Journal of Statistical Physics, Vol. 103, Nos. 1/2, 2001

Critical Metastability and Destruction of the Splitting in Non-Autonomous Systems

Vincenzo Grecchi[1] **and Andrea Sacchetti**[2]

Received January 7, 2000; final August 1, 2000

We study a periodically driven double well model. As in the case of autonomous models, previously treated in a joint paper with A. Martinez,(7) we have the destruction of the splitting for critical metastability. The relevance of the model for the understanding of the red shift in the inversion line of the molecule of ammonia is shortly discussed. We show that, in order to have a reasonable behavior of the metastability as a function of the frequency, a non-monochromatic perturbation is needed.

KEY WORDS: Resonances; localization; double well; time dependent Hamiltonian; periodic external fields; quantum stability.

1. INTRODUCTION

In this paper we study the splitting destruction in a symmetric double well model subjected to an external time dependent perturbation. The physical motivation is the problem of the localization of symmetric molecules induced by collisions, as observed in the ammonia molecule NH_3, with the associated "red shift effect," i.e., the gradual vanishing of the splitting for increasing pressure (see ref. 8, see also the recent review paper on this problem by Wightman(15)). Although our model is not completely physical, in fact, we assume that the perturbation is periodic and we don't take into account non-linear effects, it can be useful in order to understand if the phenomenon of the vanishing of the splitting is already present in the simplest

[1] Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, I-40127 Bologna, Italy; e-mail: Grecchi@dm.unibo.it
[2] Dipartimento di Matematica, Università di Modena e Reggio Emilia, Via Campi 213/B, I-41100 Modena, Italy; e-mail: Sacchetti@unimo.it

0022-4715/01/0400-0339$19.50/0

cases. We underline that our research is not in disagreement with the notion of de-coherence (see ref. 6 for a review), very useful for understanding the classical behavior of certain microscopic systems. Indeed, since our non-autonomous model yields localization of the states, as in the classical case, it can be considered as an explicit model for de-coherence.

Here, we make use of new techniques for the analysis of non-autonomous Hamiltonian systems recently developed by Soffer and Weienstein.[14] By assuming that the state is initially prepared on the two ground states of the double well, we compute the solution of the time-dependent Schrödinger equation with the rigorous control of the error (see Theorem 2 in Section 3). In particular, we obtain that the time behavior of the wave function, for times of the order of the unperturbed beating period, is described by means of two complex eigenvalues of a matrix independent of time. The imaginary part of the eigenvalues is related to the metastability of the state. Splitting vanishing occurs when these eigenvalues coincide.

As appears in an explicit model (see Sections 5 and 6), for a fixed strength of the perturbation, of the order of the square root of the unperturbed splitting, we have increasing metastability for increasing frequency of the periodic perturbation in a certain range of values (see Theorem 5 in Section 6). What it is peculiar to our research is the relevant role of the metastability and the existence of a critical value for the metastability in order to have the vanishing of the splitting. The general rule we have found can be easily understood and can be stated in a simple way: *the critical value of the mean life is equal to the beating period of the unperturbed double well.*

We underline that localization on one well, which is easy to obtain with simpler static models,[4] appears here when the strength of the perturbation is much larger than the square root of the splitting; in contrast, for perturbations with strength smaller than the square root of the splitting (Theorem 3), we observe the unperturbed beating effect.

We remark that, in order to obtain the desired results, we don't make use of resonance effects and we don't need to assume that the external perturbation is monochromatic. Thus, our research is completely different from others (see refs. 3 and 10, see also ref. 9 and the references therein) on the same subject of destruction of the splitting.

The paper is organized as follows. In Section 2 we state the principal assumptions on the potential and we introduce the notation. In Section 3 we state our main results (Theorems 1 and 2). In Section 4 we give the proof of the theorems. In Section 5 we introduce a simple one-dimensional model satisfying the technical assumptions of Section 2. In Section 6 we explicitly compute the wave function for the model given in Section 5 and we obtain the vanishing of the splitting (Theorem 5) for some values of the frequency and of the strength of the perturbation.

2. ASSUMPTIONS AND NOTATIONS

In this paper we consider the Schrödinger equation

$$i\hbar\dot{\phi} = (H_\rho + W)\,\phi, \qquad \phi(t) \in \mathcal{H}, \quad t \in \mathrm{R} \tag{1}$$

where $\mathcal{H}$ is the Hilbert space $L^2(\mathrm{R}^n, dx)$, $n \geqslant 1$,

$$H_\rho = -\frac{\hbar^2}{2m}\Delta + V_\rho$$

$\{H_\rho\}_{\rho \in \mathcal{I}}$ is a family of self-adjoint (time-independent) operators on the domains $\mathcal{D}_\rho \subset \mathcal{H}$, $\mathcal{I} \subseteq \mathrm{R}^+$ and $+\infty \in \bar{\mathcal{I}}$ where $\bar{\mathcal{I}}$ denotes the closure of $\mathcal{I}$, and W is a *time-dependent* perturbation.

Hypothesis H1: Assumption on *V*. We assume that the spectrum of H_ρ is given by a $\sigma(H_\rho) = \{\lambda_1^\rho, \lambda_2^\rho\} \cup [0, +\infty)$ where $\lambda_{1,2}^\rho$ are two negative simple eigenvalues $\lambda_1^\rho < \lambda_2^\rho < 0$ such that

$$\lim_{\rho \to +\infty,\, \rho \in \mathcal{I}} \lambda_1^\rho = \lim_{\rho \to +\infty,\, \rho \in \mathcal{I}} \lambda_2^\rho = \tilde{\lambda} < 0 \tag{2}$$

Remarks.

— We denote $\omega^\rho = \frac{1}{2}(\lambda_2^\rho - \lambda_1^\rho)$ and $\Omega^\rho = \frac{1}{2}(\lambda_2^\rho + \lambda_1^\rho)$, from (2) it follows that $\omega^\rho \to 0$ and $\Omega^\rho \to \tilde{\lambda}$ as $\rho \to +\infty$, $\rho \in \mathcal{I}$;

— In general, the above assumption is satisfied when V_ρ is a symmetric double well potential such that:

$$\lim_{|x| \to \infty} V_\rho(x) = 0$$

In particular, for a suitable choice of the potential we have that H_ρ has only two eigenvalues and, in the limit of large barrier between the wells, we have that $\omega^\rho \sim e^{-c\rho_A}$ where the Agmon distance ρ_A between the wells goes to infinity (see, for instance, ref. 7 and the references therein).

Hypothesis H2: Assumptions on *W*. The time-dependent perturbation $W \equiv W(t, x)$ has the form

$$W(t, x) = \eta g(x) + \varepsilon v(\mu t)\, f(x), \quad \varepsilon, \eta > 0$$

where ε and η are small parameters and the frequency μ is a parameter belonging to a set $\mathcal{M} \subset (0, +\infty)$. We assume that:

(i) $f(x)$ and $g(x)$ are real-valued piece-wise continuous functions with compact support contained in a compact set $\mathscr{U}$;

(ii) $v(t)$ is a real-valued periodic function, with period L, such that $c_0 = 0$ (i.e., the mean value of $v(t)$ is zero) and $c_n = 0$ for any $|n| > N$, for some positive integer N, where c_n, $n \in \mathbb{Z}$, are the Fourier coefficients of $v(t)$;

(iii) the frequencies $\mu \in \mathscr{M}$ are non-resonant, that is there exists $d > 0$ such that

$$|n\mu + \Omega^\rho| > d, \quad \forall n = 0, \pm 1, \ldots, \pm N, \quad \mu \in \mathscr{M} \qquad \text{and} \quad \rho \in \mathscr{I}$$

In particular we have that $|n\mu + \Omega^\rho| \in [d, D]$ for any n, μ and ρ and some $D > d$.

Notation.

— For the sake of simplicity, we make the choice of units such that $\hbar = 1$, $2m = 1$ and $L = 2\pi$;

— $\|\cdot\|_{\mathscr{H}}$ and $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ denote the norm and the scalar product of the Hilbert space $\mathscr{H}$, $\langle x \rangle = \sqrt{1 + |x|^2}$;

— $\chi_A(x)$ denotes the characteristic function on the set A, i.e., $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \neq A$;

— we drop the dependence on ρ, x, μ and t when this does not cause misunderstanding, in particular we denote ω^ρ and Ω^ρ by ω and Ω;

— we set $\beta = \max(\eta, \varepsilon)$ and $\tilde{\beta} = \max(\beta, \omega)$, we denote by C a generic positive constant independent of t, η, ε and ρ which need not have the same value throughout the paper;

— we denote by P_c the projection operator on the eigenspace associated to the essential spectrum of H_ρ, $\sigma_{\text{ess}}(H_\rho) = [0, +\infty)$: i.e., $P_c = 1 - \psi_1^\rho \langle \psi_1^\rho, \cdot \rangle - \psi_2^\rho \langle \psi_2^\rho, \cdot \rangle$ where $\psi_{1,2}^\rho$ are the normalized eigenvectors of H_ρ associated to $\lambda_{1,2}^\rho$;

— we formally denote

$$K_m = [H_\rho - (\Omega + m\mu + i0)]^{-1} = w - \lim_{\xi \to 0^+} [H_\rho - (\Omega + m\mu + i\xi)]^{-1}$$

- let M be a generic 2×2 matrix, with elements denoted by $M_{\pm,\pm}$, and let A be a generic column matrix, with elements denoted by $A_\pm$, i.e.:

$$M = \begin{pmatrix} M_{+,+} & M_{+,-} \\ M_{-,+} & M_{-,-} \end{pmatrix} \qquad \text{and} \quad A = \begin{pmatrix} A_+ \\ A_- \end{pmatrix}$$

$\ulcorner M \urcorner$ and $\ulcorner A \urcorner$ respectively denote $\ulcorner M \urcorner = \max |M_{\pm,\pm}|$ and $\ulcorner A \urcorner = \max |A_{\pm}|$

— let $M(t)$ be a matrix-valued function defined for $t \geqslant 0$, then we denote

$$[M](t) = \sup_{0 \leqslant \tau \leqslant t} \ulcorner M(\tau) \urcorner$$

— let $\varphi: \mathsf{R}^+ \to \mathscr{H}$ be a vector-valued function defined for $t \geqslant 0$ and such that $\langle x \rangle^{\pm\sigma} \varphi(t) \in \mathscr{H}$, where $\sigma > 0$ is fixed, then we denote

$$(\varphi)^{\pm}(t) = \sup_{0 \leqslant \tau \leqslant t} \|\langle x \rangle^{\pm\sigma} \varphi(\tau)\|_{\mathscr{H}}$$

By definition $[M](t)$ and $(\varphi)^{\pm}(t)$ are monotone non-decreasing functions.

