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MULTIPLE OPTIMAL SOLUTIONS IN THE PORTFOLIO SELECTION MODEL WITH SHORT-SELLING

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In this paper an extension of the Lintner model (1965) is considered: the problem of portfolio optimization is studied when short-selling is allowed through the mechanism of margin requirements. This induces a non-linear constraint on the wealth. When interest on deposited margin is present, Lintner ingeniously solved the problem by recovering the unique optimal solution of the linear model (no margin requirements). In this paper an alternative and more realistic approach is explored: the nonlinear constraint is maintained but no interest is perceived on the money deposited against short-selling. This leads to a fully non-linear problem which admits multiple and unstable solutions very different among themselves but corresponding to similar risk levels. Our analysis is built on a seminal idea by Galluccio, Bouchaud and Potters (1998), who have re-stated the problem of finding solutions of the portfolio optimization problem in futures markets in terms of a spin glass problem. In order to get the best portfolio (i.e. the one lying on the efficiency frontier), we have to implement a two-step procedure. A worked example with real data is presented.

1. Introduction

From the very beginning of capital market theory unrestricted short-sales with full use of the proceeds have been taken into account. A major contribution in this field had been given by Lintner [1,2] using a model where the short-sale proceeds plus 100% margin are deposited with the stock owner, who pays interest on the deposited funds. Assuming the same interest rate for lending and borrowing, this type of short-selling does not affect the efficiency of the market portfolio; the traditional CAPM remains valid. This paper would like to present an extended version of the Lintner's model [1], still assuming the possibility of long-buying/short-selling but dropping the assumption on interest earned on deposit. This problem is very similar to that of portfolio selection in future markets, with some relevant limitations: the model refers to one-period decision, and the question of updating the constraint because of price variation is ignored. Clearly in this perspective we should imagine a daily adjustment of the portfolio, because of the double constraint of the margin requirement and the rate of return.

As already pointed out in a recent paper by Galluccio, Bouchaud and Potters [3], in the case of portfolio problem with a non-linear constraint, the very concept of rational decision making becomes questionable, due to the presence of concomitant nonunique and unstable optimal portfolios. More general cases with convex non-linear constraints in the case of margin accounts or in international capital adequacy regulation have been considered in references [4,5,6].

Here we put forward the analysis, by explicitely considering the differences between the Lintner's model, which despite having a non-linear wealth constraint admits a unique optimal solution, and the full non-linear problem (no interest perceived on the money deposited against short-selling). By considering the issue of constructing the efficient frontier, we show that a very large number of (quasi) equilibrium solutions coexists in the model and we argue how any procedure developed to reach a decision regarding the structure of the portfolio must face this problem. To be a little bit more precise, if we have N assets available to the investor, for a fixed expected return R, the number $n(N,R)$ of risk *local* minima grows exponentially in the number of assets, i.e.:

$$n(N,R) \sim e^{\omega(R)N} \tag{1.1}$$

where $\omega(R)$ is a positive number depending on expected portfolio return. Since we have this multiple equilibrium solution (however let us notice that the equilibria are not equivalent among themselves respect to the level of risk), we have to implement a second step in the solution procedure in order to get the *global* risk minimum.

The plan of the paper is the following. In the next Section we introduce a model of portfolio optimization with short-selling allowed and we explain the non-linear constraint that is involved. A review of Lintner's model is presented and its differences compared to our case are enlightened. In Section 3 we discuss the solution of the model. First we treat a simplified case, where the portfolio return is not fixed, showing how multiple solutions appear in connection with the randomness of empirical correlation matrices constructed from financial return series. Then the general procedure and analytical calculation needed to construct the *efficient frontier* is obtained. We deal with a concrete example in Section 4, considering a portfolio of 16 common stocks traded on the Nasdaq Market and solving it by means of computer calculations. We show explicitly the multiple equilibrium solutions and we discuss the distribution of local risk minima. We calculate the efficient frontier and a related averaged frontier. This last curve, as discussed in the paper, represents the decision that an investor will most likely take by searching the optimal solution with standard methods of combinatorial optimization. We will also reconstruct and discuss the frontier corresponding to the "worst of the best decisions", namely the local minimum with higher risk (at fixed return). As we will see, the risk function has an exponential number of local minima and one needs to select by hand the lowest one. Moreover, there is the possibility that portfolios completely different among themselves correspond almost to the same value of the risk. Finally, summarizing Conclusions will follow as usual.

2. The Model

Let us start from the standard definitions [7]. We consider a hedge-fund as rational investor agent who has to choose how to allocate his wealth on an *unlevered* portfolio $\mathcal{P}$ composed of N risky assets (indexed by the subscript i which takes values $i = 1, \ldots, N$), each of them having expected return r_i. Eventually a risk-less asset with fixed rate of return R_f can be considered, obtaining a so called *levered* portfolio $\mathcal{B}$, which is a linear combination of the optimum unlevered portfolio and the risk-less asset in certain proportions. Let w the ratio of the wealth invested in risky assets to the total wealth invested. As usual, the measure of the unlevered portfolio risk is given by the standard deviation $\sigma_{\mathcal{P}}$, i.e. by the square root of the variance

