On the Form Factor for the Unitary Group

Mirko Degli Esposti* Andreas Knauf†

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Abstract

We study the combinatorics of the contributions to the form factor of the group $U(N)$ in the large N limit. This relates to questions about semi-classical contributions to the form factor of quantum systems described by the unitary ensemble.

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*Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato, 5, I-40127 Bologna, Italy, e-mail: desposti@dm.unibo.it

†Mathematisches Institut der Universität Erlangen-Nürnberg. Bismarckstr. 1 1/2, D-91054 Erlangen, Germany. e-mail: knauf@mi.uni-erlangen.de

1 Introduction

The *form factor* associated to a self–adjoint operator H is a real–valued function describing statistical properties of its spectrum. For sake of simplicity we assume that H acts on finite–dimensional Hilbert space and thus has eigenvalues $E_1, \ldots, E_N \in \mathbb{R}$. Then we consider the Fourier transform of the measure

$$\mu := \frac{1}{N} \sum_{j,k=1}^{N} \delta_{E_j - E_k}$$

and obtain the form factor

$$K(t) := \int_{\mathbb{R}} \exp(-itE)\, d\mu(E) = \frac{1}{N} \left| \mathrm{tr}(U(t)) \right|^2,$$

with the unitary time evolution $U(t) := \exp(-iHt)$ generated by H.

It is an empirical fact and a physical conjecture (see Bohigas, Giannoni and Schmit [BGS] and also [Ha]) that most form factors encountered in physical quantum systems resemble the form factor associated to a so–called random matrix ensemble (see Mehta [Me]).

The simplest of these is the so–called *unitary ensemble* on which we shall concentrate below. This is given by the unitary group $U(N)$ equipped with Haar probability measure μ_N. Its form factor is defined as

$$K_N(t) := \frac{1}{N} \left\langle |\mathrm{tr}(U^t)|^2 \right\rangle_N \qquad (t \in \mathbb{Z}), \tag{1.1}$$

with the expectation $\langle f \rangle \equiv \langle f \rangle_N := \int_{U(N)} f\, d\mu_N$ of a continuous function $f : U(N) \to \mathbb{C}$.

As the map $U \mapsto \mathrm{tr}(U^t)$ is a class function on the unitary group, we can apply Weyl's integration formula

$$\int_{U(N)} f\, d\mu_N = \frac{1}{N!} \int_{\mathbb{T}^N} f \Delta^2\, d\nu_N \tag{1.2}$$

to evaluate (1.1). In (1.2) $f : U(N) \to \mathbb{C}$ is assumed to be a class function. $\mathbb{T}^N \subseteq U(N)$ is a maximal torus and may be identified with the subgroup of diagonal matrices. $d\nu_N$ denotes Haar measure on $\mathbb{T}^N$. Finally for $h := \mathrm{diag}(h_1, \ldots, h_N) \in \mathbb{T}^n \subseteq U(N)$

$$\Delta(h) := \sum_{1 \le j < k \le N} |h_j - h_k|$$

is the modulus of Vandermonde's determinant for $h_1, \dots, h_N$. The combinatorial factor $N!$ is the order of the symmetric group S_N making its appearance as the Weyl group, see e.g. Fulton and Harris [FH].

With these data, the form factor is evaluated:

$$K_N(t) = \begin{cases} N & , \quad t = 0 \\ |t|/N & , \quad 0 < |t| \le N \\ 1 & , \quad N < |t| \end{cases} \qquad (t \in \mathbb{Z}). \tag{1.3}$$

This calculation is based on the eigenvalues $h_1, \dots, h_N$ of the unitary matrix.[1]

In [Be] Berry proposed a semiclassical evaluation of the form factor for quantum systems, based on the periodic orbits of the principal symbol (Hamiltonian function) of the Hamiltonian operator. For the different random matrix ensembles he derived in the range $0 < t \ll N$ the leading order of $K_N(t)$, which is linear in t/N.

More precisely, semiclassical theory based on the Gutzwiller trace formula provides a link between spectral quantities of the quantum Hamiltonian and properties of the chaotic dynamics of the corresponding classical system. In this approach the spectral two-point correlation function and its Fourier transform, namely the form factor, are calculated by approximating the density of states using the trace formula. This formula expresses them by sums over contributions from pairs of classical periodic trajectories.

If one includes only pairs of equal or time-reversed orbits (the so called "diagonal approximation") then the form factor agrees with random matrix theory, asymptotically close to the origin (long-range correlations).

A more systematic approach will require a complete control of all the other contributions. A first step towards an understanding of the "off-diagonal" contributions have been achieved in [BK]. But only recently, beginning with the article [SR] by Sieber and Richter, contributions involving pairs of periodic orbits were systematically considered in order to explain higher order terms in t/N.

In particular, for the geodesic flow on constant negative curvature, a particular family of pairs of periodic orbits have been presented in [SR] and [Si], which turned out to be relevant for the first correction to the diagonal approximation for the spectral form factor. These orbits pairs are given by trajectories which exhibit self-intersection with small intersection angles. This result has been generalized recently to more general uniformly hyperbolic dynamical systems [Sp].

[1] As the N eigenvalues of $U \in U(N)$ have mean distance $2\pi/N$, note that the natural argument of the form factor would be t/N instead of t. However, in order to simplify notation, we use the parameter $t \in \mathbb{Z}$.

The combinatories, however, turned out to be highly nontrivial. These combinatorial difficulties in handling high order corrections to the semiclassical expression of the form factor persist also in the context of quantum graphs, see Kottos and Smilansky [KS1, KS2], where these off-diagonal contributions have been explored up to the third order [Ber1, BSW1, BSW2].

To our opinion the complex combinatorics should first be studied in the simplest situation possible, that is, on the group level. Here the unitary ensemble is the simplest one, since the case of the orthogonal or symplectic ensemble involves additional elements like the Brauer algebra, see Diaconis and Evans [DE].

Now we collect the main points of the article.

We want to compare the form factor $K_N(t)$ with the diagonal contribution

$$\frac{t}{N}\Delta_N^{\max}(t) \quad \text{with} \quad \Delta_N^{\max}(t) := \sum_{i_1,\ldots,i_k=1}^{N} \left\langle \prod_{k=1}^{t} |U_{i_k i_{k+1}}|^2 \right\rangle_N \tag{1.4}$$

(note that only sum over *one* t–tuple of indices in $\Delta_N^{\max}$, hence the name *diagonal contribution*).

The expectation values of products of matrix entries in (1.4) and in (1.1) can be evaluated using the well–known formula (2.1), that is, by summing class functions on the symmetric group S_t.

So in **Sect. 2** we introduce some notation concerning the symmetric group.

In **Sect. 3** we discuss the relation between the class functions V and $\mathcal{N}$ on S_t used in (2.1). As stated in Prop. 3.3 they are mutual inverses in the group algebra of S_t.

As a simple by–product, this leads to a re–derivation of Eq. (1.2) in the linear regime $|t| \leq N$ (Remarks 3.6).

In **Sect. 4** we study the relation between a natural metric on S_t and the joint operation on the associated partition lattice $\mathcal{P}_t$ (Prop. 4.1).

Sect. 5 starts by a (partial) justification of our above definition (1.4), and an estimate of its contributions in terms of formula (2.1). Here the interplay between the partition lattice and cyclic permutation becomes essential (Prop. 5.4). Although Prop. 5.4 is a statement about the $N \to \infty$ limit, we present evidence for our conjecture 5.8 which is a uniform in $N \geq t$ version of Prop. 5.4.

In **Sect. 6** we first prove that only derangements (that is, fixed point free permutations) are involved in the diagonal approximation (Prop. 6.2). Then we estimate the number of contributions to $\Delta_N^{\max}$ with a given power of N (Prop. 6.3).

This leads us to our main result in **Sect. 7**: Assuming Conjecture 5.8, there exists a subinterval $I := [\varepsilon, C - \varepsilon] \subset [0, 1]$ such that the diagonal approximation converges uniformly to the form factor if $t/N \in I$ (Thm. 7.1).

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2 Generalities on the Symmetric Group

As already mentioned in the Introduction, the symmetric group S_t of permutations of the set $[t] := \{1, \ldots, t\}$ plays an important rôle in the analysis of the unitary ensemble.

We begin by introducing some notation, see Sagan [Sag] for more information. For $\sigma \in S_t$ the *cycle length* of $i \in [t]$ is the smallest $n \in \mathbb{N}$ with $\sigma^n(i) = i$. i is a *fixed point* of σ if $n = 1$. The *cycle* of i is given by $(i, \sigma(i), \ldots, \sigma^{n-1}(i))$, and can be interpreted as the group element of S_t which permutes the $\sigma^k(i)$ in the prescribed order, leaving the other elements of $[t]$ fixed.

$e \in S_t$ denotes the identity element.

Writing a group element $\sigma \in S_t \setminus \{e\}$ as a product of disjoint cycles $\sigma = \sigma^{(1)} \cdot \ldots \cdot \sigma^{(k)}$, we sometimes omit the fixed points.

Two lattices are associated with the symmetric group S_t:

- The *partition lattice* $\mathcal{P}_t$ of *set partitions* $p = \{a_1, \ldots, a_k\}$, with *atoms* or *blocks* $a_l \subseteq [t]$ $\big(a_l \cap a_m = \emptyset$ for $l \neq m,\ a_l \neq \emptyset$ and $\bigcup_{l=1}^k a_l = [t]\big)$.

  $p \in \mathcal{P}_t$ is called *finer* than $q \in \mathcal{P}_t$ (and q *coarser* than p, denoted by $p \preccurlyeq q$) if every block of p is contained in a block of q.

  The *meet* $p \vee q$ of $p, q \in \mathcal{P}_t$ is the unique finest element coarser than p and q.

  We define the *rank* $|p|$ of the partition $p = \{a_1, \ldots, a_k\} \in \mathcal{P}_t$ by $|p| := k$ (note that this is called the *corank* in [Ai]).

- The *dominance order* $\mathcal{D}_t$ of *number partitions* $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{D}_t$ of $t \in \mathbb{N}$ $\big($with $\lambda_l \in \mathbb{N},\ \lambda_{l+1} \leq \lambda_l$ and $\sum_{l=1}^k \lambda_l = t\big)$. The map

  $$\mathcal{P}_t \to \mathcal{D}_t \quad , \quad \{a_1, \ldots, a_k\} \mapsto (|a_1|, \ldots, |a_k|)$$

  induces an order relation and a rank function on $\mathcal{D}_t$.

See Aigner [Ai] for more information.

Each permutation $\sigma \in S_t$ partitions $[t]$ into atoms belonging to the same cycle of σ. Thus we have a map

$$S_t \to \mathcal{P}_t \quad , \quad \sigma \mapsto \hat{\sigma}.$$

If the context is clear, we omit the hat. In particular $|\sigma| := k$ if $\sigma = \sigma^{(1)} \cdot \ldots \cdot \sigma^{(k)}$ is the disjoint cycle decomposition of σ (including fixed points!).

