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Absolute Continuity of the Floquet Spectrum for a Nonlinearly Forced Harmonic Oscillator

Sandro Graffi[1,⋆], **Kenji Yajima**[2,⋆⋆]

[1] Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40127 Bologna, Italy. E-mail: graffi@dm.unibo.it

[2] Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-7815, Japan. E-mail: yajima@ms.u-tokyo.ac.jp

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Abstract: We prove that the Floquet spectrum of the time periodic Schrödinger equation $i\dfrac{\partial u}{\partial t} = -\dfrac{1}{2}\Delta u + \dfrac{1}{2}x^2 u + 2\varepsilon(\sin t)x_1 u + \mu V(t,x)u$, corresponding to a mildly nonlinear resonant forcing, is purely absolutely continuous for μ suitably small.

1. Introduction and Statement of the Result

It is well known [HLS] that the spectrum of the Floquet operator of the resonant, linearly forced Harmonic oscillator

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + \frac{1}{2}x^2 u + 2\varepsilon(\sin t)x_1 u, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \ \ \varepsilon > 0$$

is purely absolutely continuous. We show in this paper that the absolute continuity of the Floquet spectrum persists under time-periodic perturbations growing no faster than linearly at infinity provided the resonance condition still holds. Thus we consider the time-dependent Schrödinger equation

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + \frac{1}{2}x^2 u + 2\varepsilon(\sin t)x_1 u + \mu V(t,x)u \tag{1.1}$$

and suppose that $V(t,x)$ is a real-valued smooth function of (t,x), 2π-periodic with respect to t, increasing at most linearly as $|x|$ goes to infinity:

$$|\partial_x^\alpha V(t,x)| \le C_\alpha, \quad |\alpha| \ge 1. \tag{1.2}$$

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Under this condition Eq. (1.1) generates a unique unitary propagator $U(t,s)$ on the Hilbert space $L^2(\mathbb{R}^n)$. The Floquet operator is the one-period propagator $U(2\pi,0)$ and we are interested in the nature of its spectrum. It is well known that the long time behaviour of the solutions of (1.1) can be characterized by means of the spectral properties of the Floquet operator ([KY]). Our main result in this paper is the following theorem.

Theorem 1.1. *Let V be as above. Then, for $|\mu| \sup_{t,x} |\partial_{x_1} V(t,x)| < \varepsilon$, the spectrum of the Floquet operator $U = U(2\pi,0)$ is purely absolutely continuous.*

Remark. The above result can be understood in terms of the classical resonance phenomenon. If $V = 0$ all motions generated by the classical Hamiltonian $\frac{1}{2}(p^2 + x^2) + 2\varepsilon x_1 \sin t$ undergo a resonance between the proper frequency of the harmonic motions and the frequency of the linear forcing term: as a consequence, for any given initial condition the classical motion eventually diverges to infinity by oscillations of linearly increasing amplitude. The quantum counterpart of this phenomonon is the absolute continuity of the Floquet spectrum[HLS]. One might ask whether this absolute continuity is stable under perturbations which destroy the linearity of the forcing potential. Theorem 1.1 establishes the stability under perturbations which make the forcing a non-linear one but do not destroy the resonance phenomenon because all initial conditions still diverge by oscillations to infinity. Therefore, the fact that all initial conditions for the corresponding classical motions satisfy the resonance condition seems an almost necessary condition for the spectral absolute continuity of the Floquet spectrum. Indeed it is known that the Schrödinger equation $i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + \varepsilon|x|^\alpha u + \mu V(\omega t, x)u$, with bounded $V(t+2\pi, x) = V(t,x)$, $\alpha > 2$, $\omega \in \mathbb{R}$ whose classical counterpart admits but a dense set of resonant initial conditions, has no absolutely continuous part in its Floquet spectrum if $V \in C^2$([H]); moreover it has pure point spectrum for a large set of non-resonant ω for μ small and $V \in C^r$ (r suitably large), provided V satisfies a supplementary condition on its matrix elements([DS]).

Notation. We use the vector notation: for the multiplication operator X_j by the variable x_j and the differential operator $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1,\dots,n$, we denote $X = (X_1,\dots,X_n)$ and $D = (D_1,\dots,D_n)$. For a measurable function W and a set of commuting selfadjoint operators $\mathcal{H} = (\mathcal{H}_1,\dots,\mathcal{H}_n)$, $W(\mathcal{H})$ is the operator defined via functional calculus. We have the identity

$$\mathcal{U}^* W(\mathcal{H})\mathcal{U} = W(\mathcal{U}^*\mathcal{H}\mathcal{U}) \tag{1.3}$$

for any unitary operator $\mathcal{U}$.

