Distributional Borel Summability for Vacuum Polarization by an External Electric Field*

Emanuela Caliceti
Dipartimento di Matematica, Università di Bologna
40127 Bologna, Italy

Abstract

It is proved that the divergent perturbation expansion for the vacuum polarization by an external constant electric field in the pair production sector is Borel summable in the distributional sense.

1 Introduction and statement of the results

Since 1970 the standard method to deal with divergent perturbation theory in quantum mechanics (QM) and quantum field theory (QFT) has been Borel summability [1]. For convenience of exposition, let us first recall its definition ([2] ; a classical reference is [3]; general references especially dealing with QM and QFT are e.g.[4], [5], [6]):

Definition 1.1 *Consider the formal power series* $\sum_{n=0}^{\infty} a_n z^n$. *The series*

$$B(t) := \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{1.1}$$

is called Borel transform of $\sum_{n=0}^{\infty} a_n z^n$. *Assume:*

1. *$B(t)$ has a positive radius of convergence $C > 0$;*
2. *$B(t)$, a priori holomorphic for $|t| < C$, admits analytic continuation at least to a neighbourhood of the positive real axis;*

*Partially supported by Università di Bologna. Funds for selected research topics.

3. *There is $R > 0$ such that the Laplace-Borel integral*

$$f(z) := \int_0^\infty B(zu)\, e^{-u}\, du \tag{1.2}$$

converges for $z \in C_R$ and defines an analytic function therein. Here C_R is the disk of radius $R/2$ tangent to the imaginary axis at the origin, defined by $C_R := \{z \in \mathbf{C} : \mathrm{Re} z^{-1} > R^{-1}\}$.

Then we say that $\sum_{n=0}^{\infty} a_n z^n$ is Borel summable to $f(z)$ for $z \in C_R$.

Remarks.

1. If (1.1) is inserted into (1.2), and summation is formally interchanged with integration, we see that $f(z)$ admits the given formal power series as an asymptotic expansion as $z \to 0_+$. An expression equivalent to (1.2) is

$$\frac{1}{z} \int_0^\infty B(t)\, e^{-t/z}\, dt \tag{1.3}$$

2. When the series $\sum_{n=0}^{\infty} a_n z^n$ has a positive radius of convergence the Laplace-Borel integral converges in the Borel polygon and yields therefore the analytic continuation of the sum $f(z)$ outside the circle of convergence if this is strictly contained in the Borel polygon;

3. In most applications, given a formal power series (example: a perturbation expansion) there is a natural candidate to the sum (example, the physical solution). A criterion is thus needed to check whether a formal power series representing the asymptotic expansion of a function is actually Borel summable to that function. The standard one is the Watson-Nevanlinna theorem (see [3], and also [7] for a new presentation in full generality of the original paper by Nevanlinna).

The vacuum polarization by a constant, external (i.e., non quantized) electromagnetic field admits a well known exact solution ([8]; see also [5]) which generates a divergent power series if expanded in powers of the fine structure constant. Historically, the pure magnetic case has been the first example where the Borel summability (of order 2) has been proved [9] (see also [10]) through a direct verification of Properties 1-3 above. (Here also the Stieltjes summability holds[11]; it entails the convergence of the Padé approximants). It is interesting to remark that the pure electric case, which has been recently reconsidered also to discuss this point[10], represents instead a typical example where Borel summability cannot hold, because the Borel transform has singularities along the positive real axis.

To better clarify this point, consider the formal expansion $\sum_{n=0}^{\infty} n!z^n$. Its Borel transform is

$$B(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}.$$

$B(t)$ is analytic on the whole of $\mathbf{C}$ except for the simple pole at $t = 1$. Consider now the function

$$F(z) := \int_0^{\infty} \frac{e^{-t}}{1-zt}\, dt \qquad z \in \mathbf{C},\ z \notin]0, +\infty[\tag{1.4}$$

$F(z)$ is clearly holomorphic in $\{z \in \mathbf{C} : 0 < |z|;\ 0 < \arg z < 2\pi\}$ and its formal expansion at $z = 0$ is $\sum_{n=0}^{\infty} n!z^n$. The non-existence of the integral (1.4) for $z \in]0, +\infty[$ is due to the pole of the Borel transform $B(t)$ at $t = 1$. Indeed this phenomenon occurs whenever the coefficients a_n have a constant sign, because in that case $B(t)$ has a singularity at $t = C$, where C is the radius of convergence of $B(t)$ (see e.g.[12]).