$$\sigma^2_{\mathcal{P}} = \sum_{i,j=1}^{N} C_{ij} p_i p_j = p^T C p \tag{2.2}$$

and the unlevered portfolio return is the weighted mean of single assets returns

$$R_{\mathcal{P}} = \sum_{i=1}^{N} p_i r_i = p^T r \tag{2.3}$$

where we have introduced the vectorial notation with ^{T} to denote the transposed. Here $p = (p_1, p_2, \ldots, p_N)$ are the ratio of the investment in the i-th risky asset to the investment in all risky assets and C_{ij} is the matrix of returns variances ($i = j$) and covariances ($i \neq j$). Obviously, the levered portfolio variance and return are

$$\sigma^2_{\mathcal{B}} = w^2 \sigma^2_{\mathcal{P}} \tag{2.4}$$

$$R_{\mathcal{B}} = (1 - w) R_f + w R_{\mathcal{P}} \tag{2.5}$$

The complete portfolio problem is assigned by specifying the wealth constraint. The simplest case is the linear budget constraint for the unlevered case: $\sum_{i=1}^{N} p_i = 1$. We will instead be interested in the much more complex wealth constraint

$$\sum_{i=1}^{N} \gamma_i \mid p_i \mid = 1 \tag{2.6}$$

where γ_i is the (fixed) margin constraint and $p_i > 0$ or $p_i < 0$, depending on the sign of the contract (buy or sell respectively). As mentioned in the introduction, this corresponds to a first order approximation in the case of future markets, where the short-sale problem is regulated through the mechanism of margin accounts. The margin is assumed to be the same for each asset, fixed for all operations, and it does not change over time according to price variations of the underlying assets. Moreover, the problem of issuing futures on behalf of the financial institution is not considered. Without loss of generality we can set $\gamma_i = 1$, so that, introducing the vector s whose components are $s_i = sign(p_i)$, the budget constrain becomes:

$$\sum_{i=1}^{N} | p_i | = p^T s = 1 \tag{2.7}$$

We remember the reader that the *sign* function is defined as $sign(x) = 1$ if $x > 0$ and $sign(x) = -1$ if $x < 0$.

2.1. *Review of Lintner's model*

The solution of the portfolio optimization problem corresponds to finding the most efficient investment strategy, i.e. to evaluate proportions p that minimize the risk $\sigma^2_{\mathcal{P}}$ for a given return $R_{\mathcal{P}}$. In other words, we would like to calculate the *efficient frontier*, that represents the relationship between the risk of the portfolio and the expected return of the portfolio itself having the best utility for the investor. Thus to get one point on the efficient frontier, we minimize risk subject to the return being some level ($R_{\mathcal{P}} = R$) plus the restriction on the wealth. Then, by varying the return R, we obtain the entire efficient set. An equivalent procedure is to fix the risk-less interest rate to some value ($R_f = r^*$) and maximize the ratio θ of excess return (expected return minus risk-free rate) to standard deviation under the wealth constraint. Then, by varying r^*, we obtain the whole efficient frontier [8,9]. As an example, the simple case of a linear wealth constraint on an unlevered portfolio ($\sum_{i=1}^{N} p_i = 1$) can be easily solved. We have

$$R_f = R_f \cdot 1 = R_f \sum_{i=1}^{N} p_i = R_f p^T e \tag{2.8}$$

where we have defined the vector $e = (1, 1, \ldots, 1)$. The function θ reads

$$\theta = \frac{R_{\mathcal{P}} - R_f}{\sigma_{\mathcal{P}}} = \frac{p^T(r - R_f e)}{\sqrt{p^T C p}} \tag{2.9}$$

and the maximization equations are

$$-\lambda C p + r - R_f e = 0 \tag{2.10}$$

where λ is a constant. Defining a new variable $z = \lambda p$, the solution is

$$z = C^{-1}(r - R_f e) \tag{2.11}$$

where C^{-1} is the inverse of the correlations matrix. Finally the optimum proportions p to invest is obtained by re-scaling z in such a way to satisfy the wealth constraint

$$p_i = \frac{z_i}{\sum_{i=1}^{N} z_i} \tag{2.12}$$

The Lintner's approach to short-selling is basically equivalent to the linear case. He assumed that when an investor sells stock short, cash is not received but rather is held as collateral. Furthermore the investor must put up an additional amount of cash equal to the amount he/she sells short. This leads to the same wealth constraint as Eq. (2.7). In addition, the short-seller receives interest on both the money put up against short-sales and the money received from the short-sale. As a consequence the portfolio return in the Lintner model is

$$R_{\mathcal{P}} = \sum_{p_i>0} p_i r_i + \sum_{p_i<0} p_i (r_i - 2R_f) \tag{2.13}$$

Using the wealth constraint we can write

$$R_f = R_f \cdot 1 = R_f \sum_{i=1}^{N} \mid p_i \mid = \sum_{p_i>0} p_i R_f - \sum_{p_i<0} p_i R_f \tag{2.14}$$

so that the excess return is

$$R_{\mathcal{P}} - R_f = \sum_{i=1}^{N} p_i (r_i - R_f) = p^T (r - R_f e) \tag{2.15}$$

