilon$, *which transforms (1.1) into the system*

$$\mathrm{i}\dot{x} = A_F x, \quad A_F := \mathrm{diag}(\lambda_1^\infty, \lambda_2^\infty, \lambda_3^\infty, ...) \ ; \tag{1.6}$$

2. *For any initial datum* ψ_0 *the solution* $\psi(t)$ *of (1.1) is almost-periodic with frequencies* $2\pi/\lambda_1^\infty, 2\pi/\lambda_2^\infty, \dots ; \omega_1, \dots , \omega_n$, *i.e. has the form*

$$\psi(t) = \sum_{i=0}^{\infty} \phi_i^0(\omega t) e^{i\lambda_i^\infty t} \tag{1.7}$$

where $\{\phi_i^0(\omega t)\}_{i=1}^\infty$ *are the components of* $U_\epsilon(\omega t)\psi_0$ *along the eigenvector basis of* A.

The above result can be equivalently formulated in terms of Floquet spectrum ([21], and [12] for the quasi-periodic case). Consider indeed on $\mathcal{K} := \mathcal{H} \otimes L^2(\mathbb{T}^n)$ the Floquet Hamiltonian operator

$$K_F := -\mathrm{i} \sum_{l=1}^{n} \omega_l \frac{\partial}{\partial \phi_l} + A + \epsilon P(\phi) \ . \tag{1.8}$$

The maximal operator in $\mathcal{K}$ generated by the differential expression (1.8), still denoted K_F, is self-adjoint by A3, which makes $A + \epsilon P(\omega t)$ self-adjoint on $D(A)$ for all t. Then:

Corollary 1.2. *For* $|\epsilon| \le \epsilon_*$ *and* $\omega \in \Pi^\epsilon$ *the spectrum of* K_F *is pure point; its eigenvalues are* $\nu_{j,k} := \lambda_j^\infty + k \cdot \omega$, $j = 0, 1, 2 \dots$, $k \in \mathbb{Z}^n$.

Remark 1. 1. This corollary extends to unbounded and quasiperiodic perturbations the analogous result valid for operators K_F with $P(\phi)$ periodic and differentiable in ϕ as a bounded operator in $\mathcal{H}$ [5, 6]. The gap condition is the same by condition A1, but here the analyticity of the perturbation is required.

2. The KAM methods of [5, 6], first implemented in [2] (see also [3]) made possible to strengthen for small coupling the original result of [10] (see also [14],[17]) from absence of absolutely continuous spectrum to absence of continuous spectrum. Here too the set Π^ϵ is the set of all frequencies fulfilling a diophantine condition with respect to the differences $\lambda_i - \lambda_j$. Moreover, a result of the type of Corollary 1.1 up to an error of order $\exp 1/\epsilon*$ has been proved in [11] for a class of bounded perturbations via the Nekhoroshev technique.

3. Our proof extends to infinite dimensional spaces the KAM technique to eliminate the time dependence of quasiperiodically forced ordinary linear differential equations [1, 13, 20]. The main technical point is that the relevant homological equation has variable coefficients but can be solved by a technique developed by Kuksin[16] in the context of his analysis of the KdV equation by KAM theory.

As in [3, 5, 6, 10, 14, 17, 11] the main motivation for this corollary is the (Floquet) spectral analysis for the time dependent Schrödinger equation in dimension one, namely:

Theorem 1.2. *Consider the time dependent Schrödinger equation*

$$H(t)\psi(x,t) = i\partial_t\psi(x,t),\ x\in\mathbb{R};\quad H(t) := -\frac{d^2}{dx^2} + Q(x) + \epsilon V(x,\omega t),\ \epsilon\in\mathbb{R} \tag{1.9}$$

and the corresponding Floquet Hamiltonian (1.8) under the following conditions:

1. $Q(x)\in C^\infty(\mathbb{R};\mathbb{R})$, $Q(x)\sim |x|^\alpha$ *for some* $\alpha>2$ *as* $|x|\to\infty$*;*
2. $V(x,\phi)$ *is a* $C^\infty(\mathbb{R};\mathbb{R})$*-valued holomorphic function of* $\phi\in\mathbb{T}^n$*, with* $|V(x,\phi)||x|^{-\beta}$ *bounded as* $|x|\to\infty$ *for some* $\beta<\dfrac{\alpha-2}{2}$*.*

Then there is $\epsilon^*>0$ *such that the spectrum of* K_F *is pure point for all* $|\epsilon|<\epsilon^*$*,* $\omega\in\Pi^\epsilon$*.*

Remark 2.
1. We prove the result in the more general case where V is a $C^\infty(\mathbb{R}^2;\mathbb{R})$-valued holomorphic function $V(x,\xi;\phi)$ of $\phi\in\mathbb{T}^n$ with $|V(x,\xi;\phi)|(|\xi|^2+|x|^\alpha)^{-\delta/d}$ bounded as $|\xi|+|x|\to\infty$. Here $V(\phi)$ is realized as a pseudodifferential operator family in $L^2(\mathbb{R})$ of class G^β_ρ (see e.g.[19], Chapter 8) of Weyl symbol V.
2. For $\alpha=4$ we get $\beta<1$. Hence the quantum version of the original Duffing oscillator $H(t) = -\dfrac{d^2}{dx^2} + x^4 + \epsilon x\sin(\omega t)$ lies just outside the validity range of this corollary.
3. In the periodic case ($n=1$) we see that, as in classical mechanics (see e.g.[7], Chapt.5.13) not even an unbounded perturbation delocalizes the system if its strength ϵ is too small and its frequency is not too close to a resonant one. There is no diffusion (for ϵ small enough) in the classical counterpart of (1.9) even for resonant values of ω, but there are chaotic regions in phase space localized around the resonant actions. In this case it is still unkown whether or not the quantum Floquet spectrum is pure point even for bounded perturbations. On the other hand for $0<\alpha\le 2$, when condition (1.2) is not satisfied, the nature of the Floquet spectrum is still unknown apart the globally resonant case[8],[9].
4. In the quasiperiodic case ($n\ge 2$) the quantized system behaves as in the periodic one even though in the classical counterpart of (1.9) there are no topological obstructions to the growth of energy.