If we could perform the change of variable (1.3) for $F(z)$, as we can when the conditions of Definition 1.1 are satisfied, we could write

$$F(z) = \frac{1}{z} \int_0^{\infty} \frac{1}{1-t} e^{-(t/z)}\, dt \tag{1.5}$$

However (1.5) is only a formal writing because the integral in the r.h.s. diverges for all $z \in \mathbf{C}$. Nevertheless (1.5) could make sense if the Borel transform $B(t) = (1-t)^{-1}$ is regarded as an object more general than a function, for example a distribution. More precisely, in this case we can look at the boundary values $B(t \pm i0) = \dfrac{1}{1-t \pm i0} : t \geq 0$ of the holomorphic function $B(t)$, as tempered distributions:

$$B(t \pm i0) = \frac{1}{1-t \pm i0} := \lim_{\epsilon \to 0_+} \frac{1}{1-t \pm i\epsilon} = PP\left(\frac{1}{1-t}\right) \pm i\delta(t-1) \tag{1.6}$$

Here $PP\left(\dfrac{1}{1-t}\right)$ is the Cauchy principal-value distribution supported at 1. Note that $B(t - i0) = \overline{B(t + i0)}$. Next remark that the function

$$\Phi(z) := \frac{1}{z} \int_0^{\infty} B(t + i0) e^{-t/z}\, dt \tag{1.7}$$

exists, is analytic for $z \in C_R\ \forall\, R > 0$, i.e. in the half-plane $\mathrm{Re} z > 0$, concides with $F(z)$ for $z \in \{z \in \mathbf{C} : \mathrm{Im} z > 0; \mathrm{Re} z > 0\}$ and admits $\sum_{n=0}^{\infty} n!z^n$ as formal expansion at $z = 0$. It is called the *upper sum* of the series (see also the Remarks after Definition 1.2

below). Since the divergent series is real for $z \in [0,+\infty[$, so must be its sum provided it exists in any sense. Therefore the natural candidate for the Borel sum is

$$f(z) := \frac{1}{z}\int_0^\infty PP\left(\frac{1}{1-t}\right) e^{-t/z}\,dt = \frac{1}{z}\int_0^\infty \frac{1}{2}\{B(t+i0)+\overline{B(t+i0)}\}e^{-t/z}\,dt \qquad (1.8)$$

In other words $f(z) = \frac{1}{2}\{\Phi(z)+\overline{\Phi(\overline{z})}\}\forall\, z : \mathrm{Re}z > 0$. In particular $f(z) = \mathrm{Re}\Phi(z)$ for $z \in [0,+\infty[$.

This example shows that an extension of the Borel method to the case where the Borel transform admits singularities along the positive real axis has to allow for Borel transforms in the sense of distributions. In turn, distributions are particular cases of the hyperfunctions, defined as boundary values of holomorphic functions. The extension, called distributional Borel summability, has been developed in [13]. Let us recall here the definition and some of the main results.

Definition 1.2 *Consider again the formal power series* $\sum_{n=0}^\infty a_n z^n$ *and its Borel transform* $B(t)$ *as in Definition 1.1, with radius of convergence* $C > 0$. *Assume:*

1. $B(t)$ *admits analytic continuation to the intersection of some neighbourhood of* $\mathbf{R}_+$ *with* $\mathbf{C}_+ := \{t \in \mathbf{C} : \mathrm{Im}t > 0\}$*;*

2. *The boundary value distribution* $B(t+i0)$ *exists* $\forall\, t \geq 0$*;*

3. *Let* $PP(B(t)) := \frac{1}{2}\{B(t+i0)+\overline{B(t+i0)}\}$, $t \geq 0$. *Then there exists* $R > 0$ *such that the Laplace-Borel integral*

$$f(z) := \frac{1}{z}\int_0^\infty PP(B(t))e^{-t/z}\,dt \qquad (1.9)$$

converges for $z \in C_R$, C_R *as in Definition 1.1.*

Then we say that the formal power series $\sum_{n=0}^\infty a_n z^n$ *is Borel summable in the distributional sense to* $f(z)$ *for* $z \in C_R$. *The distribution* $PP(B(t)) : t \in \mathbf{R}_+$ *is called distributional Borel transform of* $\sum_{n=0}^\infty a_n t^n$.

Remarks.

1. The distribution $PP(B(t))$ coincides with the holomorphic function $B(t)$, the Borel transform, for $0 \leq t < C$;

2. The Laplace Borel integrals

$$\Phi(z) := \frac{1}{z}\int_0^\infty B(t+i0)e^{-t/z}\,dt \tag{1.10}$$

$$\overline{\Phi}(\overline{z}) := \int_0^\infty \overline{B(t+i0)}e^{-t/z}\,dt \tag{1.11}$$

exist separately in C_R as analytic functions and uniquely define the "upper" and "lower" sum, respectively. Then $f(z) = \{\Phi(z) + \overline{\Phi}(\overline{z})\}/2$ for all $z \in C_R$.In particular, $f(z) = \mathrm{Re}\Phi(z)$, $\forall\, z \in C_R \cap \mathbf{R}_+$.

3. As the ordinary Borel sum, the distributional Borel one is unique. We note for further reference that this method singles out also a unique function with zero asymptotic power series expansion, the so-called "discontinuity", uniquely defined by

$$d(z) := \Phi(z) - \overline{\Phi}(\overline{z}) = \frac{1}{z}\int_0^\infty \{B(t+i0) - \overline{B(t+i0)}\}e^{-t/z}\,dt, \quad \forall\, z \in C_R$$

In particular $d(z) = 2i\mathrm{Im}\Phi(z)$, $\forall\, z \in C_R \cap \mathbf{R}_+$.