We state now our main assumptions:

Hypothesis H3: Time-decay assumptions. There exist $\sigma > 0$ and $r > 2$ such that for any ϕ, such that $\langle x \rangle^{\sigma} \phi \in \mathscr{H}$, the following estimates uniformly hold with respect to $\rho \in \mathscr{I}$, λ and $t \geqslant 0$:

$$\|\langle x \rangle^{\sigma} [H_\rho - \lambda]^{-1} P_c \phi\|_{\mathscr{H}} \leqslant C \|\langle x \rangle^{\sigma} \phi\|_{\mathscr{H}}, \qquad \forall \lambda \in [-D, -d] \tag{3}$$

$$\|\langle x \rangle^{-\sigma} e^{-iH_\rho t} P_c \phi\|_{\mathscr{H}} \leqslant C \langle t \rangle^{-r+1} \|\langle x \rangle^{\sigma} \phi\|_{\mathscr{H}} \tag{4}$$

$$\|\langle x \rangle^{-\sigma} e^{-iH_\rho t} [H_\rho - (\lambda + i0)]^{-1} P_c \phi\|_{\mathscr{H}} \leqslant C \langle t \rangle^{-r+1} \|\langle x \rangle^{\sigma} \phi\|_{\mathscr{H}} \tag{5}$$

for any $\lambda \in [d, D]$. Moreover, we assume also that for any ϕ_1 and ϕ_2 with compact support then

$$|\langle \phi_1, [H_\rho - \lambda]^{-1} P_c \phi_2 \rangle_{\mathscr{H}}| \leqslant C, \qquad \forall \lambda \in [-D, -d] \cup [d, D] \tag{6}$$

Remarks.

— In fact, in order to prove our main result stated below it is sufficient to assume the weaker condition that (4) and (5) uniformly hold for any $t \in [0, T]$ where $T = T(\rho) = 2\pi/\omega$.

— For any fixed ρ, condition (4) is true for any $n \geqslant 7$ provided that $|V_\rho(x)| \leqslant C \langle x \rangle^{-s}$ for some $s > 0$ (see Theorem 2.1 in ref. 12)), and it is generically true for any n (see ref. 11 and Theorem 7.6 in ref. 12) provided that $\lambda = 0$ is neither an eigenvalue nor a resonance of H_ρ (property (5) can be proved as a consequence of (4) as done in Appendix A in ref. 13). We discuss in Section 5 the validity of the time-decay assumptions *uniformly with respect to* ρ for an explicit model.

3. MAIN RESULTS

Let $\psi_{1,2} \in \mathscr{H}$ be the normalized eigenvectors of H associated to $\lambda_{1,2}$ (let us drop the dependence on ρ); let

$$\psi_{\pm} = \frac{1}{\sqrt{2}} (\psi_1 \pm \psi_2)$$

be the *single-well ground states*, they are such that

$$\langle \psi_{\pm}, \psi_{\pm} \rangle_{\mathscr{H}} = 1, \qquad \langle \psi_{\pm}, \psi_{\mp} \rangle_{\mathscr{H}} = 0 \qquad \text{and} \qquad H\psi_{\pm} = \Omega\psi_{\pm} - \omega\psi_{\mp}$$

The solution $\phi(t) \in \mathscr{H}$ of the time-dependent Schrödinger equation (1) can be written as

$$\phi(t) = a_{+}(t)\, \psi_{+} + a_{-}(t)\, \psi_{-} + \phi_c(t) \tag{7}$$

where $\phi_c = P_c \phi$, that is

$$\langle \phi_c(t), \psi_{\pm} \rangle_{\mathscr{H}} = 0, \qquad \forall t \in \mathrm{R}$$

We assume that the state is initially prepared on the two single-well ground states:

Hypothesis H4. The initial state $\phi^0 = \phi(0)$ is such that $\phi^0_c = P_c \phi^0 = 0$.

By substituting ϕ by (7) in Eq. (1) and projecting the resulting equation on $\psi_{\pm}$ and on the eigenspace associated to the essential spectrum, we obtain the following system of equations for $a_{\pm}$ and ϕ_c:

$$\begin{cases} i\dot{a}_{+} = \Omega a_{+} - \omega a_{-} + a_{+} \langle \psi_{+}, W\psi_{+} \rangle_{\mathscr{H}} + a_{-} \langle \psi_{+}, W\psi_{-} \rangle_{\mathscr{H}} \\ \qquad + \langle \psi_{+}, W\phi_c \rangle_{\mathscr{H}} \\ i\dot{a}_{-} = -\omega a_{+} + \Omega a_{-} + a_{+} \langle \psi_{-}, W\psi_{+} \rangle_{\mathscr{H}} \\ \qquad + a_{-} \langle \psi_{-}, W\psi_{-} \rangle_{\mathscr{H}} + \langle \psi_{-}, W\phi_c \rangle_{\mathscr{H}} \\ i\dot{\phi}_c = H_\rho \phi_c + a_{+} P_c W\psi_{+} + a_{-} P_c W\psi_{-} + P_c W\phi_c \end{cases}$$

satisfying to the initial conditions

$$a^0_{\pm} = \langle \psi_{\pm}, \phi^0 \rangle_{\mathscr{H}}, \qquad \phi^0_c = 0$$

Let $A_{\pm}(t) = a_{\pm}(t)\, e^{i\Omega t}$, A be the column matrix with elements $A_{\pm}$, $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ be the first Pauli matrix, F and G be the 2×2 symmetric matrices with elements

$$F_{\pm,\pm} = \langle \psi_{\pm}, f\psi_{\pm} \rangle_{\mathscr{H}} \qquad \text{and} \qquad G_{\pm,\pm} = \langle \psi_{\pm}, g\psi_{\pm} \rangle_{\mathscr{H}}$$

R_c be the column matrix with elements $R_{c,\pm}(t) = e^{i\Omega t} \langle \psi_\pm, W\phi_c \rangle_{\mathscr{H}}$; then the above system can be written in the form

$$\begin{cases} i\dot{A} = -\omega\sigma_1 A + \eta G A + \varepsilon v(\mu t)\, F A + R_c \\ i\dot{\phi}_c = H_\rho \phi_c + a_+ P_c W \psi_+ + a_- P_c W \psi_- + P_c W \phi_c \end{cases} \tag{8}$$

It is a matter of integration by parts and use of the second differential equation of the system (8), to obtain the following preliminary result.

Theorem 1. Let $\beta = \max(\varepsilon, \eta)$ and

$$\Lambda^m_{\pm,\pm}(\mu) = \langle f\psi_\pm, K_m P_c f\psi_\pm \rangle_{\mathscr{H}}, \quad m \neq 0 \tag{9}$$

$$\Lambda^0_{\pm,\pm} = \langle g\psi_\pm, K_0 P_c g\psi_\pm \rangle_{\mathscr{H}} \tag{10}$$

Let

$$\phi(t) = A_+(t)\, e^{-i\Omega t} \psi_+ + A_-(t)\, e^{-i\Omega t} \psi_- + \phi_c(t)$$

be the solution of Eq. (1) where $\phi_c(t) = P_c \phi(t)$. Then $A = \binom{A_+}{A_-}$ is the solution of the equation

$$\dot{A} = -iMA + iM^{\text{per}}(\mu t)\, A + R \tag{11}$$

where $M^{\text{per}}(t)$ is a periodic 2×2 matrix, with period 2π, with mean value zero and such that $\ulcorner M^{\text{per}}(t) \urcorner \leqslant C\beta$ for some positive constant C independent of ρ and t; M is 2×2 matrix independent of t given by

$$M = -\omega\sigma_1 + \eta G + \varepsilon^2 U + \eta^2 S \tag{12}$$

where the elements of U and S are given by

$$U_{\pm,\pm} = -\sum_{m \neq 0,\, m = -N}^{N} |c_m|^2\, \Lambda^m_{\pm,\pm}(\mu), \quad S_{\pm,\pm} = -\Lambda^0_{\pm,\pm}$$

$R = R(\dot{A}, A, t)$ is a remainder term.

We consider, for the present, the linear differential equation with periodic coefficient

$$\dot{B} = [-iM + iM^{\text{per}}(\mu t)]\, B, \qquad B(0) = A(0) \tag{13}$$

It is well know (see Theorem 5.1, Chap. 3, ref. 5) that the solution of this equation has the form $B(t) = P(\mu t)\, e^{-iNt} A(0)$ for some constant matrix N and periodic matrix P. We assume that

Hypothesis H5. Let $n_{1,2}$ be the eigenvalues of the matrix N and let $\gamma = \max\{\Im n_1, \Im n_2\}$, we assume, in the limit of small perturbation and large ρ, that $\gamma \leqslant C\omega$ and $|P(t)| \leqslant C$, for any t, for some $C > 0$ independent of ρ, ε and η.

We state now our main result:

Theorem 2. Let $\mu \in \mathscr{M}$ be fixed, let $\beta = \max(\eta, \varepsilon)$ and $\tilde{\beta} = \max(\omega, \beta)$. From the above Hypotheses H1, H2, H3, H4, and H5, in the limit of small perturbation and large ρ, $\rho \in \mathscr{I}$, such that $\beta^3/\omega \to 0$, then it follows that the solution of (11) is given by

$$A(t) = B(t) + R_A(t) \tag{14}$$

where $B(t)$ is the solution of Eq. (13) and where $R_A(t)$ is a remainder term satisfying to the following uniform estimate

$$\ulcorner R_A(t) \urcorner \leqslant C \ulcorner A(0) \urcorner \tilde{\beta} \beta^2/\omega, \qquad \forall t \in [0, T], \quad T = \frac{2\pi}{\omega} \tag{15}$$

for some positive constant C independent of ε, η and ρ. Moreover, it follows also the estimate $(\phi_c)^-(t) \leqslant C\beta \ulcorner A(0) \urcorner$ for any $t \in [0, T]$.

4. PROOF OF THE THEOREMS

The proof of the theorems follows the line of the paper,[14] adapted here to our model. In particular, we obtain the estimate (15) uniformly with respect to ρ.

