Construction of $\mathcal{PT}-$asymmetric non-Hermitian Hamiltonians with $\mathcal{CPT}-$symmetry

Emanuela Caliceti

Dipartimento di Matematica, Università di Bologna,

I-40127 Bologna, Italy

e-mail: caliceti@dm.unibo.it

Francesco Cannata

Dipartimento di Fisica dell'Università and Istituto Nazionale di Fisica

Nucleare, I-40126 Bologna, Italy

e-mail: cannata@bo.infn.it

Miloslav Znojil

Ústav jaderné fyziky AV ČR, 250 68 Řež, Czech Republic

e-mail: znojil@ujf.cas.cz

and

Alberto Ventura

Ente Nuove Tecnologie, Energia e Ambiente, and Istituto Nazionale di Fisica

Nucleare, I-40126 Bologna, Italy

e-mail: ventura@bologna.enea.it

Abstract

We show that the first-order differential choice of the charge operator $\mathcal{C}$ remains compatible with a fairly broad class of $\mathcal{CPT}-$symmetric Hamiltonians even when we violate both their $\mathcal{T}-$symmetry (i.e., Hermiticity) and $\mathcal{PT}-$symmetry (i.e., parity-pseudo-Hermiticity). We provide an explicit class of such Hamiltonians and embed them in a supersymmetric scheme. A particular example of this class is discussed in order to scrutinize in detail its possible quasi-Hermiticity.

PACS 02.30.Tb 03.65Ca 03.65.Db 03.65.Ge

1 Introduction and Motivation

The majority of the recent studies of non-Hermitian Hamiltonians with real spectra employs the postulate $H = \mathcal{PT}H\mathcal{PT}$ of $\mathcal{PT}$ symmetry, where $\mathcal{P}$ denotes parity, while $\mathcal{T}$ is time-reversal [1, 2].

An inseparable part of the appeal of these models is the expectation that the reality of the spectrum proves sufficient for a recovery of a full consistency of the physical interpretation of the theory [3, 4, 5, 6, 7].

1.1 Symmetries and their breakdown

Self-adjointness of the anharmonic-oscillator type

$$H^{(AHO)}(f,g) = -\frac{d^2}{dx^2} + x^2 + f\,x^3 + g\,x^4 \tag{1}$$

means that we choose $g > 0$ and a real f, or trivial $f = g = 0$. Then, in terms of an antilinear involution with the property $\mathcal{T}^2 = 1$ we may write $H^\dagger \equiv \mathcal{T}\,H\,\mathcal{T}$. This means that the current rule $H^\dagger = H$ is equivalent to a certain involutive time-reversal symmetry of our Hamiltonians,

$$\mathcal{T}\,H - H\,\mathcal{T} = 0\,. \tag{2}$$

However, Hamiltonians (1) are usually diagonalized by means of perturbation theory, where we usually violate the rule (2) and assume that f and g are complex, lying in a certain domain in the complex plane where the power series in f and/or g converge. In addition the AHO energies $E_n^{(AHO)}(f,g)$ are often formed as sets of the values of *a single* analytic function on different Riemann sheets (consult, e.g., ref. [8] at $g = 0$). In this respect, this enhances an interest in non-Hermitian models.

1.2 Non-self-adjoint $\mathcal{PT}-$symmetric models and possible extensions

The insight gained by the transition to non-Hermitian Hamiltonians seems more than compensated by the mathematical difficulties which it evokes. Still, the lasting absence of the imaginary parts of the energies $E_n^{(AHO)}(f,g)$ of all the levels $n = 0, 1, \ldots$.was noticed long ago [9] for the purely imaginary cubic perturbations, while, later, a non-perturbative proof of the reality of the spectrum appeared, in ref. [10], for certain quartic non-Hermitian AHO models. All of them share the property $H^\dagger = \mathcal{P}\,H\,\mathcal{P} \neq H$,i.e., a "weaker-Hermiticity" symmetry

$$\mathcal{PT}\,H - H\,\mathcal{PT} = 0\,. \tag{3}$$

Recently, the emergence of further rigorous proofs of the reality of spectra in power-law models [11, 12] re-confirmed the impact of the concept of the $\mathcal{PT}$ symmetry. At the same time, a further weakening of the standard Hermiticity seems to be a natural next step of the study, i.e. to investigate a generic $\mathcal{FT}$ symmetry, where $\mathcal{F}$ can be assumed to be Hermitian.

$$\mathcal{FT}H = H\mathcal{FT} \iff \mathcal{F}H^\dagger = H\mathcal{F}, \tag{4}$$

which, for $\mathcal{F} = 1$, becomes Hermiticity and for $\mathcal{F} = \mathcal{P}$ becomes $\mathcal{PT}$ symmetry. Eq.(4) implies that if H has an eigenvalue E, then E^* is also an eigenvalue unless $\mathcal{F}\psi^* = 0$, so eigenvalues are either real, or enter in complex conjugate pairs [4].