Examples 2.1
1. $\sigma = (124)(3) \in S_4$ and $\rho = (142)(3) \in S_4$ have the set partition $\hat{\sigma} = \hat{\rho} = \{\{1,2,4\},\{3\}\} \in \mathcal{P}_4$ and number partition $[\sigma] = [\rho] = (3,1) \in \mathcal{D}_4$ (Here $[\sigma] := \{\alpha^{-1}\sigma\alpha \mid \alpha \in S_t\}$ denotes the conjugacy class of $\sigma \in S_t$).
2. $\sigma = (12)(34) \in S_4$ and $\rho = (13)(24) \in S_4$ have rank $|\sigma| = |\rho| = 2$, whereas $|\sigma \vee \rho| = |\{\{1,2,3,4\}\}| = 1$.

The importance of the dominance order $\mathcal{D}_t$ for the symmetric group is obvious, as the elements of $\mathcal{D}_t$ naturally enumerate the conjugacy classes of S_t. Thus they also enumerate the irreducible representations and their characters

$$\chi_\lambda : S_t \to \mathbb{R} \qquad (\lambda \in \mathcal{D}_t).$$

In the present context the importance of the partition lattice $\mathcal{P}_t$ comes from the following identity:

Lemma 2.2 *For all $t, N, k \in \mathbb{N}$ and $\pi_1, \ldots, \pi_k \in S_t$*

$$\sum_{(i_1,\ldots,i_t)\in[N]^t} \prod_{l=1}^{k} \left(\prod_{j=1}^{t} \delta_{i_j}^{i_{\pi_l(j)}} \right) = N^{|\pi_1 \vee \ldots \vee \pi_k|}.$$

From now on our standing assumption relating the groups S_t and $U(N)$ is $t \leq N$. Then the following important formula can be found in Samuel [Sam], see also Brouwer and Beenakker [BB]:

$$\left\langle U_{a_1b_1} \ldots U_{a_sb_s} \overline{U}_{\alpha_1\beta_1} \ldots \overline{U}_{\alpha_t\beta_t} \right\rangle_N = \delta_t^s \sum_{\sigma,\pi \in S_t} V_N(\sigma^{-1}\pi) \prod_{k=1}^{t} \delta_{a_k}^{\alpha_{\sigma(k)}} \, \delta_{b_k}^{\beta_{\pi(k)}}, \quad (2.1)$$

where for $N \geq t$ the class function $V \equiv V_N : S_t \to \mathbb{R}$ is given by

$$V_N := \sum_{\lambda \in \mathcal{D}_t} \frac{\chi_\lambda(e)}{t! f_\lambda(N)} \, \chi_\lambda. \quad (2.2)$$

f_λ is a polynomial in N of order t vanishing at certain integers:

$$f_\lambda(N) := \sum_{\sigma\in S_t} \frac{\chi_\lambda(\sigma)N^{|\sigma|}}{\chi_\lambda(e)} = \prod_{i=1}^{k} \frac{(N+\lambda_i-i)!}{(N-i)!} \qquad (\lambda\in\mathcal{D}_t) \qquad (2.3)$$

(see Appendix A of [Sam]).

Recalling the correspondence between irreducible representations of S_t and conjugacy classes of S_t, i.e. ordered number-partitions $\lambda = (\lambda_1,\ldots,\lambda_k) \in \mathcal{D}_t$, $\lambda_1 \geq \ldots \geq \lambda_k$, of t, by evaluating Frobenius' Formula the dimension $\chi_\lambda(e)$ of the representation appearing in (2.2) equals

$$\chi_\lambda(e) = t!\,\frac{\prod_{i<j}(\lambda_i-\lambda_j+j-i)}{\prod_i(\lambda_i+k-i)!} \qquad (\lambda\in\mathcal{D}_t),$$

see [FH], Eq. (4.11).

3 The Linear Regime of the Form Factor

Next we decompose the form factor $K_N(t)$ into a sum of products of the class functions V_N and $\mathcal{N}:\sigma\mapsto N^{|\sigma|}$ on S_t. This will allow us to compare it with the diagonal contribution to be introduced in Section 5.

As a side effect, we will re–derive its concrete form (1.3) for $|t|\leq N$. Since $K_N(0) = N^{-1}(\mathrm{tr}(\mathbb{1}_N))^2 = N$ and $K_N(-t) = K_N(t)$, we effectively only need to consider the regime $0 < t \leq N$ where K_N is linear.

Evaluating $\mathrm{tr}(U^t)$ as $\sum_{i\in[N]^t}\prod_{k=1}^{t} U_{i_k i_{k+1}}$ in (1.1), we get a cyclic ordering of the sub–indices, given by the circular permutation

$$\tau := (1,2,\ldots,t)\in S_t.$$

Conjugation of $\sigma\in S_t$ by τ will be denoted by $\sigma_+ := \tau^{-1}\sigma\tau$.

Given $t\in\mathbb{N}$ and the permutation group S_t, we denote by

$$\mathcal{C}_t := \{\sigma\in S_t \mid |\sigma| = 1\}$$

the set of *circular permutations*. This subset is of cardinality

$$|\mathcal{C}_t| = (t-1)!,$$

and every $\sigma\in\mathcal{C}_t$ can be written in the form $\sigma = \pi^{-1}\tau\pi$, $\pi\in S_t$.

Lemma 3.1 *The sets*

$$M(\phi,\phi') := \left\{(\pi,\sigma)\in S_t\times S_t \mid \phi=\pi^{-1}\sigma_+,\ \phi'=\pi^{-1}\sigma\right\} \qquad (\phi,\phi'\in S_t)$$

are of size

$$|M(\phi,\phi')| = \begin{cases} t & ,\ \phi'\tau\phi^{-1}\in\mathcal{C}_t \\ 0 & ,\ \textit{otherwise}\end{cases}$$

and form a partition of $S_t\times S_t$.

Proof:

1. By definition of $M(\phi,\phi')$ any given pair $(\pi,\sigma)\in S_t\times S_t$ lies in exactly one subset $M(\phi,\phi')\subseteq S_t\times S_t$, namely in $M(\pi^{-1}\sigma_+,\pi^{-1}\sigma)$.

2. If $(\pi,\sigma)\in M(\phi,\phi')$ then $\phi'\tau\phi^{-1}=\pi^{-1}\tau\pi\in\mathcal{C}_t$. Then the t different

$$(\tau^l\pi,\tau^l\sigma)\in S_t\times S_t \qquad (l=0,\ldots,t-1)$$

   are in $M(\phi,\phi')$, too. As thus there are exactly $\mathcal{C}_t\times S_t=(t-1)!\times t!$ pairs $(\phi,\phi')\in S_t\times S_t$ with cardinality of the corresponding atoms $|M(\phi,\phi')|\geq t$, but $S_t\times S_t=t!\times t!$, their cardinality must be exactly t. $\square$

Proposition 3.2 *For all* $t\leq N\in\mathbb{N}$, *the form factor (1.1) equals*

$$K_N(t) = \frac{t}{N}\cdot\sum_{\substack{\phi,\phi'\in S_t\\ \phi'\tau\phi^{-1}\in\mathcal{C}_t}} V_N(\phi')N^{|\phi|} \tag{3.1}$$

$$= \frac{t}{N}\cdot\sum_{\gamma\in\mathcal{C}_t}\sum_{\phi\in S_t} V_N(\gamma\phi\tau^{-1})N^{|\phi|}\,. \tag{3.2}$$

Proof: Using sub-indices (mod t),

$$\left\langle\left|\mathrm{tr}(U^t)\right|^2\right\rangle_N = \sum_{\substack{i_1,\ldots,i_t\\ j_1,\ldots,j_t}}\left\langle U_{i_1i_2}\cdots U_{i_ti_1}\overline{U}_{j_1j_2}\cdots\overline{U}_{j_tj_1}\right\rangle_N$$

$$= \sum_{\pi,\sigma\in S_t} V_N(\pi^{-1}\sigma)\cdot\sum_{\substack{i_1,\ldots,i_t\\ j_1,\ldots,j_t}}\prod_{k=1}^t\delta_{i_k}^{j_{\pi(k)}}\cdot\delta_{i_{k+1}}^{j_{\sigma(k+1)}}$$

$$= \sum_{\pi,\sigma\in S_t} V_N(\pi^{-1}\sigma)\cdot N^{|\pi^{-1}\sigma_+|}\,, \tag{3.3}$$

since

$$\begin{aligned}
\sum_{\substack{i_1,\ldots,i_t\\ j_1,\ldots,j_t}} \prod_{k=1}^{t} \delta_{i_k}^{j_{\pi(k)}} \cdot \delta_{i_{k+1}}^{j_{\sigma(k+1)}} &= \sum_{\substack{i_1,\ldots,i_t\\ j_1,\ldots,j_t}} \prod_{k=1}^{t} \delta_{i_k}^{j_{\pi(k)}} \cdot \delta_{i_{\tau(k)}}^{j_{\sigma\tau(k)}} = \sum_{\substack{i_1,\ldots,i_t\\ j_1,\ldots,j_t}} \prod_{k=1}^{t} \delta_{i_k}^{j_{\pi(k)}} \cdot \delta_{i_k}^{j_{\sigma_+(k)}} \\
&= \sum_{\substack{i_1,\ldots,i_t\\ j_1,\ldots,j_t}} \prod_{k=1}^{t} \delta_{i_k}^{j_{\pi(k)}} \cdot \delta_{j_k}^{j_{\pi^{-1}\sigma_+(k)}} = \sum_{j_1,\ldots,j_t} \prod_{k=1}^{t} \delta_{j_k}^{j_{\pi^{-1}\sigma_+(k)}}.
\end{aligned}$$

In the last step of (3.3) we used Lemma 2.2. Eq. (3.1) now follows from Lemma 3.1. In (3.1) we can write $\phi' = \phi\tau^{-1}\gamma$ for a unique $\gamma \in \mathcal{C}_t$. This implies the second equation. □

To further evaluate these expressions for the form factor, we remind the reader of some general group theoretical notions.

Let G be a finite group with normalized counting measure, that is, the inner product

$$\langle f_1, f_2\rangle := \frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)}f_2(g) \qquad (f_1, f_2 \in L^2(G)). \tag{3.4}$$

The characters of the irreducible representations are orthonormal w.r.t. this inner product and form a basis of the subspace of class functions.

On $L^2(G)$ we have the unitary operators of left and right translations, given by

$$R_h f(g) := f(gh) \quad , \quad L_h f(g) := f(hg) \qquad (g, h \in G).$$

For irreducible characters $\chi_\mu, \chi_\lambda : G \to \mathbb{C}$ one has

$$\langle \chi_\lambda, L_g\chi_\mu\rangle = \langle \chi_\lambda, R_g\chi_\mu\rangle = \delta_{\lambda\mu}\frac{\chi_\lambda(g)}{\chi_\lambda(e)} \qquad (g \in G), \tag{3.5}$$

see Curtis and Reiner [CR], Eq. (31.16).

We now consider the group algebra $K[S_t]$ of the symmetric group, K denoting a field, i.e. the K–vectorspace $\{f : S_t \to K\}$, with multiplication of $f, g \in K[S_t]$ given by

$$f * g(\alpha) := \sum_{\sigma\in S_t} f(\sigma)g(\sigma^{-1}\alpha) = \sum_{\sigma\in S_t} f(\alpha\sigma^{-1})g(\sigma) \qquad (\alpha \in S_t)$$

and neutral element $\mathbb{1}_e \in K[S_t]$.