Proof of Theorem 1. From the second equation of (8) we can write

$$\phi_c(t) = -i[\phi_0(t) + \phi_+(t) + \phi_-(t) + \phi_d(t)$$

where

$$\phi_0(t) = e^{iHt} \phi_c(0) \equiv 0$$

$$\phi_d(t) = \int_0^t e^{-H(t-s)} P_c W(s, \cdot)\, \phi_c(s)\, ds \tag{16}$$

$$\phi_\pm(t) = \int_0^t e^{-iH(t-s)} P_c W(s, \cdot)\, a_\pm(s)\, \psi_\pm\, ds \tag{17}$$

If we set $h_n(x) = f(x)$, $\gamma_n = \varepsilon c_n$, $n \neq 0$, and $h_0(x) = g(x)$, $\gamma_0 = \eta$, then we can write

$$W(t, x) = \sum_{n=-N}^{N} \gamma_n e^{in\mu t} h_n(x) \tag{18}$$

From this fact and assuming, for the present, that $\Omega + n\mu \notin \sigma_{\text{ess}}(H)$, for any $n = 0, \pm 1, ..., \pm N$, then it follows that

$$\begin{aligned}\phi_{\pm}(t) &= \sum_{n=-N}^{N} \gamma_{-n} \int_0^t e^{-i[H(t-s) + (\Omega + n\mu) s]} P_c h_{-n} A_{\pm}(s)\, \psi_{\pm}\, ds \\ &= \sum_{n=-N}^{N} i\gamma_{-n} \Big\{ -e^{-i(\Omega + n\mu) t} K_n P_c h_{-n} \psi_{\pm} A_{\pm}(t) \\ &\quad + e^{-iHt} K_n P_c h_{-n} \psi_{\pm} A_{\pm}(0) \\ &\quad + \int_0^t e^{-i[H(t-s) + (\Omega + n\mu) s]} K_n P_c h_{-n} \dot{A}_{\pm}(s)\, \psi_{\pm}\, ds \Big\}\end{aligned} \tag{19}$$

by integrating by parts. If n is such that $\Omega + n\mu \in \sigma_{\text{ess}}(H)$, the above formula is still true; indeed, it follows in the same way by taking $n\mu + i\xi$, $\xi > 0$, and the limit $\xi \to 0^+$. Let

$$R_{c,\pm} = e^{i\Omega t} \langle \psi_{\pm}, W\phi_c \rangle_{\mathscr{H}} = \alpha_{\pm,+} + \alpha_{\pm,-} + \alpha_{d,\pm} \tag{20}$$

where $\alpha_{\pm,+} = -ie^{i\Omega t} \langle \psi_{\pm}, W\phi_+ \rangle_{\mathscr{H}}$, $\alpha_{\pm,-} = -ie^{i\Omega t} \langle \psi_{\pm}, W\phi_- \rangle_{\mathscr{H}}$ and $\alpha_{d,\pm} = -ie^{i\Omega t} \langle \psi_{\pm}, W\phi_d \rangle_{\mathscr{H}}$. From this and from (17) we have that

$$\begin{aligned}\alpha_{\pm,+} &= \sum_{m=-N}^{N} -ie^{i(\Omega + m\mu) t} \gamma_m \langle \psi_{\pm}, h_m \phi_+ \rangle_{\mathscr{H}} \\ &= \alpha^0_{\pm,+} A_+(t) + \alpha^1_{\pm,+} A_+(t) + \alpha^2_{\pm,+} A_+(0) + \alpha^3_{\pm,+}\end{aligned}$$

where

$$\alpha^0_{\pm,+} = -\sum_{m=-N}^{N} \gamma_m \gamma_{-m} \langle h_m \psi_{\pm}, K_m P_c h_{-m} \psi_+ \rangle_{\mathscr{H}}$$

$$\alpha^1_{\pm,+} = -\sum_{n,m=-N,\, m \neq n}^{N} \gamma_m \gamma_{-n} e^{-i(n-m)\mu t} \langle h_m \psi_{\pm}, K_n P_c h_{-n} \psi_+ \rangle_{\mathscr{H}}$$

$$\alpha^2_{\pm,+} = \sum_{m,n=-N}^{N} \gamma_m \gamma_{-n} e^{i(\Omega + m\mu) t} \langle h_m \psi_{\pm}, e^{-iHt} K_n P_c h_{-n} \psi_+ \rangle_{\mathscr{H}}$$

$$\begin{aligned}\alpha^3_{\pm,+} &= \sum_{m,n=-N}^{N} \gamma_m \gamma_{-n} e^{i(\Omega + m\mu) t} \\ &\quad \times \int_0^t \langle h_m \psi_{\pm}, e^{-i[H(t-s) + (\Omega + m\mu) s]} K_n P_c h_{-n} \psi_+ \rangle_{\mathscr{H}} \dot{A}_+(s)\, ds\end{aligned}$$

and a similar result for $\alpha_{\pm,-}$ follows. From this equation then (11) follows where the time independent matrix M is obtained by collecting the terms $\omega\sigma_1$, ηG and $\alpha^0_{\pm,\pm}$, the periodic matrix $M^{\text{per}}(t)$ is obtained by collecting the terms $v(\mu t)\,F$ and $\alpha^1_{\pm,\pm}$, the remainder term R is obtained by collecting the terms $\alpha^2_{\pm,\pm}$, $\alpha^3_{\pm,\pm}$ and $\alpha_{d,\pm}$:

$$R_\pm = \alpha^2_{\pm,+} A_+(0) + \alpha^2_{\pm,-} A_-(0) + \alpha^3_{\pm,+} + \alpha^3_{\pm,-} + \alpha_{d,\pm} \tag{21}$$

We conclude the proof of Theorem 1 by underlining that $M^{\text{per}}(t)$ is periodically dependent on t, with mean value 0 and it is such that $\ulcorner M^{\text{per}}(t) \urcorner \leqslant C\beta$ for some $C>0$ independent of ε, η and ρ because of (6).

Proof of Theorem 2. Solution of (11), satisfying to the initial condition $A(0)$, is given by (14) where the term $R_A(t)$ is given by (see Theorem 3.1, Chap. 3, ref. 5)

$$R_A(t) = P(t) \int_0^t e^{-N(t-\tau)} P^{-1}(\tau)\, R(\tau)\, d\tau$$

In order to obtain a bound of the remainder term R_A we give a sequence of technical Lemmas.

Lemma 1. There exists a positive constant C, independent of t, such that

$$\ulcorner R_A(t) \urcorner \leqslant Ct[R](t) \qquad \text{and} \qquad [R_A](t) \leqslant Ct[R](t), \quad \forall t \in [0, T] \tag{22}$$

Proof. Let $\gamma = \max\{\Im n_1, \Im n_2\}$, where $n_{1,2}$ are the two eigenvalues of N. From the definition of R_A, from the facts that $\gamma \leqslant C\omega$ and $\ulcorner P(t) \urcorner \leqslant C$ for any t, then it follows that

$$\ulcorner R_A(t) \urcorner \leqslant C \int_0^t e^{\gamma(t-\tau)} \ulcorner R(\tau) \urcorner\, d\tau \leqslant Ct[R](t)$$

proving the first inequality of (22). The second one immediately follows as a result of the first one, indeed

$$[R_A](t) = \sup_{\tau \in [0,t]} \ulcorner R_A(\tau) \urcorner \leqslant \sup_{\tau \in [0,t]} C\tau[R](\tau) \leqslant Ct[R](t)$$

Lemma 2. For any ϕ: $\mathsf{R} \to \mathscr{H}$, such that $\langle x \rangle^\sigma \phi(t) \in \mathscr{H}$, and $|\lambda| \in [d, D]$ there exists a positive constant C independent of t and ρ such that

$$\int_0^t \| \langle x \rangle^{-\sigma} e^{-iH(t-s)} P_c \phi(s) \|_{\mathscr{H}}\, ds \leqslant C(\phi)^+(t)$$

and

$$\int_0^t \|\langle x\rangle^{-\sigma} e^{-iH(t-s)}[H-(\lambda+i0)]^{-1} P_c\phi(s)\|_{\mathscr{H}}\, ds \leqslant C(\phi)^+(t)$$

Proof. From (4) it follows that

$$\int_0^t \|\langle x\rangle^{-\sigma} e^{-iH(t-s)} P_c\phi(s)\|_{\mathscr{H}}\, ds \leqslant C\int_0^t \langle t-s\rangle^{-r+1}\, ds(\phi)^+(t)$$
$$\leqslant C(\phi)^+(t)$$

for any t since $r>2$. In the same way the second estimate follows from (5), when $\lambda\in[d, D]$, and from (3) and (4) when $\lambda\in[-D, -d]$.

Lemma 3. For any $t>0$ there exists a positive constant C independent of t, β and ρ such that

$$(\phi_c)^-(t)\leqslant C\beta[A](t) \tag{23}$$

Proof. Given the definition of ϕ_c, given formulas (16) and (17), given the Schwartz inequality and given the first estimate of Lemma 2, it follows that

$$\begin{aligned}\|\langle x\rangle^{-\sigma}\phi_c(\tau)\|_{\mathscr{H}} &\leqslant \|\langle x\rangle^{-\sigma}\phi_+(\tau)\|_{\mathscr{H}} + \|\langle x\rangle^{-\sigma}\phi_-(\tau)\|_{\mathscr{H}}\\ &\quad + \|\langle x\rangle^{-\sigma}\phi_d(\tau)\|_{\mathscr{H}}\\ &\leqslant C[A](\tau)\{\sup_{0\leqslant s\leqslant\tau}\|\langle x\rangle^{\sigma} W(s,x)\,\psi_+\|_{\mathscr{H}}\\ &\quad + \sup_{0\leqslant s\leqslant\tau}\|\langle x\rangle^{\sigma} W(s,x)\,\psi_-\|_{\mathscr{H}}\}\\ &\quad + C\sup_{0\leqslant s\leqslant\tau}\|\langle x\rangle^{\sigma} W(s,x)\,\phi_c\|_{\mathscr{H}}\\ &\leqslant C\beta[A](\tau) + C\sup_{0\leqslant s\leqslant\tau}\|\langle x\rangle^{\sigma} W(s,x)\,\phi_c\|_{\mathscr{H}}\\ &= C\beta[(\phi_c)^-(\tau)+[A](\tau)]\end{aligned}$$

since

$$\sup_{0\leqslant s\leqslant\tau}\|\langle x\rangle^{\sigma} W(s,x)\,\psi_\pm\|_{\mathscr{H}} \leqslant C\beta \sup_{0\leqslant s\leqslant\tau}\|\langle x\rangle^{\sigma}\chi_{\mathscr{U}}(x)\,\psi_\pm\|_{\mathscr{H}}\leqslant C\beta$$

and

$$\sup_{0\leqslant s\leqslant \tau} \|\langle x\rangle^{\sigma} W(s,x)\,\phi_c\|_{\mathscr{H}} \leqslant \sup_{0\leqslant s\leqslant \tau} \|\langle x\rangle^{\sigma} W(s,x)\langle x\rangle^{\sigma}\|_{\mathscr{H}}\cdot \|\langle x\rangle^{-\sigma}\phi_c\|_{\mathscr{H}}$$

$$\leqslant \beta C \sup_{0\leqslant s\leqslant \tau} \|\langle x\rangle^{-\sigma}\phi_c\|_{\mathscr{H}} = \beta C(\phi_c)^{-}(\tau)$$

because $W(t,x)$ has compact support contained in $\mathscr{U}$ for any t. If we remark that $(\phi_c)^{-}(t)$ and $[A](t)$ are monotone non-decreasing functions then (23) follows.

