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On the spectrum of Farey and Gauss maps

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Abstract
In this paper we introduce Hilbert spaces of holomorphic functions given by generalized Borel and Laplace transforms which are left invariant by the transfer operators of the Farey map and its induced transformation, the Gauss map, respectively. By means of a suitable operator-valued power series we are able to study simultaneously the spectrum of both these operators along with the analytic properties of associated dynamical ζ-functions. This construction establishes an explicit connection between previously unrelated results of Mayer and Rugh.

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1. Introduction

The spectral analysis of transfer operators for smooth uniformly expanding maps of the unit interval [0, 1] is now fairly well understood (see [C], [Ba1]). The spectrum depends crucially on the function space considered which is in general a Banach space. For Banach spaces of sufficiently regular functions, e.g. the space $\mathcal{C}^k$ of k-times differentiable functions on [0, 1] with $k \geqslant 0$, the transfer operator is *quasi-compact*. This means that its spectrum is made out of a finite or at most countable set of isolated eigenvalues with finite multiplicity (the discrete spectrum) and its complementary, the essential spectrum. The latter is a disk whose radius is a function both of k and the expanding constant ρ of the map (see, e.g. [CI]), in such a way that if we let $\rho \to 1$ from above (e.g. approaching an intermittency transition) the essential spectral radius tends to coincide with the spectral radius itself. In particular, in order to understand the nature of the spectrum lying under the 'essential spectrum rug', we have to consider increasingly smooth test functions as ρ approaches 1. This suggests, for instance, that for a type 1 intermittency model at the tangent bifurcation point (see [PM]), one should consider suitable spaces of analytic functions. In this paper we construct a Hilbert space $\mathcal{H}_0$ of

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analytic functions which is left invariant by the transfer operator $\mathcal{P}$ of the Farey map (see below for definitions), a prototype of smooth *intermittent interval map*, having a neutral fixed point at the origin. As a result, the spectrum of $\mathcal{P}$ when acting on $\mathcal{H}_0$ turns out to be the interval [0, 1] with embedded eigenvalues 0 and 1, plus a finite or countably infinite set of eigenvalues of finite multiplicity. The latter is conjectured to be empty. This would improve for this example a previous result obtained by Rugh in a more general framework [Rug]. The above and related achievements are obtained by (a slightly modified version of) an inducing procedure which was introduced for the first time in the pioneering study [P1] (see also [PS], [HI], [Is]) for a rather general class of intermittent interval maps. The main tool in this construction is an operator-valued function $\mathcal{Q}_z$ which enjoys simple algebraic relations both with $\mathcal{P}$ and the transfer operator $\mathcal{Q}$ of the Gauss map, the latter being obtained by inducing the Farey map with respect to the first passage time a subset of [0, 1] away from the neutral fixed point. The spectral properties of $\mathcal{Q}_z$ when acting on a Hilbert space $\mathcal{H}_1 \subset \mathcal{H}_0$ are then suitably translated into those of $\mathcal{Q}$ in $\mathcal{H}_1$ as well as $\mathcal{P}$ in $\mathcal{H}_0$. The paper is organized as follows. Section 2 is devoted to introduce the Farey–Gauss pair, briefly discussing some (mostly known) properties of these maps and of their invariant measures and ending with a short account of their intimate connection with number theory. Further material on these general facts can be found in [Bi], [Ki], [F], [Ma2]. The main results are contained in the two subsequent sections. Section 3 deals with the spectral analysis of transfer operators. We first introduce the operator-valued function $\mathcal{Q}_z$ and establish simple algebraic identities (proposition 3.1). We then extend to $\mathcal{Q}_z$ some previous results of Mayer and Roepstorff (see [MaR1], [MaR2]) for the Gauss transfer operator $\mathcal{Q}$ obtaining as a by-product an analytic continuation of $\mathcal{Q}_z$ outside the unit disk which is crucial to exploit the above identities for spectral analysis purposes (proposition 3.2). The main results on the spectrum of $\mathcal{P}$ (theorems 3.3 and 3.4) are then obtained by combining these identities with an explicit integral representation of $\mathcal{P}$ on the Hilbert space $\mathcal{H}_0$ (theorem 3.2). In section 4 we apply the construction of the previous section to study analytic properties of the dynamical ζ-functions [Ba2] for the Farey–Gauss pair. The role of $\mathcal{Q}_z$ is here played by a two-variable ζ-function $\zeta_2(s, z)$ which simply relates to the Farey and Gauss ζ's (proposition 4.3) and whose analytic structure is directly connected to the spectrum of $\mathcal{Q}_z$ (theorem 4.5). As a result, the ζ-function of the Farey map turns out to extend meromorphically to the cut plane $\mathbb{C} \setminus [1, \infty)$ (corollary 4.3).

Finally, we point out that some generalized version (involving a 'temperature' parameter β) of these functions were previously studied in [Ma1], [Ma2], [Ma3] for the Gauss map and in [D] for the Farey map paired with an induced version conjugated to the Gauss map[1]. In the more general context of piecewise analytic map with a neutral fixed point results yielding meromorphic continuation to the cut plane for ζ-functions as well as regularized Fredholm determinants were obtained in [Rug].

2. Preliminaires

We shall first consider the *Farey map* of the interval [0, 1] into itself defined as

$$F(x) = \begin{cases} F_0(x), & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ F_1(x), & \text{if } \frac{1}{2} < x \leqslant 1, \end{cases} \tag{2.1}$$

[1] The inducing construction used in [D], the same as in [P1], is only slightly different from that used here and several quantities, e.g. operators and ζ-functions, dealt with there are closely related to those discussed here. I thank one of the referees for having let me know about this work.

where

$$F_0(x) := \frac{x}{1-x} \qquad \text{and} \qquad F_1(x) := F_0(1-x) = \frac{1}{F_0(x)} = \frac{1-x}{x}\cdot \tag{2.2}$$

The inverse branches are

$$\begin{aligned} \Psi_0(x) &\equiv F_0^{-1}(x) = \frac{x}{1+x} = \frac{1}{2} - \frac{1}{2}\left(\frac{1-x}{1+x}\right), \\ \Psi_1(x) &\equiv F_1^{-1}(x) = \frac{1}{1+x} = \frac{1}{2} + \frac{1}{2}\left(\frac{1-x}{1+x}\right). \end{aligned} \tag{2.3}$$

For $x \neq 0$ the map $\Psi_0(x)$ is conjugated to the right translation $x \to S(x) = x+1$, i.e.

$$\Psi_0 = J \circ S \circ J \qquad \text{with } J(x) = J^{-1}(x) = \frac{1}{x}. \tag{2.4}$$

This yields for the n-iterate

$$\Psi_0^n(x) = J \circ S^n \circ J(x) = \frac{x}{1+nx}\cdot \tag{2.5}$$

Moreover, $\Psi_1(x)$ satisfies

$$\Psi_1(x) = J \circ S(x). \tag{2.6}$$

2.1. The induced map

Let $\mathcal{A} = \{A_n\}_{n\geqslant 1}$ be the countable partition of $[0, 1]$ given by $A_n = [1/(n+1), 1/n]$. Setting $A_0 = [0, 1]$, it is easy to check that $F(A_n) = A_{n-1}$ for all $n \geqslant 1$. Let X be the residual set of points in $[0, 1]$ which are not preimages of 1 with respect to the map F_0, namely $X = (0, 1] \setminus \{1/n\}_{n\geqslant 1}$. The *first passage time* $\tau : X \to \mathbb{N}$ in the interval A_1 is defined as

$$\tau(x) = 1 + \min\{n \geqslant 0 \;:\; F^n(x) \in A_1\} = \left[\frac{1}{x}\right], \tag{2.7}$$

where $[a]$ is the integer part of a. We see that A_n is the closure of the set $\{x \in X \;:\; \tau(x) = n\}$. On the other hand, the *return time* function $r : A_1 \to \mathbb{N} \cup \{\infty\}$ in the interval A_1 is given by

$$r(x) = \min\{n \geqslant 1 \;:\; F^n(x) \in A_1\} = \tau \circ F(x). \tag{2.8}$$

We now consider the map $G : X \to X$ obtained from F by inducing with respect to the first passage time τ, i.e.

$$G(x) = F^{\tau(x)}(x), \tag{2.9}$$

which can be extended to all of $[0, 1]$ setting $G(0) = 1$, $G(1) = 0$,

$$\lim_{x\uparrow 1/n} G(x) = 0, \qquad \lim_{x\downarrow 1/n} G(x) = 1, \qquad n > 1,$$

and whenever $x \in (1/(n+1), 1/n)$ we have, using (2.5),

$$G(x) \equiv G_n(x) = F^n(x) = F_1 \circ F_0^{n-1}(x) = \frac{1}{x} - n = \frac{1}{x} - \tau(x). \tag{2.10}$$