More specifically we use the field $K := \mathbb{C}(N)$ of rational functions and denote by $\mathcal{N} \in K[S_t]$ the monomial-valued function

$$\mathcal{N}(\alpha) := N^{|\alpha|} \qquad (\alpha \in S_t)$$

(which, like $\mathbb{1}_e$, is a class function).

Proposition 3.3 $V = \mathcal{N}^{-1}$.

Proof: We have, using Eq. (2.3)

$$V(\phi) = \sum_{\lambda \in \mathcal{D}_t} \frac{\chi_\lambda(\phi)\chi_\lambda(e)}{t! f_\lambda(N)} = \frac{1}{t!} \sum_{\lambda \in \mathcal{D}_t} \frac{\chi_\lambda(\phi)(\chi_\lambda(e))^2}{\sum_{\sigma \in S_t} \chi_\lambda(\sigma) N^{|\sigma|}}$$

Thus (as $|\sigma^{-1}| = |\sigma|$) we must prove that

$$\sum_{\lambda \in \mathcal{D}_t} \frac{\sum_{\omega \in S_t} \chi_\lambda(\alpha\omega) N^{|\omega|}}{\sum_{\sigma \in S_t} \chi_\lambda(\sigma) N^{|\sigma|}} (\chi_\lambda(e))^2 = t! \mathbb{1}_e(\alpha). \tag{3.6}$$

In order to show that the l.h.s. is in fact independent of N (if $N \geq t$ so that the denominator does not vanish!), for $\sigma \in S_t$ we sum over the conjugacy class $[\sigma] \subseteq S_t$ of σ, using that $|\rho\sigma\rho^{-1}| = |\sigma|$.

More specifically we claim the existence of a constant $C_\lambda(\alpha)$ such that for all $\sigma \in S_t$

$$\sum_{\tilde{\sigma} \in [\sigma]} \chi_\lambda(\alpha\tilde{\sigma}) = C_\lambda(\alpha) \sum_{\tilde{\sigma} \in [\sigma]} \chi_\lambda(\tilde{\sigma}). \tag{3.7}$$

Equivalently we show that

$$\mathcal{L}_\lambda(\sigma) \equiv \mathcal{L}_\lambda(\alpha, \sigma) := \sum_{\rho \in S_t} \chi_\lambda(\alpha\rho\sigma\rho^{-1})$$

equals $C_\lambda(\alpha) \cdot r_\lambda(\sigma)$ with $r_\lambda(\sigma) := \sum_{\rho \in S_t} \chi_\lambda(\rho\sigma\rho^{-1}) = t!\chi_\lambda(\sigma)$.

Now for $\alpha \in S_t$

$$\begin{aligned} \mathcal{L}_\lambda(\sigma) &= \sum_{\rho \in S_t} \chi_\lambda((\rho^{-1}\alpha\rho)\sigma) \\ &= \frac{|S_t|}{|[\alpha]|} \sum_{\pi \in [\alpha]} \chi_\lambda(\pi\sigma) = \frac{t!}{|[\alpha]|} \sum_{\pi \in [\alpha]} L_\pi \chi_\lambda(\sigma). \end{aligned}$$

$\mathcal{L}_\lambda$ being a class function, we write it in the form

$$\mathcal{L}_\lambda = \sum_{\mu \in \mathcal{D}_t} d_\mu \chi_\mu \tag{3.8}$$

and determine the coefficients d_μ using the orthonormality relation $\langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda\mu}$. By Eq. (3.5)

$$\langle L_\pi \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda\mu} \frac{\chi_\lambda(\pi)}{\chi_\lambda(e)}$$

which leads to

$$d_\mu = \delta_{\lambda\mu} \frac{t!}{|[\alpha]| \chi_\lambda(e)} \sum_{\pi \in [\alpha]} \chi_\lambda(\pi).$$

Inserting this into (3.8) we see that $C_\lambda(\alpha)$ in (3.7) equals

$$C_\lambda(\alpha) = \frac{1}{|[\alpha]| \chi_\lambda(e)} \sum_{\pi \in [\alpha]} \chi_\lambda(\pi).$$

Using:

$$\sum_{\sigma \in S_t} \chi_\lambda(\alpha\sigma) N^{|\sigma|} \; = \; C_\lambda(\alpha) \; \sum_{\sigma \in S_t} \chi_\lambda(\sigma) N^{|\sigma|} \, ,$$

the l.h.s. of (3.6) equals

$$\begin{aligned}
&\frac{1}{|[\alpha]|} \sum_{\lambda \in \mathcal{D}_t} \sum_{\pi \in [\alpha]} \chi_\lambda(\pi) \chi_\lambda(e) \\
&= \frac{1}{|[\alpha]|} \sum_{\pi \in [\alpha]} \sum_{\lambda \in \mathcal{D}_t} \chi_\lambda(\pi) \chi_\lambda(e) = \frac{1}{|[\alpha]|} \sum_{\pi \in [\alpha]} \mathbb{1}_e(\pi) \sum_{\lambda \in \mathcal{D}_t} (\chi_\lambda(e))^2 \\
&= \frac{1}{|[\alpha]|} \mathbb{1}_e(\alpha) t! = \mathbb{1}_e(\alpha) t!,
\end{aligned} \tag{3.9}$$

using the identity

$$\sum_{\lambda \in \mathcal{D}_t} \chi_\lambda(e)^2 \; = \; t!$$

in (3.9), see Chapter 5.2 in [Sag] and Rains [Ra]. This proves (3.6). $\square$

We redefine the inner product (3.4) on $\mathbb{C}[S_t]$ omitting the factor $1/|S_t| = 1/t!$:

$$\langle f_1, f_2 \rangle := \sum_{\sigma \in S_t} \overline{f_1(\sigma)} f_2(\sigma) \qquad (f_1, f_2 : S_t \to \mathbb{C}). \tag{3.10}$$

So the irreducible characters are now of norm $t!$. Anyhow we are now more interested in the following sets of functions:

Instead of considering the field $\mathbb{C}(N)$ of rational functions in the variable N we will now specialize the value $N \in \mathbb{N}$, $N \geq t$.

Define for $\sigma \in S_t$ the translates of $\mathcal{N}$:

$$\hat{q}_\sigma \in \mathbb{C}[S_t] \quad , \quad \hat{q}_\alpha(\sigma) := N^{|\sigma^{-1}\alpha|} \qquad (\alpha \in S_t).$$

Similarly we define the translates

$$V_\sigma \in \mathbb{C}[S_t] \quad , \quad V_\alpha(\sigma) := V_N(\sigma^{-1}\alpha) \qquad (\alpha \in S_t)$$

of V_N.

Lemma 3.4 *For $N \geq t$ the $\hat{q}_\sigma$, $\sigma \in S_t$ form a basis of the vector space $\mathbb{C}[S_t]$, with dual basis V_σ, $\sigma \in S_t$.*

Proof: Considered as rational functions, for $\alpha, \beta \in S_t$ the inner product equals

$$\begin{aligned}\langle V_\alpha, \hat{q}_\beta \rangle &= \sum_{\sigma \in S_t} V_N(\sigma^{-1}\alpha) N^{|\sigma^{-1}\beta|} = \sum_{\sigma \in S_t} V_N(\alpha^{-1}\sigma) N^{|\sigma^{-1}\beta|} \\ &= \sum_{\rho \in S_t} V_N(\rho) N^{|\rho^{-1}(\alpha^{-1}\beta)|} = V * \mathcal{N}(\alpha^{-1}\beta) = \delta_\alpha^\beta .\end{aligned}$$

Specializing the value of N, this duality relation is true as long as the rational functions are defined. By inspection of the definition (2.2) of V_N (in particular of the f_λ defined in (2.3)) this is the case as long as $N \geq t$. As the number of the V_α and of the $\hat{q}_\beta$ both equals $\dim(\mathbb{C}[S_t]) = t!$, these are indeed bases. $\square$

Corollary 3.5 *For $t \leq N$*

$$\sum_{\sigma \in S_t} V_N(\alpha\sigma^{-1}) N^{|\sigma|} = 1\!\!1_e(\alpha) \qquad (\alpha \in S_t).$$

Remarks 3.6

1. Corollary 3.5 allows us to regain formula (1.3), i.e.

$$K_N(t) = \frac{t}{N} \quad \text{for} \quad 0 < t \leq N.$$

Using Prop. 3.2 we have

$$\begin{aligned}K_N(t) &= \frac{t}{N} \sum_{\gamma \in C_t} \sum_{\phi \in S_t} V_N(\gamma\phi\tau^{-1}) N^{|\phi|} \\ &= \frac{t}{N} \sum_{\gamma \in C_t} \sum_{\sigma \in S_t} V_N(\sigma) N^{|\gamma^{-1}\sigma\tau|} = \frac{t}{N} \sum_{\gamma \in C_t} 1\!\!1_\gamma(\tau) = \frac{t}{N}.\end{aligned}$$

2. As $V_N : S_t \to \mathbb{R}$ is a class function, we can also use the notation

$$V_N : \mathcal{D}_t \to \mathbb{R} \quad , \quad V_N([\sigma]) := V_N(\sigma).$$

Then we can calculate V_N using Prop. 3.3. Some examples:

- for $t = 1$ we have $V_N(1) = \frac{1}{N}$;
- for $t = 2$ and denominator $D_2 := N(N^2 - 1)$ we have

$$V_N(1,1) = \frac{N}{D_2} \quad , \quad V_N(2) = -\frac{1}{D_2};$$

- for $t = 3$ and $D_3 := N^3(N^2 - 1)(N^2 - 4)$ we have

$$V_N(1,1,1) = \frac{N^4 - 2N^2}{D_3} \quad , \quad V_N(2,1) = -\frac{N^3}{D_3} \quad \text{and} \quad V_N(3) = \frac{2N^2}{D_3}.$$

3. The large N asymptotics of $V_N : \mathcal{D}_t \to \mathbb{R}$ for $\lambda = (\lambda_1, \ldots, \lambda_k)$ is given by

$$V_N(\lambda) \sim (-1)^{t-k} N^{k-2t} \prod_{l=1}^{k} C_{\lambda_l} \qquad (N \to \infty) \tag{3.11}$$

with the Catalan number $C_l := \binom{2l-2}{l-1}/l$, see [Sam].