The function θ is the same as in the linear case and it is an homogeneous function of degree zero, so that we can proceed as above. The only change that is needed is a different re-scaling at the end in such a way that the proportions satisfy the non-linear constraint:

$$p_i = \frac{z_i}{\sum_{i=1}^{N} \mid z_i \mid} \tag{2.16}$$

2.2. *Differences between ours and Lintner's model*

Despite the fact that we use the same wealth constraint as Lintner did, the different definition of return we are using leads us to completely different results. This is because the assumption on gaining interests on money deposited against short-selling is basically equivalent to consider the linear case. In the case we are considering it is not possible to restate the optimization procedure in terms of the linear case, since we are eliminating the assumption on interest on deposited money. Using Eq. (2.3) for the portfolio return and Eq. (2.14), the excess return is

$$R_{\mathcal{P}} - R_f = \sum_{p_i>0} p_i (r_i - R_f) + \sum_{p_i<0} p_i (r_i - R_f) = p^T (r - R_f s) \tag{2.17}$$

so that the function θ takes the form

$$\theta = \frac{R_{\mathcal{P}} - R_f}{\sigma_{\mathcal{P}}} = \frac{p^T(r - R_f s)}{\sqrt{p^T C p}} \tag{2.18}$$

so that the maximization equations are

$$-\lambda C p + r - R_f s = 0 \tag{2.19}$$

The solution for p is not unique and can not simply be found by inverting the matrix C because of the presence of the unknown vector s, whose components depends on p itself. One should then compute all the possible s values and solve Eq.(2.19) for any of them. Valid solutions are just those for which $s = sign(p)$ actually holds.

3. Solution of the Model

In this Section we will study in details how our model necessarily admits multiple solutions. Before describing the construction of the efficient frontier, we will first consider the problem of finding the minimum of the variance subjected to the only budget constraint (no fixed portfolio return).

3.1. *Multiple solutions and the randomness of correlation matrix*

We want to minimize the portfolio variance Eq.(2.2) with the non-linear constrain given by Eq.(2.7). We introduce a Lagrangian function with one Lagrange multiplier μ:

$$L(p, \mu) = p^T C p - \mu(p^T s - 1) \tag{3.20}$$

Differentiating with respect to the $N+1$ unknowns p and μ we obtain the following equations for the extreme points

$$p = \frac{1}{2}\mu C^{-1} s \tag{3.21}$$

$$p^T s = 1 \tag{3.22}$$

Inserting Eq.(3.21) in Eq.(3.22) we can solve for μ and then for p

$$\mu = \frac{2}{s^T C^{-1} s} \tag{3.23}$$

$$p = \frac{1}{s^T C^{-1} s} C^{-1} s \tag{3.24}$$

Applying the *sign* function to both sides of the last Equation, we finally obtain

$$s = sign(C^{-1} s) \tag{3.25}$$

where we have used the fact that, since C is a positive definite matrix, the same is true for C^{-1}, so that $s^T C^{-1} s > 0$ for every value of s.

The original problem has thus been mapped in finding the solution of Eq. (3.25): once the s_i that solve this Equation are known, the shares p_i can be calculated using Eq. (3.24), while the portfolio variance is given by

$$\sigma_{\mathcal{P}}^2 = \frac{1}{s^T C^{-1} s} \tag{3.26}$$

Equations like (3.25) have been widely studied in spin glass theory, a branch of statistical physics [10,11]. It is well-known that Eq. (3.25) admits, for a generic random matrix C, an exponential number of solution. Moreover these solutions are "chaotic", i.e. they are completely different one from another and they completely change varying the inout parameters and the number of degrees of freedom.

In the portfolio optimization problem the covariance matrix C is not *a priori* random, but it is constructed from the historical datas. Nevertheless, since historical prices/returns movements are generated by market fluctuations, they can be read as realization of a stochastic process, so that the correlation matrix C can be seen as a generic realization of some specific random matrix ensemble. More informations can be obtained by evaluating the significance of C not in terms of individual elements, but in terms of a matrix as a whole, i.e. by examining the spectrum of the eigenvalues/eigenvectors of the matrix itself. The main aim here is, roughly speaking, to separate the randomness contained in the data from the *real market* information. Recently it has been found by two indipendent groups [12,13,14,15] that empirical covariance matrices extracted from financial return series contains such a high amount of noise that, apart from a few large eigenvalues and the corresponding eigenvectors, their structure can essentially be regarded as random. In [12], e.g., it is reported that in the covariance matrix of 500 assets choosen on the S&P, only 6% of the eigenvectors, which are responsible for 26% of the total volatility, appear to carry some information, while the remaining 94% of the spectrum can be fitted by that of a random matrix drawn from an appropriate random ensemble. From this point of view, it should be stressed that Markovitz's portfolio scheme, based on a purely historical determination of the correlation matrix, proves particularly weak, since the elements of the matrix itself are dominated by noise. Notwithstanding, simulations experiments with random matrices [16] show that, in the context of the classical portfolio problem (minimizing the portfolio variance under linear constraints) noise has relatively little effect. To leading order the solutions are determined by the stable, large eigenvalues, and the displacement of the solution due to noise is rather small. The picture is completely different, however, if we attempt to minimize the variance under non-linear constraint, like those we have in the problem of short-selling with margin account. In this problem the presence of noise in the correlation matrix leads to serious instability and a high degree of degeneracy of the solutions. This will be explicitely shown in Section 4.

All this said about correlation matrix, we can borrow some results from physics of spin glasses and directly draw some first conclusions from Eq. (3.25)(see also [3]):

- At variance with the linear or the Lintner case, where we always find a minimization equation that admits a unique solution, in the present case we have an exponential number of portfolios for which the risk function has a (local) minimum. So we face the embarrassment of which solution to choose and we need to calculate by hand the portfolio variance on each solution to find the *global* minimum.

- We can have very different portfolios corresponding to (local) risk minima having almost the same risk value.
- Adding one asset to the portfolio it radically changes the shape of efficient portfolios.