2. The formal construction

Without loss of generality equation (1.1) can be written as a first-order system in ℓ^2:

$$\mathrm{i}\dot{x} = (A + \epsilon P(\omega t))x \ , \qquad x \in \ell^2 \tag{2.1}$$

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, ...) \ , \quad \lambda_i \in \mathbb{R} \ , \quad \lambda_i > 0 \tag{2.2}$$

where λ_i and $P(\omega t) \equiv P(\omega_1 t, \omega_2 t, ..., \omega_n t)$ fulfill conditions A1-A3.

The key point of any KAM method is the construction of a coordinate transformation mapping the original problem into a new one of the same form with a much smaller size of the perturbation, typically the square of the original one. Here we construct and estimate, by an algorithm very close to that of [11], a unitary operator which maps (2.1) into an equation of the same form but with a perturbation of order ϵ^2.

In this Section we describe the procedure; in Sect. 3 we work out the estimates, and in Sect.4 we set up the iterative scheme and prove its convergence.

Let $B(\phi_1, ..., \phi_n) \in \mathcal{B}^0$ be anti-selfadjoint $\forall\, \phi \in \mathbb{T}^n$. Given the unitary operator $e^{\epsilon B(\phi)}$, for fixed $\omega \in \Pi$ perform the change of basis $x = e^{\epsilon B(\omega t)} y$. Substitution in (2.1) yields

$$\mathrm{i}\dot{y} = (A + \tilde{P}^1(\omega t))y \tag{2.3}$$

The new perturbation $\tilde{P}^1$ is (the explicit dependence of B on t is omitted):

$$\begin{aligned} \tilde{P}^1 &:= \epsilon \left\{ [A, B] - \mathrm{i}\dot{B} + P \right\} \\ &+ \left(e^{-\epsilon B} A e^{\epsilon B} - A - \epsilon[A, B]\right) + \epsilon\left(e^{-\epsilon B} P e^{\epsilon B} - P\right) - \mathrm{i}\epsilon\left(e^{-\epsilon B} \dot{B} e^{\epsilon B} - \dot{B}\right) . \end{aligned} \tag{2.4}$$

If B makes the curly bracket vanish $\tilde{P}^1$ becomes of order ϵ^2. Hence we study the equation

$$[A, B] - \mathrm{i}\dot{B} + P = 0 \ . \tag{2.5}$$

Taking its matrix elements between the eigenvectors of A this equation becomes

$$-\mathrm{i} \sum_{l=1}^{n} \omega_l \frac{\partial}{\partial \phi_l} B_{ij} + (\lambda_i - \lambda_j) B_{ij} = P_{ij} \ , \tag{2.6}$$

Expand both sides in Fourier series, i.e. write

$$B_{ij} = \sum_{k \in Z^n} \hat{B}_{ijk} \mathrm{e}^{\mathrm{i}k\cdot\phi}, \qquad P_{ij} = \sum_{k \in Z^n} \hat{P}_{ijk} \mathrm{e}^{\mathrm{i}k\cdot\phi}.$$

Equating the Fourier coefficients of both sides (2.6) becomes

$$(\omega \cdot k + \lambda_i - \lambda_j) \hat{B}_{ijk} = \hat{P}_{ijk} \ .$$

Clearly this equation cannot be solved when $i = j$ and $k = 0$. Assuming now ω such that $\omega \cdot k + \lambda_i - \lambda_j \neq 0$ when $i \neq j$ or $k \neq 0$, the natural definition of B would be the operator with matrix elements defined as

$$
\begin{aligned}
B_{ij} &:= \sum_{k\in Z^n} \frac{\hat{P}_{ijk}}{\omega\cdot k+\lambda_i-\lambda_j} \mathrm{e}^{\mathrm{i}k\cdot\phi} \ , \quad i\neq j \\
B_{ii} &:= \sum_{k\in Z^n-\{0\}} \frac{\hat{P}_{iik}}{\omega\cdot k} \mathrm{e}^{\mathrm{i}k\cdot\phi}
\end{aligned}
\tag{2.7}
$$

The second line in (2.4) is of order ϵ^2 only if the operator B is bounded. However P is not bounded; as a consequence the operator $\mathrm{diag}(B_{ii})$ is in general unbounded, and the above definition cannot yield the desired result. The idea is therefore to define B by the first of (2.7) with $B_{ii}=0$; one can guess that, since the denominators $\omega\cdot k+\lambda_i-\lambda_j$ tend to infinity as i or j diverge, it should be possible to generate a bounded B even if P is unbounded. In the next section we will prove that this is actually the case.

With the above definition of B the curly bracket in (2.4) turns out to be the operator $\epsilon\,\mathrm{diag}(P_{ii})$, and hence in terms of the variables y the equation takes the form.

$$
\mathrm{i}\dot{y} = (A^1+\epsilon^2 P^1(\omega t))y \ ,
$$

with $A^1 = A+\epsilon\mathrm{diag}(P_{ii}(\omega t))$. This system is defined only for ω in the subset of Π where the denominators in (2.7) do not vanish. In the next section we will assume a diophantine type condition also for such denominators, to be valid on a Cantor subset of Π. Then it will turn out that P^1 depends in a Lipschitz way on ω in such a subset.

Iterating the construction, we see that the operator A is replaced by the operator A^1 which depends also on the angles ϕ. As we shall see, this is precisely the point where Kuksin's result[16] enters in a critical way.

3. Squaring the order of the perturbation

Keeping in mind the discussion of the preceding section we first set some notation, and then construct and estimate the transformation squaring the order of the perturbation.

Let $\mathbb{T}^n_s$ be the complexified torus with $|\mathrm{Im}\phi_i|\leq s$. If f is an analytic function from $\mathbb{T}^n_s$ to a Banach space (in what follows $\mathbb{C}$ or the complexification of $\mathcal{B}^\delta$), we denote

$$
\|f\|_s = \sup_{\phi\in\mathbb{T}^n_s} \|f(\phi)\|
$$

For $\mathcal{B}^\delta$-valued functions we use the particular symbol

$$
\|f\|_{\delta,s} := \sup_{\phi\in\mathbb{T}^n_s} \|f(\phi)\|_\delta \ .
$$

Let Π^- be a closed nonempty subset of Π of positive measure. If f has an additional (Lipschitz continuous) dependence on $\omega\in\Pi^-$ we define the norm

$$
\|f\|^{\mathcal{L}}_s := \|f\|_s + \sup_{\phi\in\mathbb{T}^n} \sup_{\omega,\omega'\in\Pi^-} \frac{\|f(\phi,\omega)-f(\phi,\omega')\|}{|\omega-\omega'|} \ .
$$

In particular for $\mathcal{B}^\delta$-valued functions we use the notation $\|.\|^{\mathcal{L}}_{\delta,s}$.