Lemma 4. For β small enough it follows that

$$[R](t) \leqslant C\beta^2(\langle t\rangle^{-r+1} \ulcorner A(0)\urcorner + \tilde{\beta}[A](t)), \qquad \text{where} \quad \tilde{\beta} = \max(\omega, \beta) \tag{24}$$

for some positive constant C independent of β, ρ and t.

Proof. In order to prove (24) we separately estimate the terms $\alpha^2_{\pm,\pm}$, $\alpha^3_{\pm,\pm}$ and $\alpha_{d,\pm}$ defined in the proof of Theorem 1. In order to consider the term $\alpha^2_{\pm,\pm}$ we set $\varphi_{n,\pm} = K_n P_c h_{-n}\psi_{\pm}$. Let, for the present, be n such that $n\mu + \Omega \leqslant -d$, from (3) it follows that $\langle x\rangle^{\sigma}\varphi_{n,\pm}\in\mathscr{H}$ since $\langle x\rangle^{\sigma} h_n\psi_{\pm}\in\mathscr{H}$; from this fact and from (4) it follows that

$$\begin{aligned}
&|\langle\langle x\rangle^{\sigma} h_m\psi_{\pm}, \langle x\rangle^{-\sigma} e^{-iHt}P_cK_nP_ch_{-n}\psi_{\pm}\rangle_{\mathscr{H}}|\\
&\quad\leqslant \|\langle x\rangle^{\sigma} h_m\psi_{\pm}\|_{\mathscr{H}}\, \|\langle x\rangle^{-\sigma} e^{-iHt}P_c\varphi_{n,\pm}\|_{\mathscr{H}}\\
&\quad\leqslant C\,\|\langle x\rangle^{\sigma} h_m\psi_{\pm}\|_{\mathscr{H}}\, \|\langle x\rangle^{\sigma} K_nP_ch_{-n}\psi_{\pm}\|_{\mathscr{H}}\,\langle t\rangle^{-r+1}\\
&\quad\leqslant C\langle t\rangle^{-r+1}
\end{aligned}$$

The same result directly follows from (5) in the case $\Omega + n\mu \geqslant d$. Therefore, we can conclude that

$$|\alpha^2_{\pm,\pm}| \leqslant C\beta^2\langle t\rangle^{-r+1} \tag{25}$$

For what regards the term $\alpha^3_{\pm,\pm}$ we remark that from (11) it follows that

$$\ulcorner \dot{A}(s)\urcorner \leqslant C\tilde{\beta}\ulcorner A(s)\urcorner + \ulcorner R(s)\urcorner, \qquad \text{where} \quad \tilde{\beta} = \max(\omega, \xi)$$

then, applying the result of Lemma 2 to the vector $e^{-i(\Omega+n\mu)s}\dot{A}_{\pm}(s)\,h_{-n}\psi_{\pm}$, we obtain that

$$|\alpha^3_{\pm,\pm}| \leqslant C\beta^2 \sum_{n=-N}^{N} (\dot{A}_{\pm}(s)\,h_{-n}\psi_{\pm})^{+}(t) \leqslant C\beta^2(\tilde{\beta}[A](t) + [R](t)) \tag{26}$$

For what concerns the last term $\alpha_{d,\pm}$ we apply Lemma 2 to $\phi = W\phi_c$ obtaining

$$|\alpha_{d,\pm}| = \left| \left\langle \langle x\rangle^{\sigma} W\psi_{\pm}, \langle x\rangle^{-\sigma} \int_0^t e^{-iH(t-s)} P_c W(s,\cdot)\,\phi_c\,ds \right\rangle_{\mathscr{H}} \right|$$

$$\leqslant C\beta \int_0^t \|\langle x\rangle^{-\sigma} e^{-iH(t-s)} P_c W(s,\cdot)\,\phi_c\|_{\mathscr{H}}\,ds$$

$$\leqslant C\beta^2 (\chi_{\mathscr{U}}\phi_c)^{+}(t) \leqslant C\beta^2 (\phi_c)^{-}(t)$$

since $\|\langle x\rangle^{\sigma} \chi_{\mathscr{U}} \phi_c\|_{\mathscr{H}} \leqslant \|\langle x\rangle^{\sigma} \chi_{\mathscr{U}} \langle x\rangle^{\sigma}\|_{\mathscr{H}} \cdot \|\langle x\rangle^{-\sigma} \phi_c\|_{\mathscr{H}}$. From this and from Lemma 3 we finally obtain

$$|\alpha_{d,\pm}| \leqslant C\beta^3 [A](t) \tag{27}$$

Collecting the results (25), (26) and (27) and relation (21) we obtain

$$\ulcorner R(t) \urcorner \leqslant \beta^2 C[\langle t\rangle^{-r+1} \ulcorner A(0) \urcorner + [R](t) + \tilde{\beta}[A](t)]$$

from which Lemma 4 follows for β small enough.

Lemma 5. In the limit of small perturbation and large ρ, such that $\beta^3/\omega \ll 1$, then there exists a positive constant C independent of ρ, ε and η such that

$$[A](t) \leqslant C \ulcorner A(0) \urcorner, \qquad \forall t \in [0, T], \qquad T = \frac{2\pi}{\omega} \tag{28}$$

Proof. From Eqs. (14), (22) and (24) and recalling that $\gamma \leqslant C\omega$ then it follows that for any $0 \leqslant t \leqslant T = 2\pi/\omega$

$$[A](t) \leqslant C[\sup_{0\leqslant\tau\leqslant t} e^{\gamma\tau} \ulcorner A(0) \urcorner + [R_A](t)]$$

$$\leqslant C[\ulcorner A(0) \urcorner + t[R](t)]$$

$$\leqslant C[\ulcorner A(0) \urcorner + \beta^2 t(\langle t\rangle^{-r+1} \ulcorner A(0) \urcorner + \tilde{\beta}[A](t))]$$

$$\leqslant C[\ulcorner A(0) \urcorner (1 + \beta^2 \omega^{r-2}) + \tilde{\beta}\beta^2 \omega^{-1} [A](t)]$$

for some positive constant C. From this the result follows since $r > 2$ and $\tilde{\beta}\beta^2\omega^{-1} = \max(\beta^2, \beta^3\omega^{-1}) \ll 1$.

Collecting all these results we are able to prove Theorem 2; indeed, from Eqs. (22), (24) and (28) it follows that

$$\begin{aligned} \ulcorner R_A(t) \urcorner &\leqslant Ct[R](t) \\ &\leqslant C\beta^2(\langle t\rangle^{-r+2} \ulcorner A(0)\urcorner + \tilde{\beta} t[A](t)) \\ &\leqslant C\tilde{\beta}\beta^2\omega^{-1} \ulcorner A(0)\urcorner, \qquad \forall t\in[0,T] \end{aligned}$$

Finally, the bound of $(\phi_c)^-$ it follows as a direct result of Lemmas 3 and 5.

5. THE EXPLICIT MODEL

In this section we consider the following explicit one-dimensional model: the double well potential is given by means of two attractive delta functions at $x=\pm a$ with negative strength $-b$ and one repulsive delta function at $x=0$ with positive strength ρ:

$$V_\rho = -b\,\delta(x-a) - b\,\delta(x+a) + \rho\,\delta(x)$$

where $a>0$ and $b>0$ are fixed and $\rho>0$ is large enough. We recall that the limit case $\rho=+\infty$ corresponds to the case of two attractive δ functions at $x=\pm a$ with Dirichlet condition at $x=0$; that is $H_\infty = H_D^+ \oplus H_D^-$ where $H_D^\pm$ is the Schrödinger operator on the half line $\mathsf{R}^\pm$ with Dirichlet condition at $x=0$.[(2)] We prove now that this model satisfies to the assumptions H1 and H3 of Section 2, and in particular the uniform bounds (3)–(6), provided that $ab>1$ and $\rho\in\mathscr{I}=[\tilde{\rho},+\infty)$ with $\tilde{\rho}>0$ large enough. Let

$$y_1=a, \qquad y_2=0, \qquad y_3=a, \qquad G_k(x)=\frac{i}{2k}e^{ik|x|}, \qquad \Im k>0$$

$$\Gamma_{\rho,k} = \begin{pmatrix} -\left(-\frac{1}{b}+\frac{i}{2k}\right) & -\frac{i}{2k}e^{ika} & -\frac{i}{2k}e^{i2ka} \\ -\frac{i}{2k}e^{ika} & -\left(\frac{1}{\rho}+\frac{i}{2k}\right) & -\frac{i}{2k}e^{ika} \\ -\frac{i}{2k}e^{i2ka} & -\frac{i}{2k}e^{ika} & -\left(-\frac{1}{b}+\frac{i}{2k}\right) \end{pmatrix} \tag{29}$$

$$\mathscr{K}_\rho(x,y;k) = \frac{i}{2k}e^{ik|x-y|} - \frac{1}{4k^2}\sum_{r,s=1}^{3} (\Gamma_{\rho,k}^{-1})_{rs}\, e^{ik|x-y_r|} e^{ik|y-y_s|}$$

we recall that the resolvent operator $[H_\rho - z]^{-1}$, where $H_\rho = -d^2/dx^2 + V_\rho$ and $z = k^2$, is an integral operator with kernel $\mathscr{K}_\rho$ (see ref. 2):

$$([H_\rho - k^2]^{-1}\,\phi)(x) = \int_{\mathsf{R}} \mathscr{K}_\rho(x, y; k)\,\phi(y)\,dy \tag{30}$$

In order to study the spectrum of H_ρ we introduce the functions $\mathsf{f}(w)$, such that $\mathsf{f}(0) = 1$ and $w\mathsf{f}(w) = e^w - 1$ for $w \neq 0$, $\mathsf{h}(w) = 1/ab - \mathsf{f}(w)$ and $\mathsf{g}_\rho(w) = \mathsf{h}(w) - 1/a\rho\,[w\mathsf{f}(w) + 2 + w/ab]$. By means of a simple computation it follows that

$$\det \Gamma_{\rho,k} = \frac{a^3}{w}\,\mathsf{h}(w)\,\mathsf{g}_\rho(w), \qquad \text{where} \quad w = i2ka$$

and that the elements of the inverse matrix $\Gamma^{-1}_{\rho,k}$ are given by

$$\begin{aligned}
(\Gamma^{-1}_{\rho,k})_{1,1} &= (\Gamma^{-1}_{\rho,k})_{3,3} = \frac{-\mathsf{f}(w) + 1/a(1/b - 1/\rho) - w/a^2\rho b}{a\mathsf{g}_\rho(w)\,\mathsf{h}(w)} \\
(\Gamma^{-1}_{\rho,k})_{1,2} &= (\Gamma^{-1}_{\rho,k})_{2,1} = (\Gamma^{-1}_{\rho,k})_{2,3} = (\Gamma^{-1}_{\rho,k})_{3,2} = -\frac{e^{w/2}}{a\mathsf{g}_\rho(w)} \\
(\Gamma^{-1}_{\rho,k})_{2,2} &= \frac{2 + w/ab + w\mathsf{f}(w)}{a\mathsf{g}_\rho(w)} \\
(\Gamma^{-1}_{\rho,k})_{1,3} &= (\Gamma^{-1}_{\rho,k})_{3,1} = \frac{e^w}{a^2\rho\,\mathsf{g}_\rho(w)\,\mathsf{h}(w)}
\end{aligned} \tag{31}$$

The discrete spectrum of H_ρ is given by the negative eigenvalues $\lambda = -w^2/4a^2$ where w are the real and negative solutions of the equations $\mathsf{h}(w) = 0$ and $\mathsf{g}_\rho(w) = 0$. If $ab > 1$ and ρ is large enough then the equation $\mathsf{h}(w) = 0$ has only one real and negative solution w_1 independent of ρ and the equation $\mathsf{g}_\rho(w) = 0$ has only one real and negative solution w_2^ρ such that $w_2^\rho - w_1 = \mathcal{O}(\rho^{-1})$. Therefore hypothesis H_1 follows.