In other words the induced map is the celebrated *Gauss map*

$$G(x) = \begin{cases} \left\{\dfrac{1}{x}\right\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \tag{2.11}$$

where $\{a\}$ denotes the fractional part of a. It has countably many inverse branches Φ_n given by

$$\Phi_n(x) = G_n^{-1}(x) = \frac{1}{x+n}, \qquad n \geqslant 1. \tag{2.12}$$

2.2. Invariant measures

It is an easy task to verify that the σ-finite absolutely continuous measure

$$\nu(\mathrm{d}x) \equiv e(x)\,\mathrm{d}x = \frac{1}{\log 2}\frac{\mathrm{d}x}{x} \tag{2.13}$$

is invariant for the dynamical system ([0, 1], F). Note that $\nu(A_n) = (\log 2)^{-1} \log(1 + 1/n)$ and $\nu([0, 1]) = \infty$. Let $B_n = \{x \in A_1 : r(x) = n\}$. Using (2.8) we have $\overline{F_1(B_n)} = A_n$. We now show that $\nu(A_n) = \sum_{k\geqslant n} \nu(B_k)$. Indeed, for $n = 1$ we have $\sum_{k\geqslant 1} \nu(B_k) = \nu(A_1) = 1$. Moreover, since ν is F-invariant, $\nu(A_n) = \nu(F^{-1}(A_n)) = \nu(A_{n+1})+\nu(B_{n+1})$, and the assertion follows by induction. Therefore, the expected return time is infinite:

$$\nu_{A_1}(r) = \int_{A_1} r(x)\,\nu\,(\mathrm{d}x) = \sum_{n\geqslant 1} n\,\nu(B_n) = \sum_{n\geqslant 1} \nu(A_n) = \nu([0, 1]) = \infty, \tag{2.14}$$

where ν_{A_1} is the conditional probability measure defined as $\nu_{A_1}(E) = \nu(E \cap A_1)/\nu(A_1)$. It is known that in this situation there is the coexistence of two different statistics for the dynamical system (F, [0, 1]): besides ν, the ergodic means $(1/n)\sum_{i=0}^{n-1} \delta_{F^i(x)}$ converge weakly to the Dirac delta at 0 (see [Me], [HY]).

Let ρ be the probability measure obtained by pushing forward ν with F_1, i.e.

$$\rho(E) = ((F_1)_*\,\nu)(E) = (\nu \circ \Psi_1)(E). \tag{2.15}$$

Reasoning as above one readily verifies that the converse relation is

$$\nu(E) = \sum_{n\geqslant 0} (\rho \circ \Psi_0^n)(E). \tag{2.16}$$

In particular we have $\nu(A_n) = \sum_{l\geqslant n} \rho(A_l)$ and $\rho(A_n) = \rho(F_1(B_n)) = \nu(B_n)$, where B_n is as above. We then have

$$\rho(E) = (\nu \circ \Psi_1)(E) = \sum_{n\geqslant 0} (\rho \circ \Psi_0^n \circ \Psi_1)(E) = \rho(G^{-1}E), \tag{2.17}$$

which says that ρ is G-invariant. Moreover ρ is ergodic with respect to G (see, e.g. [Bi]). Setting $h(x) = \rho(\mathrm{d}x)/\mathrm{d}x$, we get

$$h = |\Psi_1'| e \circ \Psi_1, \qquad e = \sum_{k=0}^{\infty} (\Psi_0^k)' h \circ \psi_0^k, \tag{2.18}$$

which gives the well-known result

$$h(x) = \frac{1}{\log 2}\frac{\mathrm{d}x}{(1 + x)}. \tag{2.19}$$

The primitive $H(x)$ of $h(x)$, with $H(0) = 0$, is $H(x) = \log(1 + x)/\log 2$. Setting $q_n := H(1/(n + 1)) = (\log 2)^{-1} \log(1 + 1/(n + 1))$, we have $\nu(A_n) = q_n$ and $\rho(A_n) = q_{n-1} - q_n$. We see that q_n is a (strict) Kaluza sequence, i.e. for all $n \geqslant 1$

$$1 = q_0 > q_1 > \cdots > q_n > 0 \qquad \text{and} \qquad q_n^2 < q_{n-1}\,q_{n+1}. \tag{2.20}$$

Finally, by (2.7), (2.8), (2.14) and (2.15), we have

$$\rho(\tau) = ((F_1)_*\nu)(\tau) = \nu(\tau \circ F_1) = \nu(r) = \infty. \tag{2.21}$$

On the other hand, we have the following lemma.

Lemma 2.1. *The function* $\log \tau$ *is in* $L_1(\rho)$ *and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \tau(G^j(x)) = \rho(\log \tau) = K, \qquad \rho - \text{a.e.}, \tag{2.22}$$

where the positive constant K *is defined by*

$$e^K = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\log k/\log 2}. \tag{2.23}$$

Proof. We have

$$\begin{aligned}\rho(\log \tau) &= \sum_{k=1}^{\infty} \rho(A_k) \log k = \sum_{k=1}^{\infty} (q_{k-1} - q_k) \log k \\ &= \sum_{k=1}^{\infty} \frac{\log k}{\log 2} \log \left(\left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{k+1}\right)^{-1} \right) \\ &= \sum_{k=1}^{\infty} \frac{\log k}{\log 2} \log \left(1 + \frac{1}{k(k+2)}\right) = K < \infty.\end{aligned}$$

This computation shows both that $\log \tau \in L_1(\rho)$ and the last equality in (2.22). The first equality in (2.22) now follows from the ergodic theorem [Bi]. □

The constant K which appears above is known in number theory as *Khinchin's constant*. This is not a coincidence, as we now briefly explain.

2.3. Connection with number theory

The Farey sum over two rationals a/b and a'/b' is the mediant operation given by [HR]

$$\frac{a''}{b''} = \frac{a + a'}{b + b'}. \tag{2.24}$$

It is easy to see that a''/b'' falls in the interval $(a/b, a'/b')$. Now, having fixed $n \geqslant 0$, let $\mathcal{F}_n$ be the ascending sequence of irreducible fractions between 0 and 1 obtained inductively in the following way. Set first $\mathcal{F}_0 = (\frac{0}{1}, \frac{1}{1})$. Then $\mathcal{F}_n$ is obtained from $\mathcal{F}_{n-1}$ by inserting among each pair of consecutive rationals a/b and a'/b' in $\mathcal{F}_{n-1}$ their mediant a''/b'' as above. Thus $\mathcal{F}_1 = (\frac{0}{1}, \frac{1}{2}, \frac{1}{1})$, $\mathcal{F}_2 = (\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1})$, $\mathcal{F}_3 = (\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1})$ and so on. The elements of $\mathcal{F}_n$ are called Farey fractions. The name of the map F can be related to the easily verified observation that the set of pre-images $\bigcup_{k=0}^{n+1} F^{-k}\{0\}$ coincides with $\mathcal{F}_n$ for all $n \geqslant 0$. In particular, this implies that $\bigcup_{k=0}^{\infty} F^{-k}\{0\} = \mathbb{Q} \cap [0, 1]$ (notice that the same is true for the induced map: $\bigcup_{k=0}^{\infty} G^{-k}\{0\} = \mathbb{Q} \cap [0, 1]$).

On the other hand, we recall that every real number $0 < x < 1$ has a continued fraction expansion of the form [Ki]

$$x = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cdots}}} = [k_1, k_2, k_3, \ldots], \tag{2.25}$$

with $k_i \in \mathbb{N}$. By applying Euclid's algorithm one sees that the above expansion terminates if and only if x is a rational number. There is an intimate connection between the partial quotients

$k_1, k_2, \ldots$ and the Gauss map G. Indeed, given x as above we can write

$$x = \frac{1}{\frac{1}{x}} = \frac{1}{\left[\frac{1}{x}\right] + \left\{\frac{1}{x}\right\}} = \frac{1}{k_1 + G(x)} = \frac{1}{k_1 + \frac{1}{\frac{1}{G(x)}}}$$

$$= \frac{1}{k_1 + \frac{1}{\left[\frac{1}{G(x)}\right] + \left\{\frac{1}{G(x)}\right\}}} = \frac{1}{k_1 + \frac{1}{k_2 + G^2(x)}} = \cdots \tag{2.26}$$

Therefore, $k_1 = [1/x]$, $k_2 = [1/G(x)]$, $k_3 = [1/G^2(x)]$ and so on. Alternatively,

$$\text{if } x = [k_1, k_2, k_3, \ldots], \qquad \text{then } G(x) = [k_2, k_3, \ldots]. \tag{2.27}$$