One could, generically, construct many operators $\mathcal{F}$ which would be, via eq. (4), compatible with a given Hamiltonian H. Among all the possible choices of these metric operators, a privileged position is occupied by the positive bounded Hermitian ones (i.e., by $\mathcal{F} = \eta_+ > 0$ in the extremely convenient notation of ref. [4]).

The reason is that, in the light of the arguments presented in full detail in the review paper [5], only the η_+-pseudo-Hermiticity is really compatible with the standard probabilistic interpretation of the mean values of the standard quantum operators of the observable quantities. For this reason, the η_+-pseudo-Hermiticity also fully deserves to be assigned the special name of the so called quasi-Hermiticity [5].

2 $\mathcal{CPT}-$symmetric models which are non-self-adjoint and non-$\mathcal{PT}-$symmetric

2.1 Breakdown of $\mathcal{PT}-$symmetry

Factorization of the operator $\mathcal{F}$ in eq.(4) into the product $\mathcal{F} \equiv \mathcal{CP}$, with $\mathcal{F} = \mathcal{F}^\dagger$, leads us to consider a $\mathcal{CPT}$-symmetric quantum mechanics, where, conventionally, $\mathcal{C}$[7] can be called "charge conjugation" operator. We emphasize that no involutory property like $\mathcal{C}^2 = 1$ will be postulated. Naturally, our interest will be focused on the possibility that $\mathcal{CPT}$-symmetric quantum mechanics violates $\mathcal{PT}, \mathcal{P}, \mathcal{T}$. We intend to proceed constructively. With this purpose we select the class of the one-body Hamiltonians in one dimension,

$$H = -\frac{d^2}{dx^2} + V(x), \qquad x \in \mathbf{R}, \tag{5}$$

and restrict our attention to one of the simplest possible forms of the charge (or rather quasi-charge) operator,

$$\mathcal{C} = \frac{d}{dx} + w(x),$$

where the requirement of the Hermiticity of its product $\mathcal{CP} \equiv \mathcal{F} = \mathcal{F}^\dagger$ with $\mathcal{P} = \mathcal{P}^\dagger$, such that $\mathcal{P}^2 = 1$ reads $\mathcal{CP} = \mathcal{P}^\dagger\mathcal{C}^\dagger = \mathcal{P}\,\mathcal{C}^\dagger$. This condition is equivalent to the postulate of the definite spatial symmetry properties of the complex function $w(x) = \sigma(x) + i\,\alpha(x)$ in $\mathcal{C}$,

$$\mathcal{R}e\ w(x) = \sigma(x) = \sigma(-x), \qquad \mathcal{I}m\ w(x) = \alpha(x) = -\alpha(-x), \qquad x \in \mathbf{R}$$

This means that we may contemplate a family of power-law examples with $\sigma(x) = \sigma_n(x) = \mu_n x^{2n}$ and $\alpha(x) = \alpha_n(x) = \nu_n x^{2n-1}$ for illustration, with real $\mu_n,\ \nu_n$ and any choice of the index $n = 1, 2\ldots$

2.2 $\mathcal{F}-$pseudo-Hermiticity

Our Hamiltonian H is assumed compatible with the $\mathcal{CPT}-$symmetry condition at any "allowed" product $\mathcal{F} = \mathcal{CP}$,

$$H\,\mathcal{F} = \mathcal{F}\,\mathcal{T}\,H\,\mathcal{T} \equiv \mathcal{F}\,H^\dagger\,. \tag{6}$$

A verification of this condition will be the core of our present construction. It necessitates a direct insertion of $\mathcal{C}$ and $\mathcal{P}$ and a decomposition of our Hamiltonian in the sum

$$H = -\frac{d^2}{dx^2} + \Sigma(x) + K(x) + i\,S(x) + i\,D(x), \tag{7}$$

where the separate components of the complex potential $V(x)$ may be chosen as real and exhibiting definite parities,

$$\Sigma(x) = \Sigma(-x), \qquad K(x) = -K(-x), \qquad S(x) = S(-x), \qquad D(x) = -D(-x)\,. \tag{8}$$