Let us now include our system into a more general framework, which, by the above discussion, is convenient for the iteration scheme. Consider in ℓ^2 the equation

$$\mathrm{i}\dot{x} = (A^- + P^-(\omega t))x \tag{3.1}$$

under the following conditions

H1)

$$A^- = \mathrm{diag}(\lambda_1^-(\omega) + \mu_1^-(\omega t, \omega), \lambda_2^-(\omega) + \mu_2^-(\omega t, \omega), \lambda_3^-(\omega) + \mu_3^-(\omega t, \omega), ...) \ , \tag{3.2}$$

Here:

H1.a) $\forall i = 1, \dots$ $\lambda_i^-(\omega)$ is positive and Lipschitz continuous w.r.t. $\omega \in \Pi^-$; moreover

$$\lambda_i^- \sim i^d \ ,$$

uniformly in $\omega \in \Pi^-$. Hence there is $C_\lambda^- > 0$ independent of ω such that

$$\left|\lambda_i^- - \lambda_j^-\right| \geq C_\lambda^- |i^d - j^d| \ . \tag{3.3}$$

H1.b) There is $C_\omega^- > 0$ suitably small and $\delta < d-1$ such that

$$\sup_{\omega,\omega' \in \Pi^-} \frac{|\lambda_i^-(\omega) - \lambda_i^-(\omega')|}{|\omega - \omega'|} \leq C_\omega^- i^\delta \tag{3.4}$$

H1.c) $\forall i = 1, \dots$ $\mu_i^-(\omega) : \mathbb{T}^n_s \times \Pi^- \to \mathcal{R}$ is analytic w.r.t. ϕ, Lipschitz continuous w.r.t. ω, and has zero average, i.e.

$$\int_{\mathbb{T}^n} \mu_i(\phi, \omega)\, d\phi = 0.$$

Moreover it fulfills the estimates

$$\|\mu_i\|_s \leq C_\mu^- i^\delta \tag{3.5}$$

$$\sup_{\phi \in \mathbb{T}^n_s} \sup_{\omega,\omega' \in \Pi^-} \frac{|\mu_i^-(\omega, \phi) - \mu_i^-(\omega', \phi)|}{|\omega - \omega'|} \leq C_\omega^- i^\delta \tag{3.6}$$

H2) The operator valued function $P^- : \mathbb{T}^n_s \times \Pi^- \to \mathcal{B}^\delta$ is analytic with respect to $\phi \in \mathbb{T}^n_s$ and Lipschitz continuous w.r.t. $\omega \in \Pi^-$.

H3) there exist $\gamma^- > 0$ and $\tau > n + 1 + s2/(d-1)$ such that, for any $\omega \in \Pi^-$, one has

$$|\omega \cdot k| \geq \frac{\gamma^-}{|k|^\tau} \ , \quad \forall k \in \mathbb{Z}^n - \{0\} \ , \tag{3.7}$$

$$|\lambda_i - \lambda_j + \omega \cdot k| \geq \frac{\gamma^- |i^d - j^d|}{1 + |k|^\tau} \ , \forall k \in \mathbb{Z}^n \ , \quad i \neq j \tag{3.8}$$

Remark 1. In the next section we will prove that it is possible to construct a set Π^- of positive measure such that also the original system (1.1) fulfills the above assumption.

Let now

$$B : \mathbb{T}^n_s \ni (\phi_1, ..., \phi_n) \mapsto B(\phi_1, ..., \phi_n) \in \mathcal{B}^0 \tag{3.9}$$

be an analytic map with $B(\phi_1, ..., \phi_n)$ anti-selfadjoint for each real value of $(\phi_1, ..., \phi_n)$. Consider the corresponding unitary operator $e^{B(\phi_1,...,\phi_n)}$, and (as above) for any $\omega \in \Pi^-$ consider the unitary change of basis $x = e^{B(\omega t)}y$. Substitution in equation 3.1 yields

$$\mathrm{i}\dot{y} = (A^+ + P^+(\omega t))y \tag{3.10}$$

$$A^+ := A^- + \mathrm{diag}(P^-). \tag{3.11}$$

Here $\mathrm{diag}(P^-)$ is the diagonal matrix formed by the diagonal elements of P^-, that is $\mathrm{diag}(P^-) := \mathrm{diag}(P^-_{11}(\omega t), P^-_{22}(\omega t), P^-_{33}(\omega t)...)$.

The new perturbation P^+ is given by (the explicit dependence of B on t is omitted):

$$\begin{aligned} P^+ := \Big\{[A^-, B] - \mathrm{i}\dot{B} + (P^- - \mathrm{diag}(P^-))\Big\} + \\ + \left(e^{-B}A^-e^B - A^- - [A^-, B]\right) + \left(e^{-B}P^-e^B - P^-\right) - \mathrm{i}\left(e^{-B}\dot{B}e^B - \dot{B}\right). \end{aligned} \tag{3.12}$$

According to the standard procedure we subtract the mean of the perturbation. Namely, we write $A^+ = \mathrm{diag}(\lambda_i^+ + \mu_i^+(\omega t))$ where $\lambda_i^+ = \lambda_i^- + \overline{P_{ii}(\phi)}$ (the overline denotes angular average). Hence the functions $\mu^+(\phi)$ have zero average; the quantities λ_i^+ are independent of ϕ and by A3 fulfill the estimate $|\lambda_i^+ - \lambda_i^-| \le C_\mu^- i^\delta$.

The main step of the proof is to construct B so as to make the curly bracket in (3.12) vanish, i.e. to solve for the unknwon B the equation

$$[A^-, B] - \mathrm{i}\dot{B} + (P^- - \mathrm{diag}(P^-)) = 0 \ , \tag{3.13}$$

The procedure explained in the previous section has to be modified since now the eigenvalues of A^- depend also on the angles ϕ. The construction is based on a lemma by Kuksin [16] that we now summarize.