2. Proof of the Theorem

It is well known ([Ya]) that the nature of the spectrum of the Floquet operator U is the same (apart from multiplicities) as that of the Floquet Hamiltonian formally given by

$$\mathcal{K}u = -i\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u + \frac{1}{2}x^2 u + 2\varepsilon(\sin t)x_1 u + \mu V(t,x)u \tag{2.1}$$

on the Hilbert space $\mathbf{K} = L^2(\mathbb{T}) \otimes L^2(\mathbb{R}^n)$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the circle. More precisely, if $\mathcal{K}$ is the generator of the one-parameter strongly continuous unitary group $\mathcal{U}(\sigma)$, $\sigma \in \mathbb{R}$, defined by

$$(\mathcal{U}(\sigma)u)(t) = U(t, t-\sigma)u(t-\sigma), u = u(t, \cdot) \in \mathbf{K}, \tag{2.2}$$

then, $\mathcal{U}(2\pi) = e^{-i2\pi\mathcal{K}}$ is unitarily equivalent to $\mathbf{1} \otimes U(2\pi, 0)$. We set

$$\mathbf{D} \equiv C^\infty(\mathbb{T}, \mathcal{S}(\mathbb{R}^n)).$$

It is easy to see that:

1. The function space $\mathbf{D}$ is dense in $\mathbf{K}$.
2. $\mathbf{D}$ is invariant under the action of the group $\mathcal{U}(\sigma)$.
3. $\mathbf{D}$ is a subset of the domain $D(\mathcal{K})$ of $\mathcal{K}$ and, for $u \in \mathbf{D}$, $\mathcal{K}u$ is given by the right-hand side of (2.1).

It follows that $\mathbf{D}$ is a core for $\mathcal{K}$ ([RS]) and $\mathcal{K}$ is the closure of the operator defined by (2.1) on $\mathbf{D}$.

We introduce four unitary operators $\mathcal{U}_0 \sim \mathcal{U}_3$ on $\mathbf{K}$ and successively transform $\mathcal{K}$ by $\mathcal{U}_j$ as follows: Write H_0 for the selfadjoint operator on $L^2(\mathbb{R}^n)$ defined by

$$H_0 = -\frac{1}{2}\Delta + \frac{1}{2}x^2 - \frac{1}{2}$$

with the domain $D(H_0) = \{u \in L^2(\mathbb{R}^n) : D^2u, x^2u \in L^2(\mathbb{R}^n)\}$ and define

$$\mathcal{U}_0 u(t, \cdot) = e^{-itH_0}u(t, \cdot), \quad u \in \mathbf{K}. \tag{2.3}$$

Proposition 2.1. (1) *The operator* $\mathcal{U}_0$ *is well defined on* $\mathbf{K}$ *and is unitary.*
(2) $\mathcal{U}_0$ *maps* $\mathbf{D}$ *onto itself.*
(3) *For* $u \in \mathbf{D}$, $\mathcal{K}_1 \equiv \mathcal{U}_0^*\mathcal{K}\mathcal{U}_0$ *is given by*

$$\mathcal{K}_1 u = -i\frac{\partial u}{\partial t} + 2\varepsilon \sin t(X_1 \cos t + D_1 \sin t)u + \mu V(t, X\cos t + D\sin t)u + \frac{u}{2}. \tag{2.4}$$

(4) $\mathbf{D}$ *is a core of* $\mathcal{K}_1$.

Proof. It is well-known that $\sigma(H_0) = \{0, 1, \dots\}$ and we have $e^{-2\pi niH_0} = \mathbf{1}$. Hence (2.3) defines a unitary operator on $\mathbf{K}$. We have $\mathcal{S}(\mathbb{R}^n) = \cap_{k=1}^\infty D(H_0^k)$ and (2) follows. (3) follows from the identity (1.3) and the well-known formulae

$$e^{itH_0}Xe^{-itH_0} = X\cos t + D\sin t, \quad e^{itH_0}De^{-itH_0} = -X\sin t + D\cos t.$$

Since $\mathbf{D}$ is a core of $\mathcal{K}$ and $\mathcal{U}_0$ maps $\mathbf{D}$ onto itself, $\mathbf{D}$ is also a core for $\mathcal{K}_1$. □

Note that for any linear function $aX + bD + c$ of X and D, and W satisfying (1.2), $W(aX+bD+c)$ is the pseudo-differential operator with the Weyl symbol $W(ax+b\xi+c)$ ([Hö]).