Farey fractions have close relationships with continued fractions. Let us say that a Farey fraction has order n if it belongs to $\mathcal{F}_n \backslash \mathcal{F}_{n-1}$. Given $n \geqslant 1$ there are exactly 2^{n-1} Farey fractions of order n (they form the set $F^{-(n+1)}\{0\}$) and it is possible to show (see below equation (2.28)) that the integers k_i in their (finite) continued fraction expansion sum up to $n+1$. Furthermore, it is easy to realize that all Farey fractions which fall in the interval $(1/(n+1), 1/n)$ have order greater than or equal to $n+1$, whereas their continued fraction expansion starts with $k_1 = n$. Thus, the map F acts on Farey fractions by reducing their order of one unit. We can write an explicit expression for the action of F on continued fraction expansions. Indeed, if $\frac{1}{2} < x \leqslant 1$ then $k_1 = 1$ and $F(x) = 1/x - k_1 = G(x)$. If instead $0 < x \leqslant \frac{1}{2}$, then $k_1 > 1$ and $F(x) = 1/(1/x - 1)$. Therefore,

$$\text{if } x = [k_1, k_2, k_3, \ldots], \qquad \text{then } F(x) = [k_1 - 1, k_2, k_3, \ldots], \tag{2.28}$$

with $[0, k_2, k_3, \ldots] \equiv [k_2, k_3, \ldots]$ (compare to (2.27)). Now, it is well known that for almost all $x \in (0, 1)$, the arithmetic mean of the partial quotients is infinite (see, e.g. [Ki]), i.e.

$$\lim_{n\to\infty} \frac{k_1 + \cdots + k_n}{n} = \infty \quad \text{(a.e.)}. \tag{2.29}$$

From the above discussion and (2.7) we get $k_l = [1/G^{l-1}(x)] = \tau(G^{l-1}(x))$, which for $l > 1$ is the time between the $(l-1)$st and the lth passage in A_1 of the orbit of x with F. Therefore, the total number S_n of iterates of F needed to observe n passages in A_1, i.e. the function

$$S_n(x) = \tau(x) + \tau(G(x)) + \cdots + \tau(G^{n-1}(x)), \tag{2.30}$$

satisfies

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \infty \quad \text{(a.e.)}. \tag{2.31}$$

Since ρ is absolutely continuous with respect to the Lebesgue measure on [0, 1], the properties expressed by (2.21) and (2.31) can be regarded as an instance of validity of the ergodic theorem for the non-integrable function τ. One can actually say more. As a consequence of ([Ki], theorem 30) we have that for almost all $x \in (0, 1)$ the inequality

$$S_n(x) \geqslant n \log n \tag{2.32}$$

is satisfied for an infinite number of values of n. On the other hand, lemma 2.1 can now be rephrased by saying that the geometric mean of the partial quotients has a certain finite value (a.e.). This, in turn, is a corollary of a theorem of Khinchin ([Ki], theorem 35), which says that for any function $f(k)$ defined on the positive integers and satisfying $f(k) = \mathcal{O}(k^p)$ with $0 \leqslant p < \frac{1}{2}$ we have, for almost all $x \in (0, 1)$,

$$\left| \frac{1}{n} \sum_{j=i}^{n} f(k_j) - \sum_{k=1}^{\infty} \frac{f(k)}{\log 2} \log\left(1 + \frac{1}{k(k+2)}\right) \right| \leqslant \epsilon(n), \tag{2.33}$$

where the error function $\epsilon(n)$ is any positive function decreasing to zero as $n \to \infty$ so that $\sum n^{-2}\epsilon^{-2}(n) < \infty$. Lemma 2.1 then corresponds to the choice $f(k) = \log k$.

3. Transfer operators

We start by establishing some formal algebraic relations between the transfer operators $\mathcal{P}$ and $\mathcal{M}$ associated to the maps F and G, respectively (see [Ba1]). They describe the action of the differentiable dynamical systems F and G on the density f of a measure absolutely continuous measure with respect to Lebesgue by

$$\begin{aligned}\mathcal{P}f(x) &= (\mathcal{P}_0 + \mathcal{P}_1)f(x) =: |\Psi_0'(x)| \cdot f(\Psi_0(x)) + |\Psi_1'(x)| \cdot f(\Psi_1(x)) \\ &= \left(\frac{1}{x+1}\right)^2 \left[f\left(\frac{x}{x+1}\right) + f\left(\frac{1}{x+1}\right)\right],\end{aligned} \tag{3.34}$$

and

$$\mathcal{Q}\, f(x) = \sum_{n=1}^{\infty} Q_n\, f(x) =: \sum_{n=1}^{\infty} |\Phi_n'(x)| \cdot f(\Phi_n(x)) = \sum_{n=1}^{\infty} \left(\frac{1}{x+n}\right)^2 f\left(\frac{1}{x+n}\right). \tag{3.35}$$

We first notice that

$$Q_n f(x) = \mathcal{P}^n(f \cdot \chi_n)(x) = \mathcal{P}_1\mathcal{P}_0^{n-1} f(x), \tag{3.36}$$

where χ_n is the indicator function of A_n. Let $\mathcal{S}f(x) := f \circ S(x) = f(x+1)$ be the shift operator. Note by (2.4) and (2.6), we have

$$\mathcal{P}_1\, \mathcal{P}_0 f(x) = \mathcal{S}\, \mathcal{P}_1\, f(x), \tag{3.37}$$

and therefore (3.36) yields

$$Q_n f(x) = \mathcal{P}_1\mathcal{P}_0^{n-1} f(x) = \mathcal{S}^{n-1}\, \mathcal{P}_1\, f(x). \tag{3.38}$$

More generally, for $z \in \mathbb{C}$, we shall consider a formal operator-valued power series $\mathcal{Q}_z$ defined by

$$\mathcal{Q}_z f(x) = \sum_{n=1}^{\infty} z^{\tau(\Phi_n(x))} |\Phi_n'(x)| \cdot f(\Phi_n(x)) = z\, \mathcal{P}_1 (1 - z\mathcal{P}_0)^{-1} f(x) \tag{3.39}$$

so that $\mathcal{Q}_1 \equiv \mathcal{Q}$. The following operator relations are in force and are independent of the function space the operators are acting on.

Proposition 3.1. *Let $z \in \mathbb{C}$ be such that (3.39) is absolutely convergent. Then we have*

$$(1 - \mathcal{Q}_z)(1 - z\mathcal{P}_0) = 1 - z\mathcal{P} \tag{3.40}$$

and

$$(1 - z\,\mathcal{S})(1 - \mathcal{Q}_z) = 1 - z\,\tilde{\mathcal{P}}\,. \tag{3.41}$$

where $\tilde{\mathcal{P}} = \mathcal{S} + \mathcal{P}_1$.

Remark 1. As already remarked in section 1, an inducing procedure closely related to that used here and leading to the study of the operator-valued function $\mathcal{M}_z = (1 - z\mathcal{P}_0)^{-1} z\mathcal{P}_1$ has been introduced in Prellberg's thesis [P1] (see also [PS] and [D]), where algebraic identities closely related to those stated above have been used to achieve a deep understanding of the thermodynamic formalism for intermittent interval maps. Notice that $\mathcal{P}_1\, \mathcal{M}_z = \mathcal{Q}_z\, \mathcal{P}_1$. Similar constructions have been used in [HI] and [Is].

Proof of proposition 3.1. Using the first identity in (3.38), we get

$$\begin{aligned}(1-\mathcal{Q}_z)(1-z\mathcal{P}_0) &= \left(1-\sum_{n=1}^{\infty} z^n \mathcal{P}_1\mathcal{P}_0^{n-1}\right)(1-z\mathcal{P}_0)\\ &= 1-z\mathcal{P}_0-\sum_{n=1}^{\infty} z^n\mathcal{P}_1\mathcal{P}_0^{n-1}+\sum_{n=1}^{\infty} z^{n+1}\mathcal{P}_1\mathcal{P}_0^{n}\\ &= 1-z\mathcal{P}_0-z\mathcal{P}_1 = 1-z\mathcal{P}.\end{aligned}$$

In a similar way, using the second identity in (3.38) one shows (3.41). □

Corollary 3.1. *Let $z \neq 0$ be such that (3.39) is absolutely convergent and assume that the kernel of $1 - z\mathcal{P}_0$ is empty. Then 1 is an eigenvalue of $\mathcal{Q}_z$ if and only if z^{-1} is an eigenvalue both of $\mathcal{P}$ and $\tilde{\mathcal{P}}$, and they have the same geometric multiplicity. Furthermore, the corresponding eigenfunctions e_z of $\mathcal{P}$ and h_z of $\tilde{\mathcal{P}}$ and $\mathcal{Q}_z$ are related by $h_z = (1 - z\mathcal{P}_0)e_z$ or else $e_z = \sum_{k=0}^{\infty} z^k \mathcal{P}_0^k h_z$.*

Proof. Assume that $\mathcal{Q}_z h_z = h_z$. From (3.40) it then follows that $(1 - z\mathcal{P}) \sum_{k=0}^{\infty} z^k \mathcal{P}_0^k h_z = 0$. Conversely, assume that $z\mathcal{P}e_z = e_z$, then we have $(1 - \mathcal{Q}_z)(1 - z\mathcal{P}_0)e_z = 0$. In the same way, from (3.41) it follows that $\mathcal{Q}_z h_z = h_z$ if and only if $\tilde{\mathcal{P}} h_z = z^{-1} h_z$. □

Remark 2. As it will be clear in the following the condition on the emptyness of the kernel of $1 - z\mathcal{P}_0$ is plainly satisfied in the function space $\mathcal{H}_0$ considered below (cf (3.69)).