4 The Rank Function and the Join of Partitions

It is useful to give a geometric meaning to our estimates. For this purpose we equip the symmetric group S_t with the metric

$$d : S_t \times S_t \to \{0, 1, \ldots, t\} \quad , \quad d(\sigma, \gamma) = t - |\sigma\gamma^{-1}|.$$

The easiest way to visualize this metric is to consider the $\binom{t}{2}$–regular *Cayley graph* (S_t, E_t) having the symmetric group as its vertex set, and edge set

$$E_t := \left\{ (\rho, \rho') \in S_t \times S_t \mid \rho^{-1}\rho' \text{ is a transposition } \right\}.$$

Proposition 4.1 *1. $d(\sigma, \gamma)$ is the distance between the vertices σ and γ on the Cayley graph (S_t, E_t). So in particular the metric d is invariant under the left and right self-actions of S_t.*

2. $$|\pi \vee \sigma| = \min\left\{|\pi\mu^{-1}| \mid \mu \preccurlyeq \sigma\right\} \qquad (\pi, \sigma \in S_t).$$
So in particular
$$d(\pi, \sigma) \leq t - |\pi \vee \sigma|.$$

3. $$d(\rho, \rho') \geq \big|\, |\rho \vee \sigma| - |\rho' \vee \sigma| \,\big| \qquad (\rho, \rho', \sigma \in S_t).$$

Proof:

1. For $\rho := \sigma\gamma^{-1}$ with disjoint cycle decomposition $\rho = \rho_1 \cdot \ldots \cdot \rho_k$ we have $d(\sigma, \gamma) = d(\rho, e) = t - k = \sum_{i=1}^{k}(l_i - 1)$, l_i being the length of ρ_i. Exactly $l - 1$ transpositions are needed to form a cycle of length l.

2. Let $(c_1, \ldots, c_m)$, $c_k \subseteq \{1, \ldots, t\}$ be the partition corresponding to the cycles of π. We consider the graph (V, E) with vertex set $V := \{c_1, \ldots, c_m\}$ and edges $\{c_i, c_j\} \in E$ for which there are elements $e_i \in c_i$, $e_j \in c_j$ which belong to the same cycle of σ.

   Choose for each connected component of (V, E) a spanning tree and representatives $\{e_i, e_j\}$ of its edges. Then by construction the product μ_0 of the transpositions (e_i, e_j) meets $\mu_0 \preccurlyeq \sigma$, and $|\pi\mu_0^{-1}| = |\pi \vee \sigma|$. Any $\mu \preccurlyeq \sigma$ can be written in the form $\mu = \rho\mu_0$ with $\rho \preccurlyeq \sigma$. As no cycles of $\pi\mu_0^{-1}$ can be joined by right multiplication with $\rho^{-1} \preccurlyeq \sigma$, the statement follows.

3. By symmetry of the metric d we assume $|\rho \vee \sigma| \geq |\rho' \vee \sigma|$ and choose $\mu_0 \preccurlyeq \sigma$ so that $|\rho' \vee \sigma| = |\rho'\mu_0^{-1}|$. Then, again by Part 2 of the proposition
$$0 \leq |\rho \vee \sigma| - |\rho' \vee \sigma| \leq |\rho\mu_0^{-1}| - |\rho'\mu_0^{-1}|.$$
   By Part 1 of the proposition
$$|\rho\mu_0^{-1}| - |\rho'\mu_0^{-1}| \leq d(\rho\mu_0^{-1}, \rho'\mu_0^{-1}) = d(\rho, \rho')$$
   since multiplication by a transposition changes the number of cycles by one, and since d is invariant under right multiplication. □

Remark 4.2 As the elements $\rho = \sigma = (12)(34)$, $\rho' = (13)(24)$ of S_4 show, in general the inequality $\big|\,|\rho \vee \sigma| - |\rho' \vee \sigma|\,\big| \leq \big|\,|\rho| - |\rho'|\,\big|$ does **not** hold. The reverse inequality is wrong, too in general.

5 The Diagonal Contribution

We now define and study the *diagonal approximation* for the unitary ensemble.

Setting $[N] := \{1, \dots, N\}$, the diagonal contribution is defined by:

$$\Delta_N(t) := \sum_{i \in [N]^t} \mathrm{per}(i) \left\langle \prod_{k=0}^{t-1} \left| U_{i_k i_{k+1}} \right|^2 \right\rangle, \tag{5.1}$$

where $\mathrm{per}(i)$ denotes the period of i. In fact (see (2.1)), only those terms of the form factor

$$K_N(t) = \sum_{i,j \in [N]^t} \left\langle \prod_{k=0}^{t-1} U_{i_k i_{k+1}} \overline{U}_{j_k j_{k+1}} \right\rangle$$

can be non-zero for which the sets

$$m_i(r) := \{k \in [t] \mid i_k = r\}$$

have equal multiplicity ($|m_i(r)| = m_i(r)$ for all $r \in [N]$).

In this case, if only multiplicities $|m_i(r)| \leq 1$ occur for i, there is a unique permutation σ with $j_{\sigma(k)} = i_k$ (and $\pi := \tau\sigma\tau^{-1}$ with $j_{\pi(k)+1} = i_{k+1}$), but in general we have

$$K_N(t) = \sum_{i,j \in [N]^t} \sum_{\kappa^{(1)}, \kappa^{(2)} \in S(m;(1)) \times \ldots \times S(m;(N))} V\left((\sigma \kappa^{(1)})^{-1} \pi \kappa^{(2)} \right).$$

As the dominant (in N) contributions are the ones with $V(e)$, i.e. $\sigma = \tau^l$ for some l, we call the sum

$$\Delta_N(t) = \sum_{i,j \in [N]^t \exists l \text{ with } j_{k+l} = i_k} \left\langle \prod_{k=0}^{t-1} U_{i_k i_{k+1}} \overline{U}_{j_k j_{k+1}} \right\rangle \tag{5.2}$$

$$= \sum_{i \in [N]^t} \mathrm{per}(i) \left\langle \prod_{k=0}^{t-1} \left| U_{i_k i_{k+1}} \right|^2 \right\rangle. \tag{5.3}$$

the *diagonal contribution*.

If $|m_i(r)| \leq 1$, then only terms $V(e)$ occur in $\left\langle \prod_{k=0}^{t-1} \left| U_{i_k i_{k+1}} \right|^2 \right\rangle$.

The number of all terms in $\sum_{i_1,\ldots,i_t}$ being N^t, for $k|t$

$$I_k := \{(i_1,\ldots,i_t) \mid i_{l+k} = i_l\}$$

is the set of terms with $\mathrm{per}(i_1,\ldots,i_t)|k$. So the number of terms with $\mathrm{per}(i) < t$ equals

$$\sum_{r>1, r|t} |I_{t/r}|\mu(r),$$

with the Möbius μ function. As $|I_k| = N^k$, this is only of order $N^{t/2}\log(t)$ and thus negligible compared to $|I_t| = N^t$ as $N \to \infty$.

For these reasons, we just define and study a function similar to (5.1) but replacing the period by its maximal value t. In fact for simplicity of notation we use the constant one instead:

$$\Delta_N^{\max}(t) := \sum_{i_1,\ldots,i_t} \left\langle \prod_{k=0}^{t-1} \left|U_{i_k i_{k+1}}\right|^2 \right\rangle$$

A basic manipulation yields:

Proposition 5.1

$$\Delta_N^{\max}(t) = \sum_{\pi,\sigma \in S_t} V_N(\pi^{-1}\sigma) N^{|\pi \vee \sigma_+|}, \tag{5.4}$$

Proof: Using (2.1),

$$\begin{aligned} \Delta_N^{\max}(t) &= \sum_{i \in [N]^t} \sum_{\pi,\sigma \in S_t} V_N(\pi^{-1}\sigma) \cdot \prod_{k=0}^{t-1} \delta_{i_k}^{i_{\pi(k)}} \cdot \delta_{i_{k+1}}^{i_{\sigma(k+1)}} \\ &= \sum_{i \in [N]^t} \sum_{\pi,\sigma \in S_t} V_N(\pi^{-1}\sigma) \cdot \prod_{k=0}^{t-1} \delta_{i_k}^{i_{\pi(k)}} \cdot \delta_{i_k}^{i_{\tau^{-1}\sigma\tau(k)}}. \end{aligned} \tag{5.5}$$

Lemma (2.2) now gives the result. □

A first easy observation is that for *bounded* t

$$\Delta_N^{\max}(t) = 1 + \mathcal{O}(1/N^2).$$

This follows by inserting (3.11) into (5.4).

Remark 5.2 In general for $0 < t \leq N$ **neither** $K_N(t) = \frac{1}{N}\Delta_N(t)$ **nor** $K_N(t) = \frac{t}{N}\Delta_N^{\max}(t)$, although both equations hold for $N = 1$ and $N = 2$. Already for $t = 3 \leq N$

$$\Delta_N(3) = \frac{3N^3}{D_3}(N^4 - 7N^2 + 4N + 2) \neq 3 = NK_N(3)$$

and

$$\Delta_N^{\max}(3) = \frac{N^3}{D_3}(N^4 - 3N^2 - 6N + 8) \neq 1 = N\frac{K_N(3)}{3},$$

with denominator $D_3 = N^3(N^2-1)(N^2-4)$.

So the diagonal approximation is not exact.

Remark 5.3 Note that we can also write [Ta]:

$$\Delta_N^{\max}(t) = \operatorname{tr}\left\langle M^t \right\rangle_N,$$

where M is the doubly stochastic matrix with elements $M_{i,j} := |U_{i,j}|^2$. In [Ber] it has been shown that if $\lambda_1 = 1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of M ($|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|$), then

$$\langle|\lambda_j|\rangle_N \to 0, \qquad j = 2, \ldots, N.$$

Moreover, the expectation of the second eigenvalue $\langle|\lambda_2|\rangle$ is of order $1/\sqrt{N}$. This upper bound is not helpful to show convergence of the trace to 1, namely this does not allow to bound expectations of positive powers $\langle|\lambda_j|^t\rangle$ [2].

This last remark force us to investigate more carefully each single contribution to the diagonal contribution. In particular, as we will see in the rest of the paper, the combinatorics become essential in order to bound uniformly the difference of the diagonal approximation to the form factor. This difference become large

[2]for a generic random variable X with values in $[0,1]$, like $X := |\lambda_2|$ the lower resp. upper bounds

$$\mathbb{E}(X)^t \leq \mathbb{E}(X^t) \leq \mathbb{E}(X) \qquad (t \in \mathbb{N})$$

following from Jensen's inequality (applied to the convex function $x \mapsto x^t$) resp. convolution inequality are optimal in general. This can be seen (for $c = \mathbb{E}(X) \in [0,1]$) by considering the cases of X distributions δ_c respectively $(1-c)\delta_0 + c\delta_1$.Even for an a.c. distribution like in our case the fall-off in t is not exponential.

above the regime $t = N$. According to (3.11) the terms in the sum (5.4) have fluctuating sign:

$$\operatorname{sign}\left(V_N(\pi^{-1}\sigma)\right) = (-1)^{d(\pi,\sigma)}.$$

This makes it advisable to perform a partial summation before estimating terms in absolute value. We thus rewrite the sum over π in (5.4) in the form of an inner product:

$$\Delta_N^{\max}(t) \;=\; \sum_{\sigma\in S_t} \langle V_N, \hat{p}_\sigma\rangle \quad \text{with} \quad \hat{p}_\sigma(\alpha) := N^{|\sigma\alpha^{-1}\vee\sigma_+|}. \tag{5.6}$$

Proposition 5.4 *There exists a function $C_t : S_t \to \{0,1\}$ such that*

$$\langle V_N, \hat{p}_\sigma\rangle_N = N^{|\sigma\vee\sigma_+|-t}(C_t(\sigma) + \mathcal{O}(1/N)) \qquad (\sigma \in S_t).$$

Proof: For $\sigma \in S_t$ the symmetric group is partitioned into the sets

$$B_n := \{\alpha \in S_t \mid |\alpha \vee \sigma_+| = |\sigma_+| - n\} \qquad (n = 0,\ldots,|\sigma|-1).$$

The metric d on S_t is then used to introduce for $\gamma \in B_n$

$$B(\gamma) := \left\{\alpha \in \bigcup_{k=0}^{n} B_k \mid |\alpha\vee\sigma_+| - |\gamma\vee\sigma_+| = d(\alpha,\gamma)\right\}.$$

Observe that by Part 3 of Lemma 4.1 we always have

$$0 \le |\alpha\vee\sigma_+| - |\gamma\vee\sigma_+| \le d(\alpha,\gamma). \tag{5.7}$$

In particular γ is the only element in $B(\gamma)\cap B(n)$. This enables us to define for $n = 0,\ldots,|\sigma|-1$

$$C_t(\gamma) := 1 - \sum_{\alpha\in B(\gamma)\setminus\{\gamma\}} C_t(\alpha) \qquad (\gamma \in B_n), \tag{5.8}$$

and the approximants

$$\tilde{p}_\sigma : S_t \to \mathbb{R} \quad , \quad \tilde{p}_\sigma := \sum_{\gamma\in S_t} C_t(\gamma)\, N^{|\gamma\vee\sigma_+|-t}\widehat{q}_{\gamma^{-1}\sigma} \qquad (\sigma \in S_t)$$

of the functions $\hat{p}_\sigma$.