This simplifies the form of $H^\dagger$ and reduces our main compatibility constraint (6) to the condition of vanishing of the operator $H\,\mathcal{F} - \mathcal{F}\,H^\dagger$. In principle, it should be a linear differential operator of the third order but once we re-write it in the form of a product, $[H\,\mathcal{C} - \mathcal{C}\,(\mathcal{P}H^\dagger\mathcal{P}\,)]\,\mathcal{P}$, we immediately see that the coefficients at the third and at the second derivative are identically zero. A condition of the vanishing of the coefficient at the first derivative remains nontrivial and relates the potentials K and S to the choice of $\mathcal{C}$,

$$\begin{aligned} K(x) &= \frac{d}{dx}\,\sigma(x), \\ S(x) &= \frac{d}{dx}\,\alpha(x). \end{aligned} \tag{9}$$

In this way we are left with the condition which connects the two complex functions $V(x)$ and $w(x)$. Its separation into its real and imaginary part proves encouragingly simple and splits into the two ordinary and real differential equations

$$\begin{aligned} \frac{d}{dx}\,\Sigma(x) &= 2\,\sigma(x)\,\frac{d}{dx}\,\sigma(x) - 2\,\alpha(x)\,\frac{d}{dx}\,\alpha(x)\,, \\ \frac{d}{dx}\,D(x) &= 2\,\sigma(x)\,\frac{d}{dx}\,\alpha(x) + 2\,\alpha(x)\,\frac{d}{dx}\,\sigma(x)\,. \end{aligned} \tag{10}$$

Both of them exhibit a correct parity symmetry and, luckily, they also admit an entirely elementary integration,

$$\begin{aligned} \Sigma(x) &= \sigma^2(x) - \alpha^2(x) + \omega\,, \\ D(x) &= 2\,\sigma(x)\,\alpha(x)\,, \end{aligned} \tag{11}$$

introducing just a single real integration constant ω. One can rewrite the previous equations supersymmetrically [13], because it is not difficult to check that H of eq. (7) satisfies, with $\omega = 0$

$$H = \mathcal{F}\mathcal{F}^*, \qquad H^\dagger = \mathcal{F}^*\mathcal{F}, \tag{12}$$

where $\mathcal{F}^*$ is the complex conjugate of $\mathcal{F}$. We introduce the super-charges

$$Q \equiv \begin{pmatrix} 0 & \mathcal{F} \\ 0 & 0 \end{pmatrix}, \qquad \tilde{Q} \equiv \begin{pmatrix} 0 & 0 \\ \mathcal{F}^* & 0 \end{pmatrix}. \tag{13}$$

It is easy to check that $Q \neq \tilde{Q}^\dagger$, insofar as $\mathcal{F}^\dagger \neq \mathcal{F}^*$, $Q^2 = \tilde{Q}^2 = 0$, and $[Q, H] = \left[\tilde{Q}, H\right] = 0$. The super-Hamiltonian, $\mathcal{H}$, not necessarily positive, reads

$$\mathcal{H} \equiv \left\{Q, \tilde{Q}\right\} = \begin{pmatrix} \mathcal{F}\mathcal{F}^* & 0 \\ 0 & \mathcal{F}^*\mathcal{F} \end{pmatrix}. \tag{14}$$

3 Discussion, conclusion and perspectives

Let us now examine in more detail the case $n = 1$, with $\sigma(x) = \mu_1 x^2$, $\alpha(x) = \nu_1 x$. $\mathcal{F}$ is not bounded and not positive; however, it is invertible in $L^2(\mathbf{R})$ and, from eq.(4), one can derive that

$$H^\dagger \mathcal{F}^{-1} = \mathcal{F}^{-1} H. \tag{15}$$

Supersymmetrically, one can define new conserved charges

$$Q^{-1} \equiv \begin{pmatrix} 0 & 0 \\ \mathcal{F}^{-1} & 0 \end{pmatrix}, \qquad \tilde{Q}^{-1} \equiv \begin{pmatrix} 0 & \mathcal{F}^{*-1} \\ 0 & 0 \end{pmatrix}. \tag{16}$$

such that

$$\left[Q^{-1}, H\right] = \left[\tilde{Q}^{-1}, H\right] = 0, \tag{17}$$

with Q^{-1} being not the standard inverse operator, but satisfying

$$\left\{Q, Q^{-1}\right\} = I\!I, \qquad \left\{\tilde{Q}, \tilde{Q}^{-1}\right\} = I\!I, \tag{18}$$

with $I\!I$ the identity operator.