Remark 3. Setting $z = 1$ in proposition 3.39 we recover (9) with $e \equiv e_1$ and $h \equiv h_1$. In particular, we see that the Gauss probability density (2.19) is a fixed point both of $\mathcal{Q}$ and $\tilde{\mathcal{P}}$.

Having fixed an open connected domain $\Omega \subset \mathbb{C}$, let $\mathcal{H}(\Omega)$ be the Fréchet space of functions which are holomorphic in Ω with the topology generated by the family of sup norms on compact subsets of Ω. Moreover, let $A_\infty(\Omega) \subset \mathcal{H}(\Omega)$ denote the Banach space given by the subset of functions in $\mathcal{H}(\Omega)$ having continuous extension to $\bar{\Omega}$, endowed with the norm

$$\|f\| = \sup_{w\in\bar{\Omega}} |f(w)|, \tag{3.42}$$

where $w = x + \mathrm{i}y$. Let first $\mathcal{Q}_z$ act on the Banach space $A_\infty(D)$ with $D = \{w \in \mathbb{C} : |w - 1| < 1\}$. It is easy to verify that $\Phi_n(\bar{D}) \subset D$ for all $n \in \mathbb{N}$. Standard arguments (see [Ma2]) then imply that whenever the power series in (3.39) is uniformly convergent, $\mathcal{Q}_z$ defines a nuclear operator of order zero on $A_\infty(D)$.

Lemma 3.2. *The power series of $\mathcal{Q}_z : A_\infty(D) \to A_\infty(D)$ has radius of convergence bounded from below by 1 and, moreover, it converges absolutely at every point of the unit circle.*

Proof. The radius of convergence of $\mathcal{Q}_z$ is $\lim_{n\to\infty} \|Q_n\|^{-1/n}$ (here $\|\ \|$ denotes the operator norm as well). We have $\sup_{w\in\bar{D}} |Q_n f(w)| \leqslant C\, n^{-2}\, \|f\|$ and therefore $\|Q_n\| \leqslant C\, n^{-2}$. □

We now introduce a subspace of $A_\infty(D)$ on which the action of $\mathcal{Q}_z$ will turn out to be particularly expressive. This is achieved via a generalized Laplace transform.

Definition 3.1. *Let $\mathcal{H}_1$ denote the Hilbert space of all complex-valued functions f which have a representation as generalized Laplace transform*

$$f(w) = (\mathcal{L}[\varphi])(w) := \int_0^\infty \mathrm{e}^{-tw}\, \varphi(t)\, \mathrm{d}m(t), \tag{3.43}$$

where $\varphi \in L_2(m)$ and $\mathrm{d}m$ *is the measure on* $\mathbb{R}^+$ *given by*

$$\mathrm{d}m(t) = \frac{t}{e^t - 1}\,\mathrm{d}t. \tag{3.44}$$

As a Hilbert space $\mathcal{H}_1$ *is endowed with the inner product*

$$(f_1, f_2) = \int_0^\infty \overline{\varphi_1(t)}\,\varphi_2(t)\,\mathrm{d}m(t) \qquad \textit{if } f_i = \mathcal{L}\,[\varphi_i]. \tag{3.45}$$

Remark 4. Putting $z = 1$ we see that the G-invariant density h can be represented as $h = (\log 2)^{-1}\,\mathcal{L}((1 - \mathrm{e}^{-t})/t)$.

The following proposition generalizes corresponding results obtained by Mayer and Roepstorff (see [MaR1], [MaR2]) for the operator $\mathcal{Q}$.

Proposition 3.2. *For each $z \neq 0$ with $|z| \leqslant 1$, the space $\mathcal{H}_1$ is invariant under $\mathcal{Q}_z$. More precisely, we have*

$$\mathcal{Q}_z\mathcal{L}[\varphi] = \mathcal{L}[z\,(1 - M)(1 - zM)^{-1}\,\mathcal{K}\varphi], \tag{3.46}$$

where $M : L_2(m) \to L_2(m)$ is the multiplication operator

$$M\varphi(t) = \mathrm{e}^{-t}\varphi(t) \tag{3.47}$$

and $\mathcal{K} : L_2(m) \to L_2(m)$ is the integral operator

$$(\mathcal{K}\varphi)(t) = \int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}}\,\varphi(s)\,\mathrm{d}m(s) \tag{3.48}$$

and J_p denotes the Bessel function of order p.

Proof. Letting $f = \mathcal{L}\,[\varphi]$, we have from (3.39) and (3.38)

$$\mathcal{Q}_z f(w) = \sum_{n=1}^{\infty} \frac{z^n}{(w+n)^2} \int_0^\infty \mathrm{d}m(t)\,\mathrm{e}^{-t/(w+n)}\,\varphi(t). \tag{3.49}$$

Clearly, for $|z| \leqslant 1$, the sum $\sum_{n=1}^{\infty}(z^n/(w+n)^2)\mathrm{e}^{-t/(w+n)}$ is uniformly convergent in $t \in \mathbb{R}^+$. Therefore, interchanging summation and integration we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{z^n}{(w+n)^2}\,\mathrm{e}^{-t/(w+n)} &= \sum_{k\geqslant 0} \frac{(-t)^k}{k!} \sum_{n=1}^{\infty} \frac{z^n}{(w+n)^{2+k}} \\ &= \sum_{k\geqslant 0} \frac{(-t)^k}{k!}\,z\,\Phi(z, k+2, w+1),\end{aligned} \tag{3.50}$$

where $\Phi(z, a, b) = \sum_{n=0}^{\infty} z^n/(b+n)^a$ is the Lerch transcendental function which, for $\Re a > 1$, possesses the integral representation

$$q\Phi(z, a, b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^a} = \frac{1}{\Gamma(a)} \int_0^\infty \frac{s^{a-1}\mathrm{e}^{-(b-1)s}}{\mathrm{e}^s - z}\,\mathrm{d}s. \tag{3.51}$$

This yields

$$\Phi(z, k+2, w+1) = \frac{1}{(k+1)!} \int_0^\infty \frac{s^{k+1}\mathrm{e}^{-ws}}{\mathrm{e}^s - z}\,\mathrm{d}s\,\cdot \tag{3.52}$$

Noting that

$$\sum_{k\geqslant 0} \frac{(-st)^k}{(k+1)!\,k!} = \frac{J_1(2\sqrt{st})}{\sqrt{st}}, \tag{3.53}$$

where $J_1(x)$ is the Bessel function of the first kind, we have thus found that

$$
\begin{aligned}
\mathcal{Q}_z f(w) &= \int_0^\infty \mathrm{d}s\, \frac{zs}{\mathrm{e}^s - z}\, \mathrm{e}^{-ws} \int_0^\infty \mathrm{d}m(t)\, \frac{J_1(2\sqrt{st})}{\sqrt{st}} \varphi(t) \\
&= \int_0^\infty \mathrm{d}m(s)\, \mathrm{e}^{-ws} (z\,(1-M)(1-zM)^{-1}\, \mathcal{K}\varphi)(s) \\
&= (\mathcal{L}\,[z\,(1-M)(1-zM)^{-1}\, \mathcal{K}\varphi])(w).
\end{aligned} \tag{3.54}
$$

Notice that for each $t \in \mathbb{R}^+$ the function $J_1(2\sqrt{st})/\sqrt{st}$ is uniformly bounded and continuous for $s \in \mathbb{R}^+$. It is then an easy task to verify that for $\varphi \in L_2(m)$ and for $|z| \leqslant 1$ the function $(1-M)(1-zM)^{-1}\,\mathcal{K}\varphi$ is in $L_2(m)$ as well. □

Remark 5. As already remarked in [MaR1], the integral operator $\mathcal{K}$ is symmetric. Therefore, the above proposition with $z = 1$ yields $\mathrm{sp}\,(\mathcal{Q}) \subset \mathbb{R}$.

But we can say more. Indeed, the operator $(1 - zM)$ is invertible in $L_2(m)$ with bounded inverse provided $1/z \notin [0, 1]$. Therefore, for any $\varphi \in L_2(m)$ the integral in (3.46) converges uniformly in any compact region of the complex z-plane not containing points of the ray $(1, +\infty)$. Moreover, it has been proved in [MaR1] that the operator $\mathcal{K}$ is compact (actually trace-class) in $L_2(m)$. Therefore, as long as $(1 - zM)$ has bounded inverse the operator $(1-M)(1-zM)^{-1}\,\mathcal{K}$ is compact as well (being the composition of a compact operator with a bounded operator). Proposition 3.2 and the above observations prove the following result.