On the n–dimensional torus consider the equation

$$-\mathrm{i}\sum_{k=1}^n \omega_k \frac{\partial}{\partial \phi_k}\chi(\phi) + E_1\chi(\phi) + E_2 h(\phi)\chi(\phi) = b(\phi) \ . \tag{3.14}$$

Here χ denotes the unknown, while b, h denote given analytic functions on $\mathbb{T}^n_s$. h has zero average; E_1, E_2 are positive constants and $\|h\|_s \le 1$. Concerning the frequency vector $\omega = (\omega_1, ..., \omega_n)$ the assumptions are:

$$|\omega \cdot k| \ge \frac{\gamma_2}{|k|^\tau} \ , \forall k \in \mathbb{Z}^n - \{0\} \ , \quad |\omega \cdot k + E_1| \ge \frac{\gamma_1}{1 + |k|^\tau} \ , \forall k \in \mathbb{Z}^n \ . \tag{3.15}$$

The final hypothesis is an order assumption on the magnitude of the different parameters, namely: given $0 < \theta < 1$ and $C > 0$ we assume

$$E_1^\theta \ge C E_2 \tag{3.16}$$

Lemma 3.1. *(Kuksin) Under the above assumptions equation (3.14) has a unique analytic solution χ which for any $0<\sigma<s$ fulfills the estimate*

$$\|\chi\|_{s-\sigma} \le C_1 \frac{1}{\gamma_1\sigma^{a_1}} \exp\left(\frac{C_2}{\gamma_2^{a_2}\sigma^{a_3}}\right) \|b\|_s \,. \tag{3.17}$$

Here a_1, a_2, a_3, C_1, C_2 constants independent of $E_1, E_2, \sigma, s, \gamma_1, \gamma_2, \omega$.

To apply this lemma to the construction and estimation of B, denote $\mathcal{G}$ the Banach space of all bounded operators B in ℓ^2 such that $A^{-\delta/d}BA^{\delta/d}$ extends to a bounded linear operator. The norm in $\mathcal{G}$ is denoted

$$\|B\|^{\mathcal{G}} := \max\left\{\|B\|_0, \|A^{-\delta/d}BA^{\delta/d}\|_0\right\}. \tag{3.18}$$

Moreover for the $s-$ norms of an analytic function on the torus taking values in $\mathcal{G}$ (possibly Lipschitz-continuous on $\omega\in\Pi^-$) we will use the notations

$$\|B\|^{\mathcal{G}}_s \ , \qquad \|B\|^{\mathcal{G},\mathcal{L}}_s \ .$$

In what follows the notation $a \le\cdot b$ stands for "there exists a constant C independent of $C^{\pm}_\omega, C^{\pm}_\mu, \gamma^{\pm}, s, \sigma, i, j, K$ (some of these parameters will be defined later on) such that $a \le Cb$. Equivalently we will use the notation $b \cdot\ge a$.

Lemma 3.2. *Let $\frac{\delta}{d-1} < \theta < 1$, $\gamma_* > 0$, $C^*_\omega > 0$, and $C^* > 0$ be fixed. Assume that*

$$C^* > \frac{C^-_\mu}{C^-_\lambda} \ , \quad \gamma \ge \gamma_* \ , \quad C^-_\omega \le C^*_\omega \ . \tag{3.19}$$

Then for any $0<\sigma<s$ equation (3.13) has a unique solution $B\in\mathcal{G}$ analytic on $\mathbb{T}^n_{s-\sigma}$, fulfilling the estimate

$$\|B\|^{\mathcal{G},\mathcal{L}}_{s-\sigma} \le\cdot \frac{1}{\sigma^{b_1}} \exp\left(\frac{c}{\sigma^{b_2}}\right) \left\|P^-\right\|^{\mathcal{L}}_{\delta,s} . \tag{3.20}$$

Here c, b_1, b_2 are constants depending only $\theta, n, \tau, \delta, C^*, \gamma_*, C^*_\omega$.

Proof. Taking matrix elements among eigenvectors of A^-, equation (3.13) becomes

$$-\mathrm{i}\sum_{k=1}^n \omega_k \frac{\partial}{\partial\phi_k} B_{ij} + (\lambda^-_i - \lambda^-_j)B_{ij} + (\mu^-_i(\phi) - \mu^-_j(\phi))B_{ij} = P_{ij}, \quad i\ne j \tag{3.21}$$

The first inequality of (3.19) ensures that (3.16) holds with a suitable C independent of all the relevant constants. Then a direct application of Kuksin's Lemma yields that (3.13) has a unique analytic solution fulfilling the estimate

$$\|B_{ij}\|_{s-\sigma} \le\cdot \frac{1}{\gamma|i^d - j^d|} \frac{1}{\sigma^{a_1}} \exp\left(\frac{c}{\gamma^{a_2}\sigma^{a_3}}\right) \|P_{ij}\|_s \tag{3.22}$$

To estimate of the sup norm of B we use Lemma 5.2. To this end, first remark that $|i^d - j^d| \ge |i-j|(i^\delta + j^\delta)$. Then consider the infinite matrices of elements

$$\frac{P_{ij}}{(i^\delta + j^\delta)} \ , \quad \frac{P_{ij}}{j^\delta}\frac{i^\delta}{(i^\delta + j^\delta)}$$

Assumption H2 entails a fortiori that these infinite matrices represent bounded operators in ℓ^2. Then Lemma 5.2 yields the estimate of the sup norm of B and of $A^{-\delta/d}BA^{\delta/d}$, i.e. one has

$$\|B\|^{\mathcal{G}}_{s-2\sigma} \leq\cdot \frac{1}{\sigma^{a_1+n}} \exp\left(\frac{c}{\sigma^{a_3}}\right) \left\|P^-\right\|_{\delta,s} \tag{3.23}$$

after redefinition of σ as 2σ and of the constant c. To obtain the estimate of the Lipschitz norm we proceed as follows. Given a function B of ω set

$$\Delta B := B(\omega) - B(\omega'). \tag{3.24}$$

Applying the operator Δ to (3.21) one gets that ΔB_{ij} fulfills an analogous equation. Hence by Kuksin's Lemma its solution ΔB can be estimated by the same argument applied in estimating B. Dividing by $|\omega-\omega'|$ and applying again Lemma 5.2 one gets

$$\left\|\frac{\Delta B}{\Delta\omega}\right\|_{s-3\sigma} \leq\cdot \left[\|P\|^{\mathcal{L}}_{\delta,s} + \frac{1}{\sigma^{a_1}} \exp\left(\frac{c}{\sigma^{a_3}}\right) \left\|P^-\right\|^{\mathcal{L}}_{\delta,s}\right]$$

whence the proof redefining σ as 3σ and taking the sup as above. □

We are now ready to state and prove the main result of this section.