While $\mathcal{F}^{-1}$ is a bounded Hermitian operator acting on $L^2(\mathbf{R})$ [14], the positivity requirement presents problems, in general; however, evaluation of matrix elements of eq. (15) between eigenstates of the Hermitian operator $\mathcal{F}^{-1}$ yields

$$\begin{aligned} \left\langle i \left| H^\dagger \mathcal{F}^{-1} \right| k \right\rangle &= \left\langle i \left| \mathcal{F}^{-1} H \right| k \right\rangle , && (19) \\ \lambda_k \left\langle i \left| H_1^\dagger \right| k \right\rangle &= \lambda_i \left\langle i \left| H_1 \right| k \right\rangle , && (20) \end{aligned}$$

where H_1 reads

$$\begin{aligned} H_1 &= -\frac{d^2}{dx^2} + \mu_1 x^4 - \nu_1^2 x^2 + 2\mu_1 x + i\nu_1 + 2i\mu_1\nu_1 x^3 \qquad (21) \\ &\equiv H_R + i\nu_1 + 2i\mu_1\nu_1 x^3 = H_R + i\nu_1 V_I \, . \end{aligned}$$

We can rewrite eq. (20) as

$$\lambda_k \left(H_R^{ik} - i\nu_1 V_I^{ik} \right) = \lambda_i \left(H_R^{ik} + i\nu_1 V_I^{ik} \right) , \qquad (22)$$

where $H_R^{ik} = A + iB$ and $V_I^{ik} = C + iD$ are complex numbers. Thus, by equating real and imaginary parts of both sides of eq. (22), one gets

$$\frac{\lambda_k}{\lambda_i} = \frac{A^2 - \nu_1^2 D^2}{(A + \nu_1 D)^2} = \frac{B^2 - \nu_1 C^2}{(B - \nu_1 C)^2} , \qquad (23)$$

and, if $\lambda_k/\lambda_i < 0$, one can argue that the H_R^{ik}'s are strongly suppressed for small values of ν_1. In this case, the two sectors corresponding to positive and negative eigenvalues are expected to be, in practice, dynamically decoupled; thus, $\mathcal{F}^{-1}$ seems to be not too unreasonable as a candidate metric operator, at least for small values of ν_1.

Coming now to the properties of Hamiltonian (21), one can separate the $\mathcal{PT}$-symmetric and antisymmetric parts as

$$H_1 = H_1^{\mathcal{PT}} + 2\mu_1 x + i\nu_1 \, , \qquad (24)$$

where

$$H_1^{\mathcal{PT}} = -\frac{d^2}{dx^2} + \mu_1 x^4 + \nu_1 x^2 + 2i\mu_1\nu_1 x^3 \, . \qquad (25)$$

$H_1^{\mathcal{PT}}$ is well controlled from a mathematical point of view, so that our proposal opens a way to study some additional Hamiltonians enriching the class of the recent popular non-Hermitian versions by addition of the non-$\mathcal{PT}$-symmetric Stark-like term.

In general,for all $\mu, \nu \in \mathbf{R}$ let $H(\mu,\nu)$ denote the Schrödinger operator in $L^2(\mathbf{R})$ defined by

$$H(\mu,\nu) = -\frac{d^2}{dx^2} + \mu^2 x^4 + 2i\mu\nu x^3 - \nu^2 x^2 + 2\mu x \tag{26}$$

on the domain $D = H^2(\mathbf{R}) \cap L^2_4(\mathbf{R})$.

Then $H(\mu,\nu)$ is a closed operator with compact resolvents and therefore discrete spectrum. In fact the operator $-\frac{d^2}{dx^2} + \mu^2 x^4$ enjoyes such properties (see [19]), which extend to the analytic family of operators

$$H_g(\mu,\nu) = -\frac{d^2}{dx^2} + \mu^2 x^4 + g(2i\mu\nu x^3 - \nu^2 x^2 + 2\mu x) \tag{27}$$

for $g \in \mathbf{C}$ and in particular to the original operator $H(\mu,\nu)$ for $g = 1$ (for more details on the theory of analytic families of operators see [17] or [18]).