3.2. *Constructing the efficient frontier*

In the previous Section we have shown how multiple solutions naturally arise in the risk minimization procedure subjected to the non-linear wealth constraint. This does not say nothing about the efficient frontier (even if it shows the instability of rational investment decisions). To completely solve the problem, we have to repeat the minimization of the variance fixing the average return to the value R with an extra Lagrange multiplier ν.
Thus the problem is now to minimize Eq.(2.2) subject to Eq.(2.7) and to the additional constraint

$$R_{\mathcal{P}} = p^T r = R \tag{3.27}$$

We introduce the Lagrangian function

$$L(p,\mu) = p^T C p - \mu(p^T s - 1) - \nu(p^T r - R) \tag{3.28}$$

Differentiating with respect to the $N+2$ unknowns p, μ and ν we obtain

$$\begin{aligned} p &= \frac{1}{2}\mu C^{-1} s + \frac{1}{2}\nu C^{-1} r \\ p^T s &= 1 \\ p^T r &= R \end{aligned} \tag{3.29}$$

Inserting the first equation in the second and the third, we can solve for μ and ν, and then for p. Defining

$$\begin{aligned} \alpha &= s^T C^{-1} s \\ \beta &= r^T C^{-1} s \\ \gamma &= r^T C^{-1} r \end{aligned} \tag{3.30}$$

we obtain the following expressions:

$$
\begin{aligned}
\mu &= 2\frac{\gamma - R\beta}{\alpha\gamma - \beta^2} \\
\nu &= 2\frac{R\alpha - \beta}{\alpha\gamma - \beta^2} \\
p &= \frac{\gamma - R\beta}{\alpha\gamma - \beta^2}C^{-1}s + \frac{R\alpha - \beta}{\alpha\gamma - \beta^2}C^{-1}r
\end{aligned}
\tag{3.31}
$$

Applying the *sign* function to both sides of the last Equation and remembering that $s = sign(p)$ by definition, we finally obtain

$$
s = sign\left(\frac{\gamma - R\beta}{\alpha\gamma - \beta^2}C^{-1}s + \frac{R\alpha - \beta}{\alpha\gamma - \beta^2}C^{-1}r\right) \tag{3.32}
$$

This is the basic equation that substitute Eq.(3.25) in the case of a fixed average return R.

The general procedure for tracing the N-stocks efficient frontier in the case of *futures markets* thus can be summarized as follows:

1. Fix a certain value of the average expected portfolio return R.
2. For this return R solve the system of N equations (3.32) for the vector $s = (s_1, s_2, \ldots, s_N)$. In general the number of solutions n will be exponential in number of assets N: $n \sim e^{\omega N}$, where the exponential rate $\omega = \omega(R)$ depends on the fixed return R.
3. Calculate the value of the proportions investment $p = (p_1, p_2, \ldots, p_N)$ corresponding to each solution of step 2 through formula (3.31) and then the associated risk. Select the lowest value of the risk and the corresponding optimum portfolio investment.
4. Increase the return R by a certain (constant) amount and repeat the entire procedure from step 2 through 4.

4. A Worked Example

In this Section we explicitly treat an example with real data: we demonstrate the presence of multiple solutions in our problem and we calculate the *efficient frontier* by means of computer calculations.

4.1. *Data*

We considered the case of a portfolio consisting of 16 risky assets. These risky assets are some common stocks traded on the Nasdaq, in the period October 1, 1998 - November 13, 2000. An historical record of daily prices of these stocks for

Table 1. Rate of Return.

	Name	*Symbol*	*Return*
1	Adobe	ADO	0.004818
2	Amazon	AMA	0.002838
3	Ameritrade	AME	0.005522
4	Aol	AOL	0.003374
5	Appmc	APP	0.008313
6	Broadvis	BRO	0.008397
7	Cisco	CIS	0.002844
8	Cmgi	CMG	0.004129
9	Dell	DEL	0.000263
10	Doubleclick	DOU	0.004788
11	Ebay	EBA	0.005765
12	Inktomy	INK	0.004144
13	Intel	INT	0.001683
14	Jdsuni	JDS	0.005999
15	Microsoft	MIC	0.000843
16	Oracle	ORA	0.004045
**	Nasdaq Index	NAS	0.001362

the $T = 553$ trading days of the period was used to estimate the relevant parameters - the mean return r_i and the variance/covariance matrix C_{ij}. The data source is *DataStream*. Calling $x(i,k)$ the price of the i-th asset (where $i = 1, \dots, N$) at the k-th day (where $k = 1, \dots, T$), the daily rates of return are:

$$r(i,k) = \frac{x(i,k+1) - x(i,k)}{x(i,k)} \tag{4.33}$$

while the formula used to estimate average returns and covariances are:

$$r_i = \frac{1}{T-1} \sum_{k=1}^{T-1} r(i,k) \tag{4.34}$$

$$C_{ij} = \frac{1}{T-1} \sum_{k=1}^{T-1} \left[r(i,k) - r_i\right]\left[r(j,k) - r_j\right] \tag{4.35}$$

The estimated mean returns are given in Table (1), which also lists the stocks by name, while the variance/covariance matrix is split in Tables (2) and (3).

4.2. *Multiple solutions*

First of all we analyzed the whole set of possible choices that one obtains for a fixed value of the return on the portfolio composed by $N = 16$ assets. We choose to fix the portfolio return to the value of the average daily Nasdaq index return in the period we considered: $R = R_{NAS} = 0.0014$ (i.e. 0.14%). Applying the solving technique described in the previous Section, we found the solutions $\hat{s} = (\hat{s}_1, \hat{s}_2, \dots, \hat{s}_N)$ of the Eq. (3.32) doing an exhaustive enumeration of all the 2^N possible values of the N-dimensional vector s. We found 6675 $\hat{s}^{(j)}$ vectors that satisfy Eq.(3.32).