- Next we prove that C_t only takes the values 0 and 1. This follows from the definition (5.8), if we can show that each γ has exactly one *predecessor* in

$$P := \{\alpha \in S_t \mid C_t(\alpha) = 1\},$$

that is, $|B(\gamma) \cap P| = 1$. This is done by induction in n, with

$$\gamma \in P \cap B_n \qquad (n = 0, \ldots, |\sigma| - 1)$$

and noting that $P \cap B_0 = B_0$ (the $\gamma \in B_0$ are their own predecessors so that $C_t(\gamma) = 1$).

- For the induction step we use the directed graph (S_t, E) with vertex set S_t and edges

$(\alpha, \beta) \in E \Leftrightarrow d(\alpha, \beta) = 1$ and $\alpha \in B_n, \beta \in B_{n-1}$ for some $n \in \{1, \ldots, |\sigma|-1\}$.

By the triangle inequality for $\gamma \in B(n)$ the set $B(\gamma)$ contains all $\alpha \in B_k,\ 0 \leq k \leq n$ for which there exists a directed chain

$\gamma = c_n, c_{n-1}, \ldots, c_k = \alpha$ from γ to α with $c_l \in B_l$ and $(c_l, c_{l-1}) \in E$

$(l = k + 1, \ldots, n)$.

Conversely all $\alpha \in B(\gamma)$ are of that form. Namely for $\alpha \in B(\gamma) \cap B_k$ we know that $d(\alpha, \gamma) = n - k$ so that there exist $c_n, \ldots, c_k \in S_t$ with $c_k = \gamma,\ c_k = \alpha$ and $d(c_{l-1}, c_l) = 1$. As $||c_{l-1} \vee \sigma_+| - |c_l \vee \sigma_+|| \leq d(c_{l-1}, c_l) = 1$ and $|c_n \vee \sigma_+| = n,\ |c_k \vee \sigma_+| = k$, we conclude $|c_l \vee \sigma_+| = l$ so that $(c_l, c_{l-1}) \in E$.

- This shows that $\alpha \in P$ if there does not exist an edge $(\alpha, \beta) \in E$, and thus $|B(\gamma) \cap P| \geq 1$ (as every directed chain starting at γ ends somewhere).

To prove that $|B(\gamma) \cap P| = 1$, we need a more precise characterization of the predecessors $\alpha \in P$. As there does not exist an edge of the form (α, β) in E, for all neighbors $\beta \in S_+$ of α (i.e. $d(\alpha, \beta) = 1$) we have $|\beta \vee \sigma_+| \leq |\alpha \vee \sigma_+|$. In other words if β differs from α by a transposition, and if two blocks of $\hat{\sigma}_+ \in P_t$ belong to the same block of $\alpha \vee \sigma_+$, then they belong to the same block of $\beta \vee \sigma_+$.

- We model this by considering for given $\sigma \in S_t$ the directed multigraph

$$G_\alpha = (V_\alpha, \mathcal{E}_\alpha)$$

associated to $\alpha \in S_t$. The vertex set of $G(\alpha)$ equals $V_\alpha := \{\hat{\sigma}_{+,1}, \ldots, \hat{\sigma}_{+,m}\}$, with $\hat{\sigma}_{+,1}, \ldots, \hat{\sigma}_{+,m} \subseteq [N]$ the blocks of the set partition $\hat{\sigma}_+ \in P_t$.

The multiplicity of the directed edge $(\hat{\sigma}_{+,i}, \hat{\sigma}_{+,j}) \in V_\alpha \times V_\alpha$ is given by

$$\mathcal{E}_\alpha : V_\alpha \times V_\alpha \backslash \Delta \to \mathbb{N}_0 \ , \ \mathcal{E}_\alpha(\hat{\sigma}_{+,i}, \hat{\sigma}_{+,j}) := \left|\{(u,v) \in \hat{\sigma}_{+,i} \times \hat{\sigma}_{+,j} \mid \alpha(u) = v\}\right|.$$

The in- and outdegrees of the blocks $\hat{\sigma}_{+,i}$ coincide, that is $\mathcal{E}^+_\alpha(\hat{\sigma}_{+,i}) = \mathcal{E}^-_\alpha(\hat{\sigma}_{+,i})$ for

$$\mathcal{E}^+_\alpha(\hat{\sigma}_{+,i}) := \sum_{\hat{\sigma}_{+,j}} \mathcal{E}_\alpha(\hat{\sigma}_{+,i}, \hat{\sigma}_{+,j}) \quad , \quad \mathcal{E}^-_\alpha(\hat{\sigma}_{+,i}) := \sum_{\hat{\sigma}_{+,j}} \mathcal{E}_\alpha(\hat{\sigma}_{+,i}, \hat{\sigma}_{+,j}).$$

Henceforth we omit the superscripts $\pm$ and simply refer to the *degree* $\mathcal{E}_\alpha(\hat{\sigma}_{+,i}) = \mathcal{E}^\pm_\alpha(\hat{\sigma}_{+,i})$ of the block.

- All ancestors $\alpha \in P$ have multigraphs G_α which have two–connected components, that is, the number of connected components cannot be increased by reducing a single degree $\mathcal{E}(\hat{\sigma}_{+,i})$ by one. This can be seen by noticing that for every $\alpha \in S_t$ the number of connected components of G_α equals $|\alpha \vee \sigma_+|$, and using that the $\alpha \in P$ don't have neighbors β with $|\beta \vee \sigma_+| = |\alpha \vee \sigma_+| + 1$.

- To prove $|B(\gamma) \cap P| = 1$, we assume that $\alpha^{(1)}, \alpha^{(2)} \in B(\gamma) \cap P$. So there exist directed chains $\gamma = c^{(i)}_n, c^{(i)}_{n-1}, \ldots, c^{(i)}_{k^{(i)}} = \alpha^{(i)}$ (with $\left(c^{(i)}_l, c^{(i)}_{l-1}\right) \in E$) from γ to $\alpha^{(i)}$, $i = 1, 2$, and we are to show that $\alpha^{(1)} = \alpha^{(2)}$. In each step the number $|c^{(1)}_l \vee \sigma_+| = |c^{(2)}_l \vee \sigma_+| = l$ of connected components of the multigraphs $G_{c^{(i)}_l}$ is reduced by one. That is, all connected components of the multigraph G_γ are broken into their two–connected subcomponents:

$$\mathcal{E}_{\alpha^{(i)}} \leq \mathcal{E}_\gamma \quad \text{and} \quad \mathcal{E}_{\alpha^{(i)}}(\hat{\sigma}_{+,j}) \neq 1 \qquad (i = 1, 2,\ j = 1, \ldots, m).$$

In fact this shows that $\mathcal{E}_{\alpha^{(1)}} = \mathcal{E}_{\alpha^{(2)}}$ so that the multigraphs of $\alpha^{(1)}$ and $\alpha^{(2)}$ coincide.

The multiplicity $\mathcal{E}_\gamma(\hat{\sigma}_{+,i}\hat{\sigma}_{+,j})$ of a directed edge of G_α is reduced only if $\mathcal{E}_\gamma(\hat{\sigma}_{+,i}, \hat{\sigma}_{+,j}) = 1$. So not only $\mathcal{E}_{\alpha^{(1)}}(\hat{\sigma}_{+,i}, \hat{\sigma}_{+,j}) = \mathcal{E}_{\alpha^{(2)}}(\hat{\sigma}_{+,i}, \hat{\sigma}_{+,j})$ but the chains connecting γ with $\alpha^{(1)} = \alpha^{(2)}$.

- We now know that $C_t(\alpha)$ only takes the values 0 and 1, and that $\tilde{p}_\sigma = \sum_{\gamma\in P} N^{|\gamma\vee\sigma_+|-t}\hat{q}_{\gamma^{-1}\sigma}$. This implies

$$\langle V_N, \tilde{p}_\sigma\rangle_N = \sum_{\gamma\in P} N^{|\gamma\vee\sigma_+|-t}\,\langle V_N, \hat{q}_{\gamma^{-1}\sigma}\rangle_N = C_t(\sigma)N^{|\sigma\vee\sigma_+|-t}.$$

It remains to show that

$$\langle V_N, \hat{p}_\sigma\rangle_N = \langle V_N, \tilde{p}_\sigma\rangle_N + \mathcal{O}(N^{|\sigma\vee\sigma_+|-t-1}).$$

But, denoting the unique predecessor of $\beta\in S_t$ by $P(\beta)$ (that is $\{P(\beta)\} = B(\beta)\cap P$), we have

$$\begin{aligned}\tilde{p}_\sigma(\beta^{-1}\sigma) &= \sum_{\gamma\in P} N^{|\gamma\vee\sigma_+|-d(\gamma,\beta)} \\ &= N^{|P(\beta)\vee\sigma_+|-d(P(\beta),\beta)} + \sum_{\substack{\gamma\in P\\ \gamma\neq P(\beta)}} N^{|\gamma\vee\sigma_+|-d(\gamma,\beta)}.\end{aligned}$$

By definition of $B(\gamma)$ the exponent of the first term equals

$$\big|P(\beta)\vee\sigma_+\big| - d\big(P(\beta),\beta\big) = |\beta\vee\sigma_+|,$$

whereas the exponents of the second term are smaller:

$$|\gamma\vee\sigma_+| - d(\gamma,\beta) < |\beta\vee\sigma_+|$$

on the other hand

$$\hat{p}_\sigma(\beta^{-1}\sigma) = N^{|\beta\vee\sigma_+|},$$

proving the claim. □

If $\sigma\in S_t$ consists of a single nontrivial cycle, the estimate of Prop. 5.4 can be replaced by an identity (Prop. 5.6 below).