In the above example we have:

$$d(z) = \frac{1}{z}\int_0^\infty 2i\delta(t-1)e^{-t/z}\,dt, \quad = \frac{2i}{z}e^{-1/z}$$

4. The analogue of the Watson-Nevanlinna criterion has also been established [13]. Its conditions have been verified to prove the distributional Borel summability in a number of physically interesting cases which generate constant sign divergent perturbation expansions. Examples include the Rayleigh-Schrödinger perturbation theory for Stark effect[14] and the odd anharmonic oscillators[15], which are summable to the resonances, and a variant of the bound state perturbation theory for the double well quartic oscillator[16].

Let us now proceed to state the result of this paper. Its proof is to be described in the next section.

The effective action for the vacuum polarization by a uniform electric field can be obtained as a particular case from the Schwinger solution[8, 5] valid for a general external constant electromagnetic field, and reads:

$$S(\alpha) = -\frac{1}{8\pi^2}\int_0^\infty \frac{e^{-is}}{s^3}\left\{(2\sqrt{\pi\alpha}s)\coth(2\sqrt{\pi\alpha}s) - 1 - \frac{4\pi\alpha s^2}{3}\right\} ds \tag{1.12}$$

Here α is the fine structure constant, and without loss the electron mass m and the strenght E of the field are set equal to 1. Notice that (1.12) defines an analytic function of α for $-\pi < \arg\alpha < \pi$. It can be easily checked (see also Lemma 2.2 below) that $S(\alpha)$ admits the following formal expansion in power series of α

$$S(\alpha) \sim -\frac{1}{8\pi^2}\sum_{n=2}^{\infty}(16\pi)^n B_{2n}\frac{(2n-3)!}{(2n)!}(-\alpha)^n = \sum_{n=2}^{\infty} a_n\alpha^n, \tag{1.13}$$

$$a_n := -\frac{(-1)^n}{8\pi^2}(16\pi)^n B_{2n}\frac{(2n-3)!}{(2n)!} \tag{1.14}$$

where $\{B_{2n}\} : n = 0, 1, \dots$ is the sequence of the Bernoulli numbers. Now (see e.g.[17])

$$B_0 = 1, B_{2n} = 2(-1)^{n+1}\frac{(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty} m^{-2n}, n = 1, 2, \dots \tag{1.15}$$

Hence $a_n > 0$ for all $n \in \mathbf{N}$ and $a_n \sim (2n)!$ as $n \to \infty$. Then we can state the main result of this paper:

Theorem 1.3 *The perturbation expansion (1.13) is Borel summable in the distributional sense to* $\frac{1}{2}\{S(\alpha) + \overline{S}(\alpha)\} = \mathrm{Re}S(\alpha)$ *for any* $0 \le \alpha < +\infty$. *More precisely* $S(\alpha)$ *and* $\overline{S}(\overline{\alpha})$ *are the upper and lower sum of* $\sum_{n=0}^{\infty} a_n\alpha^n$ *for* $\mathrm{Re}\alpha > 0$, *respectively.*

1. The effective action $S(\alpha)$ is complex-valued, while the perturbation expansion is real. As already remarked (and will become evident in the course of the proof) the distributional Borel sum uniquely determines also the imaginary part $\mathrm{Im}S(\alpha) = -\frac{i}{2}d(\alpha)$, which has zero power series expansion in α. This is a point of some importance because the imaginary part is proportional to the pair creation rate.

2. The numerical resummation procedure recently introduced by [18] which also yields the imaginary part, admits a rigorous justification within the present method if the convergence (in measure) can be proved of the Padé approximants for the distributional Borel transform.

2 Proof of the distributional summability

Consider the effective action (1.12). First of all notice that for $0 < \arg\alpha < \pi$ we can rotate the integration path in (1.12) and choose the half-line $\Gamma := \{s \in \mathbf{C} : s = -it, 0 \le t < +\infty\}$, i.e. the negative imaginary axis. Now $\coth(ix) = -i\cot(x), \forall\, x \in A := \{x \in \mathbf{C} : x \neq k\pi\,; \forall\, k \in \mathbf{Z}\}$. Hence:

$$S(\alpha) = \frac{1}{8\pi^2}\int_0^{\infty}\frac{e^{-t}}{t^3}\left\{2\sqrt{\pi\alpha}t\cot(2\sqrt{\pi\alpha}t) - 1 + \frac{4\pi\alpha t^2}{3}\right\}dt \tag{2.1}$$

The proof of Theorem 1.3 is based on the fundamental criterion for distributional Borel summability (see [13], Theorem 1). Let us report here the part relevant to our purpose.