Table 2. Covariance matrix.

	ADO	AMA	AME	AOL	APP	BRO	CIS	CMG
ADO	0.0017655	0.0006738	0.0005875	0.0004569	0.0010777	0.0007906	0.0006333	0.0009492
AMA	0.0006738	0.0039831	0.0015211	0.0014186	0.0009082	0.0013937	0.0008632	0.0025806
AME	0.0005875	0.0015211	0.0059604	0.0014733	0.0009082	0.0013879	0.0007159	0.0021351
AOL	0.0004569	0.0014186	0.0014733	0.0018082	0.0006482	0.0010780	0.0006245	0.0017036
APP	0.0010777	0.0009082	0.0009082	0.0006482	0.0038179	0.0016104	0.0011169	0.0017814
BRO	0.0007906	0.0013937	0.0013879	0.0010780	0.0016104	0.0052910	0.0008336	0.0020747
CIS	0.0006333	0.0008632	0.0007159	0.0006245	0.0011169	0.0008336	0.0011323	0.0011854
CMG	0.0009492	0.0025806	0.0021351	0.0017036	0.0017814	0.0020747	0.0011854	0.0051837
DEL	0.0004815	0.0006627	0.0006693	0.0006179	0.0006952	0.0006867	0.0007166	0.0011147
DOU	0.0006174	0.0019525	0.0022296	0.0014429	0.0013858	0.0020133	0.0007361	0.0029053
EBA	0.0007485	0.0022906	0.0016129	0.0013104	0.0012851	0.0017050	0.0009150	0.0025681
INK	0.0006859	0.0019766	0.0015506	0.0012455	0.0013270	0.0020950	0.0009573	0.0026140
INT	0.0005465	0.0006466	0.0005004	0.0005156	0.0009470	0.0006798	0.0006638	0.0009792
JDS	0.0007854	0.0008836	0.0009315	0.0007894	0.0016879	0.0013783	0.0010121	0.0015445
MIC	0.0003891	0.0005764	0.0004947	0.0004240	0.0005583	0.0006681	0.0004818	0.0007499
ORA	0.0007064	0.0008636	0.0007249	0.0005809	0.0011963	0.0009879	0.0007990	0.0012780

Table 3. Covariance matrix *(Continued)*.

	DEL	DOU	EBA	INK	INT	JDS	MIC	ORA
ADO	0.0004815	0.0006174	0.0007485	0.0006859	0.0005465	0.0007854	0.0003891	0.0007064
AMA	0.0006627	0.0019525	0.0022906	0.0019766	0.0006466	0.0008836	0.0005764	0.0008636
AME	0.0006693	0.0022296	0.0016129	0.0015506	0.0005004	0.0009315	0.0004947	0.0007249
AOL	0.0006179	0.0014429	0.0013104	0.0012455	0.0005156	0.0007894	0.0004240	0.0005809
APP	0.0006952	0.0013858	0.0012851	0.0013270	0.0009470	0.0016879	0.0005583	0.0011963
BRO	0.0006867	0.0020133	0.0017050	0.0020950	0.0006798	0.0013783	0.0006681	0.0009879
CIS	0.0007166	0.0007361	0.0009150	0.0009573	0.0006638	0.0010121	0.0004818	0.0007990
CMG	0.0011147	0.0029053	0.0025681	0.0026140	0.0009792	0.0015445	0.0007499	0.0012780
DEL	0.0014036	0.0007637	0.0008658	0.0008183	0.0007496	0.0006910	0.0005324	0.0005532
DOU	0.0007637	0.0057720	0.0018463	0.0020844	0.0006473	0.0009491	0.0006221	0.0006542
EBA	0.0008658	0.0018463	0.0050324	0.0023282	0.0006089	0.0010614	0.0005742	0.0008820
INK	0.0008183	0.0020844	0.0023282	0.0046071	0.0007030	0.0012190	0.0005463	0.0009930
INT	0.0007496	0.0006473	0.0006089	0.0007030	0.0011612	0.0008158	0.0004691	0.0006545
JDS	0.0006910	0.0009491	0.0010614	0.0012190	0.0008158	0.0025830	0.0005088	0.0010459
MIC	0.0005324	0.0006221	0.0005742	0.0005463	0.0004691	0.0005088	0.0008008	0.0004617
ORA	0.0005532	0.0006542	0.0008820	0.0009930	0.0006545	0.0010459	0.0004617	0.0019352

Table 4. Some portfolios corresponding to fixed return $R = R_{NAS} = 0.0014$. The first column is the "best" portfolio corresponding to the global risk minimum $\hat{\sigma}_{MIN} \approx 0.0090$. The last column is the "worst" portfolio corresponding to the highest of the local risk minima $\hat{\sigma}_{MAX} \approx 0.0162$. In the middle columns there are some portfolios having risk around the average risk value $\hat{\sigma}_{AVE} \approx 0.0100$.