Lemma 3.3. *Consider the system (3.1) within the stated assumptions. Assume furthermore that also (3.19) holds. Then there exists an anti-selfadjoint operator $B \in \mathcal{G}$ analytically depending on $\phi \in \mathbb{T}^n_{s-\sigma}$, and Lipschitz continuous in $\omega \in \Pi^-$ such that*

1. *B fulfills the estimate (3.20);*
2. *For any $\omega \in \Pi^-$ the unitary operator $e^{B(\omega t)}$ transforms the system (3.1) into the system (3.10);*
3. *The new perturbation P^+ fulfills the estimate*

$$\left\|P^+\right\|^{\mathcal{L}}_{\delta,s-\sigma} \leq\cdot \left(\left\|P^-\right\|^{\mathcal{L}}_{\delta,s}\right)^2 \exp\left(\frac{c}{\sigma^{b_1}}\right) \tag{3.25}$$

4. *For any positive K such that $(1+K^\tau) < \dfrac{\gamma^-}{\|P^-\|_{\delta,s}}$, there exists a closed set $\Pi^+ \subset \Pi^-$ and a $d_4 > 1$ (independent of K) fulfilling*

$$\left|\Pi^- - \Pi^+\right| \leq\cdot \gamma^- \left(1 + \frac{1}{K^{d_4}}\right) \tag{3.26}$$

5. *If $\omega \in \Pi^+$ then assumptions H1-H3 above are fulfilled also by A^+ provided the constants are replaced by the new ones defined by*

$$\gamma^+ = \gamma^- - \left\|P^-\right\|_{\delta,s}(1+K^\tau), \quad C^+_\mu = C^-_\mu + \left\|P^-\right\|_{\delta,s}, \tag{3.27}$$

$$C^+_\omega = C^-_\omega + \left\|P^-\right\|^{\mathcal{L}}_{\delta,s}, \quad C^+_\lambda = C^-_\lambda - 2\left\|P^-\right\|_{\delta,s}. \tag{3.28}$$

Proof. The estimates on B are an obvious consequence of Lemma 3.2 above. The estimate (3.25) is an immediate consequence of Lemmas 5.3 and 5.4. Concerning (3.27) and (3.28) the only nontrivial fact to be proved is the existence of a set Π^+ such that, for $\omega \in \Pi^+$ (3.7) and (3.8) are fulfilled with the new value of γ. Since (3.7) obviously holds, we examine (3.8). First remark that one has

$$|\lambda_i^- - \lambda_i^+| \leq \left\|P^-\right\|_{\delta,s} i^\delta \ ;$$

therefore, for $|k|\leq K$ we can write, by (3.8) and the inequality $|i^d-j^d|\geq(i^\delta+j^\delta)$:

$$\begin{aligned}\left|\lambda_i^+ - \lambda_j^+ - \omega\cdot k\right| &\geq \left|\lambda_i^- - \lambda_j^- - \omega\cdot k\right| - \left\|P^-\right\|_{\delta,s}(i^\delta+j^\delta)\\ &\geq \frac{\gamma^- - \|P^-\|_{\delta,s}(1+K^\tau)}{1+|k|^\tau}|i^d-j^d| \ .\end{aligned}$$

Hence (3.8) is satisfied for such values of k. Fix i,j,k and set:

$$\mathcal{R}_{ijk}(\alpha) := \left\{\omega\in\Pi \ : \ \left|\lambda_i^+ - \lambda_j^+ - \omega\cdot k\right| \leq \alpha\right\} \ i\neq j \tag{3.29}$$

$$\Pi^+ := \Pi^- - \bigcup_{|k|\geq K} \mathcal{R}_{ijk}\left(\frac{\gamma|i^d-j^d|}{1+|k|^\tau}\right) \ . \tag{3.30}$$

By Lemma 5.5 the set (3.29) is nonempty only if $|k|\geq|i^d-j^d|(C_\lambda^- - \gamma^-)$, and by Lemma 5.6, one has

$$\left|\mathcal{R}_{ijk}\left(\frac{\gamma|i^d-j^d|}{1+|k|^\tau}\right)\right| \leq\cdot \frac{\gamma|i^d-j^d|}{(1+|k|^\tau)|k|} \ .$$

Since $|i^d-j^d|\geq|i-j|(i^{d-1}+j^{d-1})$, the cardinality of the set $\{(i,j) \mid |i^d-j^d|\leq L\}$ is bounded by an absolute constant times $L^{2/(d-1)}$. Hence if $\tau>n+1+2/(d-1)$ one has

$$\begin{aligned}\left|\bigcup_{ijk:|k|\geq K}\mathcal{R}_{ijk}\left(\frac{\gamma|i^d-j^d|}{1+|k|^\tau}\right)\right| &\leq\cdot \sum_{|k|\geq K,|i^d-j^d|\leq C|k|}\frac{\gamma|i^d-j^d|}{(1+|k|^\tau)|k|}\\ &\leq\cdot\gamma\sum_{s\geq K}\frac{1}{s^{\tau-n+1-2/(d-1)}}\leq\cdot\frac{\gamma}{K^{d_4}} \ ,\end{aligned} \tag{3.31}$$

and this proves the assertion. □

4. Iteration

In this section we set up the iteration needed to prove the stated results. First we preassign the values of the various constants occurring in the iterative estimates. Hence we keep ϵ, K, s and γ fixed and define, for $l\geq1$,

$$\epsilon_l := \epsilon^{(4/3)^l} \ , \quad \sigma_l := \frac{s}{4l^2} \ , \quad s_l = s_{l-1}-\sigma_l \ , \quad K_l := lK \tag{4.1}$$

$$\gamma_l = \gamma_{l-1} - 4\epsilon_l(1+K_l^\tau) \ , \quad C_{\mu,l} = C_{\mu,l-1}+\epsilon_l \ , \tag{4.2}$$

$$C_{\lambda,l} = C_{\lambda,l-1} - 2\epsilon_l \ , \quad C_{\omega,l} = C_{\omega,l-1}+\epsilon_l \ . \tag{4.3}$$

The initial values of the sequences are chosen as follows:

$$\gamma_0 := \gamma \ , s_0 = s, \quad C_{\mu,0} := 0 \ , \quad C_{\lambda,0} := C_\lambda \ , \quad C_{\omega,0} := 0 \ .$$