If we now introduce a further perturbation parameter $\gamma \in \mathbf{C}$ only in the linear term:

$$H_\gamma(\mu,\nu) = -\frac{d^2}{dx^2} + \mu^2 x^4 + 2i\mu\nu x^3 - \nu^2 x^2 + 2\mu\gamma x, \tag{28}$$

then $H_0(\mu,\nu)$ is $\mathcal{PT}$-symmetric with real spectrum (see [12]).

The spectral analysis for the complete operator $H_\gamma(\mu,\nu)$ for $\gamma \in \mathbf{R}$ can be performed in the framework of perturbation theory around $\gamma = 0$. More precisely, referring to recent results detained in [16] and [15], it is possible to prove that for fixed μ and ν there exists $\delta > 0$ such that the eigenvalues of $H_\gamma(\mu,\nu)$ are real and represent a sequence of analytic functiones $E_n(\gamma)$ for $\gamma \in]-\delta,\delta[$. For such values of γ each eigenvalue $E_n(\gamma)$ is the sum of the corresponding Rayleigh-Schrödinger perturbation expansion around the eigenvalue $E_n(0)$ of $H_0(\mu,\nu)$.

Yet to be explored is the usefulness of trying second and higher order derivatives in the ansatz for the $\mathcal{C}$ operator, with the possibility of non-linear algebraic structures [20].

Acknowledgements

M. Z. appreciates the hospitality of Dipartimento di Matematica, Università di Bologna where this work has been initiated. Partially, his participation was also supported by GA AS CR, grant Nr. 104 8302 and from the budget of the AS CR projects K1048102, K1010104 and AV0Z1048901. The authors wish to thank Prof. S. Graffi for useful discussions and for reading the manuscript.

References

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 4243.

[2] F. Cannata, G. Junker and J. Trost, Phys. Lett. A 246 (1998) 219;

F. M. Fernández, R. Guardiola, J. Ros and M. Znojil, J. Phys. A: Math. Gen. 31 (1998) 10105;

G. Lévai, F. Cannata and A. Ventura, Phys. Lett. A 300 (2002) 271 and J. Phys. A: Math. Gen. 35 (2002) 5041.

[3] M. Znojil, Conservation of pseudo-norm in PT symmetric quantum mechanics (arXiv: math-ph/0104012, unpublished).

[4] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205, 2814 and 3944 and Czech. J. Phys. 53 (2003) 1079;

A. Mostafazadeh, A Critique of PT-Symmetric Quantum Mechanics (arXiv: quant-ph/0310164, unpublished).

[5] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.

R.Kretschmer, L. Szymanowski, Phys.Lett.A 325 (2004) 112.

[6] M. Znojil, Phys. Lett. A 259 (1999) 220.

[7] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89 (2002) 270401 and 92 (2004) 119902;

C. M. Bender, P. N. Meisinger and and Q. Wang, J. Phys. A: Math. Gen. 36 (2003) 1973.

Z. Ahmed, Phys.Lett. A 310 (2003) 39.

[8] G.Alvarez, J. Phys. A: Math. Gen. 27 (1995) 4589.

[9] E. Caliceti, S. Graffi and M. Maioli, Commun. Math. Phys. 75 (1980) 51.

[10] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. 26 (1993) 5541.

[11] P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Gen. 34 (2001) 5679.

[12] K.C. Shin, Comm. Math. Phys. 229 (2002) 543

[13] A.Andrianov, F.Cannata, J.P.Dedonder, M.V.Ioffe, Int. J.Mod. Phys. A 14 (1999). 2675.

M.Znojil, F.Cannata, B.Bagchi, R.Roychoudhury, Phys.Lett. B 483 (2000) 284.

[14] E. Caliceti, S. Graffi, J. Phys. A: Math. Gen. 37 (2004) 2239.

[15] E. Caliceti, S. Graffi, in preparation.

[16] E. Caliceti, S. Graffi and J. Sjöstrand, *Spectra of $\mathcal{PT}$-Symmetric operators and perturbation theory*, in preparation.

[17] T. Kato, *Perturbation Theory for linear Operators*, 2nd edn (Springer, Berlin, 1976).

[18] M. Reed, B. Simon *Methods of Modern Mathematical Physics* vol IV (Academic Press, New York, 1978).

[19] B. Simon, Ann. Phys.(N. Y.) 58 (1970) 76.

[20] A. A. Andrianov and F. Cannata, *Nonlinear supersymmetry for spectral design in quantum mechanics*, submitted to J. Phys. A: Math. Gen. and references therein.