Risk	0.0090	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0162
ADO	0.0630	-0.0070	0.0780	0.0890	0.1070	0.0990	0.1100	0.0720
AMA	-0.0460	-0.0520	-0.0390	-0.0260	-0.0320	-0.0380	-0.0080	0.0250
AME	0.0150	0.0210	0.0330	-0.0020	-0.0160	0.0170	0.0330	0.0310
AOL	0.0820	0.0950	-0.0280	-0.0110	0.1480	0.0900	-0.0180	0.1250
APP	0.0320	0.0600	0.0380	0.0480	0.0660	0.0800	0.0720	0.0280
BRO	0.0390	0.0340	0.0340	0.0260	0.0360	0.0350	0.0410	0.0550
CIS	0.1230	-0.1610	-0.1770	-0.1570	-0.1280	-0.1720	-0.1460	0.0670
CMG	-0.0450	0.0030	0.0170	-0.0390	-0.0330	0.0020	-0.0440	0.0120
DEL	-0.1380	-0.1600	-0.1520	-0.0800	-0.0390	0.0410	0.1070	0.0000
DOU	0.0220	-0.0230	-0.0170	0.0320	-0.0150	-0.0280	-0.0100	-0.1100
EBA	0.0410	0.0490	0.0520	0.0510	0.0460	0.0320	0.0340	0.0420
INK	-0.0300	-0.0320	-0.0290	-0.0250	-0.0280	-0.0300	0.0180	-0.1640
INT	0.0890	0.0970	0.1010	-0.1090	-0.0760	-0.1470	-0.1140	0.0560
JDS	-0.0440	0.0580	0.0630	0.0870	0.0860	0.0820	0.0930	0.0100
MIC	-0.1490	0.0900	0.0950	0.1400	-0.1080	0.0800	-0.1360	0.1720
ORA	0.0420	0.0570	0.0470	0.0760	-0.0370	-0.0260	-0.0170	0.0300

From them we calculate the corresponding proportions $\hat{p}^{(j)}$ using Eq.(3.31) and the risk values $\hat{\sigma}^{(j)}$ using Eq.(2.2). Here the index j labels all the local risk minima, $j = 1, 2, \ldots, 6675$. The risk ranges between $\hat{\sigma}_{MIN} \approx 0.0090$ and $\hat{\sigma}_{MAX} \approx 0.0162$ and has an *averaged risk* $\hat{\sigma}_{AVE} \approx 0.0100$, where the averaged value is obviously defined as

$$\hat{\sigma}_{AVE} = \frac{1}{6675} \sum_{j=1}^{6675} \hat{\sigma}^{(j)}$$

We show how the risk values are distributed in Fig.(1), where we plot their probability density function, i.e. the histogram normalized to have area 1. We see that most of the risk local minima have a risk value higher than the global risk minimum $\hat{\sigma}_{MIN}$.

It is important to remark that the local minimum which corresponds to the maximum of the distribution shown in Fig. (1) are in fact the solutions that one would obtain with very "high probability". To be more precise, if one consider portfolios of some bigger dimension, just say 100 assets, then it would be impossible to compute exactly all the local minimum but instead one should rely on other methods. For example (but not only) one could implement a Monte Carlo algorithm to find approximated solutions [17]. Unfortunately the biggest majority of such solutions will fall in a very narrow neighborhood of the average risk, so that it would almost impossible to end up on the efficient frontier (see also section). Moreover we observe that the proportions $\hat{p}$ can be very different among themselves and from $\hat{p}_{MIN}$. This is shown in Table (4) where are reported the portfolio p_{MIN} (first column) corresponding to $\hat{\sigma}_{MIN}$, the portfolio $\hat{p}_{MAX}$ (last column) corresponding to $\hat{\sigma}_{MAX}$ and some portfolios $\hat{p}$ (middle columns) having a risk value around $\hat{\sigma}_{AVE}$.

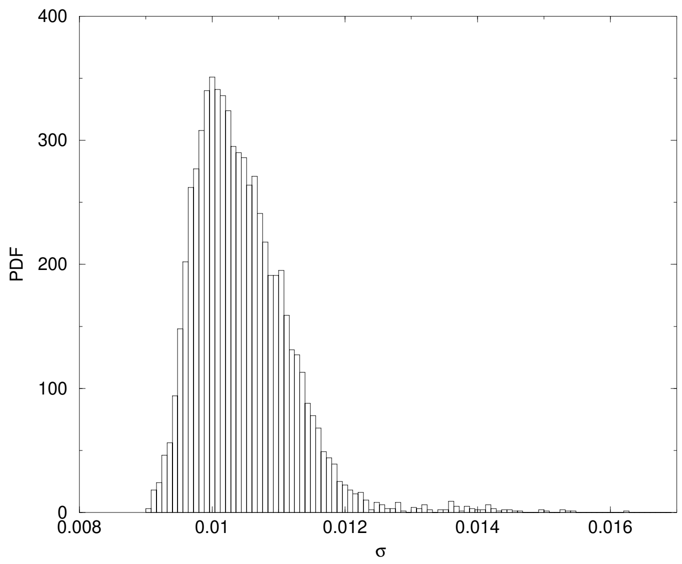

Fig. 1. Histogram of the local risk minima. The number of assets is $N = 16$ and the fixed portfolio return is the Nasdaq return $R = R_{NAS} = 0.0014$. *PDF* stands for Probability Density Function.