Theorem 2.1 *Let* $\sum_{n-0}^{\infty} a_n z^n$ *be a formal power series and* $B(t) = \sum_{n-0}^{\infty} \frac{a_n}{n!} t^n$ *its Borel transform. Assume:*

1. $B(t)$ *is convergent for* $|t| < \rho$ *for some* $\rho > 0$*;*
2. $B(t)$ *admits an analytic continuation to the region* $\Omega_\rho := \{t \in \mathbf{C} : \mathrm{Im}t > 0; \mathrm{Re}t > -\rho\}$*;*
3. *There are* $A > 0$*,* $R > 0$ *such that*

$$|B(t + i\eta_0)| \leq A\eta_0^{-1} \exp[t/R], \quad \forall\, t > 0, \quad \forall \eta_0 \in]0, \rho[. \tag{2.2}$$

Then the boundary value distributions $B(t + i0)$ *and* $PP(B(t)) = \frac{1}{2}\{B(t + i0) + \overline{B(t + i0)}\}$ *exist for all* $t \geq 0$ *and the integral*

$$\frac{1}{z} \int_0^\infty PP(B(t)) e^{-t/z}\, dt \tag{2.3}$$

defines a real-analytic function $f(z)$ *in* C_R*; moreover*

$$\Phi(z) := \frac{1}{z} \int_0^\infty B(t + i0) e^{-t/z}\, dt \tag{2.4}$$

is analytic in C_R *and fulfills the estimates*

$$|\Phi(z) - \sum_{n=0}^{N-1} a_n z^n| \leq C_0 c(\epsilon)^N N! |z|^N, \quad N = 1, 2, \ldots \tag{2.5}$$

uniformly in $C_{R,\epsilon} := \{z \in C_R : \mathrm{arg}z \geq -\pi/2 + \epsilon\}$.

Remarks

1. $f(z)$ is the distributional Borel sum, and $\Phi(z)$ the upper sum of $\sum_{n-0}^{\infty} a_n z^n$. They are both uniquely determined by conditions 1 and 2, together with the imaginary part $[\Phi(z) - \overline{\Phi}(\overline{z})]/2i$.
2. The estimate (2.2) makes the distribution $B(t)$ locally of order 1. However it is not a priori tempered because it might grow faster than any polynomial at infinity.

Let us now proceed to apply this theorem to our case.

Lemma 2.2 *For any α such that $0 < \arg\alpha < \pi$ set $\beta = \sqrt{\alpha}$ and*

$$\Phi(\beta) := S(\beta^2) = \frac{1}{8\pi^2}\int_0^\infty \frac{e^{-t}}{t^3}\left[2\sqrt{\pi}\beta t \cot(2\sqrt{\pi}\beta t) - 1 + \frac{4\pi\beta^2 t^2}{3}\right] dt \tag{2.6}$$

Then $\Phi(\beta)$ is analytic for $0 < \arg\beta < \pi/2$ and admits the following formal expansion in powers of β:

$$\Phi(\beta) \sim \frac{1}{8\pi^2}\sum_{n=2}^\infty (-16\pi)^n B_{2n}\frac{(2n-3)!}{(2n)!}\beta^{2n} = \sum_{n=2}^\infty a_n\beta^{2n} \tag{2.7}$$

where $\{B_{2n}\}$ is the sequence of the Bernoulli numbers defined by (1.15).

Proof

(2.7) corresponds to (1.13) with $\alpha = \beta^2$. Let us work it out for the sake of completeness. First recall that

$$x\cot x = \sum_{n=0}^\infty (-1)^n \frac{2^{2n}}{(2n)!}B_{2n}x^{2n}, \quad |x| < \pi \tag{2.8}$$

where $B_0 = 1$ and B_{2n} is given by the expression (1.15) (see e.g.[17]). Then

$$2\sqrt{\pi}\beta t\cot(2\sqrt{\pi}\beta t) = \sum_{n=0}^\infty (-1)^n\frac{2^{2n}}{(2n)!}B_{2n}2^{2n}\pi^n\beta^{2n}t^{2n} \tag{2.9}$$

Since $B_0 = 1$ and $B_2 = 1/6$ we have

$$2\sqrt{\pi}\beta t\cot(2\sqrt{\pi}\beta t) - 1 + \frac{4}{3}\pi\beta^2t^2 = \sum_{n=2}^\infty(-1)^n\frac{(16\pi)^n}{(2n!)}B_{2n}\beta^{2n}t^{2n} \tag{2.10}$$

Hence

$$\Phi(\beta) = \frac{1}{8\pi^2}\int_0^\infty e^{-t}\sum_{n=2}^\infty(-1)^n\frac{(16\pi)^n}{(2n!)}B_{2n}\beta^{2n}t^{2n-3}\,dt =$$

$$\frac{1}{8\pi^2}\sum_{n=2}^\infty(-1)^n\frac{(16\pi)^n}{(2n!)}B_{2n}\beta^{2n}\int_0^\infty e^{-t}t^{2n-3}\,dt = \frac{1}{8\pi^2}\sum_{n=2}^\infty(-1)^n\frac{(16\pi)^n(2n-3)!}{(2n!)}B_{2n}\beta^{2n}$$

and this concludes the proof of the Lemma.