We prepare this by a sum rule for the class function $\mathcal{N}$:

Lemma 5.5 *For all $k\in\mathbb{N}$*

$$\sum_{\sigma\in S_k} N^{|\sigma|} = \prod_{l=0}^{k-1}(N+l). \tag{5.9}$$

Proof: For $k = 1$ both sides equal N. So assume the formula to hold for $k-1$, so that

$$\sum_{\tilde{\sigma}\in S_k, \tilde{\sigma}(k)=k} N^{|\tilde{\sigma}|} = N \prod_{l=0}^{k-2}(N+l). \tag{5.10}$$

The group elements $\sigma \in S_k$ either have k as a fixed point s or can uniquely be written in the form

$$\sigma = (l,k)\tilde{\sigma}$$

with $l \in \{1,\dots,k-1\}$ and $\tilde{\sigma}(k) = k$. As in the second case $|\sigma| = |\tilde{\sigma}| - 1$,

$$\sum_{l=1}^{k-1} \sum_{\tilde{\sigma}\in S_k, \tilde{\sigma}(k)=k} N^{|\tilde{\sigma}|} = (k-1) \prod_{l=0}^{k-2}(N+l). \tag{5.11}$$

Adding the contributions (5.10) and (5.11) yields (5.9). $\square$

We now decompose $\hat{p}_\sigma$ in the form

$$\hat{p}_\sigma = \sum_{\gamma\in S_t} c_\gamma \hat{q}_\gamma \quad \text{with} \quad c_\gamma := \langle V_\gamma, \hat{p}_\sigma \rangle .$$

Proposition 5.6 *For a cycle* $\sigma = (i_1{+}1,\dots,i_k{+}1) \in S_t$ *(and* $\sigma_+ = (i_1,\dots,i_k)$*)*

$$\hat{p}_\sigma = \left(\prod_{l=1}^{k-1}(N+l)\right)^{-1} \sum_{\gamma' \preccurlyeq \sigma_+} \hat{q}_{\gamma'\sigma}. \tag{5.12}$$

Proof: • We evaluate both sides on $\tilde{\alpha} \in S_t$ and write $\tilde{\alpha}$ as $\tilde{\alpha} = \alpha\sigma$ to simplify expressions. Then

$$\hat{p}_\sigma(\alpha\sigma) = N^{|\alpha^{-1}\vee\sigma_+|} \quad \text{and} \quad \hat{q}_{\gamma'\sigma}(\alpha\sigma) = N^{|\alpha^{-1}\gamma'|}. \tag{5.13}$$

• Next we write α as a product of disjoint cycles z_j and note that

$$|z_j\alpha^{-1} \vee \sigma_+| - |z_j\alpha^{-1}\gamma'| = |\alpha^{-1} \vee \sigma_+| - |\alpha^{-1}\gamma'|$$

if z_j and σ_+ are disjoint. We thus can reduce α to a product of cycles intersecting σ_+.

• So we assume w.l.o.g. that all cycles z_j of α intersect $\sigma_+ = (i_1,\dots,i_k)$:

$$z_j = \big(i_{\pi(1)}, \tilde{z}_1, \dots, \tilde{z}_2, i_{\pi(2)}, \tilde{z}_3, \dots, \tilde{z}_{2s-2}, i_{\pi(s)}, \tilde{z}_{2s-1}, \dots, \tilde{z}_{2s}\big)$$

with $\tilde{z}_n \in \{1, \ldots t\} \backslash \{i_1, \ldots, i_k\}$. Then

$$\begin{aligned}\left(i_{\pi(s)}, i_{\pi(s-1)}, \ldots, i_{\pi(1)}\right) z_j &= \left(i_{\pi(1)}, \tilde{z}_1, \ldots, \tilde{z}_2\right)\left(i_{\pi(2)}, \tilde{z}_3, \ldots, \tilde{z}_4\right) \ldots \\ &\quad \left(i_{\pi(s)}, \tilde{z}_{2s-1}, \ldots, \tilde{z}_{2s}\right)\end{aligned}$$

is a product of disjoint cycles intersecting the cycle σ_+ only at $i_{\pi(n)}$. Furthermore

$$\left(i_{\pi(s)}, i_{\pi(s-1)}, \ldots, i_{\pi(1)}\right) \preccurlyeq \sigma_+$$

so that the map

$$\gamma' \mapsto \left(i_{\pi(s)}, i_{\pi(s-1)}, \ldots, i_{\pi(1)}\right) \gamma'$$

simply permutes the γ' in $\sum_{\gamma' \preccurlyeq \sigma_+} \hat{q}_{\gamma'\sigma}$. This allows to reduce to the case of simple intersections.

• We thus assume w.l.o.g. that the cycles z_j in the decomposition of α intersect σ_+ exactly in one point, say i_j. Under this assumption, by Lemma 5.5 and (5.13)

$$\begin{aligned}&\left(\prod_{l=1}^{k-1}(N+l)\right)^{-1} \sum_{\gamma' \preccurlyeq \sigma_+} \hat{q}_{\gamma'\sigma}(\alpha\sigma) \\ &= \left(\prod_{l=1}^{k-1}(N+l)\right)^{-1} N^{|\alpha^{-1}|-k} \sum_{\gamma \in S_k} N^{|\gamma|} = N^{|\alpha^{-1}|-k+1} \\ &= N^{|\alpha^{-1} \vee \sigma_+|} = \hat{p}_\sigma(\alpha\sigma),\end{aligned}$$

proving the assertion. □

Corollary 5.7 *For a cycle $\sigma = (i_1, \ldots, i_k) \in S_t$ of length k*

$$\langle V_N, \hat{p}_\sigma \rangle = \begin{cases} 1 & , k = 1 \quad \textit{that is} \quad \sigma = e \\ 0 & , 1 < k < t \\ \left(\prod_{l=1}^{t-1}(N+l)\right)^{-1} & , k = t \end{cases} . \tag{5.14}$$

Proof: This follows from Prop. 5.6 with $\langle V_N, \hat{q}_\sigma \rangle = \delta_{e,\sigma}$ (Lemma 3.4), remarking that only for $k = 1$ or $k = t$ there is a $\gamma' \preccurlyeq \sigma_+$ with $\gamma'\sigma = e$. □

This result and numerical experiments support the following conjecture (compare with Prop. 5.4):

Conjecture 5.8 There exists a constant $C_1 \geq 1$ such that for all $t \leq N \in \mathbb{N}$

$$|\langle V_N, \hat{p}_\sigma \rangle_N| \leq C_1 N^{|\sigma \vee \sigma_+| - t} \qquad (\sigma \in S_t).$$

6 Derangements and Circular Order

We now show that, apart from the identity, only the *derangements*, that is the fixed-point free permutations

$$D_t := \{\sigma \in S_t \mid \sigma(k) \neq k \quad \text{for all} \quad k \in \{1, \ldots, t\}\}$$

contribute in the sum (5.6).

This will follow from a statement of independent interest:

Proposition 6.1 *For $k = 1, \ldots, t+1$ denote by ${S_t}^{(k)}$ the subgroup*

$${S_t}^{(k)} := \{\sigma \in S_{t+1} \mid \sigma(k) = k\},$$

and by $I_k : S_t \to {S_t}^{(k)}$ the isomorphism induced by the injection

$$\tilde{I}_k : \{1, \ldots, t\} \hookrightarrow \{1, \ldots, t+1\} \quad , \quad \tilde{I}_k(i) = \begin{cases} i & , 1 \leq i < k \\ i+1 & , i \geq k. \end{cases} .$$

Then for $\sigma = I_{k+1}(\tilde{\sigma})$ and

$$\hat{p}_{\tilde{\sigma}} = \sum_{\tilde{\gamma} \in S_t} c_{\tilde{\gamma}} \, \hat{q}_{\tilde{\gamma}\tilde{\sigma}}$$

we have

$$\hat{p}_\sigma = \sum_{\tilde{\gamma} \in S_t} c_{\tilde{\gamma}} \, \hat{q}_{I_k(\tilde{\gamma})\sigma}.$$

Proof: • For $\beta \in {S_t}^{(k)} \subset S_{t+1}$, that is $\beta = I_k(\tilde{\beta})$ with $\tilde{\beta} \in S_t$

$$\begin{aligned} \hat{p}_\sigma\left(\beta\sigma\right) &= N^{\left|I_k(\tilde{\beta}) \vee \sigma_+\right|} = N^{\left|I_k(\tilde{\beta}) \vee (I_{k+1}(\tilde{\sigma}))_+\right|} = N^{\left|I_k(\tilde{\beta}) \vee I_k(\tilde{\sigma}_+)\right|} \\ &= N^{\left|\tilde{\beta} \vee \tilde{\sigma}_+\right|+1} = N \hat{p}_{\tilde{\sigma}}(\tilde{\beta}\tilde{\sigma}) \end{aligned}$$

and similarly

$$\begin{aligned} \hat{q}_{I_k(\tilde{\gamma})\sigma}\left(\beta\sigma\right) &= N^{|I_k(\tilde{\gamma})(I_k(\tilde{\beta}))^{-1}|} = N^{|I_k(\tilde{\gamma}\tilde{\beta}^{-1})|} = N^{|\tilde{\gamma}\tilde{\beta}^{-1}|+1} \\ &= N \hat{q}_{\tilde{\gamma}\tilde{\sigma}}(\tilde{\beta}\tilde{\sigma}). \end{aligned}$$

• The other elements of S_{t+1} can be uniquely written as a product of a transposition $(l, k) \in S_{t+1}$ and $\beta = I_k(\tilde{\beta}) \in {S_t}^{(k)}$. In that case a similar argument leads to

$$\hat{p}_\sigma(\beta\sigma) = \hat{p}_{\tilde{\sigma}}(\tilde{\beta}\tilde{\sigma}) \quad \text{and} \quad \hat{q}_{I_k(\tilde{\gamma})\sigma}\left(\beta\sigma\right) = \hat{q}_{\tilde{\gamma}\tilde{\sigma}}(\tilde{\beta}\tilde{\sigma}).$$

• So in any case the proportionality factor does not depend on $\tilde{\gamma}$. $\square$

Proposition 6.2 *For all $t \le N \in \mathbb{N}$*

$$\langle V_N, \hat{p}_\sigma \rangle = 0 \quad \textit{for} \quad \sigma \in S_t \setminus D_t\,,\, \sigma \neq e. \tag{6.1}$$

Proof: Lemma 3.4 implies the formula

$$\langle V_N, \hat{q}_\sigma \rangle = \delta_e^\sigma.$$

So (6.1) is equivalent to show that for these σ in the base decomposition

$$\hat{p}_\sigma = \sum_{\gamma \in S_t} c_\gamma \hat{q}_{\gamma\sigma}$$

of $\hat{p}_\sigma$ the coefficient $c_{\sigma^{-1}}$ equals zero. These σ have a fixed point $k+1$ (mod t) which has the additional property that k (mod t) is *not* a fixed point. So $\sigma = I_{k+1}(\tilde{\sigma})$ with $\tilde{\sigma} \in S_{t-1},\ \tilde{\sigma}(k) \neq k$. The base decomposition $\hat{p}_{\tilde{\sigma}} = \sum_{\tilde{\gamma} \in S_{t-1}} c_{\tilde{\gamma}} \hat{q}_{\tilde{\gamma}\tilde{\sigma}}$ leads to $\hat{p}_\sigma = \sum_{\tilde{\gamma} \in S_{t-1}} c_{\tilde{\gamma}} \hat{q}_{I_k(\tilde{\gamma})\sigma}$, see Prop. 6.1.