We underline that when $ab > 1$ then there exist a positive constant C and $\tilde{\rho} > 0$ such that

$$|\mathsf{h}(0)| > C \qquad \text{and} \qquad |\mathsf{g}_\rho(0)| > C, \qquad \forall \rho \in \mathscr{I} = [\tilde{\rho}, +\infty) \tag{32}$$

in particular $\lambda = 0$ is neither an eigenvalue nor a resonance; moreover $\sigma_{\mathrm{ess}}(H_\rho) = \sigma_{\mathrm{ac}}(H_\rho) = [0, +\infty)$ (see ref. 2 again). We remark also that from the above formulas (31) it follows that $\Gamma^{-1}_\infty - \Gamma^{-1}_\rho = \mathcal{O}(\rho^{-1})$ and

$$\|[H_\infty - k^2]^{-1} - [H_\rho - k^2]^{-1}\| \leqslant C_k \rho^{-1}, \qquad \Im k > 0$$

therefore H_ρ converges to H_∞ as $\rho \to +\infty$ in norm resolvent sense. Hence, conditions (3) and (6) are uniformly true for any ρ large enough.

In order to prove the bounds (4) and (5) uniformly with respect to ρ we observe that a direct computation gives that

$$\sum_{i=1}^{3} (\Gamma_{\rho,k}^{-1})_{j,i} = w \mathscr{F}_j^\rho(w), \qquad j = 1, 2, 3 \tag{33}$$

where we set

$$\mathscr{F}_1^\rho(w) = \mathscr{F}_3^\rho(w) = \frac{a\rho\, \mathsf{f}(w)\, \mathsf{f}(w/2) - 2/ab - \rho/b\, \mathsf{f}(w/2) + 2\mathsf{f}(w)}{2a^2 \rho\, \mathsf{g}_\rho(w)\, \mathsf{h}(w)} \tag{34}$$

and

$$\mathscr{F}_2^\rho(w) = \frac{a\rho\, \mathsf{h}(w)[-\mathsf{f}(w/2) + 1/ab + \mathsf{f}(w)]}{a^2 \rho\, \mathsf{g}_\rho(w)\, \mathsf{h}(w)} \tag{35}$$

Let γ be an anti-clockwise curve, surrounding the absolute continuous spectrum $\sigma_{ac}(H_\rho) = [0, +\infty)$, with endpoints $+\infty + i0$ and $+\infty - i0$; the spectral theorem gives that

$$\begin{aligned}
(e^{-itH_\rho} P_c \phi)(x) &= \int_\gamma e^{-izt} ([H_\rho - z]^{-1} \phi)(x)\, dz \\
&= i \int_{\mathsf{R}+i0} 2k e^{-ik^2 t}\, dk \int_{\mathsf{R}} \mathscr{K}_\rho(x, y; k)\, \phi(y)\, dy \\
&= \int_{\mathsf{R}} \mathscr{U}_\rho^t(x, y)\, \phi(y)\, dy
\end{aligned}$$

where

$$\mathscr{U}_\rho^t(x, y) = i \int_{\mathsf{R}+i0} 2k e^{-ik^2 t} \mathscr{K}_\rho(x, y; k)\, dk$$

We prove that:

Lemma 6. Let $a, b > 0$ such that $ab > 1$ and let $\tilde{\rho} > 0$ be large enough in order to have (32). Then, the following asymptotic behavior uniformly holds for any $\rho \geqslant \tilde{\rho}$:

$$\mathscr{U}_\rho^t(x, y) = \mathscr{O}([ut^{-1/2}]^3) \qquad \text{where} \quad u = \max(\langle x \rangle, \langle y \rangle) \tag{36}$$

Proof. In order to prove the above proposition we set $u_{r,s} = |x - y_r| + |y - y_s|$ and we remark that the Fourier transform of e^{-ik^2t} is $\sqrt{\pi}\, e^{iu^2/4t}/\sqrt{it}$, then

$$\mathscr{U}^t_\rho(x, y) = i \int_{\mathbb{R}+i0} e^{-ik^2t} e^{ik\,|x-y|}\, dk + \mathscr{V}^t_\rho(x, y)$$

$$= \frac{i\sqrt{\pi}}{\sqrt{it}}\, e^{i\,|x-y|^2/4t} + \mathscr{V}^t_\rho(x, y)$$

$$\mathscr{V}^t_\rho(x, y) = \frac{1}{2} \sum_{r,s=1}^{3} \int_{\mathbb{R}+i0} e^{-ik^2t} e^{ik(|x-y_r|+|y-y_s|)} (\Gamma^{-1}_{\rho,k})_{r,s} \frac{dk}{k}$$

$$= \frac{1}{2} \sum_{r,s=1}^{3} \int_{\mathbb{R}+i0} e^{-ik^2t} e^{iku_{r,s}} (\Gamma^{-1}_{\rho,k})_{r,s} \frac{dk}{k}$$

By means of the McLaurin series $(\Gamma^{-1}_{\rho,k})_{r,s} = (\Gamma^{-1}_{\rho,0})_{r,s} + k(\Gamma^{-1}_{\rho,0})'_{r,s} + (\Gamma^{-1}_{\rho,k})^R_{r,s}$, where $(\Gamma^{-1}_{\rho,k})^R_{r,s}$ has a double (or higher) zero at $k = 0$, we have that $\mathscr{V}^t_\rho = \mathscr{V}^t_{0,\rho} + \mathscr{V}^t_{1,\rho} + \mathscr{V}^t_{R,\rho}$ where

$$\mathscr{V}^t_{0,\rho}(x, y) = \frac{1}{2} \sum_{r,s=1}^{3} (\Gamma^{-1}_{\rho,0})_{r,s} \int_{\mathbb{R}+i0} e^{-ik^2t} e^{iku_{r,s}} \frac{dk}{k}$$

$$\mathscr{V}^t_{1,\rho}(x, y) = \frac{1}{2} \sum_{r,s=1}^{3} (\Gamma^{-1}_{\rho,0})'_{r,s} \int_{\mathbb{R}+i0} e^{-ik^2t} e^{iku_{r,s}}\, dk$$

$$\mathscr{V}^t_{R,\rho}(x, y) = \frac{1}{2} \sum_{r,s=1}^{3} \int_{\mathbb{R}+i0} (\Gamma^{-1}_{\rho,k})^R_{r,s}\, e^{-ik^2t} e^{iku_{r,s}} \frac{dk}{k}$$

here $'$ denotes the derivative with respect to k. For what concerns the computation of the first term $\mathscr{V}^t_{0,\rho}$ we remark that the Cauchy theorem (where we perform the change of the path of integration $k \to e^{-i\pi/4}k$) gives:

$$\int_{\mathbb{R}+i0} e^{-ik^2t} e^{iku} \frac{dk}{k} = \int_{\mathbb{R}+i0} e^{-k^2t} e^{e^{i\pi/4}ku} \frac{dk}{k}$$

$$= -i\pi e^{iu^2/4t} \mathscr{W}(e^{i\pi/4}u/2\sqrt{t})$$

where $\mathscr{W}(\tau)=(i/\pi)\int_{\mathbf{R}} e^{-s^2}(ds/(\tau-s))$, $\Im\tau>0$ (see formula (7.1.4), ref. 1) is such that (see formula (7.1.8), ref. 1)

$$H^t_{r,s}(x,y)=e^{iu^2_{r,s}/4t}\mathscr{W}(e^{i\pi/4}u_{r,s}/2\sqrt{t})-1=\left[1+\frac{iu^2_{r,s}}{4t}+\mathcal{O}(u^4/t^{-2})\right]$$

$$\times\left[1+\frac{ie^{i\pi/4}u_{r,s}}{2\sqrt{t}\,\Gamma(3/2)}+\frac{(ie^{i\pi/4}u_{r,s})^2}{(2\sqrt{t})^2\,\Gamma(2)}+\mathcal{O}([u/\sqrt{t}]^3)\right]-1$$

$$=\frac{ie^{i\pi/4}u_{r,s}}{2\sqrt{t}\,\Gamma(3/2)}+\mathcal{O}([ut^{-1/2}]^3)$$

as $ut^{-1/2}\to 0$. From this and from (33) it follows that

$$\mathscr{V}^t_{0,\rho}(x,y)=\frac{\pi}{2i}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})_{r,s}\,e^{iu^2_{r,s}/4t}\mathscr{W}(e^{i\pi/4}u_{r,s}/2\sqrt{t})$$

$$=\frac{\pi}{2i}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})_{r,s}[1+H^t_{r,s}(x,y)]$$

$$=\frac{\pi}{2i}\sum_{r=1}^{3}(w\mathscr{F}^{\rho}_{r}(w))_{w=0}+\frac{\pi}{2i}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})_{r,s}\,H^t_{r,s}(x,y)$$

$$=\frac{\pi}{2i}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})_{r,s}\frac{ie^{i\pi/4}(|x-y_r|+|y-y_s|)}{2\sqrt{t}\,\Gamma(3/2)}$$

$$+\mathcal{O}([ut^{-1/2}]^3)$$

$$=\frac{\sqrt{\pi}\,e^{i\pi/4}}{2\sqrt{t}}\left[\sum_{r=1}^{3}(|x-y_r|+|y-y_r|)(w\mathscr{F}^{\rho}_{\tau}(w))_{w=0}\right]$$

$$+\mathcal{O}([ut^{-1/2}]^3)$$

$$=\mathcal{O}([ut^{-1/2}]^3)$$

where $u=\max(\langle x\rangle,\langle y\rangle)$. For what concerns the computation of the term $\mathscr{V}^t_{1,\rho}(x,y)$ it follows that:

$$\mathscr{V}^t_{1,\rho}(x,y)=\frac{1}{2}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})'_{r,s}\int_{\mathbf{R}+i0}e^{-ik^2t}e^{iku_{r,s}}\,dk$$

$$=\frac{\sqrt{\pi}}{2\sqrt{it}}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})'_{r,s}\,e^{iu^2_{r,s}/4t}$$