From a practical point of view, this *quasi-degeneracy of the risk value with corresponding proportions* $\hat{p}$ *very far from each other* is the most interesting finding of this paper. In order to illustrate better this point, we further investigate the differences between solutions. Let us denote the vector sign of the global minimum solution $s^{MIN} = \{s_1^{MIN}, \ldots, s_N^{MIN}\}$ and by $\hat{s} = \{\hat{s}_1, \ldots, \hat{s}_N\}$ another generic local minimum solution. A simple number, which describe how different this two solutions are, can be defined basically by counting how many s_j's one must changes in sign in order to go from one configuration to the other. We call this number $m(s^{MIN}, \hat{s})$. It can be computed by the following formula

$$m(s^{MIN}, \hat{s}) = \frac{1}{2}\left(N - \sum_{k=1}^{N} s_k^{MIN} \hat{s}_k\right)$$

and it ranges from 0 (identical sign vectors) to N (opposite sign vectors).
In the top side of Fig.(2) we plot the probability distribution function of the numbers m_j's, obtained by measuring the number of different signs between any single local minimum and the global solution: $m_j = m(s^{MIN}, \hat{s}^{(j)})$, where $\hat{s}^{(j)}$ runs over all possible 6675 solutions. In particular, this histogram show clearly that *most of* the solutions are in turn very different from the one we would like to calculate a priori, i.e. s^{MIN}.
Actually, a little bit more than this can be said. Roughly speaking, *in general* two different local solution might have almost the same risk level but in a strategic-

economic context they can be totally different. This is shown in the bottom side of Fig.(2) , where the histogram of the number of different signs between all possible solutions is shown. Summarizing, a multiple choice is available to the investor and an irreducible component of arbitrariness is present in the final decision.

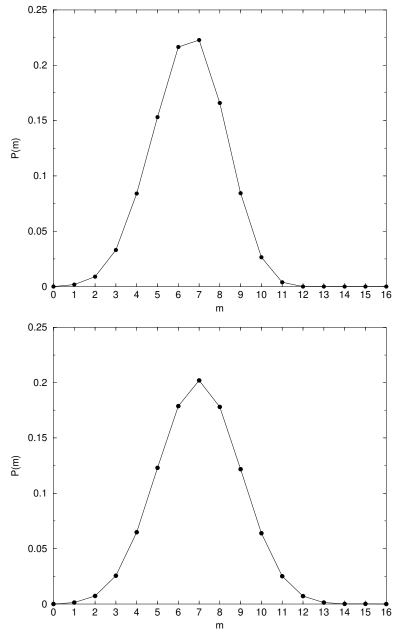

Fig. 2. **Top:** the distribution of the number of different signs between the configuration s^{MIN} corresponding to the risk global minimum and the others configurations $\hat{s}^{(j)}$ corresponding to the risk local minima. **Bottom:** the distribution of the number of different signs between all the configuration $\hat{s}^{(j)}$ (including s^{MIN}) corresponding to the risk local minima.

4.3. *Exponential growth of solutions in the number of assets*

The multiplicity of solutions increases with the number of assets N. We performed numerical experiments varying N from 5 to 16 keeping the average return R fixed to R_{NAS}. For each value we calculate the number of local risk minima $n(N)$, i.e. the number of solutions of Eq. (3.32). As it is clear from Fig.(3) this number growths exponentially with the number of assets (note the lin-log scale in the graph). The best numerical fit yields $n(N) \sim \exp(0.69N)$. We note that essentially the same value $n_{NY}(N) \sim \exp(0.68N)$ has been found in reference [3] for the case of 20 assets of the New York Stock Exchange with no fixed portfolio return.
From the economic point of view, the consequence of this growth is that the arbitrariness degree enlightened above get even bigger by increasing the portfolio dimension. Moreover we have verified (we do not report data for the sake of space)

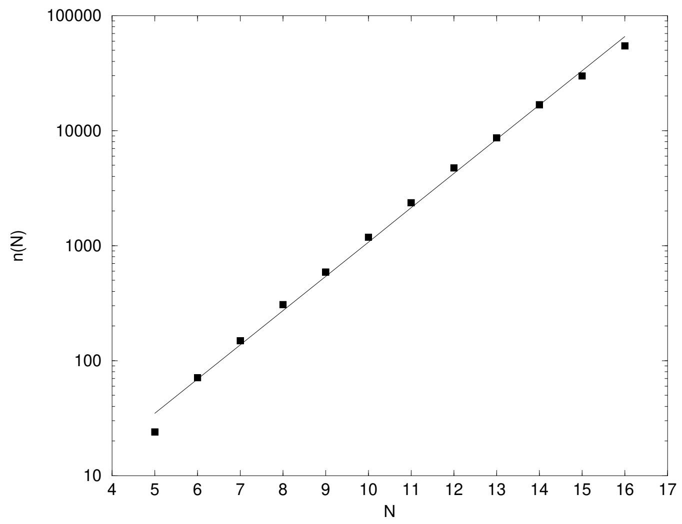

Fig. 3. The number of local risk minima versus the number of assets N. The fixed average return is the Nasdaq return $R_{NAS} = 0.0014$. The solid line is the best numerical fit $n(N) \sim \exp(0.69N)$.

that the multiple solutions have a "chaoticity" property, in the sense that a small change of the correlation matrix C, or the addition of an extra asset, completely changes the values of optimal proportions. On the other hand the number of possible decisions decreases for increasing value of the fixed return R under which minimization is performed. We argue this fixing $N = 16$ and calculating the number of risk local minima varying the return R in the range $[0, 0.003]$. In Fig.(4) we plot the number of solutions of Eq.(3.32) corresponding to each R value. We see that the local risk minimums decreases for increasing value of the return, becoming zero at $R \sim 0.0026$. Above these threshold we do not find from Eq.(3.32) any portfolios satisfying both the budget and the return constrains. One should look for risk minima on the border of the manifold where Lagrange optimization is performed. We did not investigated this point further.