Proposition 4.1. *There exist $\epsilon_* = \epsilon_*(\gamma) > 0$ and, for any $l \geq 1$, a closed set $\Pi_l^\gamma \subset \Pi$ such that, if $|\epsilon| < \epsilon_*$, one can construct for $\omega \in \Pi_l^\gamma$ a unitary transformation U_ϵ^l, analytic and quasiperiodic in t with frequencies ω, mapping the system (2.1) into the system*

$$\mathrm{i}\dot{x} = (A^l + P^l(\omega t))x \tag{4.4}$$

where:

1. *$U_\epsilon^l(\omega t)$ is as follows: $U_\epsilon^l(\phi) = \mathrm{e}^{B_\epsilon^1(\phi)}\mathrm{e}^{B_\epsilon^2(\phi)}...\mathrm{e}^{B_\epsilon^l(\phi)}$, and the anti-selfadjoint operators $B_\epsilon^j \in \mathcal{G}$, j=1,...,l depend analytically on $\phi \in \mathbb{T}^n_{s-\sigma_l}$, are Lipschitz continuous in $\omega \in \Pi_l^\gamma$ and fulfill (3.20) with P_{l-1}, σ_l in place of P^-, σ, respectively.*
2. *A^l has the form of (3.2) with the upper index "minus" replaced by l, i.e.*
$$A^l = \mathrm{diag}(\lambda_1^l(\omega) + \mu_1^l(\omega t, \omega), \lambda_2^l(\omega) + \mu_2^l(\omega t, \omega), \lambda_3^l(\omega) + \mu_3^l(\omega t, \omega), ...) \ , \tag{4.5}$$
3. *The corresponding λ_i^l and μ_i^l fulfill conditions H1, H2, H3 of the previous section, provided λ_i^-, μ_i^- are replaced by λ_i^l, μ_i^l, respectively.*
4. *The following estimates hold*
$$\left\| P^l \right\|_{\delta, s_l} \leq \epsilon_l \ , \quad \left\| B_\epsilon^l \right\|_{\delta, s_{l+1}}^{\mathcal{G},\mathcal{L}} \leq \epsilon_l \ , \quad |\Pi_l^\gamma - \Pi_{l+1}^\gamma| \leq \gamma_l \left(1 + \frac{1}{(lK)^{d_4}} \right) . \tag{4.6}$$

Proof. We proceed by induction applying Lemma 3.3. First we want to apply it to the original system (2.1) to the effect of obtaining a system of the form (4.4) with $l = 1$. To this end remark that (2.1) satisfies all the assumptions of Lemma (3.3) except the nonresonance conditions (3.7) and (3.8) on the frequencies. We have to restrict the set of the frequencies. Define therefore

$$\Pi_0^\gamma := \Pi - \bigcup_{ijk} \mathcal{R}_{ijk} \left(\frac{\gamma |i^d - j^d|}{1 + |k|^\tau} \right)$$

and remark that, by Lemma 5.6, $|\Pi - \Pi_0^\gamma| \leq \cdot \gamma$. Hence we can apply Lemma 3.3 and the starting point of our induction procedure is established.

To go from step l to step $l+1$ one has to verify that the assumptions of Lemma 3.3 are satisfied for any l. More specifically, defining $\gamma^* := \gamma/2$ and fixing C^* and C^*_ω we must verify that (3.19) holds. It is easy to check that this is true provided ϵ is smaller than a constant which in particular vanishes as $\gamma \to 0$. Then it is immediately realized that the conclusions of Lemma 3.3 imply the thesis if ϵ is small enough (independently of l). □

Proof of Theorem 1.1 Proposition 4.1 ensures the existence of $\epsilon^* > 0$ such that, for $|\epsilon| < \epsilon^*(\gamma)$, $\lim_{l\to\infty} \gamma_l = \gamma^\infty$, $\gamma^\infty > \gamma/2$, and $\lim_{l\to\infty} s_l = s/2$. This entails the uniform convergence of the operator valued sequence of functions U_ϵ^l on $\mathbb{T}^n_{s/4}$. Hence the limit, denoted $U_\epsilon^\infty(\omega t)$, will be analytic and quasi-periodic. Moreover, writing $A_\infty := \mathrm{diag}(\lim_{l\to\infty}(\lambda_i^l + \mu_i^l))$, one has

$$\lim_{l\to\infty} \|A_l(\phi) - A_\infty(\phi)\|_\delta = 0$$

uniformly on $\mathbb{T}^n_{s/4}$. This proves T1 and T2. The first three estimates of T3 are also clearly implied by the above convergence. Set now $\Pi^\gamma = \bigcap_{l=1}^{\infty} \Pi_l^{\gamma/2}$. By the third of (4.6) we have

$$|\Pi - \Pi^\gamma| \leq \cdot \gamma_0 = \gamma$$

Denote now $\gamma(\epsilon^*)$ the inverse function of $\gamma \mapsto \epsilon^*(\gamma)$, and define $\Pi^\epsilon := \Pi^{\gamma(\epsilon)}$. Then the fourth estimate of assertion T3 follows. □

Proof of Corollaries 1.1 and 1.2 Integration of (1.4) yields:

$$\chi_i(t) = \chi_i(0) e^{\mathrm{i}\lambda_i^\infty t} e^{iF_i^\infty(t)}, \qquad F_i^\infty(t) := \sum_{k \in \mathbb{Z}^n - \{0\}} \frac{\mu_{i,k}^\infty}{\omega \cdot kt} (e^{\mathrm{i}\omega \cdot k} - 1), \quad i = 0, 1, \ldots$$

where $\mu_{i,k}^\infty, k = \mathbb{Z}^n$, are the Fourier coefficients of $\mu_i(\phi)$. Setting $x_i := \mathrm{e}^{\mathrm{i}F_i^\infty(t)} \chi_i$ we get $\mathrm{i}\dot{x}_i = \lambda_i^\infty x_i$. Formula (1.7) follows taking $\chi = U_\epsilon \phi$. Moreover it is trivially verified that $\phi_i^0(\omega t) e^{i\lambda_i^\infty t}$ solves (1.1) if and only if $\lambda_i^\infty + \langle k, \omega \rangle$ is an eigenvalue of (1.8). □