Theorem 3.1. *The operator-valued function $z \to \mathcal{Q}_z$, when acting on $\mathcal{H}_1$, can be analytically continued to the entire z-plane with a cross cut along the ray $(1, +\infty)$, and for each z in this domain is isomorphic to the operator*

$$
\mathcal{K}_z := z\,(1-M)(1-zM)^{-1}\,\mathcal{K} \tag{3.55}
$$

acting on $L_2(m)$. They are both compact operators.

Remark 6. The relevance of the above result issues from the following observation: the spectral radius of $\mathcal{P}$ in any reasonable Banach space of functions is equal to 1 (see [Ba1], [C]) so that according to proposition 3.1 there are no z-values with $|z| < 1$ such that 1 is an eigenvalue of $\mathcal{Q}_z$. Therefore, if we aim to exploit the identities in proposition 3.1 in order to investigate the spectrum of $\mathcal{P}$ (when acting upon a suitable function space, see below) it is necessary to have some analytic continuation of $\mathcal{Q}_z$ outside the unit disk. We point out that proposition 3.1 and corollary 3.1 remain valid when $\mathcal{Q}_z$ is analytically continued across the cut $(1, +\infty)$.

Remark 7. Putting

$$
H_\delta := \{w \in \mathbb{C} : \Re w > \delta\}, \tag{3.56}
$$

one sees that a function $f = \mathcal{L}\,[\varphi\,]$ with $\varphi \in L_2(m)$ can be extended to a function holomorphic in the half-plane $H_{-1/2}$.

If, in addition, f is an eigenfunction corresponding to a non-zero eigenvalue λ of $\mathcal{Q}_z$ in $\mathcal{H}_1$, for some non-zero $z \in \mathbb{C} \setminus (1, \infty)$, then

$$
\lambda\,\varphi(t) = (\mathcal{K}_z\,\varphi)(t) = \left(\frac{1 - \mathrm{e}^{-t}}{1/z - \mathrm{e}^{-t}}\right) \int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}}\, \varphi(s)\, \mathrm{d}m(s). \tag{3.57}
$$

Since the integral on the right-hand side is bounded for all $t \in [0, \infty)$ the function $\varphi(t)$ is bounded as well in this domain and therefore f is holomorphic in the half-plane H_{-1}.

Putting together proposition 3.1 along with standard arguments (see [DS], chapter VII) we get the following corollary.

Corollary 3.2. *The operator-valued function $z \to (1-\mathcal{Q}_z)^{-1}$, when acting on $\mathcal{H}_1$, is analytic in the open unit disk $\{z : |z| < 1\}$ and can be meromorphically continued to the entire z-plane with a cross cut along the ray $[1, +\infty)$. It has a pole whenever $\mathcal{K}_z$ has 1 as an eigenvalue.*

Now, from proposition 3.1 we obtain the following formal relation for the resolvent $\mathcal{R}_\lambda$ of $\mathcal{P}$:

$$\mathcal{R}_\lambda \equiv (\lambda - \mathcal{P})^{-1} = (\lambda - \mathcal{P}_0)^{-1}(1 - \mathcal{Q}_{1/\lambda})^{-1}. \tag{3.58}$$

The analytic properties of the first factor on the right-hand side can be understood in terms of the spectrum of the operator $\mathcal{P}_0$ when acting on a suitable function space invariant under the action of $\mathcal{P}$. A calculation along the same lines as in the proof of proposition 3.2 shows that, for $f \in \mathcal{H}_1$ with $f = \mathcal{L}[\varphi]$,

$$(1 - z\,\mathcal{P}_0)^{-1} f(w) = \frac{1}{w^2}\int_0^\infty \mathrm{e}^{-t/w}\,\mathrm{e}^{t}\, z^{-1}\,(\mathcal{K}_z\varphi)(t)\,\mathrm{d}m(t). \tag{3.59}$$

We shall therefore characterize the space $\mathcal{H}_0$ to be acted on by $\mathcal{P}$ as follows.

Definition 3.2. *We denote by $\mathcal{H}_0$ the Hilbert space of all complex-valued functions f which can be represented as a generalized Borel transform:*

$$f(w) = (\mathcal{B}[\varphi])(w) := \frac{1}{w^2}\int_0^\infty \mathrm{e}^{-t/w}\,\mathrm{e}^{t}\,\varphi(t)\,\mathrm{d}m(t), \qquad \varphi \in L_2(m), \tag{3.60}$$

endowed with the inner product

$$(f_1, f_2) = \int_0^\infty \overline{\varphi_1(t)}\,\varphi_2(t)\,\mathrm{d}m(t), \qquad \textit{if } f_i = \mathcal{B}[\varphi_i]. \tag{3.61}$$

Remark 8. A function $f \in \mathcal{H}_0$ is holomorphic in the disk

$$D_1 = \left\{ w \in \mathbb{C} : \Re\frac{1}{w} > \frac{1}{2} \right\} = \{ w \in \mathbb{C} \,:\, |w-1| < 1 \}. \tag{3.62}$$

For w real and positive, a simple change of variable makes (3.60) in the form

$$f(w) = \frac{1}{w}\int_0^\infty \mathrm{e}^{-s}\,\psi(sw)\,\mathrm{d}s \qquad \text{with } \psi(t) = \left(\frac{t}{1-\mathrm{e}^{-t}}\right)\varphi(t)\,. \tag{3.63}$$

Remark 9. The F-invariant density e (see (2.13)) can be represented as

$$e = \left(\frac{1}{\log 2}\right)\mathcal{B}\left(\frac{1-\mathrm{e}^{-t}}{t}\right), \tag{3.64}$$

whereas for the G-invariant density h, we have (see also remark 4)

$$h = \left(\frac{1}{\log 2}\right)\mathcal{L}\left[\frac{1-\mathrm{e}^{-t}}{t}\right] = \left(\frac{1}{\log 2}\right)\mathcal{B}\left[\frac{(1-\mathrm{e}^{-t})^2}{t}\right]. \tag{3.65}$$

In the representation of remark 8 we have that if $f = e\log 2$ then $\psi(t) \equiv 1$ whereas for $f = h\log 2$ we find $\psi(t) = 1 - \mathrm{e}^{-t}$. Both these functions can be viewed as ordinary Borel transforms of a sequence $\{a_n\}_{n=0}^\infty$, i.e. $\psi(t) = \sum_{n=0}^\infty t^n a_n/n!$ so that by (3.63) we have $wf(w) = \sum_{n=0}^\infty w^n a_n$. In the former case we find $a_0 = 1$ and $a_n = 0$ for $n > 0$, in the latter $a_0 = 0$ and $a_n = (-1)^{n-1}$ for $n > 0$. Therefore in both cases the integral (3.63) provides a continuation of $wf(w)$ outside the disk D_1 (see [Tit1], p 164).

We now have the following lemma.

Lemma 3.3. *For all $\varphi \in L_2(m)$,*

$$\mathcal{L}[\,\varphi] = \mathcal{B}[\,(1-M)\,\mathcal{K}\,\varphi], \tag{3.66}$$

where $M\varphi(t) = \mathrm{e}^{-t}\varphi(t)$ and $\mathcal{K}$ is the symmetric integral operator defined in (3.48).

Proof. The proof is an easy calculation based on Tricomi's theorem (see [Sne], p 165):

$$\frac{1}{u^{p+1}}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t/u}\varphi(t) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tu}\int_0^\infty \mathrm{d}s\, \left(\frac{t}{s}\right)^{p/2} J_p(2\sqrt{st})\,\varphi(s), \tag{3.67}$$

with $p = 1$, and therefore we omit it. □

It is now not difficult to verify that

$$\mathcal{P}_1\,\mathcal{B}[\varphi] = \mathcal{L}\,[\varphi], \tag{3.68}$$

and

$$\mathcal{P}_0\,\mathcal{B}[\varphi] = \mathcal{B}\,[\,M\varphi\,]. \tag{3.69}$$

In addition, we have

$$\mathcal{S}\mathcal{L}[\varphi] = \mathcal{L}\,[M\varphi], \tag{3.70}$$

so that

$$\mathcal{P}_1\,\mathcal{P}_0^{n-1}\,\mathcal{B}[\varphi] = \mathcal{S}^{n-1}\,\mathcal{P}_1\mathcal{B}[\varphi] = \mathcal{L}\,[\,M^{n-1}\varphi\,], \tag{3.71}$$

and therefore

$$\mathcal{Q}_z\mathcal{B}[\varphi] = z\mathcal{L}\,[\,(1 - zM)^{-1}\varphi]. \tag{3.72}$$

We thus see that $\mathcal{P}_0$ leaves $\mathcal{H}_0$ invariant and by (3.70) its spectral properties in $\mathcal{H}_0$ are identical to those of $\mathcal{S}$ in $\mathcal{H}_1$. Moreover $\mathcal{P}_1$ maps $\mathcal{H}_0$ into $\mathcal{H}_1 \subset \mathcal{H}_0$, and the same does $\mathcal{Q}_z$ for all $z \in \mathbb{C} \setminus (1, +\infty)$. Notice that using lemma 3.3 and (3.72) we immediately recover proposition 3.2, in that

$$\mathcal{Q}_z\mathcal{L}[\varphi] = \mathcal{Q}_z\mathcal{B}[(1 - M)\,\mathcal{K}\varphi] = \mathcal{L}\,[z(1 - zM)^{-1}(1 - M)\,\mathcal{K}\varphi] \equiv \mathcal{L}\,[\mathcal{K}_z\varphi]. \tag{3.73}$$

We are now in the position to write explicit representations for $\mathcal{P}$ and its resolvent $\mathcal{R}_\lambda$ in the space $\mathcal{H}_0$.