Lemma 2.3 *Let $D := \{t \in \mathbf{C} : t \neq k\sqrt{\pi}/2,\ \forall\, k \in \mathbf{Z}\}$, and set*

$$B(t) := \frac{2\sqrt{\pi}t\cot(2\sqrt{\pi}t) - 1 + 4\pi t^2/3}{8\pi^2t^3}, \quad \forall\, t \in D \tag{2.11}$$

Then $B(t)$ is clearly analytic in D with simple poles at $t = k\sqrt{\pi}/2$, $k \in \mathbf{Z}$. Moreover:

$$B(t) = \sum_{n=2}^\infty \frac{a_n}{(2n-3)!}t^{2n-3}, \quad \forall\, t : |t| < \frac{\sqrt{\pi}}{2} \tag{2.12}$$

i.e. $B(t)$ is the Borel transform of $\sum_{n=2}^\infty a_n\beta^{2n-3}$.

Proof. To obtain (2.12) we proceed as in the previous Lemma using (2.8). More precisely:

$$B(t) = \frac{1}{8\pi^2}\sum_{n=2}^{\infty}(-1)^n \frac{2^{2n}}{(2n!)} B_{2n} 2^{2n}\pi^n t^{2n-3} = \sum_{n=2}^{\infty} \frac{a_n}{(2n-3)!} t^{2n-3},\ |t| < \sqrt{\pi}/2.$$

Proposition 2.4 *Let* $\sum_{n=2}^{\infty} a_n \beta^{2n-3}$ *be the formal power series whose coefficients* a_n *are defined by (1.13, 1.14) (see also (2.7)), and let* $B(t)$ *be its Borel transform (2.11) with the expansion (2.12). Then* $B(t)$ *satisfies the hypotheses of Theorem 2.1.*

Proof By Lemma 2.3 Conditions 1 and 2 of Theorem 2.1 are satisfied with $\rho = \sqrt{\pi}/2$. As far as Condition 3 is concerned, we will prove it in the following stronger version: for any $R > 0$ there is $A > 0$ such that

$$|B(t + i\eta_0)| \le A\eta_0^{-1} \exp(t/R), \quad \forall\, t > 0, \quad \forall \eta_0 \in]0, \sqrt{\pi}/2[. \tag{2.13}$$

To this end first let $0 < \delta < \sqrt{\pi}/2$ be fixed. Then the function $B(t + i\eta_0)$ is continuous on the compact set $K := \{z = t + i\eta_0 \in \mathbf{C} : |t| \le \delta; 0 \le \eta_0 \le \sqrt{\pi}/2\}$. Hence $B(t + i\eta_0)$ is bounded in K by some constant $c > 0$ and we can write

$$|B(t + i\eta_0)| \le c \le c\frac{\sqrt{\pi}}{2}\eta_0^{-1} \exp(t/R)$$

where the second inequality holds because $\frac{\sqrt{\pi}}{2}\eta_0^{-1} \ge 1$ and obviously we can choose R as large as we like . Hence it suffices to prove (2.13) for $t > \delta$ and $\eta_0 \in]0, \sqrt{\pi}/2[$. Now for $t > \delta$ the term $\frac{1}{6\pi|t + i\eta_0|}$, which comes from the third summand in (2.11) where we have replace t by $t + i\eta_0$, can be estimated as follows:

$$\frac{1}{6\pi|t + i\eta_0|} = \frac{1}{6\pi\sqrt{t^2 + \eta_0^2}} \le \frac{1}{6\pi\delta}$$

for $t > \delta$. Thus, this term trivially fulfills (2.13) with R as large as we like. Therefore we can restrict our attention to the term

$$B_1(t + i\eta_0) := 2\sqrt{\pi}(t + i\eta_0) \cot\{2\sqrt{\pi}(t + i\eta_0)\} - 1 \tag{2.14}$$

because the denominator $\frac{1}{8\pi^2|t + i\eta_0|^3}$ is bounded by $\frac{1}{8\pi^2}\delta^{-3}$ for $t > \delta$. Consider now the well known expansion (see e.g.[17])

$$x \cot x = 1 + 2x^2 \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2\pi^2} \tag{2.15}$$

Then, for any $R > 0$, we have to find $A > 0$ such that

$$|B_1(t+i\eta_0)| = 4\pi|t+i\eta_0|^2 \left|\sum_{n=1}^{\infty} \frac{1}{4\pi(t+i\eta_0)^2 - n^2\pi^2}\right| \leq A\eta_0^{-1}\exp(t/R) \tag{2.16}$$

$\forall\, t > \delta$, $\eta_0 \in]0, \sqrt{\pi}/2[$. First remark that

$$4\pi|t+i\eta_0|^2 = 4\pi(t^2+\eta_0^2) \leq 4\pi(t^2+1) \leq C_1\exp(t/R)$$

for a suitable constant $C_1 >$ and $R > 0$ arbitrarily large. Moreover one has

$$\left|\sum_{n=1}^{\infty} \frac{1}{4(t+i\eta_0)^2/\pi - n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{\{n^2 - 4(t^2-\eta_0^2)/\pi\}^2 + 64\eta_0^2 t^2/\pi^2}} \tag{2.17}$$