Thus if the $\tilde{\gamma} \in S_{t-1}$ term in (6.1) would be non–zero, it would be of the form $\langle V_N, \hat{p}_\sigma \rangle = c_{\tilde{\gamma}}$ for $\tilde{\gamma} \in S_{t-1}$ with $I_k(\tilde{\gamma}) = \sigma^{-1} = I_{k+1}(\tilde{\sigma}^{-1})$ or $I_{k+1}(\tilde{\sigma}) = I_k(\tilde{\gamma}^{-1})$. But this would imply $\tilde{\sigma}(k) = k$, contradicting the assumption. □

It is known that

$$|D_t| \sim \frac{|S_t|}{e} \quad \text{as} \quad t \to \infty.$$

So it could seem that we would only gain an unimportant factor $1/e$ by restricting the summation in (5.6) to the derangements (and the identity).

This is not so, since we can use the structure of the derangements under the τ action in our estimation.

For that purpose we now partition the derangements D_t by setting

$$D_t(k) := \{\sigma \in D_t \mid |\sigma \vee \sigma_+| = k\} \qquad (k = 1, \ldots, t).$$

So $D_t(k) = \emptyset$ for $k > t/2$, and we estimate the cardinalities of these sets.

Proposition 6.3 *There exists a $C_2 \ge 1$ such that for all $t \in \mathbb{N}$*

$$|D_t(k)| \le k C_2^k (t-k+1)! \qquad (k = 1, \ldots, \lfloor t/2 \rfloor).$$

Proof: Remark that the statement becomes trivial for $k = 1$ so that in the proof we assume $k \ge 2$.

- Each $\sigma \in D_t(k)$ induces a set partition

$$B(\sigma) \equiv B = (B_1, \ldots, B_k)$$

of $\{1, \ldots, t\}$ into the blocks of $\sigma \vee \sigma_+$ which is unique if you assume $|B_{l+1}| \geq |B_l|$ and $\min(B_l) \leq \min(B_{l+1})$ if $|B_{l+1}| = |B_l|$. As each B_l contains at least one cycle of σ (or rather the block corresponding to the cycle in the partition of σ), we have $|B_l| \geq 2$.

- Next we consider the intersections

$$C_{l,m} := B_l \cap B_m^+ \qquad (l, m \in \{1, \ldots, k\})$$

with the atoms $B_m^+ := \tau(B_m) = \{j + 1 \mid j \in B_m\}$ of the shifted set partition B^+. We thus get a set partition

$$C(\sigma) \equiv C = (C_{1,1}, \ldots, C_{k,k})$$

of $\{1, \ldots, t\}$ which is finer than B and B^+ but may contain empty atoms $C_{l,m}$. However, as σ is a derangement, we know that if $C_{l,m}(\sigma)$ is nonempty, it is a union of cycles of σ so that in any case $|C_{l,m}(\sigma)| \neq 1$.

- We now estimate $|D_t(k)|$ by

$$|D_t(k)| \leq \sum_{b=(b_1,\ldots,b_k)} Y(b)$$

where $2 \leq b_1 \leq \ldots \leq b_k$, $\sum_{l=1}^k b_l = t$ and

$$Y(b) := |\{\sigma \in D_t \mid |B_l(\sigma)| = b_l, \ l = 1, \ldots, k\}|.$$

- This quantity, in turn is estimated by

$$Y(b) \leq \sum_{c=(c_{1,1},\ldots,c_{k,k})} X(c) \prod_{l,m=1}^{k} c_{l,m}! \tag{6.2}$$

where now $c_{l,m} \in \{0, \ldots, b_l\} \backslash \{1\}$ with $\sum_{m=1}^k c_{l,m} = b_l$ and

$$X(c) = \left|\left\{B = (B_1, \ldots, B_k) \mid |C_{l,m}| = c_{l,m}\right\}\right|.$$

Here $\{B_1,\ldots,B_k\}$ is an arbitrary set partition of $\{1,\ldots,t\}$ with enumeration fixed by demanding $2 \le |B_1| \le \ldots \le |B_k|$ and, again, $\min(B_l) \le \min(B_{l+1})$ if $|B_l| = |B_{l+1}|$. Denoting as before by $C_{l,m}$ the intersection $B_l \cap B_m^+$ formula (6.2) follows by our above remark that all $\sigma \in D_t$ with $C_{l,m}(\sigma) = C_{l,m}$ have a cycle partition finer than $C = (C_{1,1},\ldots,C_{k,k})$ and there are $c_{l,m}!$ ways to permute the set $C_{l,m}$.

We bound $X(c)$ by considering the directed multigraph $G = G(c)$ with vertex set $V := \{1,\ldots,k\}$ and $c_{l,m}$ unlabeled directed edges from vertex l to m. Then

$$X(c) \le X_G(c), \tag{6.3}$$

where $X_G(c)$ is the number of closed Euler trails on G. This can be seen as follows:

1. The length of any closed Euler trail equals $\sum_{l,m=1}^{k} c_{l,m} = t$.
2. Any closed directed Euler trail on G (shortly called *trail* from now on) is uniquely characterized by the sequence $(v_1,\ldots,v_t)$ of vertices $v_j \in V$ it visits. This is due to our assumptions that the edges from l to m are unlabeled, and that the beginning of the closed trail is marked.
3. A set partition $B = (B_1,\ldots,B_k)$ of $\{1,\ldots,t\}$ gives rise to a sequence $(v_1,\ldots,v_t)$ of vertices $v_i \in V$, where $v_i := j$ if $i \in B_j$. Using a t–periodic notation with $v_{t+1} = v_1$, we have

   $$|\{i \in \{1,\ldots,t\} \mid (v_i, v_{i+1}) = (l,m)\}| = c_{l,m} \qquad (l,m \in \{1,\ldots,k\}).$$

   Thus B gives rise to a trail in $G(c)$ rooted at $v_1 \in V$.

It may be remarked that we have equality in (6.3) if the vertices of the directed multigraph $G(c)$ can be discerned by their outdegree, that is $b_1 < \ldots < b_k$. Then, given an Euler trail with sequence $(v_1,\ldots,v_t)$, we define the partition $(B_1,\ldots,B_k)$ by setting $B_j := \{i \in \{1,\ldots,t\} \mid v_i = j\}$.

- To get an upper bound on $X_G(c)$ we select a root vertex $j \in V$ and consider the Euler trails in $G(c)$ beginning at j. By the BEST formula their number equals

$$b_j^R T_j(c) \cdot \frac{\prod_{l=1}^{k}(b_l - 1)!}{\prod_{l,m=1}^{k} c_{l,m}!} \tag{6.4}$$

where $T_j(c)$ is the number of directed spanning trees rooted at j. (6.4) is derived from Thm. 13 of Chapter I of [Bo] by noting that, unlike here, Bollobas considers directed multigraphs with labeled edges. Here the *reduced outdegree*

$$b_l^R := \sum_{m=1}^{k} c_{l,m}^R \quad \text{with} \quad c_{l,m}^R := 1 \text{ if } c_{l,m} > 0 \text{ and } c_{l,m}^R := 0 \text{ otherwise.}$$

- The number of directed spanning trees rooted at j equals the $(k-1) \times (k-1)$–minor of the $k \times k$ degree matrix

  $$\mathrm{diag}(b_1^R, \ldots, b_k^R) - (c_{l,m}^R)_{l,m=1}^k$$

  obtained by deleting the j–th row and the j–th column. This number is known to be independent of j, and we call it $\Delta(c^R)$.

  By this remark and (6.4)

  $$X_G(c) \leq t\Delta(c^R) \frac{\prod_{l=1}^k (b_l - 1)!}{\prod_{l,m=1}^k c_{l,m}!}, \tag{6.5}$$

  since $\sum_{j=1}^k b_j^R \leq \sum_{j=1}^k b_j = t$.

- From (6.2) and (6.5) we obtain the estimate

  $$Y(b) \leq t \prod_{l=1}^{k} (b_l - 1)! \cdot \sum_{c=(c_{1,1},\ldots,c_{k,k})} \Delta(c^R). \tag{6.6}$$

  A bound on $\Delta(c^R)$ only depending on the reduced outdegrees $b_1^R, \ldots, b_k^R$ can be found in [GM]. We use it in the slightly weakened version

  $$\Delta(c^R) \leq \tfrac{1}{2} \prod_{l=1}^{k} b_l^R$$

  and thus get from (6.6)

  $$Y(b) \leq \tfrac{t}{2} \prod_{l=1}^k \left[b_l^R (b_l - 1)! \right] \cdot \sum_c 1. \tag{6.7}$$

- The cardinality $\sum_c 1$ of number partitions $c = (c_{1,1}, \ldots, c_{k,k})$ compatible with the number partition $(b_1, \ldots, b_k)$ of t is calculated as follows:

$$\sum_c 1 = \prod_{l=1}^{k} \left| \left\{ (c_{l,1}, \ldots, c_{l,k}) \mid c_{l,m} \neq 1 \text{ and } \sum_{m=1}^{k} c_{l,m} = b_l \right\} \right| .$$

But

$$\begin{aligned}
&\left| \left\{ (c_{l,1}, \ldots, c_{l,k}) \mid c_{l,m} \neq 1 \text{ and } \sum_{m=1}^{k} c_{l,m} = b_l \right\} \right| \\
&= \sum_{U \subseteq \{1,\ldots,k\}} \left| \left\{ (c_{l,1}, \ldots, c_{l,k}) \mid c_{l,m} \geq 2 \text{ if } m \in U \text{ and } c_{l,m} = 0 \right. \right. \\
&\qquad\qquad \left. \left. \text{otherwise, } \sum_{m \in U} c_{l,m} = b_l \right\} \right| \\
&= \sum_{r=1}^{\min(k, \lfloor b_l/2 \rfloor)} \sum_{|U|=r} \binom{b_l - r - 1}{r-1} \\
&= \sum_{r=1}^{\min(k, \lfloor b_l/2 \rfloor)} \binom{b_l - r - 1}{r-1} \binom{k}{r},
\end{aligned}$$

so that (6.7) reduces to

$$Y(b) \leq \frac{t}{2} \prod_{l=1}^{k} \left[b_l^R (b_l - 1)! \sum_{r=1}^{\min(k, \lfloor b_l/2 \rfloor)} \binom{b_l - r - 1}{r-1} \binom{k}{r} \right] . \tag{6.8}$$

- We bound the sums appearing in (6.8), depending on the relative size of k and b_l. Remember our assumption $k \geq 2$.

  We set $\hat{b} := \lfloor b/2 \rfloor$.