Theorem 3.2. *Let $f \in \mathcal{H}_0$, i.e. $f = \mathcal{B}\,[\varphi]$ for some $\varphi \in L_2(m)$, then*

$$\mathcal{P}f = \mathcal{B}\,[\,(M + (1 - M)\mathcal{K}\,)\varphi\,], \tag{3.74}$$

and

$$\mathcal{R}_\lambda f \equiv (\lambda - \mathcal{P})^{-1}f\, = \mathcal{B}\,[\,(1 - \mathcal{K}_{1/\lambda})^{-1}(\lambda - M)^{-1}\varphi]. \tag{3.75}$$

Remark 10. Note that for $\varphi \in L_2(m)$, the functions

$$M\,\varphi \qquad \text{and} \qquad (1 - M)\mathcal{K}\varphi \tag{3.76}$$

are bounded at infinity and therefore, by (3.74), the function $\mathcal{P}f$ with $f = \mathcal{B}\,[\varphi]$ is analytic in the half-plane H_0. In particular so is any eigenfunction of $\mathcal{P}$ in $\mathcal{H}_0$.

Proof of theorem 3.2. From (3.69) and (3.68) one obtains $\mathcal{P}f = \mathcal{B}[M\varphi] + \mathcal{L}\,[\varphi]$, so that (3.74) follows using lemma 3.3. The expression for $\mathcal{R}_\lambda$ can now be obtained directly from (3.74). But we can also make use of (3.72) and (3.54) to obtain, for a given $f = \mathcal{B}\,[\varphi]$,

$$\mathcal{Q}^n_{1/\lambda}\, f = \mathcal{L}[\mathcal{K}^{n-1}_{1/\lambda}(\lambda - M)^{-1}\varphi] \tag{3.77}$$

and therefore

$$(1 - \mathcal{Q}_{1/\lambda})^{-1}f = \mathcal{B}\,[\varphi] + \mathcal{L}[(1 - \mathcal{K}_{1/\lambda})^{-1}\,(\lambda - M)^{-1}\varphi]. \tag{3.78}$$

This expression along with (3.58), (3.59) and (3.69) yield

$$\begin{aligned}\mathcal{R}_\lambda f &= \mathcal{B}\,[\,(\lambda - M)^{-1}\,\varphi\,] + \mathcal{B}\,[\,\mathcal{K}_{1/\lambda}(1 - \mathcal{K}_{1/\lambda})^{-1}\,(\lambda - M)^{-1}\,\varphi\,]\\ &= \mathcal{B}\,[\,(1 - \mathcal{K}_{1/\lambda})^{-1}(\lambda - M)^{-1}\varphi\,].\end{aligned}$$

Using corollary 3.2 we see that $\mathcal{R}_\lambda$ extends to a meromorphic (operator-valued) function in $\bar{\mathbb{C}} \setminus [0, 1]$. □

The next theorem (partially) describes the spectrum of $\mathcal{P}$ in $\mathcal{H}_0$.

Theorem 3.3. *The spectrum of the operator* $\mathcal{P} : \mathcal{H}_0 \to \mathcal{H}_0$ *is the union of* $[0, 1]$ *and a finite or countably infinite set of eigenvalues of finite multiplicity.*

Proof. By theorem 3.2 the action of transfer operator $\mathcal{P}$ on $\mathcal{H}_0$ can be explicitly expressed in the form

$$\mathcal{P}\mathcal{B}[\varphi] = \mathcal{B}[T\varphi], \tag{3.79}$$

with

$$(T\varphi)(t) := \mathrm{e}^{-t}\varphi(t) + \int_0^\infty K(s,t)\varphi(s)\,\mathrm{d}s \tag{3.80}$$

and

$$K(s,t) = \mathrm{e}^{-t}\left(\frac{\mathrm{e}^t - 1}{\mathrm{e}^s - 1}\right)\sqrt{\frac{s}{t}}\, J_1(2\sqrt{st}). \tag{3.81}$$

It is an easy exercise to check that M when acting upon $L_2(m)$ is self-adjoint and its spectrum is the line segment $[0, 1] = \mathrm{Cl}\,\{\mathrm{e}^{-t} \ : \ t \in \mathbb{R}^+\}$ (see, e.g. [DeV]). Therefore, the spectrum of $\mathcal{P}$ in $\mathcal{H}_0$ is given by a compact perturbation of the continuous spectrum $\sigma_c = [0, 1]$. The assertion is now a consequence of theorem 5.2 in [GK]. □

We shall now characterize some properties of the eigenfuctions of $\mathcal{P}$ in $\mathcal{H}_0$. First, it is easy to see that $\lambda = 0$ is an eigenvalue of infinite multiplicity. This follows by noting that (see (2.3) and (3.34)) any function $f \in \mathcal{H}_0$ which is odd with respect to $x = \frac{1}{2}$, e.g. $f(w) = 1 - 2w = \mathcal{B}[(1-t)(1-\mathrm{e}^{-t})]$, lies in the kernel of $\mathcal{P}$.

Now suppose that $\mathcal{P}f = \lambda f$ for some $f \in \mathcal{H}_0$ and $\lambda \neq 0$, or explicitly

$$\lambda\, f(w) = \left(\frac{1}{w+1}\right)^2 \left[f\left(\frac{w}{w+1}\right) + f\left(\frac{1}{w+1}\right)\right]. \tag{3.82}$$

By remark 10 $f(w)$ extends analytically to the half-plane H_0. If we transform this equation by substituting $1/w$ for w and then dividing through w^2, we get

$$\lambda\, w^{-2} f\left(\frac{1}{w}\right) = \left(\frac{1}{w+1}\right)^2 \left[f\left(\frac{1}{w+1}\right) + f\left(\frac{w}{w+1}\right)\right]. \tag{3.83}$$

Therefore, f satisfies

$$wf(w) = \frac{1}{w} f\left(\frac{1}{w}\right) \tag{3.84}$$

for all $w \in H_0$. Note that applying (3.84) to each term on the right-hand side (3.82), one obtains

$$\lambda\, w\, f(w) = w\, f(w+1) + \frac{1}{w} f\left(1 + \frac{1}{w}\right). \tag{3.85}$$

For $\lambda = 1$ this yields $wf(w) = 1$. Note that for $f = \mathcal{B}[\varphi]$, we have

$$w^{-2} f\left(\frac{1}{w}\right) = \int_0^\infty \mathrm{e}^{-t\,w}\,\mathrm{e}^t\,\varphi(t)\,\mathrm{d}m(t) = \mathcal{B}[(1-M)\mathcal{K}M^{-1}\varphi]. \tag{3.86}$$

Therefore, the functional equation (3.84) can be written as

$$(1-M)\mathcal{K}M^{-1}\varphi = \varphi. \tag{3.87}$$

Now, given a continuous function ψ on $\mathbb{R}^+$, one can define (a version of) its *Hankel transform* (of order 1) as the integral

$$(\mathcal{J}\psi)(t) = \int_0^\infty J_1(2\sqrt{st})\sqrt{\frac{t}{s}}\,\psi(s)\,\mathrm{d}s. \tag{3.88}$$

From the estimates $J_1(t) \sim t$ as $t \to 0^+$ and $J_1(t) = \mathrm{O}(t^{-1/2})$ as $t \to \infty$ ([E], vol II), we see that the conditions on ψ sufficient to give the absolute convergence of the integral (3.88) are $\psi(t) = \mathrm{O}(t^{-\beta})$ as $t \to \infty$ with $\beta > -\frac{1}{4}$ and $\psi(t) = \mathrm{O}(t^{\alpha})$ as $t \to 0^+$ with $\alpha > -1$. The identity (3.87) then says that the function (cf remark 8)

$$\psi(t) = \left(\frac{t}{1 - \mathrm{e}^{-t}}\right)\varphi(t) \tag{3.89}$$

satisfies

$$\psi(t) = \int_0^\infty J_1(2\sqrt{st})\sqrt{\frac{t}{s}}\,\psi(s)\,\mathrm{d}s. \tag{3.90}$$

Note that the simplest solution of this equation is $\psi \equiv 1$ and corresponds to $f = e$ (more general self-reciprocal functions satisfying equations related to (3.90) are discussed, for example, in the book [Tit2]). Furthermore, putting together (3.84), (3.86) and (3.89) we have

$$f(w) = \int_0^\infty \mathrm{e}^{-t\,w}\psi(t)\,\mathrm{d}t \tag{3.91}$$

for all $w \in H_0$. Finally, one easily checks that if $\varphi \in L_2(m)$, then $\psi \in L_2(\hat{m})$ where

$$\mathrm{d}\hat{m}(t) = \frac{\mathrm{e}^{-t}(1 - \mathrm{e}^{-t})}{t\log 2}\,\mathrm{d}t.$$

We summarize the above in the following theorem.