If $t \leq \eta_0$ the right hand side of eq.(2.17) can be bounded by $\sum_{n=0}^{\infty} \frac{1}{n^2} = \pi^2/6$. Thus (2.16) holds for $R > 0$ arbitrarily large by suitably choosing $A > 0$. We are thus left with the case $t > \eta_0$, $t > \delta$, $\eta_0 \in]0, \sqrt{\pi}/2[$. In this case, setting $Q(t, \eta_0) := [2\sqrt{t^2-\eta_0^2}/\sqrt{\pi}] + 1$ ($[x]$=greatest integer $\leq x$, $x \in \mathbf{R}$) we can estimate the r.h.s. of (2.17) as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{\sqrt{\{n^2 - 4(t^2-\eta_0^2)/\pi\}^2 + 64\eta_0^2 t^2/\pi^2}} \\
&\leq \sum_{n=1}^{Q(t,\eta_0)} \frac{1}{\sqrt{\{n^2 - 4(t^2-\eta_0^2)/\pi\}^2 + 64\eta_0^2 t^2/\pi^2}} \\
&+ \sum_{n=Q(t,\eta_0)+1}^{\infty} \frac{1}{\sqrt{\{n^2 - 4(t^2-\eta_0^2)/\pi\}^2 + 64\eta_0^2 t^2/\pi^2}} \\
&\leq \sum_{n=1}^{Q(t,\eta_0)} \frac{\pi}{8t\eta_0} + \sum_{n=Q(t,\eta_0)+1}^{\infty} \frac{1}{n^2 - 4(t^2-\eta_0^2)/\pi}
\end{aligned} \tag{2.18}$$

The last inequality is a consequence of the positivity of $n^2 - 4(t^2-\eta_0^2)/\pi$ for $n \geq Q(t,\eta_0)+1$. Now the first summand in (2.18) can be bounded by:

$$\frac{\pi}{8t\eta_0}\left(\frac{2}{\sqrt{\pi}}\sqrt{t^2-\eta_0^2}+1\right) \leq \frac{\sqrt{\pi}}{4}\eta_0^{-1} + \frac{\pi}{8t\eta_0} \tag{2.19}$$

and clearly satisfies (2.13) recalling that $t > \delta$. Concering the second term in (2.18) we have

$$\begin{aligned}
\sum_{Q(t,\eta_0)+1}^{\infty} \frac{1}{n^2 - 4(t^2-\eta_0^2)/\pi} &\leq \int_{Q(t,\eta_0)+1}^{\infty} \frac{dx}{x^2 - 4(t^2-\eta_0^2)/\pi} \\
&+ \frac{1}{(Q(t,\eta_0)+1)^2 - 4(t^2-\eta_0^2)/\pi}
\end{aligned} \tag{2.20}$$

where the inequality follows by the well known comparison theorem between series with positive terms and generalized integrals. Since $[x] \leq x \,\forall\, x \geq 0$, recalling the definition of $Q(t,\eta_0)$, we can write

$$\sum_{Q(t,\eta_0)+1}^{\infty} \frac{1}{n^2 - 4(t^2-\eta_0^2)/\pi} \leq \int_{\frac{2}{\sqrt{\pi}}\sqrt{t^2-\eta_0^2}+2}^{\infty} \frac{dx}{x^2 - 4(t^2-\eta_0^2)/\pi} + \frac{1}{3} \tag{2.21}$$

because $[x]^2 = [x^2]$ and $[x]+1-x > 0$, $\forall\, x > 0$. Since the additive factor $1/3$ can be trivially absorbed in the constants, it is enough to estimate the integral in (2.21). One has:

$$\int_{\frac{2}{\sqrt{\pi}}\sqrt{t^2-\eta_0^2}+2}^{\infty} \frac{dx}{x^2 - 4(t^2-\eta_0^2)/\pi} = \frac{1}{4}\sqrt{\frac{\pi}{t^2-\eta_0^2}}\ln\left(1+\frac{2}{\sqrt{\pi}}\sqrt{t^2-\eta_0^2}\right).$$

Given $R > 0$ arbitrarily large the existence of a constant $A > 0$ such that

$$\frac{1}{4}\sqrt{\frac{\pi}{t^2-\eta_0^2}}\ln\left(1+\frac{2}{\sqrt{\pi}}\sqrt{t^2-\eta_0^2}\right) \leq A\eta_0^{-1}e^{t/R}$$

$\forall\, t > \delta$, $t > \eta_0$, $\forall\, \eta_0 \in]0, \sqrt{\pi}/2[$ is now obvious. This concludes the proof of the Proposition.