For what regards the computation of the remainder term $\mathscr{V}^{t}_{R,\rho}(x,y)$ we observe that the functions $(1/k)(\Gamma^{-1}_{\rho,k})^{R}_{r,s}$ have a zero at $k=0$ and asymptotically behaves like k^{-1} as k goes to infinity; from this fact, by integrating by parts and by means of the stationary phase theorem, as $ut^{-1/2}\to 0$, it follows that $\mathscr{V}^{t}_{R,\rho}(x,y)=\mathscr{O}(u/t^{3/2})$. Collecting all these results, it follows that:

$$\begin{aligned}\mathscr{U}^{t}_{\rho}(x,y) &= \frac{\sqrt{\pi}}{\sqrt{it}}\left[ie^{i\,|x-y|^2/4t}-\frac{1}{2}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})'_{r,s}\,e^{iu^2_{r,s}/4t}\right]+\mathscr{O}([ut^{-1/2}]^3)\\ &= \frac{\sqrt{\pi}}{\sqrt{it}}\left[i-\frac{1}{2}\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})'_{r,s}+\mathscr{O}(u^2t^{-1})\right]+\mathscr{O}([ut^{-1/2}]^3)\\ &= \mathscr{O}([ut^{-1/2}]^3)\end{aligned}$$

since (see formulas (33), (34) and (35))

$$\sum_{r,s=1}^{3}(\Gamma^{-1}_{\rho,0})'_{r,s}=2i(2\mathscr{F}_1(0)+\mathscr{F}_2(0))=2i$$

We underline that the bound (36) is uniform with respect to ρ as a result of the uniform bound (32).

From (36) it follows the uniform bound (4) for a suitable σ large enough. By applying the same arguments to

$$\begin{aligned}&e^{-itH_\rho}[H_\rho-(\lambda+i0)]^{-1}P_c\phi\\ &\quad=\int_\gamma\frac{e^{-izt}}{z-(\lambda+i0)}[H_\rho-z]^{-1}\phi\,dz,\qquad \lambda\in[d,D]\end{aligned}$$

we obtain the uniform bound (5).

6. SPLITTING VANISHING AND LOCALIZATION

In this section we explicitly compute the solutions of Eq. (11) for the double well model discussed in the previous section where, for the sake of argument, we fix $a=1$ and $b=2$; for such values we have that $\lambda_1=-w_1^2/4a^2=0.6351$. We fix also the time-dependent perturbation such that

$$g(x)=\chi_{(0,a)}(x)\qquad\text{and}\qquad f(x)=\chi_{(0,a)}(x)+\delta\chi_{(-a,0)}(x)\tag{37}$$

where $-1<\delta<1$ is a given parameter, and $v(t)$ is a kicked-type function, for instance

$$v(t)=\sum_{n=-N,\, n\neq 0}^{N} c_n e^{int}, \qquad c_n=\frac{N}{2n}\sin(n\pi/N)(1-(-1)^n), \quad N=10$$

In order to satisfies Hypothesis H2 we assume μ such that $n\mu+\lambda_1\neq 0$, $n=0, \pm 1, \pm 2,..., \pm N$, and ρ large enough.

In particular, we consider here two cases: the case of small perturbation regime, where the strength ε of the periodic part of the perturbation is of the same order of the splitting ω; and the case of critical perturbation regime, where ε is of the same order of the square root of ω. In the following, the strength η of the static part of the perturbation is assumed to be of the same order of the splitting ω.

As we will show, in the first case we observe that the periodic part of the perturbation doesn't actually affect the dynamics, that is we observe again the beating effect (see Theorem 3 and the following remark); in contrast, in the second case, we obtain the splitting vanishing for a critical value of the parameters (see Theorem 5).

We remark that in the double well model considered in the previous section and in the large barrier limit $\rho\to+\infty$ (corresponding to the Dirichlet condition at $x=0$) we have that $\lambda_{1,2}\sim\lambda_D=-w_1^2/4a^2$ and $\psi_\pm(x)\sim\psi_D(\pm x)$, where λ_D is the eigenvalue of H_D^+ with associate normalized eigenvector ψ_D. From this fact and from (37) it follows that the elements of the matrices F and G are such that

$$G_{+,+}\sim g_0, \qquad G_{+,-}\sim G_{-,+}\sim G_{-,-}\sim 0$$

and

$$F_{+,+}\sim f_0, \qquad F_{-,-}\sim\delta f_0, \qquad F_{-,+}\sim F_{+,-}\sim 0$$

in the large barrier limit, where

$$f_0=g_0=\langle\psi_D, \chi_{(0,a)}\psi_D\rangle_{\mathscr{H}}$$

In order to compute the terms (9) and (10), let

$$P_c f\psi_\pm=f\psi_\pm-\psi_+\langle\psi_+, f\psi_\pm\rangle_{\mathscr{H}}-\psi_-\langle\psi_-, f\psi_\pm\rangle_{\mathscr{H}}$$

then it follows that

$$\langle f\psi_{\pm}, K_m P_c f\psi_{+}\rangle_{\mathscr{H}} = \langle f\psi_{\pm}, K_m f\psi_{+}\rangle_{\mathscr{H}} - \langle f\psi_{\pm}, K_m\psi_{+}\rangle_{\mathscr{H}} \langle \psi_{+}, f\psi_{+}\rangle_{\mathscr{H}} - \langle f\psi_{\pm}, K_m\psi_{-}\rangle_{\mathscr{H}} \langle \psi_{-}, f\psi_{+}\rangle_{\mathscr{H}}$$

where an explicit computation gives

$$K_m\psi_{+} = \frac{2\omega}{m^2\mu^2 - \omega^2}\psi_{-} + \frac{2m\mu}{m^2\mu^2 - \omega^2}\psi_{+}$$

Recalling that $\omega \sim 0$ and $\langle \psi_{\pm}, f\psi_{\mp}\rangle_{\mathscr{H}} \sim 0$ in the large barrier limit, then

$$\langle f\psi_{+}, K_m P_c f\psi_{+}\rangle_{\mathscr{H}} \sim \langle f\psi_{+}, K_m f\psi_{+}\rangle_{\mathscr{H}} - \frac{2m\mu}{m^2\mu^2 - \omega^2} |\langle f\psi_{+}, \psi_{+}\rangle_{\mathscr{H}}|^2$$

and

$$\langle f\psi_{+}, K_m P_c f\psi_{-}\rangle_{\mathscr{H}} \sim \langle f\psi_{+}, K_m f\psi_{-}\rangle_{\mathscr{H}} - \frac{2\omega}{m^2\mu^2 - \omega^2} \langle f\psi_{-}, \psi_{-}\rangle_{\mathscr{H}} \langle \psi_{+}, f\psi_{+}\rangle_{\mathscr{H}} \sim \langle f\psi_{+}, K_m f\psi_{-}\rangle_{\mathscr{H}}$$

Therefore, in the large barrier limit

$$U_{\pm,\pm}(\mu) \sim u_{\pm,\pm}(\mu) = \sum_{m\neq 0} |c_m|^2 \langle f\psi_{\pm}, K_m f\psi_{\pm}\rangle_{\mathscr{H}} \tag{38}$$

where c_m are the Fourier coefficients of $v(t)$. Recalling that V is an even function, then we have that $K_m = \mathscr{S} K_m \mathscr{S}$ where $(\mathscr{S} f)(x) = f(-x)$; hence

$$\langle f\psi_{+}, K_m f\psi_{+}\rangle_{\mathscr{H}} = u_{+,m} \qquad \langle f\psi_{-}, K_m f\psi_{-}\rangle_{\mathscr{H}} = \delta^2 u_{+,m}$$

$$\langle f\psi_{-}, K_m f\psi_{+}\rangle_{\mathscr{H}} = \delta u_{-,m} \qquad \langle f\psi_{+}, K_m f\psi_{-}\rangle_{\mathscr{H}} = \delta u_{-,m}$$

in the large barrier limit where we set

$$u_{+,m} = \langle \chi_{(0,a)}\psi_D, K_m \chi_{(0,a)}\psi_D\rangle_{\mathscr{H}}$$

$$u_{-,m} = \langle \chi_{(0,a)}\psi_D, K_m \mathscr{S}(\chi_{(0,a)}\psi_D)\rangle_{\mathscr{H}}$$

If we set $u_\pm = \sum_{m\neq 0} |c_m|^2 u_{\pm,m}$ then the matrices M and $M^{\rm per}$ have the form

$$M \sim \omega M^0 + \varepsilon^2 M^1, \quad M^0 = \begin{pmatrix} \eta g_0/\omega & -1 \\ -1 & 0 \end{pmatrix}, \qquad M^1 = \begin{pmatrix} -u_+ & -\delta u_- \\ -\delta u_- & -\delta^2 u_+ \end{pmatrix}$$

and

$$M^{\rm per} \sim \varepsilon v(\mu t)\, M^{\rm per,\,0} + \varepsilon^2 M^{\rm per,\,1}(\mu t), \qquad M^{\rm per,\,0} = \begin{pmatrix} f_0 & 0 \\ 0 & \delta f_0 \end{pmatrix} \tag{39}$$

and $M^{\rm per,\,1}(\mu t)$ is a bounded, uniformly with respect to ρ, ε and η, periodic matrix with mean value zero.

6.1. Small Perturbation Regime

We consider here the small perturbation regime where ε is of the same order of the splitting ω. We underline that, in such a case, by means of a perturbative argument, the assumptions H5 is true.

We have the following result.

Theorem 3. Let $J(t) = e^{-i\omega M^0 t} B(0)$, then, in the limit of small perturbation and large ρ such that ε/ω and η/ω have a finite limit, we have that

$$|B(t) - J(t)| \leqslant C\omega, \quad \forall t \in [0, T]$$

for some positive constant C independent of t, ε, μ and ρ.

Proof. In order to prove this theorem we simply apply the averaging perturbative method to the Eq. (13), where the mean value of $M^{\rm per}$ is equal to zero.

Remark. As a result of Theorems 2 and 3 it follows that we observe again a beating motion with shorter period given by $2\pi/(\sqrt{\omega^2 + \eta^2 g_0^2/4})$.

6.2. Critical Perturbative Regime

We assume here that the strength ε of the periodic perturbation is of the same order of the square root of the splitting ω; in particular we assume the limit of small perturbation and large ρ such that

$$\varepsilon^2/\omega \to k \in \mathrm{R}^+ \qquad \text{and} \qquad \eta/\omega \to \zeta \in \mathrm{R} \tag{40}$$

In such a case we have that:

Theorem 4. Let $J(t)=e^{-iMt}B(0)$, then, in the limit of small perturbation and large ρ satisfying (40), we have that

$$|B(t)-J(t)|\leqslant C\sqrt{\omega}, \qquad \forall t\in[0,T] \tag{41}$$

for some positive constant C independent of t, ε, μ and ρ.