Proof of Theorem 1.2 Let A denote the maximal operator in $L^2(\mathbb{R})$ generated by the differential expression $-\frac{d^2}{dx^2} + Q(x)$. It is well known that A is self-adjoint, strictly positive and has compact resolvent and that, denoting $\lambda_i, i = 1, 2, \ldots$ its eigenvalues, one has $\lambda_i \sim i^{\frac{2\alpha}{\alpha+2}}, i \to \infty$. Hence condition A1 is fulfilled if $\alpha > 2$. A can be realized also as a pseudifferential operator of symbol $\sigma_A(x, \xi) := \xi^2 + Q(x)$ under Weyl quantization. $\sigma_A(x, \xi)$ belongs to the symbol class $\Gamma^\alpha_\rho(\mathbb{R}) := \Gamma^\alpha_\rho$ for any $0 < \rho < 1$ (notations as in [19], Sect.23). This class of symbols generates the class G^α_ρ of pseudodifferential operators in $L^2(\mathbb{R})$ under the Weyl quantization formula:

$$(Au)(x) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\xi} \sigma_A(\frac{x+y}{2}, \xi) u(y)\, dy d\xi, \qquad u \in \mathcal{S}(\mathbb{R})$$

The inverse $[A+1]^{-1}$, whose principal symbol is $\sigma_{(A+1)^{-1}}(x, \xi) = (\xi^2 + Q(x) + 1)^{-1}$, belongs to the the class $G^{-\alpha}_\rho$. The functional calculus for pseudodifferential operators (see e.g.[19], Chapt.II.10,11 or [4], Chapt.8) can be applied to operators in these classes. Hence the self-adjoint operator $A^q, q > 0$ defined by the spectral theorem can also be realized a pseudodifferential operator in $G^{\alpha q}_\rho$, with symbol in $\Gamma^{\alpha q}_\rho$. Its principal symbol is $\sigma_{A^q}(x, \xi) := (\xi^2 + Q(x))^q$, and the principal symbol of $[A^q + 1]^{-1} \in G^{-\alpha q}_\rho$ is $\sigma_{(A^q+1)^{-q}}(x, \xi) := [(\xi^2 + Q(x))^q + 1]^{-1}$. By assumption the symbol of the perturbation V belongs to Γ^β_ρ for any $0 < \rho < 1$, and hence V belongs to G^β_ρ. By the composition property, the operator $T := V[A^q + 1]^{-1}$ admits a symbol in $\Gamma^{-\alpha q + \beta}_\rho$, and it will be bounded if $-\alpha q + \beta \leq 0$ ([19], Thm. 24.3). In turn, it is enough to verify this property for the principal symbol, which in this case, by the composition formula, is given by

$$\sigma^P_T(x, \xi) = v(x, \xi; \phi)[(\xi^2 + Q(x))^q + 1]^{-1}.$$

Since here $q = \delta/d$, $|\sigma_T^P(x,\xi)|$ is bounded $\forall\,(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ if there is $D > 0$ such that $|v(x,\xi;\phi)| \le D(\xi^2 + |x|^\alpha)^{\delta/d}$. If $V \sim |x|^\beta$ as $|x| \to \infty$ the inequality is satisfied for $\beta \le \alpha\delta/d$. Now we can set $1 < d = \dfrac{2\alpha}{\alpha+2}$. Then $\delta < d-1$ means $0 < \delta < \dfrac{\alpha-2}{\alpha+2}$ and therefore $\beta < \dfrac{\alpha-2}{2}$. □

5. Technical Lemmas

Lemma 5.1. *Let f_j be analytic functions on $\mathbb{T}^n_s$. Then for any $0 < \sigma < s$ one has*

$$\Big(\sum_{j\ge1} \|f_j\|^2_{s-\sigma}\Big)^{1/2} \le \frac{4^n}{\sigma^n} \|\Big(\sum_{j\ge1} |f_j|^2\Big)^{1/2}\|_s$$

Proof. This is Lemma B.3 of [15]; we reproduce its proof here for convenience of the reader. First consider the case $n = 1$. For each $j \ge 1$ there exists a point $\phi_j \in \mathbb{T}_{s-\sigma}$ such that

$$\|f_j\|_{s-\sigma} \le |f_j(\phi_j)| \ .$$

By the Cauchy integral formula

$$f_j(\phi_j) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Gamma_\rho} \frac{f_j(\zeta)}{\zeta - \phi_j} d\zeta \ ,$$

where $0 < \rho < \sigma$, is a parameter independent of j, and $\partial\Gamma_\rho$ is the boundary of the set $\Gamma_\rho := \{\phi \ : \ -\rho < \mathrm{Re}\phi < 2\pi + \rho \ , \ -(s-\sigma+\rho) < \mathrm{Im}\phi < s - \sigma + \rho\}$. One has

$$\begin{aligned} \Big(\sum_{j\ge1} \|f_j\|^2_{s-\sigma}\Big) &\le \Big(\sum_{j\ge1} \Big|\frac{1}{2\pi \mathrm{i}} \int_{\partial\Gamma_\rho} \frac{f_j(\zeta)}{\zeta-\phi_j} d\zeta\Big|^2\Big)^{1/2} \\ \le \frac{1}{2\pi} \int_{\Gamma_\rho} \Big(\sum_{j\ge1} \Big|\frac{f_j(\zeta)}{\zeta-\phi_j}\Big|^2\Big)^{1/2} |d\zeta| &\le \frac{4}{\rho} \sup_{\mathbb{T}_s} \Big(\sum_{j\ge1} |f_j(\phi)|^2\Big)^{1/2} \ . \end{aligned} \tag{5.1}$$

Taking the limit $\rho \to \sigma$ one gets the result. The case $n > 1$ follows similarly. □

Lemma 5.2. *Let $F = (F_{ij})$ be a bounded operator on ℓ^2, and let the matrix elements (F_{ij}) be analytic functions of $\phi \in \mathbb{T}^n_s$. Let $R = (R_{ij})$ be another operator with matrix elements depending analytically on $\phi \in \mathbb{T}^n_\sigma$ and such that*

$$\sup_{\phi\in\mathbb{T}^n_s} |R_{ij}(\phi)| \le \frac{1}{|i-j|} \sup_{\phi\in\mathbb{T}^n_s} |F_{ij}(\phi)| \ , \quad i \ne j \ .$$