Corollary 2.5 *In the notations of Lemmas 2.2,2.3 and Proposition 2.4 the boundary value distributions $B(t+i0)$ and $PP(B(t))$ exist for all $t \geq 0$. Moreover the integral*

$$\frac{1}{\beta}\int_0^{\infty} B(t+i0)e^{-t/\beta}\,dt \tag{2.22}$$

defines an analytic function on $D_1 := \{\beta \in \mathbf{C} : \mathrm{Re}\beta > 0\}$, coinciding with $F(\beta) := \beta^{-3}\Phi(\beta)$ $\forall\, \beta : 0 < \arg\beta < \pi/2$. Equivalently: $\sum_{n=0}^{\infty} a_n\beta^{2n-3}$ is Borel summable in the distributional sense; its upper sum is $\beta^{-3}\Phi(\beta)$ while the distributional sum is

$$f(\beta) := \frac{1}{\beta}\int_0^{\infty} PP(B(t))e^{-t/\beta}\,dt \tag{2.23}$$

$\forall\, \beta \in D_1$; moreover

$$f(\beta) = \frac{1}{2}\{\beta^{-3}\Phi(\beta) + (\overline{\beta})^{-3}\overline{\Phi}(\overline{\beta})\}, \quad \beta \notin \mathbf{R}_+. \tag{2.24}$$

Proof. The first assertion follows from Theorem 2.1 in view of Proposition 2.4, which also guarantee the existence of (2.22) in $C_R = \{\beta \in \mathbf{C} : \mathrm{Re}\beta^{-1} > R^{-1}\}$ $\forall\, R > 0$. By the same results it suffices now to show that (2.22) coincides with $\beta^{-3}\Phi(\beta)$ for $0 < \arg\beta < \pi/2$. Indeed we have

$$\frac{1}{\beta}\int_0^{\infty} B(t+i0)e^{-t/\beta}\,dt = \frac{1}{\beta}\lim_{\epsilon\to 0_+}\int_0^{\infty} B(t+i\epsilon)e^{-t/\beta}\,dt =$$

$$\frac{1}{8\pi^2\beta}\lim_{\epsilon\to 0_+}\int_0^{\infty} e^{-t/\beta}(t+i\epsilon)^{-3}\{2\sqrt{\pi}(t+i\epsilon)\cot(2\sqrt{\pi}(t+i\epsilon)) - 1 + \frac{4}{3}\pi(t+i\epsilon)^2\}\,dt$$

Now, performing the change of variables $t = s\beta$, $s \in \Gamma_1 := \{s : s = \frac{t}{\beta}, 0 \le t < \infty\}$ we obtain:

$$\frac{1}{\beta}\int_0^\infty B(t+i0)e^{-t/\beta}\,dt =$$
$$\frac{1}{8\pi^2\beta^3}\int_{\Gamma_1} s^{-3}e^{-s}\{(2\sqrt{\pi}\beta s)\cot(2\sqrt{\pi}\beta s) - 1 + \frac{4}{3}\pi\beta^2 s^2\}\,ds =$$
$$\frac{1}{8\pi^2\beta^3}\int_0^\infty t^{-3}e^{-t}\{(2\sqrt{\pi}\beta t)\cot(2\sqrt{\pi}\beta t) - 1 + \frac{4}{3}\pi\beta^2 t^2\}\,dt \tag{2.25}$$

where the last equality follows from the analyticity of the integrand (in the variable s) in a sector containing $\mathbf{R}_+$, if $\beta \notin \mathbf{R}_+$. Now by (2.6) the integral (2.25) is precisely $\beta^{-3}\Phi(\beta)$, and this concludes the proof of the Corollary.
Remark. Notice that in the representations (2.23,2.24) we had to exclude $\beta \in \mathbf{R}$ because by (2.6) $\Phi(\beta)$ is not defined for $\beta \in \mathbf{R}$.

Now, multiplying by β^3 the functions $F(\beta) = \beta^{-3}\Phi(\beta)$ and $f(\beta)$ as well as the formal series $\sum_{n=2}^{\infty} a_n\beta^{2n-3}$ we immediately conclude

Corollary 2.6 *The formal power series* $\sum_{n=2}^{\infty} a_n\beta^{2n}$ *is Borel summable in the distributional sense* $\forall\,\beta : \mathrm{Re}\beta > 0$. *Its upper sum is* $\Phi(\beta)$, $\forall\,\beta \notin \mathbf{R}_+$, *and its distributional sum is* $\beta^3 f(\beta)$. *For* $\beta \notin \mathbf{R}_+$ *one has* $\beta^3 f(\beta) = \frac{1}{2}\{\Phi(\beta) + \overline{\Phi}(\overline{\beta})\}$.

Proof of Theorem 1.3

¿From Corollary (2.6) we obtain

$$\Phi(\beta) = \beta^2\int_0^\infty B(t+i0)e^{-t/\beta}\,dt, \quad \forall\,\beta : 0 < \arg\beta < \frac{\pi}{2} \tag{2.26}$$

where the r.h.s. is the upper sum of $\sum_{n=2}^{\infty} a_n\beta^{2n}$, $\forall\,\beta : -\frac{\pi}{2} < \arg\beta < \frac{\pi}{2}$. Now, setting $\beta = \sqrt{\alpha}$ and using (2.1) to represent $S(\alpha)$ we have

$$S(\alpha) = \sqrt{\alpha}\int_0^\infty B(t+i0)e^{-t/\sqrt{\alpha}}\,dt \tag{2.27}$$

and

$$\overline{S}(\overline{\alpha}) = \sqrt{\alpha}\int_0^\infty \overline{B(t+i0)}e^{-t/\sqrt{\alpha}}\,dt$$