Proof. In order to prove this theorem we remark that Eq. (13) takes the form

$$\begin{cases}\dot{\varphi}=\mu\\ \dot{B}=[-iM+iM^{\mathrm{per}}(\varphi)]\,B, \qquad B(0)=A(0)\end{cases} \tag{42}$$

where $M\sim\varepsilon^2[k^{-1}M^0+M^1]$, k is such that $\varepsilon^2/\omega\to k$, and $M^{\mathrm{per}}(\varphi)$ is a periodic matrix with order ε and mean value equal to zero. Let

$$D=B+\varepsilon K(\varphi)\,B, \quad K(\varphi)=\begin{pmatrix}K_{+,+}(\varphi) & K_{+,-}(\varphi)\\ K_{-,+}(\varphi) & K_{-,-}(\varphi)\end{pmatrix} \tag{43}$$

be a linear transformation where K is a bounded matrix defined below. For ε small enough it follows that this transformation is invertible with inverse

$$B=[1+\varepsilon K(\varphi)]^{-1}\,D=D+\varepsilon\tilde{K}(\varphi)\,D \tag{44}$$

for some bounded matrix $\tilde{K}$. From (42) and from (43) it follows that

$$\dot{D}=\dot{B}+\varepsilon K(\varphi)\,\dot{B}+\varepsilon\mu\,\frac{\partial K}{\partial\varphi}\,B$$

$$=\left[iM^{\mathrm{per}}(\varphi)+\varepsilon\mu\,\frac{\partial K}{\partial\varphi}\right]B+[-iM+i\varepsilon K(\varphi)\,M^{\mathrm{per}}(\varphi)]\,B-i\varepsilon K(\varphi)\,MB$$

Recalling that $M^{\mathrm{per}}(\varphi)$ is of order ε and M is of order ε^2 then the first term of the above equation is of order ε, the second one is of order ε^2 and the third one is of order ε^3. By choosing

$$K_{\pm,\pm}(\varphi)=-\frac{i}{\mu\varepsilon}\int_0^{\varphi}M^{\mathrm{per}}_{\pm,\pm}(\theta)\,d\theta \tag{45}$$

we have that the above equation takes the form

$$\begin{aligned}\dot{D} &= [-iM + i\varepsilon K(\varphi)\, M^{\mathrm{per}}(\varphi)]\, B - i\varepsilon K(\varphi)\, MB \\ &= [-iM + i\varepsilon K(\varphi)\, M^{\mathrm{per}}(\varphi)]\, D + S\, D \\ S &= \varepsilon[-iM + i\varepsilon K(\varphi)\, M^{\mathrm{per}}(\varphi)]\, \tilde{K}(\varphi) - i\varepsilon K(\varphi)\, M(1+\varepsilon \tilde{K}(\varphi))\end{aligned}$$

where the first term is of order ε^2 and the second one, that is the matrix S, is of order ε^3. We apply now the averaging perturbative method for any $t \in [0, T]$, where $T = 2\pi/\omega$ is of order $1/\varepsilon^2$, obtaining that

$$|D(t) - \exp[-i(M - \varepsilon \overline{KM^{\mathrm{per}}})\, t]\, D(0)| \leqslant C\varepsilon^2, \qquad \forall t \in [0, T] \tag{46}$$

for some positive constant C. The term $\overline{KM^{\mathrm{per}}}$ denotes the mean value of the term $K(\varphi)\, M^{\mathrm{per}}(\varphi)$; by integrating by parts we have that

$$\begin{aligned}\overline{KM^{\mathrm{per}}} &= \frac{1}{2\pi} \int_0^{2\pi} K(\varphi)\, M^{\mathrm{per}}(\varphi)\, d\varphi \\ &= \frac{-i}{\mu\varepsilon} \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\varphi} M^{\mathrm{per}}(\theta)\, d\theta \right) M^{\mathrm{per}}(\varphi)\, d\varphi \\ &= \frac{-i}{\mu\varepsilon} \frac{1}{2\pi} \left[\int_0^{\varphi} M^{\mathrm{per}}(\theta)\, d\theta \right]_0^{2\pi} \\ &\quad + \frac{i}{\mu\varepsilon} \frac{1}{2\pi} \int_0^{2\pi} M^{\mathrm{per}}(\varphi) \left(\int_0^{\varphi} M^{\mathrm{per}}(\theta)\, d\theta \right) d\varphi \\ &= -\overline{M^{\mathrm{per}} K}\end{aligned}$$

From this fact and since M^{per} and K are, up to a term of order, respectively, ε^2 and ε, diagonal matrices (see Eqs. (39) and (45)) then we have that

$$\overline{KM^{\mathrm{per}}} = \mathcal{O}(\varepsilon^2) \tag{47}$$

From (43), (44), (46) and (47) it follows that

$$\begin{aligned}|B(t) - e^{-iMt} B(0)| &\leqslant |B(t) - D(t)| \\ + |D(t) - e^{-iMt}\, D(0)| + |e^{-iMt}(D(0) - B(0))| &\leqslant C\varepsilon, \qquad \forall t \in [0, T]\end{aligned}$$

for some positive constant C.

Remarks. — We underline that from the above result the validity of assumption H5 follows too.

— As a consequence of equations (14), (15) and (41) it follows that the localization phenomenon and the beating effect depend on the eigenvalues ℓ_1 and ℓ_2 of M. In particular, if $|\Im(\ell_1-\ell_2)|/\omega$ is large enough then we have a localization result; in contrast, if $|\Im(\ell_1-\ell_2)| \ll \omega$ then we observe a beating effect depending on $\Re(\ell_1-\ell_2)$.

Now, it turns out that when the time dependent perturbation is asymettrical, i.e., $\delta \neq \pm 1$, then the phenomenon of the vanishing of the splitting occurs. More precisely:

Theorem 5. For any non-resonant $\mu^* > \mu_0$, for some $\mu_0 > 0$ and any $\rho^* \in \mathscr{I}$ large enough then there exist $\varepsilon^* = \varepsilon^*(\mu^*, \rho^*)$ and $\eta^* = \eta^*(\mu^*, \rho^*)$ such that we have the vanishing of the splitting of the two eigenvalues of M, i.e., $\ell_1 = \ell_2$.

Proof. Indeed, recalling that the leading term of M has the form

$$M \sim \begin{pmatrix} \eta g_0 - \varepsilon^2 u_+ & -\omega - \varepsilon^2 \delta u_- \\ -\omega - \varepsilon^2 \delta u_- & -\varepsilon^2 \delta^2 u_+ \end{pmatrix} \tag{48}$$

in the large barrier limit, then the eigenvalues $\ell_{1,2}$ of M are such that $\ell_{1,2} \sim \frac{1}{2} \omega L_{1,2}$ where

$$L_{1,2} = z - k u_+ (1 + \delta^2) \pm \sqrt{(z - k u_+ (1 - \delta^2))^2 + 4(1 + k\, \delta u_-)^2}$$

where $\eta/\omega \to \zeta$, $z = \zeta g_0$ and $\varepsilon^2/\omega \to k$. Hence, we have that $L_1 = L_2$ when

$$z^* = z(k^*) = 2i(1 + k^* \delta u_-) + k^* u_+ (1 - \delta^2) \tag{49}$$

and $k^* \in \mathrm{R}^+$ is such that $\Im z(k^*) = 0$, i.e.,

$$k^* = 2/[2\delta \Re u_- + (1 - \delta^2)\, \Im u_+] \tag{50}$$

In fact $\Re u_- \approx 0$ and $\Im u_+ > 0$ for μ not too small (see Fig. 1). By means of a continuity argument the vanishing of the splitting, i.e., $\ell_1 = \ell_2$, follows for some ε^* and η^*.

Remarks. — In Fig. 2 we fix $\rho = 10^5$, in such a case we have that $\lambda_2 = -0.6349$, $\lambda_1 = -0.6351$ and $\omega = 8.5 \times 10^{-5}$. For $\mu = \mu^* = 0.8$ formulas (49) and (50) give that $k^* = 0.408$ and $z^* = 3.078$. For such values of the parameters we observe the exact crossing (see Fig. 2a).

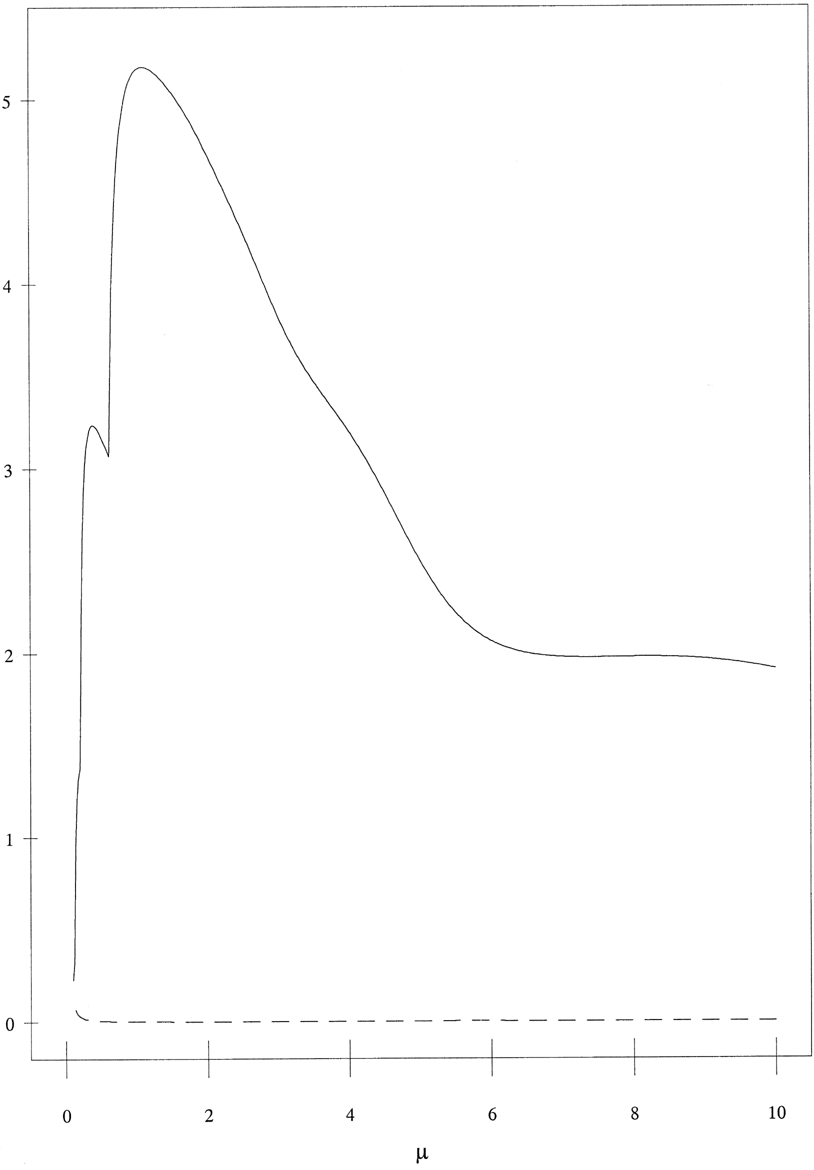

Fig. 1. Broken and full lines represent, respectively, the functions $\Re u_-(\mu)$ and $\Im u_+(\mu)$. It appears that the condition $k_+ > 0$ (see Eq. (50)) is satisfied provided μ is not too small; indeed $\Re u_-(\mu) \approx 0$ and $\Im u_+(\mu) > 0$.