Then, for any $\phi \in \mathbb{T}^n_s$, R is bounded in ℓ^2 and for any positive $\sigma < s$ it fulfills the estimate

$$\|R\|_{0,s-\sigma} \le \frac{4^{n+1}}{\sigma^n} \|F\|_{0,s} \ .$$

Proof. This is Lemma B.4 of [15]; again we reproduce its proof here for convenience of the reader. Fix $\phi \in \mathbb{T}_{s-\sigma}$. By Lemma 5.1 and the Schwarz inequality we have

$$\sum_{j\geq 1} |R_{ij}(\phi)| \leq \sum_{j\geq 1} \|R_{ij}\|_{s-\sigma} \leq \Big(\sum_{j\geq 1} \|F_{ij}\|^2_{s-\sigma}\Big)^{1/2} \Big(\sum_{j\neq i} \frac{1}{|i-j|^2}\Big)^{1/2} \\ \leq \frac{4^{n+1}}{\sigma^n} \sup_{\mathbb{T}^n_s} \Big(\sum_{j\geq 1} |F_{ij}|^2\Big)^{1/2} \leq \frac{4^{n+1}}{\sigma^n} \|F\| 0, s \ . \tag{5.2}$$

The same estimate holds for $\sum_{i\geq 1} |F_{ij}(\phi)|$. Hence, for $\phi \in \mathbb{T}^n_\sigma$

$$\|R(\phi)v\|^2 = \sum_{i\geq 1} \Big(\sum_{j\geq 1} |R_{ij}(\phi)||v_j|\Big)^2 \leq \sum_{i\geq 1} \Big(\sum_{j\geq 1} |R_{ij}(\phi)|\Big)\Big(\sum_{j\geq 1} |R_{ij}(\phi)||v_j|^2\Big) \\ \leq \Big(\sum_{j\geq 1} |R_{ij}(\phi)|\Big)\Big(\sum_{i\geq 1} |R_{ij}(\phi)|\Big)\Big(\sum_{j\geq 1} |v_j|^2\Big) \leq \Big(\frac{4^{n+1}}{\sigma^n} \|F\|_{0,s}\Big)^2 \|v\|^2 \tag{5.3}$$

which proves the result. □

Lemma 5.3. *Let $B \in \mathcal{G}$ be a bounded anti-selfadjoint operator, and let $P \in \mathcal{B}^\delta$ be a selfadjoint operator. Then $e^{-B} P e^B \in \mathcal{B}^\delta$ and, provided $\|B\|^{\mathcal{G}} \leq 1/2$, the following estimate holds*

$$\left\|e^{-B} P e^B - P\right\|_\delta \leq 4 \|P\|_\delta \|B\|^{\mathcal{G}} \tag{5.4}$$

Moreover, if both B and P are Lipschitz continuous with respect to $\omega \in \Pi$, then

$$\left\|e^{-B} P e^B - P\right\|^{\mathcal{L}}_\delta \leq 4 \|P\|^{\mathcal{L}}_\delta \|B\|^{\mathcal{G},\mathcal{L}} \tag{5.5}$$

Proof. Define $P(t) := e^{-tB} P e^{tB}$. Then $P(t)$ fulfills the linear differential equation

$$\dot{P} = [B, P] \ , \quad P(0) = P$$

whence

$$\|\dot{P}(t)\|_\delta \leq 2 \|B\|^{\mathcal{G}} \|P(t)\|_\delta \implies \|P(t)\|_\delta \leq \exp\Big(2 \|B\|^{\mathcal{G}} t\Big) \|P\|_\delta \ .$$

Then (5.4) follows on account of

$$P(t) - P = \int_0^t [B, P(s)] ds \ .$$

To obtain the Lipschitz estimate remark that (same notation as in the proof of Lemma 3.2), ΔP fulfills the equation

$$(\Delta P)^{\cdot} = [\Delta B, P] + [B, \Delta P] \ ,$$

and then proceed as in the estimation of the operator norm. □

Lemma 5.4. *Let $B \in \mathcal{G}$ be the solution of equation (3.13) and let $0 < \sigma < s/2$. Then:*

$$\left\|e^{-B} A^- e^B - A^- - [A^-, B]\right\|_{\delta, s-2\sigma} \leq\cdot \|B\|^{\mathcal{G}}_{s-\sigma} \left(\frac{1}{\sigma} \|B\|_{\delta, s-\sigma} + \left\|P^-\right\|_\delta\right) \tag{5.6}$$

$$\left\|e^{-B} A^- e^B - A^- - [A^-, B]\right\|^{\mathcal{L}}_{\delta, s-2\sigma} \leq\cdot \|B\|^{\mathcal{G},\mathcal{L}}_{s-\sigma} \left(\frac{1}{\sigma} \|B\|^{\mathcal{L}}_{\delta, s-\sigma} + \left\|P^-\right\|^{\mathcal{L}}_\delta\right) \tag{5.7}$$

Proof. The proof goes by the same argument of Lemma 5.3; just use the formula

$$e^{-B}A^{-}e^{B} - A^{-} - [A^{-}, B] = \int_0^1 ds \int_0^s e^{-s_1 B}[[A^{-}, B], B]e^{s_1 B} ds_1$$

and compute $[A^{-}, B]$ from equation (3.13). The the assertion easily follows. □

Lemma 5.5. *Assume that the sequence λ_i fulfills Assumption H1a) of Sect.3 and 3.4, and fix $\alpha < C_\lambda/2$; then if $i \neq j$ the set $\mathcal{R}_{ijk}(\alpha|i^d - j^d|)$ is empty if $|k| < (C_\lambda/2)|i^d - j^d|$.*

The proof of this Lemma is straigthforward and therefore omitted.

Lemma 5.6. *If the sequence λ_i fulfills assumption H1a) and (3.4) $\exists\, C > 0$ such that, if*

$$\frac{nC_\omega}{C_\lambda} \leq \frac{1}{2}$$

then one has

$$|\mathcal{R}_{ijk}(\alpha)| \leq \frac{C\alpha}{|k|} \ .$$

Proof. Following the proof of Lemma 5 of ref. [18] we fix $v \in \{-1, 1\}^n$ such that $v \cdot k = |k|$ and write $\omega = av + w$ with $w \in v^{\perp}$. One has that, as a function of a

$$(\omega \cdot k)\big|_s^t = |k|(t-s), \qquad (\lambda_i - \lambda_j)\big|_s^t \leq C_\omega(i^\delta + j^\delta)|v|(t-s) \ .$$

so, by Lemma 5.5, either $\mathcal{R}_{ijk}$ is empty or

$$(\omega \cdot k + \lambda_i - \lambda_j)\big|_s^t \geq |k|(t-s)\left(1 - \frac{nC_\omega 2}{C_\lambda}\right) \geq \frac{1}{2}|k|(t-s) \ ,$$

and therefore by the assumption we can conclude

$$|\mathcal{R}_{ijk}(\alpha)| \leq \frac{4}{|k|}\alpha \ .$$

□

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