$\forall\,\alpha : 0 < \arg\alpha < \pi$ (notice that (2.1) actually defines S as a holomorphic function of α for $0 < \arg\alpha < \pi$). On the other hand the original representation (1.12) for $S(\alpha)$, namely

$$S(\alpha) = -\frac{1}{8\pi^2}\int_0^\infty \frac{e^{-is}}{s^3}\left\{(2\sqrt{\pi\alpha}s)\coth(2\sqrt{\pi\alpha}s) - 1 - \frac{4\pi\alpha s^2}{3}\right\}ds$$

defines S as a holomorphic function of α for $-\pi < \arg\alpha < \pi$. Thus, it represents an analytic continuation of the l.h.s. of (2.27) across the positive real axis because $S(\alpha)$ as represented by (2.27) and (2.1) coincide for $0 < \arg\alpha < \pi$. Since the r.h.s. of (2.27) is also an analytic function of α for $-\pi < \arg\alpha < \pi$ we can write

$$\begin{aligned} S(\alpha) &= -\frac{1}{8\pi^2}\int_0^\infty \frac{e^{-is}}{s^3}\left\{(2\sqrt{\pi\alpha}s)\coth(2\sqrt{\pi\alpha}s) - 1 - \frac{4\pi\alpha s^2}{3}\right\} ds \\ &= \sqrt{\alpha}\int_0^\infty B(t+i0)e^{-t/\sqrt{\alpha}}\, dt, \quad \forall\, \alpha : -\pi < \arg\alpha < \pi \end{aligned} \tag{2.28}$$

Thus $S(\alpha)$ is the upper sum of $\sum_{n=2}^{\infty} a_n\alpha^n$ and the distributional Borel sum is given by

$$\frac{1}{2}\{S(\alpha) + \overline{S}(\overline{\alpha})\} = \sqrt{\alpha}\int_0^\infty PP(B(t))e^{-t/\sqrt{\alpha}}\, dt, \quad \forall\, \alpha : -\pi < \arg\alpha < \pi \tag{2.29}$$

In particular, the distributional Borel sum is Re$S(\alpha)$ for $\alpha > 0$. This concludes the proof of Theorem 1.3.

Remarks.

1. It follows by the above theorem that the distributional Borel sumability uniquely determines also
$$\mathrm{Im}S(\alpha) = \frac{1}{2i}[S(\alpha) - \overline{S}(\alpha)], \quad \alpha \in \mathbf{R}_+.$$
Moreover Im$S(\alpha)$ is proportional to the pair-production rate. Its explicite expression is [5]:
$$\mathrm{Im}S(\alpha) = \frac{1}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n\pi}{\alpha}\right)$$
and has zero asymptotic expansion in α.

2. Strictly speaking, the representations (2.28,2.29) yield the distributional Borel-Leroy sum of order 2 (see [13], Theorem 3) of the divergent perturbation expansion (1.13). That definition is completely equivalent ([13]) to ordinary summability in the variable $\beta = \sqrt{\alpha}$. We have preferred to proceed in this last way for convenience of exposition.

References

[1] S.Graffi, V.Grecchi and B.Simon, Phys.Letters **32 B** (1970), 631

[2] E.Borel, *Lecons sur les séries divergentes*, Gauthier-Villars, Paris 1928

[3] G.H.Hardy, *Divergent Series*, Oxford University Press 1948

[4] M.Reed and B.Simon: *Methods of Modern Mathematical Physics., II, IV*, New York: Academic Press (1978)

[5] C.Itzykson and J.B.Zuber: *Quantum Field Theory*, New York: McGraw-Hill (1980)

[6] J.Zinn-Justin: *Quantum Field Theory and Critical Phenomena*, Oxford University Press (1996)

[7] A.D.Sokal, J.Math.Phys. **21** (1980), 261

[8] J.Schwinger, Phys.Rev. **82** (1951), 664

[9] V.Ogievetski, Dokl.Akad.Nauk. SSSR **109** (1956), 919

[11] S.Graffi, J.Math.Phys. **14**, (1973), 1184

[10] G.V.Dunne and T.M.Hall, Phys.Rev. **D60** (1999), 065002

[12] E.C.Titchmarsh, *The Theory of Analytic Functions*, Oxford University Press 1957

[13] E.Caliceti, V.Grecchi and M.Maioli, Commun.Math.Phys. **104** (1986), 163

[14] E.Caliceti, V.Grecchi and M.Maioli, Commun.Math.Phys. **157** (1993), 347

[15] E.Caliceti, J.Phys.A **33** (2000), 3753

[16] E.Caliceti, V.Grecchi and M.Maioli, Commun.Math.Phys. **176** (1996), 1

[17] I.M.Gradshteyn and A.B.Ryzhik: *Tables of Integrals, Series and Products*, New York: Academic Press (1978)

[18] U.D.Jentschura, Phys.Rev.D, **62** (2000), 076001