To eliminate the term $2\varepsilon X_1 \sin t \cos t$ from $\mathcal{K}_1$, we define

$$\mathcal{U}_1 u(t, x) = e^{i\varepsilon(\cos 2t)x_1/2}u(t, x). \tag{2.5}$$

It is easy to see that $\mathcal{U}_1$ maps **D** onto itself and we have

$$\mathcal{U}_1^*\left(-i\frac{\partial}{\partial t}\right)\mathcal{U}_1 = \left(-i\frac{\partial}{\partial t}\right) - \varepsilon(\sin 2t)X_1, \quad \mathcal{U}_1^* D\mathcal{U}_1 = D + \frac{\varepsilon\cos 2t}{2}\mathbf{e}_1,$$

on **D**. It follows that $\mathcal{K}_2 \equiv \mathcal{U}_1^*\mathcal{K}_1\mathcal{U}_1$ is given by the closure of

$$\begin{aligned}\mathcal{K}_2 u = & -i\frac{\partial u}{\partial t} + 2\varepsilon(\sin^2 t)D_1 u + \varepsilon^2(\sin^2 t\cos 2t)u \\ & + \mu V(t, X\cos t + \sin t(D + \frac{\varepsilon\cos 2t}{2}\mathbf{e}_1))u + \frac{u}{2}\end{aligned} \tag{2.6}$$

defined on **D**. We write $2\varepsilon(\sin^2 t)D_1 = \varepsilon D_1 - \varepsilon(\cos 2t)D_1$ in the right side of (2.6).

Next, to eliminate the term $-\varepsilon(\cos 2t)D_1$, we define

$$\mathcal{U}_2 u(t,x) = e^{i\varepsilon(\sin 2t)D_1/2}u(t,x) = u(t, x + \varepsilon(\sin 2t)\mathbf{e}_1/2).$$

Then, $\mathcal{U}_2$ maps **D** onto itself and we have on **D**,

$$\mathcal{U}_2^*\left(-i\frac{\partial}{\partial t}\right)\mathcal{U}_2 = \left(-i\frac{\partial}{\partial t}\right) + \varepsilon(\cos 2t)D_1, \quad \mathcal{U}_2^* X\mathcal{U}_2 = X - \frac{\varepsilon\sin 2t}{2}\mathbf{e}_1.$$

It follows, also with the help of the identity (1.3), that $\mathcal{K}_3 \equiv \mathcal{U}_2^*\mathcal{K}_2\mathcal{U}_2$ is the closure of the operator given on **D** by

$$\begin{aligned}\mathcal{K}_3 u = & -i\frac{\partial u}{\partial t} + \varepsilon D_1 u + \varepsilon^2(\sin^2 t\cos 2t)u \\ & + \mu V(t, X\cos t + D\sin t - \frac{\varepsilon\sin t}{2}\mathbf{e}_1)u + \frac{u}{2}.\end{aligned} \tag{2.7}$$

Here we also used the obvious identity $\cos 2t\sin t - \cos t\sin 2t = -\sin t$.

We write now

$$(\sin^2 t)\cos 2t = \frac{1}{2}\cos 2t - \frac{1}{4}\cos 4t - \frac{1}{4},$$

and define

$$\mathcal{U}_3 u(t,x) = e^{-i\varepsilon^2(\sin 2t)/4 + i\varepsilon^2(\sin 4t)/16}u(t,x).$$

Again $\mathcal{U}_3$ maps **D** onto itself and $\mathcal{L} \equiv \mathcal{U}_3^*\mathcal{K}_2\mathcal{U}_3$ is the closure of the operator given on **D** by

$$\begin{aligned}\mathcal{L}u = & -i\frac{\partial u}{\partial t} + \varepsilon D_1 u + \frac{(2-\varepsilon^2)u}{4} \\ & + \mu V\left(t, X\cos t + D\sin t - \frac{\varepsilon\mathbf{e}_1\sin t}{2}\right)u.\end{aligned} \tag{2.8}$$

Thus, $\mathcal{K}$ is unitarily equivalent to $\mathcal{L}$ defined as the closure of the operator with domain **D** and action specified by the right side of (2.8).

Completion of the proof of the Theorem. We apply Mourre's theory of conjugate operators ([M]; see also[PSS]). We take the selfadjoint operator $\mathcal{A}$ defined by

$$\mathcal{A}u(t,x) = x_1 u(t,x)$$

with obvious domain as the conjugate operator for $\mathcal{L}$, and verify the conditions (a)–(e) of Definition 1 of [M] are satisfied.