Theorem 3.4. *If $f \in \mathcal{H}_0$ satisfies $\mathcal{P}f = \lambda f$ for some $\lambda \neq 0$ then f is the (ordinary) Laplace transform of a function $\psi \in L_2(\hat{m})$ which is self-reciprocal with respect to Hankel transform of order* 1*, namely f and ψ satisfy (3.91) and (3.90), respectively.*

Now from corollary 3.1 we know that a function $f = \mathcal{B}[\varphi]$ satisfies $\mathcal{P}f = \lambda f$ if and only if (the analytic continuation of) $\mathcal{K}_{1/\lambda} : L_2(m) \to L_2(m)$ satisfies $\mathcal{K}_{1/\lambda}\varphi = \varphi$, which can also be written as

$$(\mathcal{K}\varphi)(t) = \frac{\lambda - \mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}}\,\varphi(t) = \frac{\lambda - \mathrm{e}^{-t}}{t}\,\psi(t). \tag{3.92}$$

Expressing the integral operator $\mathcal{K}$ in terms of the Hankel transform (3.88), we get $(\mathcal{K}\varphi)(t) = (1/t)\,\mathcal{J}(\exp_{-1}\psi)(t)$, where we have defined the function $\exp_c : \mathbb{R} \to \mathbb{R}$ by $\exp_c(t) = \mathrm{e}^{ct}$. Identities (3.90) and (3.92) then yield the integral equation

$$\mathcal{J}(\exp_{-1}\psi) = (\lambda - \exp_{-1})\mathcal{J}\psi. \tag{3.93}$$

Once more, $\psi \equiv 1$ satisfies this equation with $\lambda = 1$ (recall that $\mathcal{J}\exp_{-1} = 1 - \exp_{-1}$). On the other hand, the above discussion suggests that there are no $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that (3.93) has a (non-constant) solution $\psi \in L_2(\hat{m})$. We thus are led to formulate the following conjecture.

Conjecture 1. *The only (non-zero) eigenvalue of $\mathcal{P} : \mathcal{H}_0 \to \mathcal{H}_0$ is $\lambda = 1$.*

We end this section with two additional remarks.

Remark 11. (3.92) is a particular case of the Lewis functional equation:

$$f(w) - f(w+1) = \frac{1}{w^{2(q+1)}} f\left(1 + \frac{1}{w}\right), \tag{3.94}$$

which is related to the so-called Maass cusp forms, i.e. PSL(2, $\mathbb{Z}$)-invariant eigenfunctions of the Laplacian on the Poincaré upper half-plane which vanish at the cusp (see [Le]). Another type of functions equivalent to (even) Maass forms and considered in [Le] are those satisfying an integral equation which in our notation can be written as

$$g(t) = \int_0^\infty \frac{J_{2q+1}(2\sqrt{st})}{\sqrt{st}} \left(\frac{s}{t}\right)^q g(s)\,\mathrm{d}m(s). \tag{3.95}$$

By the foregoing (see remark 9) we see that for $q = 0$ we have the relation

$$f = \mathcal{B}[g]. \tag{3.96}$$

Remark 12. In the recent work [P2], following [P1] ten years later and somehow inspired by the construction presented here, Thomas Prellberg has studied the spectrum of (a generalized version of) $\mathcal{P}$ in a space of functions which is identical to $\mathcal{H}_0$ with the exception that the measure on $\mathbb{R}^+$ is slightly different from (3.44), being given by

$$\mathrm{d}\tilde{m}(t) = t\,\mathrm{e}^{-t}\,\mathrm{d}t. \tag{3.97}$$

It is easy to see that with this new measure the operator $\mathcal{Q}_z$ is isomorphic under generalized Laplace transform (cf theorem 3.1) to $\tilde{\mathcal{K}}_z : L_2(\tilde{m}) \to L_2(\tilde{m})$ given by

$$\tilde{\mathcal{K}}_z = z\,(1 - zM)^{-1}\tilde{\mathcal{K}}, \tag{3.98}$$

where

$$\tilde{\mathcal{K}}\varphi(s) = \int_0^\infty \mathrm{d}\tilde{m}(t)\,\frac{J_1(2\sqrt{st})}{\sqrt{st}}\,\varphi(t). \tag{3.99}$$

Notice that $\mathcal{K}_1 = (1 - M)^{-1}\tilde{\mathcal{K}}$ which is not symmetric anymore (cf remark 5). On the other hand, the relation given by lemma 3.3 now writes (we keep using the symbols $\mathcal{L}$ and $\mathcal{B}$ to denote generalized Laplace and Borel transforms with respect to the measure $\tilde{m}$):

$$\mathcal{L}[\varphi] = \mathcal{B}[\tilde{\mathcal{K}}\varphi] \tag{3.100}$$

and hence the integral representation of $\mathcal{P}$ becomes

$$\mathcal{P}\mathcal{B}[\varphi] = \mathcal{B}[(M + \tilde{\mathcal{K}})\varphi], \tag{3.101}$$

which is now symmetric (cf (3.74)). Thus, everything goes as if the operators $\mathcal{P}$ and $\mathcal{Q}$ were not 'symmetrizable' both at the same time. Also notice that the function $\log 2 \cdot e$ if expressed as a generalized Borel transform now yields the function $\varphi(s) = 1/s$ which is not in $L_2(\tilde{m})$.

4. ζ-functions

We now consider the dynamical ζ-functions ζ_{F} and ζ_{G} associated to the maps F and G, respectively, and defined by the following formal series [Ba2]:

$$\zeta_{\mathrm{F}}(z) = \exp\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(F) \qquad \text{and} \qquad \zeta_{\mathrm{G}}(s) = \exp\sum_{n=1}^{\infty} \frac{s^n}{n} Z_n(G), \tag{4.102}$$

where the 'partition functions' $Z_n(F)$ and $Z_n(G)$ are given by

$$\begin{aligned} Z_n(F) &= \sum_{x=F^n(x)} \prod_{k=0}^{n-1} \frac{1}{|F'(F^k(x))|} \\ Z_n(G) &= \sum_{x=G^n(x)} \prod_{k=0}^{n-1} \frac{1}{|G'(G^k(x))|}. \end{aligned} \tag{4.103}$$

Let us first examine how $\zeta_{\mathrm{F}}(z)$ and $\zeta_{\mathrm{G}}(z)$ are related to one another. Let Per F and Per G denote the sets of all periodic points of the maps F and G, respectively. It is not difficult to realize that, as subsets of [0, 1], Per $F \setminus \{0\} =$ Per G. Accordingly, given x in either of these sets, let $p_F(x)$ and $p_G(x)$ denote the periods of x with respect to F and G, respectively. They are related by

$$p_F(x) = \tau(x) + \tau(G(x)) + \cdots + \tau(G^{p_G(x)-1}(x)). \tag{4.104}$$

Moreover from the definitions of F and G, we have

$$\prod_{k=0}^{p_F(x)-1} \frac{1}{|F'(F^k(x))|} = \prod_{k=0}^{p_G(x)-1} \frac{1}{|G'(G^k(x))|} = \prod_{k=0}^{p_G(x)-1} (G^k(x))^2 \cdot \tag{4.105}$$

Using this fact we write $Z_n(F)$ as follows:

$$Z_n(F) = 1 + \sum_{m=1}^{n} \frac{n}{m} \sum_{x=F^n(x)=G^m(x)} \prod_{k=0}^{m-1} (G^k(x))^2 \cdot \tag{4.106}$$

The second sum ranges over the $\binom{n-1}{m-1}$ ways to write the integer n as a sum of m positive integers. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(F) &= \log\left(\frac{1}{1-z}\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{m} \sum_{x=F^n(x)=G^m(x)} z^n \prod_{k=0}^{m-1} (G^k(x))^2 \\ &= \log\left(\frac{1}{1-z}\right) + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{x=G^l(x)} z^{p_F(x)} \prod_{k=0}^{l-1} (G^k(x))^2 \cdot\end{aligned}$$

We are thus led to study the 'grand partition function' $\Xi_l(z)$ given by

$$\Xi_l(z) := \sum_{x=G^l(x)} z^{p_F(x)} \prod_{k=0}^{l-1} (G^k(x))^2 = \sum_{n=0}^{\infty} z^{l+n} \sum_{x=G^l(x)=F^{l+n}(x)} \prod_{k=0}^{l-1} (G^k(x))^2. \tag{4.107}$$

The sum over periodic points yields

$$\binom{n+l-1}{l-1} = \binom{n+l-1}{n}$$

terms, corresponding to the number of ways of distributing n identical objects into l distinct boxes. According to (2.27), (2.28) and (4.107), we can also write $\Xi_l(z)$ in the following way:

$$\Xi_l(z) = \sum_{n=0}^{\infty} z^{l+n} \sum_{k_1+\cdots+k_l=n+l} \prod_{i=1}^{l} x^2_{k_i,\ldots,k_l k_1,\ldots,k_{i-1}}, \tag{4.108}$$

where $x_{k_1,\ldots,k_l} = [\overline{k_1,\ldots,k_l}]$ denotes the irrational number whose continued fraction expansion is periodic of period l and starts with the entries $k_1, \ldots, k_l$. Putting together the above observations we obtain the next result, to be compared with proposition 3.1.