4.4. *The efficient frontier*

Here we use the data concerning the 16-stocks in order to compute the efficient frontier and two more related curves. More precisely, we first allow the averaged return to range from 0 to 0.003 in 100 constant steps and for each value of the return we calculated the proportions p corresponding to the risk local minima. Then, among them we selected the one associated to the risk global minimum and the one associated to the worst choice, namely the local minimum corresponding to the portfolio at the very right tail of Fig. (1) . Moreover, for a given fixed return, we also calculate the *averaged risk* σ_{AVE}. Namely, if n is the total number of local

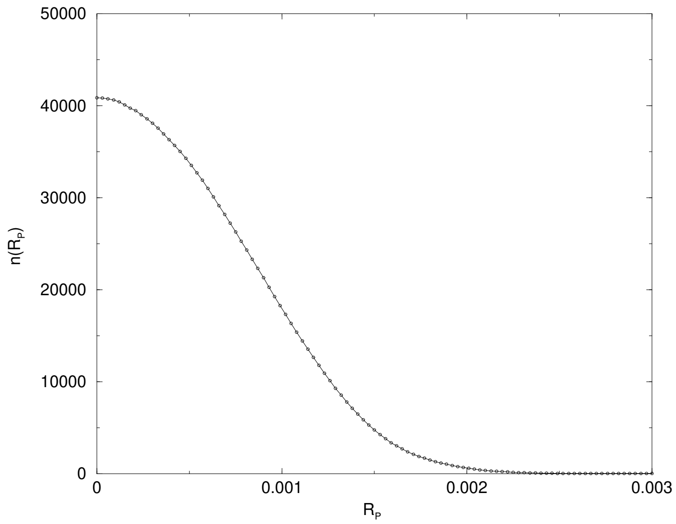

Fig. 4. The number of local risk minima versus the average portfolio return. The number of assets is fixed to $N = 16$.

minima for a fixed return R and $\sigma^{(j)}$ are the associated risks ($j = 1, \ldots, n$), we define

$$\sigma_{AVE} = \frac{1}{n} \sum_{j=1}^{n} \sigma^{(j)}$$

By varying the return R, we use this data to reproduce a kind of *averaged* efficient frontier, which is shown in Fig. (5), together with the other two. We checked that, as one could expect, the proportion of the investment associated to the smallest and greatest local minima are completely different. Furthermore also the portfolios corresponding to risk value around the average risk σ_{AVE} are very different, yielding many different equivalent investment strategies.

5. Conclusions

In this paper we have presented a model of portfolio optimization in the case of a non-linear wealth constraint, that is allowing for long-buying/short-selling of assets with a fixed margin requirement and no interest on the margin account. In this perspective, the model generalizes some relevant results originally obtained by Lintner who assumed the same non-linear constraint but a different return function, which takes into account interest on deposit and allows him to recover the linear case.
The dropping of the assumption of perceiving interest makes the optimization procedure to find the solution very difficult. Firstly, it is not possible to find a unique solution because of the presence in the minimization equation of a vector composed by a sequence of ± 1 (corresponding to buying or selling the single asset), whose or-

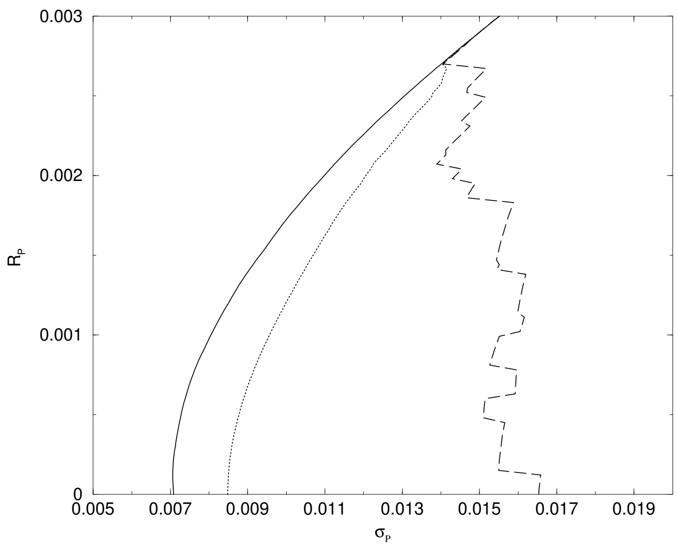

Fig. 5. Efficient frontiers for the portfolio consisting of 16 assets. The continuous line is the *"best-efficient"* frontier corresponding to the lowest value between all the risk local minima. The dotted line shows the *"average-efficient"* frontier, i.e. it correspond to the average value of the risk between all the risk local minima. The dashed line is the *"worst-efficient"* frontier, constructed by using the higher values of the risk local minima for each fixed return.

der is undefined. The consequence is that we have multiple solutions corresponding to local risk minima. The number of these minima is an exponential function of the number of assets. To face this multiplicity of optimal portfolios, we have implemented a successful numerical procedure to search for the minimum of all minima, in such a way to find the actual efficient frontier, that will allow to get the best portfolio in terms of relationship between risk and return. We have applied our model to a concrete portfolio, formed by 16 assets traded on the Nasdaq. We went through the whole two-step procedure, obtaining results which may suggest further efforts to develop the presented model.

Acknowledgment

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