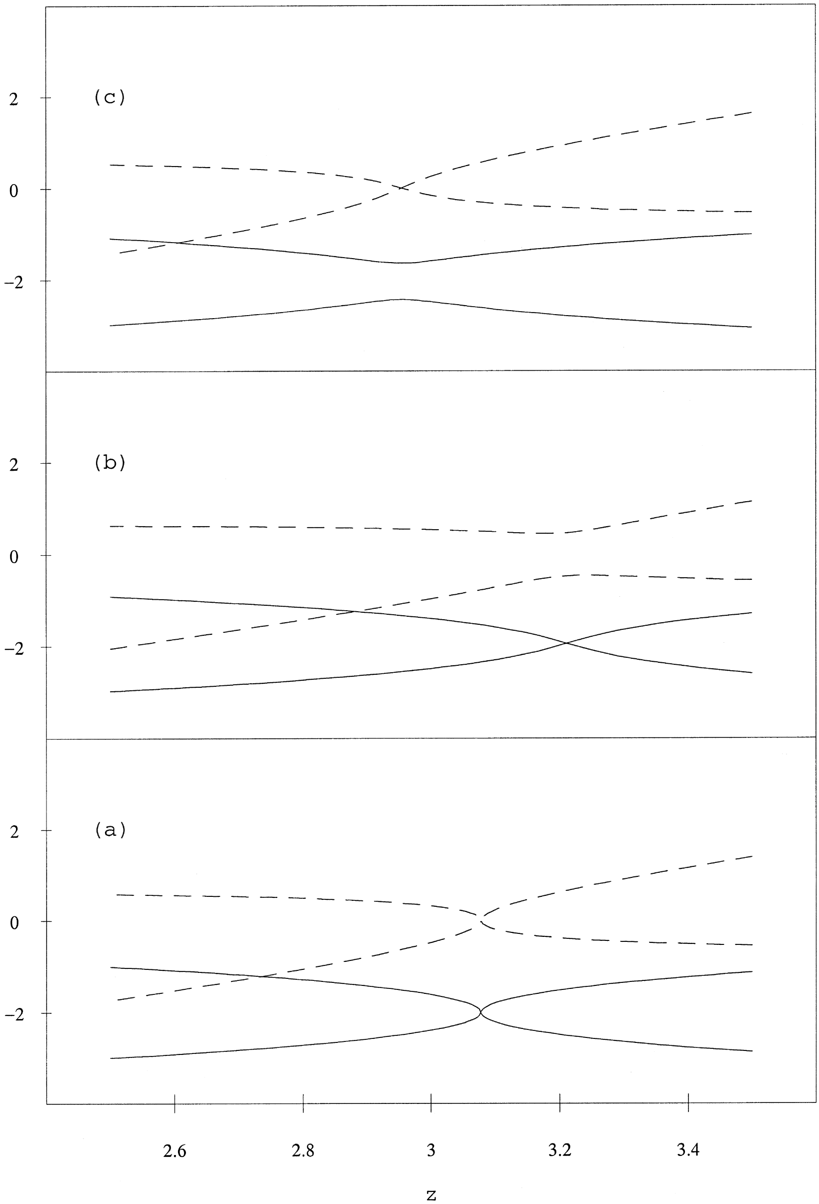

Fig. 2. We plot the real (broken line) and imaginary part (full line) of $L_{1,2}$. In figure (a), for $\mu = \mu^*$, we have an exact crossing; vanishing of the real part of the splitting and avoided crossing of the real part of the eigenvalues are observed, respectively, for $\mu > \mu^*$ (figure (c)) and $\mu < \mu^*$ (figure (b)).

— In the case of f symmetric, i.e., $\delta = 1$, we don't have the vanishing of the splitting because $\Re u_{-} = 0$ (see Fig. 1). Because the perturbation term appears at the second order, the same argument applies in the case $\delta = -1$ too.

— We remark that, in the very large perturbation regime, i.e., $\varepsilon^2 \gg \omega$ (provided that $\omega/\varepsilon^3 \ll 1$ in order to have Theorem 2), we have a localization result. In particular, by assuming $\eta = 0$ and $\delta = 0$ for the sake of argument, then we have $\ell_1 \sim 0$ and $\ell_2 \sim -\varepsilon^2 u_{+}$. From Theorem 2 we have that the leading terms of the solutions $A_{\pm}(t)$ are given by:

$$A_{\pm}(t) \sim c_{\pm,1} e^{-i\ell_1 t} + c_{\pm,2} e^{-i\ell_2 t}, \qquad \forall t \in [0, T]$$

where

$$c_{+,1} = \frac{\ell_1 A_{+}(0) - \omega A_{-}(0)}{\ell_1 - \ell_2} \qquad c_{+,2} = \frac{-\ell_2 A_{+}(0) + \omega A_{-}(0)}{\ell_1 - \ell_2}$$

$$c_{-,1} = \frac{-\omega A_{+}(0) - \ell_2 A_{-}(0)}{\ell_1 - \ell_2} \qquad c_{-,2} = \frac{\omega A_{+}(0) + \ell_1 A_{-}(0)}{\ell_1 - \ell_2}$$

From these facts it follows that, if the state is initially prepared in the left well, i.e., $A_{+}(0) = 0$ and $A_{-}(0) = 1$, it stays in the same well for a time larger than the unperturbed beating period T; that is we generically obtain the destruction of the beating effect because of localization on one well.

— We remark that a similar result has been obtained in the large perturbation limit and by neglecting the term R_c in (8) which couples the discrete spectrum with the continuous one (see ref. 9 and the references therein). In such a way we obtain the simpler two-level model

$$\begin{cases} i\dot{A}_{+} = -\omega A_{-} + \varepsilon v(\mu t) f_0 A_{+} \\ i\dot{A}_{-} = -\omega A_{+} \end{cases} \tag{51}$$

By assuming $v(t) = \cos(t)$ (for a recent study where v is a quasi-periodic function see ref. 16) the system (51) takes the form $\dot{\alpha}_{\pm} = i\omega\alpha_{\mp} e^{\pm iq(t)}$ where we set $\alpha_{+}(t) = A_{+}(t)\, e^{iq(t)}$, $\alpha_{-}(t) = A_{-}(t)$ and $q(t) = \varepsilon f_0/\mu \sin(\mu t)$. By assuming that $\omega \ll \mu$ and by means of the mean value approximation, this equation has approximate solutions of the form $\alpha_0 \cos(\tilde{\omega} t + \varphi_0)$, for some α_0 and φ_0, for any $t \in [0, c\omega^{-1}]$, for some c, where

$$\tilde{\omega} = \frac{\omega}{2\pi} \int_{\pi}^{-\pi} e^{i(\varepsilon f_0/\mu)\sin(\tau)}\, d\tau = \omega J_0\left(\frac{\varepsilon f_0}{\mu}\right)$$

J_0 is the zero-th Bessel function. It appears that for a two-level periodically driven model the beating period is given by $T_{per} = TJ_0^{-1}(\varepsilon f_0/\mu)$ where $T = 2\pi/\omega$ is the beating period of the unperturbed double-well model. In the large perturbation limit such that $\varepsilon/\mu \to \infty$ then $T_{per} \sim T\sqrt{(\varepsilon f_0/\mu)} \gg T$, in such a case we have the interruption of the beating effect; we remark that the interruption of the beating effect appears also for intermediate values of ε, such that $\varepsilon f_0/\mu$ coincides with a zero of the zero-th Bessel function.

ACKNOWLEDGMENTS

This work is partially supported by the Italian MURST and INDAM-GNFM. V. Grecchi is supported by the INFN and by the University of Bologna (funds for selected research topics); A. Sacchetti is supported by the program "Progetti di ricerca avanzata o applicata" of the University of Modena and Reggio Emilia.

REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Wiley, New York, 1972).
2. S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, New York, 1988)
3. S. H. Autler and C. H. Townes, Stark effect in rapidly varying fields, *Phys. Rev.* **100**:703–722 (1955).
4. P. Claviere and G. Jona Lasinio, Instability of tunneling and the concept of molecular structure in quantum mechanics: The case of pyramidal molecules and the enantiomer problem, *Phys. Rev. A* **33**:2245–2253 (1986).
5. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw–Hill, 1955).
6. D. Giuliani, E. Joos, G. Kiefer, J. Kupsch, I. O. Stamatescu, and H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer-Verlag, 1996).
7. V. Grecchi, A. Martinez, and A. Sacchetti, Splitting instability: the unstable double wells, *J. Phys. A: Math. and Gen.* **29**:4561–4587 (1996).
8. V. Grecchi and A. Sacchetti, Molecular localization induced by collisions, *Phys. Rev. A* **61**(52106):1–4 (2000).
9. M. Grifoni and P. Hänggi, Driven quantum tunneling, *Phys. Rep.* **304**:229–354 (1998).
10. F. Grossmann, T. Dittrich, P. Jung and P. Hänggi, Coherent destruction of tunneling, *Phys. Rev. Lett.* **67**:516–519 (1991).
11. A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time-decays of the wave functions, *Duke Math. J.* **46**:583–611 (1979).
12. M. Murata, Asymptotic expansion in time for solutions of Schrödinger-type equations, *J. Func. Anal.* **49**:10–56 (1982).

13. A. Soffer and M. I. Weinstein, Time dependent resonance theory, *Geom. and Func. Anal.* **8**:1086–1128 (1998).
14. A. Soffer and M. I. Weinstein, Nonautonomous Hamiltonians, *J. Stat. Phys.* **93**:359–391 (1998).
15. A. S. Wightman, Superselection rules; old and new, *Nuovo Cimento B* **110**:751–769 (1995).
16. W. F. Wreszinski and S. Casmeridis, Models of two-levels atoms in quasiperiodic external fields, *J. Stat. Phys.* **90**:1061–1068 (1998).