(a) $\mathbf{D} \subset D(\mathcal{A}) \cap D(\mathcal{L})$ and hence $D(\mathcal{A}) \cap D(\mathcal{L})$ is a core of $\mathcal{L}$.
(b) It is clear that $e^{i\alpha\mathcal{A}} = e^{i\alpha X_1}$ maps $\mathbf{D}$ onto $\mathbf{D}$ and that, for $u \in \mathbf{D}$, we have

$$e^{-i\alpha\mathcal{A}}\mathcal{L}e^{i\alpha\mathcal{A}}u - \mathcal{L}u = \varepsilon\alpha u - \mu V\left(t, X\cos t + D\sin t - \frac{\varepsilon\mathbf{e}_1\sin t}{2}\right)u + \mu V\left(t, X\cos t + D\sin t - \frac{(\varepsilon - 2\alpha)\mathbf{e}_1\sin t}{2}\right)u.$$

Since $V(x) - V(x + \alpha\mathbf{e}_1\sin t)$ is bounded with bounded derivatives, the right-hand side extends to a bounded operator on $\mathbf{K}$ and it is continuous with respect to α in the operator norm topology. It follows that $e^{i\alpha\mathcal{A}}$ maps the domain of $\mathcal{L}$ into itself and $\sup_{|\alpha|\le 1} \|\mathcal{L}e^{i\alpha\mathcal{A}}u\|_{\mathbf{K}} < \infty$ for any $u \in D(\mathcal{L})$.
(c) Let us verify the conditions (c'), (i), (ii), (iii) of Proposition II.1 of [M] taking $H = \mathcal{L}$, $A = \mathcal{A}$ and $\mathcal{S} = \mathbf{D}$. The verification of these conditions in turn implies (c). First remark that (i) and (ii) are a direct consequence of (a) and (b) above. Moreover for any $u \in \mathbf{D}$ we have

$$i[\mathcal{L}, \mathcal{A}]u = \varepsilon u + \mu\sin t \cdot \partial_{x_1}V\left(t, X\cos t + D\sin t - \frac{\varepsilon\sin t}{2}\mathbf{e}_1\right)u. \quad (2.9)$$

The right-hand side extends to a bounded operator in $\mathbf{K}$ which, following [M], we denote $i[\mathcal{L}, \mathcal{A}]^\circ$. The boundedness implies a fortiori Condition (iii) and hence (c) is verified.
(d) By direct computation we have for $u \in \mathbf{D}$,

$$i[[\mathcal{L}, \mathcal{A}]^\circ, \mathcal{A}]u = \mu\sin^2 t(\partial_{x_1}^2 V)\left(t, X\cos t + D\sin t - \frac{\varepsilon\sin t}{2}\mathbf{e}_1\right)u. \quad (2.10)$$

The right-hand side extends to a bounded operator on $\mathbf{K}$. It follows that $[\mathcal{L}, \mathcal{A}]^\circ D(\mathcal{A}) \subset D(\mathcal{A})$ and (2.10) holds for $u \in D(\mathcal{A})$. Hence $[[\mathcal{L}, \mathcal{A}]^\circ, \mathcal{A}]$ defined on $D(\mathcal{L}) \cap D(\mathcal{A})$ is bounded and this yields (d).
(e) The operator norm of $u \mapsto \sin t \cdot \partial_{x_1}V\left(t, X\cos t + D\sin t - \frac{\varepsilon\sin t}{2}\mathbf{e}_1\right)u$ is bounded by $\sup_{t,x} |\partial_{x_1}V(t, x)|$ by an abstract theorem of functional calculus([RS]) or by noticing that the operator is unitarily equivalent to $\sin t \cdot \partial_{x_1}V(t, x - \varepsilon\sin t\,\mathbf{e}_1/2)$ via the unitary operator $\mathcal{U}_0$. Hence if $|\mu|\|\partial_{x_1}V\|_{L^\infty} < \varepsilon$, then we have $i[\mathcal{L}, \mathcal{A}]^\circ \ge c > 0$.

Thus the conditions of [M] are satisfied and we can conclude that $\sigma(\mathcal{K}) = \sigma_{ac}(\mathcal{K})$ if $|\mu|\|\partial_{x_1}V\|_{L^\infty} < \varepsilon$ by Theorem and Proposition II.4 of [M].

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