  1. For all $k, b \geq 2$ we have the estimate

$$\begin{aligned}
&\sum_{r=1}^{\min(k,\hat{b})} \binom{b-r-1}{r-1} \binom{k}{r} \\
&\leq \sum_{r=1}^{\min(k,b-1)} \binom{b-r-1}{r-1} \binom{k}{r} \leq \sum_{r=1}^{\min(k,b-1)} \binom{b-2}{r-1} \binom{k}{k-r} \\
&= \binom{k+b-2}{k-1}
\end{aligned} \tag{6.9}$$

2. For all $k \geq b \geq 2$ and $r \leq \hat{b}$ we use the inequality $\binom{k}{r} \leq \binom{k}{\hat{b}} \leq \binom{k}{\lfloor k/2 \rfloor}$ to show

$$\begin{aligned} \sum_{r=1}^{\min(k,\hat{b})} \binom{b-r-1}{r-1}\binom{k}{r} &\leq \left(\sum_{r=1}^{\hat{b}} \binom{b-r-1}{r-1}\right)\binom{k}{\hat{b}} \\ &= \frac{g^{b-1} - \left(\frac{-1}{g}\right)^{b-1}}{\sqrt{5}}\binom{k}{\hat{b}} \end{aligned} \tag{6.10}$$

with the golden mean $g := \frac{1+\sqrt{5}}{2}$, since the sum of the binomials equals the Fibonacci numbers.

- The reduced outdegree b_l^R is bounded by

$$b_l^R = \sum_{m=1}^{k} c_{l,m}^R \leq \min(k, \hat{b}_l) \leq \frac{2k\hat{b}_l}{k+\hat{b}_l}. \tag{6.11}$$

Instead of summing (6.8) over the ensemble of $b = (b_1, \ldots, b_k)$ with $2 \leq b_1 \leq \ldots \leq b_k$ and $\sum_{l=1}^{k} b_l = t$, we shift the b_l by 2 and set for $\tilde{k} \leq k$

$$Y_k(b_1, \ldots, b_{\tilde{k}}; t) := \frac{t}{2} k^{k-\tilde{k}} \prod_{l=1}^{\tilde{k}} b_l^R (b_l+1)! \sum_{r=1}^{\min(k,\hat{b}_l)} \binom{b_l-r+1}{r-1}\binom{k}{r},$$

with $\hat{b}_l := \left\lfloor \frac{b_l}{2} \right\rfloor + 1$ and b_l^R is redefined as $\min(k, \hat{b}_l)$.

Then for $b_l \geq 0$

$$Y_G(b_1+2, \ldots, b_k+2) \leq Y_k(b_1, \ldots, b_k; t) \tag{6.12}$$

and our aim is to find a $C \geq 1$, independent of k and t, such that the recursion

$$\sum_{\substack{0 \leq b_1 \leq \ldots \leq b_{\tilde{k}+1} \\ \sum_{l=1}^{\tilde{k}+1} b_l = t-2k}} Y_k(b_1, \ldots, b_{\tilde{k}+1}; t) \leq C \cdot \sum_{\substack{0 \leq b_1 \leq \ldots \leq b_{\tilde{k}} \\ \sum_{l=1}^{\tilde{k}} b_l = t-2k}} Y_k(b_1, \ldots, b_{\tilde{k}}; t) \tag{6.13}$$

in $\tilde{k}$ holds true. Assuming (6.13), we obtain from (6.12)

$$\sum_{\substack{2 \leq b_1 \leq \ldots \leq b_k \\ \sum_{l=1}^{k} b_l = t}} Y_G(b_1, \ldots, b_k) \leq ke(ce)^{k-1}(t-k+1)!,$$

since

$$
\begin{aligned}
Y_k(t-2k;t) &\le \frac{t}{2}k^{k-1}\min\left(k, \left\lfloor \frac{t}{2}-k\right\rfloor+1\right)(t-2k+1)!\binom{k+t-2k}{k-1} \\
&= \frac{t}{2}\frac{k^k}{k!}\min\left(k\left\lfloor \frac{t}{2}-k\right\rfloor+1\right)(t-k)! \\
&\le ke^k(t-k+1)!.
\end{aligned}
$$

Using (6.9), (6.13) follows from the recursion

$$
\begin{aligned}
&\sum_{l=0}^{\hat{b}-1}\hat{l}(l+1)!\left(\sum_{r=1}^{\min(k,\hat{l})}\binom{l-r+1}{r-1}\binom{k}{r}\right)\widehat{b-l}(b-l+1)!\binom{k+b-l}{k-1} \\
&\quad\le C\cdot k(\hat{b})(b+1)!\binom{k+b}{k-1},
\end{aligned}
$$

with $\hat{l}=\left\lfloor\frac{l}{2}\right\rfloor+1$. this is equivalent to the claim

$$
\sum_{l=0}^{\hat{b}-1}\hat{l}(l+1)!\left(\sum_{r=1}^{\min(k,\hat{l})}\binom{l-r+1}{r-1}\binom{k}{r}\right)\widehat{b-l}\prod_{r=1}^{l}\frac{1}{k+b-r+1}\le Ck\hat{b}. \tag{6.14}
$$

Depending on the relative size of k and b, we estimate the l.h.s. of (6.14) in two ways:

- Using (6.9), we get the uppper bound for the l.h.s. of (6.14)

$$
\begin{aligned}
&\sum_{l=0}^{\hat{b}-1}\hat{l}(l+1)!\binom{k+l}{k-1}\widehat{b-l}\prod_{r=1}^{l}\frac{1}{k+b-r+1} \\
&\quad= k\sum_{l=0}^{\hat{b}-1}\hat{l}\widehat{b-l}\prod_{r=1}^{l}\frac{k+r}{k+b-r+1}.
\end{aligned} \tag{6.15}
$$

We write the product in (6.15) in the form

$$
\prod_{r=1}^{l}\frac{k+r}{k+b-r+1}=\exp\left(\sum_{r=1}^{l}g(r)\right)\quad\text{with}\quad g(r):=\ln\left(\frac{k+r}{k+b-r+1}\right).
$$

For $r \le l \le \widehat{b-1}$ not only $g(r) \le 0$ and $g(r) \ge g(r-1)$ but also (for all real such r)

$$g''(r) = \frac{1}{(k+b-r+1)^2} - \frac{1}{(k+r)^2} \le 0.$$

So $\sum_{r=1}^{l} g(r) \le lg\left(\frac{1+l}{2}\right) \le lg\left(\frac{\hat{b}}{2}\right)$ or

$$\prod_{r=1}^{l} \frac{k+r}{k+b-r+1} \le \lambda^l \quad \text{with} \quad \lambda := \frac{k+\frac{\hat{b}}{2}}{k+b-\frac{\hat{b}}{2}+1}. \tag{6.16}$$

For $k \le b$ we have the uniform bound $\lambda \le \frac{3}{4}$.

Inserting (6.16) in (6.15) and noting that $\widehat{b-l} \le \hat{b}$ and $\hat{l} \le \frac{l}{2}+1$, we get (6.14) with

$$C := \sum_{l=0}^{\infty} \left(\frac{l}{2}+1\right)\left(\frac{3}{4}\right)^l = 10.$$

- For $k \ge b$ we insert (6.10) in the l.h.s. of (6.14) which is thus bounded by

$$\begin{aligned}
&\sum_{l=0}^{\hat{b}-1} \hat{l}(l+1)!g^l \tbinom{k}{\hat{l}-1} \widehat{b-l} \prod_{r=1}^{l} \frac{1}{k+b-r+1} \\
&= \sum_{l=0}^{\hat{b}-1} \hat{l}\widehat{b-l}g^l \tbinom{k}{\hat{l}-1} \prod_{r=1}^{l} \frac{r+1}{k+b-r+1}.
\end{aligned}$$

By an argument similar to the one leading to (6.16)

$$\begin{aligned}
&\sum_{l=0}^{\hat{b}-1} \hat{l}\widehat{b-l}g^l \tbinom{k}{\hat{l}-1} \prod_{r=1}^{l} \frac{r+1}{k+b-r+1} \\
&= \sum_{l=0}^{\hat{b}-1} \hat{l}^2\widehat{b-l} \prod_{r=1}^{\lfloor l/2 \rfloor} \frac{g^2(r+\hat{l})}{(k+b-r+1)(k+b-r-\lfloor l/2 \rfloor+1)} \\
&\le \sum_{l=0}^{\hat{b}-1} \hat{l}^2\widehat{b-l} \prod_{r=1}^{\lfloor l/2 \rfloor} \frac{g^2}{k+b-r+1} \le \hat{b}\sum_{l=0}^{\hat{b}-1} \hat{l}^2 \left(\frac{g}{2}\right)^l \\
&\le \hat{b}\sum_{l=0}^{\infty} \left(\frac{l^2}{4}+l+1\right)\left(\frac{g}{2}\right)^l = \frac{1}{4}(161+71\sqrt{5}) < 80,
\end{aligned}$$

assuming $k \geq 4$ and treating $k = 2$ and $k = 3$ separately. $\square$

7 The Asymptotic Estimate

Now we are ready to present our asymptotic result.

Theorem 7.1 *Under the assumption of Conjecture 5.8 the form factor K_N is approximated by the diagonal contribution in the following sense:*
For all $\varepsilon > 0$ uniformly in $\frac{t}{N} \in \left[\varepsilon, \frac{e}{C_2}(1-\varepsilon)\right]$

$$\left| K_N(t) - \frac{t}{N}\Delta_N^{\max}(t) \right| \to 0 \qquad (N \to \infty).$$

Proof: As $K_N(t) = \frac{t}{N}$ for the t–values under consideration,

$$\Delta_N^{\max}(t) = \sum_{\sigma \in S_t} \langle V_N, \hat{p}_\sigma \rangle$$

(Eq. (5.6)), and $\langle V_N, \hat{p}_e \rangle = 1$ (Cor. 5.7), we need to show that

$$\sum_{\sigma \in S_t \setminus \{e\}} \langle V_N, \hat{p}_\sigma \rangle \to 0 \qquad (N \to \infty).$$

Using Prop. 6.2 this amounts to show

$$\sum_{\sigma \in D_t} \langle V_N, \hat{p}_\sigma \rangle \to 0 \qquad (N \to \infty),$$

which is implied by the asymptotic vanishing of

$$\sum_{k=1}^{\lfloor t/2 \rfloor} \sum_{\sigma \in D_t(k)} | \langle V_N, \hat{p}_\sigma \rangle | \leq C_1 \sum_{k=1}^{\lfloor t/2 \rfloor} k C_2^k (t-k+1)! N^{k-t},$$

using Conjecture 5.8 and Prop. 6.3. Under our assumptions for t/N

$$
\begin{aligned}
\sum_{k=1}^{\lfloor t/2 \rfloor} k C_2^k (t-k+1)! N^{k-t} &\le tN \sum_{k=1}^{\lfloor t/2 \rfloor} C_2^k \left(\frac{t-k+1}{N^e} \right)^{t-k+1} \\
\le \quad & N^2 \sum_{k=1}^{\lfloor t/2 \rfloor} C_2^k \left(\frac{1-\varepsilon}{C_2} \right)^{t-k+1} \\
\le \quad & N^2 C_2^{-1} \sum_{k=1}^{\lfloor t/2 \rfloor} (1-\varepsilon)^{t-k+1} \\
\le \quad & \frac{N^2}{C_2 \varepsilon} \cdot (1-\varepsilon)^{\lceil t/2 \rceil + 1} \le \frac{N^2}{C_2 \varepsilon} (1-\varepsilon)^{\varepsilon N/2} \to 0.
\end{aligned}
$$

proving the theorem. □

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