Proposition 4.3. *Consider the two-variable ζ-function given by*

$$\zeta_2(s, z) = \exp \sum_{l=1}^{\infty} \frac{s^l}{l} \Xi_l(z). \tag{4.109}$$

Then we have

$$\zeta_2(1, z) = (1-z)\,\zeta_{\mathrm{F}}(z) \qquad \text{and} \qquad \zeta_2(s, 1) = \zeta_{\mathrm{G}}(s), \tag{4.110}$$

wherever the series expansions converge absolutely.

In order to study the analytic properties of the function $\zeta_2(s, z)$, we further generalize (3.39) by introducing a family of operator-valued functions $\mathcal{Q}_{z,q}$, $q = 0, 1, \ldots$, acting as (see [Ma1] and [D] for related quantities)

$$\mathcal{Q}_{z,q} f(x) = (-1)^q \sum_{n=1}^{\infty} z^{\tau(\Phi_n(x))} |\Phi_n'(x)|^{1+q} f(\Phi_n(x)), \tag{4.111}$$

together with a family of function spaces $\mathcal{H}_{1,q} \subseteq \mathcal{H}_1$ such that a function $f \in \mathcal{H}_{1,q}$ can be represented as

$$f(w) = (\mathcal{L}_q[\varphi])(w) := \int_0^\infty \mathrm{d}m(t)\, \mathrm{e}^{-tw}\, t^q\, \varphi(t), \qquad \varphi \in L_2(m). \tag{4.112}$$

In particular, $\mathcal{Q}_{z,0} \equiv \mathcal{Q}_z$, $\mathcal{L}_0 \equiv \mathcal{L}$ and $\mathcal{H}_{1,0} \equiv \mathcal{H}_1$. We have the following result.

Proposition 4.4. *For any given $q = 0, 1, \ldots$ the operator-valued function $z \to \mathcal{Q}_{z,q}$ when acting on $\mathcal{H}_{1,q}$ can be analytically continued to the entire z-plane with a cross cut along the ray $(1, +\infty)$. For each z in this domain, we have*

$$\mathcal{Q}_{z,q}\, \mathcal{L}_q\, [\varphi] = \mathcal{L}_q\, [\mathcal{K}_{z,q}\varphi], \tag{4.113}$$

where $\mathcal{K}_{z,q} : L_2(m) \to L_2(m)$ is given by

$$(\mathcal{K}_{z,q}\varphi)(t) := (-1)^q\, z\, (1 - M)(1 - zM)^{-1} \int_0^\infty \mathrm{d}m(s)\, \frac{J_{2q+1}(2\sqrt{st})}{\sqrt{st}} \varphi(s). \tag{4.114}$$

The operators $\mathcal{Q}_{z,q} : \mathcal{H}_{1,q} \to \mathcal{H}_{1,q}$ and $\mathcal{K}_{z,q} : L_2(m) \to L_2(m)$ are both of trace class.

Proof. The first part follows from a straightforward extension to non-zero q values of the arguments of the previous section. The proof of the last assertion can be extracted from ([Ma1], theorem 3). □

Now, the trace of the operator $\mathcal{K}_{z,q}$ is easily obtained (see also [Ma1]):

$$\begin{aligned} \operatorname{tr} \mathcal{K}_{z,q} &= (-1)^q\, z \int_0^\infty \frac{J_{2q+1}(2t)}{\mathrm{e}^t - z}\, \mathrm{d}t \\ &= (-1)^q \sum_{k=1}^{\infty} z^k \int_0^\infty \mathrm{e}^{-kt}\, J_{2q+1}(2t)\, \mathrm{d}t \\ &= (-1)^q \sum_{k=1}^{\infty} z^k\, \frac{x_k^{2(q+1)}}{1 + x_k^2}, \end{aligned} \tag{4.115}$$

where the numbers $x_k = (\sqrt{k^2 + 4} - k)/2 = [k, k, k, \ldots] \equiv [\bar{k}]$ are the fixed points of $G(x)$ and the identity [GR]

$$\int_0^\infty \mathrm{e}^{-kt}\, J_p(2t)\, \mathrm{d}t = \frac{(\sqrt{k^2+4} - k)^p}{2^p\, \sqrt{k^2+4}}, \qquad p = 0, 1, \ldots \tag{4.116}$$

has been used. From (4.115) we immediately obtain the trace formula

$$\Xi_1(z) = \operatorname{tr} \mathcal{K}_{z,0} - \operatorname{tr} \mathcal{K}_{z,1}. \tag{4.117}$$

But we can say more. Indeed, a straightforward adaptation of ([Ma1], corollaries 4 and 5) to our z-dependent situation leads to the following general expressions:

$$\Xi_l(z) = \operatorname{tr} \mathcal{K}_{z,0}^l - \operatorname{tr} \mathcal{K}_{z,1}^l = \operatorname{tr} \mathcal{M}_{z,0}^l - \operatorname{tr} \mathcal{M}_{z,1}^l, \tag{4.118}$$

with

$$\operatorname{tr} \mathcal{K}_{z,q}^l = (-1)^{ql} \sum_{k_1,\ldots,k_l=1}^{\infty} z^{k_1+\cdots+k_l}\, \frac{\prod_{i=1}^l x_{k_i,\ldots,k_l k_1,\ldots,k_{l-1}}^{2(q+1)}}{1 - (-1)^l \prod_{i=1}^l x_{k_i,\ldots,k_l k_1,\ldots,k_{l-1}}^2}. \tag{4.119}$$

Formula (4.118) along with standard arguments (see [Ma1]) allow us to write the two-variables ζ-function (4.109) as a ratio of Fredholm determinants:

$$\zeta_2(s,z) = \exp \sum_{l=1}^{\infty} \frac{s^l}{l}\, \Xi_l(z) = \frac{\det(1 - s\,\mathcal{K}_{z,1})}{\det(1 - s\,\mathcal{K}_{z,0})} = \frac{\det(1 - s\,\mathcal{M}_{z,1})}{\det(1 - s\,\mathcal{M}_{z,0})}, \qquad (4.120)$$

where by definition

$$\det(1 - s\,\mathcal{K}_{z,q}) = \exp\left(- \sum_{l=1}^{\infty} \frac{s^l}{l}\, \mathrm{tr}\, \mathcal{K}_{z,q}^l \right) \qquad (4.121)$$

is in the sense of Grothendieck [G]. We have thus proved the following result.

Theorem 4.5. *Set $\mathcal{K}_z \equiv \mathcal{K}_{z,0}$, then we have:*

(a) for each $s \in \mathbb{C}$, the function $\zeta_2(s,z)$, considered as a function of the variable z, extends to a meromorphic function in the cut plane $\mathbb{C} \setminus [1,\infty)$. Its poles are located among those z-values such that $\mathcal{K}_z : L_2(m) \to L_2(m)$ has $1/s$ as an eigenvalue;
(b) for each $z \in \mathbb{C} \setminus (1,\infty)$, the function $\zeta_2(s,z)$, considered as a function of the variable s, extends to a meromorphic function in $\mathbb{C}$. Its poles are located among the inverses of the eigenvalues of $\mathcal{K}_z : L_2(m) \to L_2(m)$.

Putting together the above theorem and proposition 4.3 we obtain the following.

Corollary 4.3. *The dynamical ζ-functions ζ_F and ζ_G of the Farey and Gauss maps have the following properties:*

(a) $\zeta_F(z)$ has a meromorphic extension to the cut plane $\mathbb{C} \setminus [1,\infty)$;
(b) $\zeta_G(s)$ has a meromorphic extension to $\mathbb{C}$. All poles are real and are located among the inverses of the eigenvalues of $\mathcal{K} : L_2(m) \to L_2(m)$.

Remark 13. Statement 1 of corollary 4.3 is akin to corollary 1.3 in [Rug]. On the other hand, the validity of conjecture 1 would imply that $\zeta_F(z)$ is actually analytic in $\mathbb{C} \setminus [1,\infty)$. Statement 2 was proved by Mayer in [Ma3], where he also showed (using results from [Rue]) that the poles of $\zeta_G(s)$, if arranged in increasing absolute values and according to their order, tend to infinity exponentially fast.

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