REPLICA EQUIVALENCE IN THE EDWARDS-ANDERSON MODEL

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Abstract

After introducing and discussing the *link-overlap* between spin configurations we show that the Edwards-Anderson model has a quenched state which fulfills, in the thermodynamic limit, the replica-equivalent ansatz introduced by Parisi in the description of the mean-field spin-glass phase. Our argument is based on the property of stochastic stability.

Preliminaries

In the description of the spin glass phase the standard quantity usually investigated is the *overlap* between Ising spin configurations:

$$q(\sigma, \tau) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tau_i \ . \tag{1}$$

Within the mean-field approach such a quantity gives, in the so called quenched ensemble, a complete description of the system and it is in fact in terms of its distribution properties that the mean-field theory has been developed and understood by Parisi [MPV] and became successively accessible to a rigorous mathematical investigation starting from the seminal paper by Guerra [G]. The mean field picture is described by two main features: first the overlap quenched distribution has a support that includes the neighbor of zero where the disorder is concentrated, second the quenched distribution of the overlaps fulfills factorization-like properties [MPV] and is, in particular, completely identified by the single overlap probability distribution. The factorization properties have been distinguished in two classes, *replica equivalent* and *ultrametric* the second class being a subclass of the first (see [P1], [PRT]); in [AC], [GG] and [C] it has been shown how to derive replica equivalence from elementary thermodynamic properties involving stochastic stability or equivalently boundedness of fluctuations.

In this work we show that a finite dimensional spin glass like the Edwards-Anderson is *replica equivalent* with respect to the *bond-overlap* quenched distribution. The bond-overlap (or link-overlap as it first appeared in [MPRL1]) may be defined as follows. Consider the Edwards-Anderson d-dimensional spin-glass model defined by configurations of Ising spins σ_n, $n \in \Lambda \subset \mathbb{Z}^d$ for some d-cube Λ, interacting via the Hamiltonian

$$H_\Lambda(J,\sigma) = \sum_{(n,n')} J_{n,n'}\sigma_n\sigma'_n \ , \tag{2}$$

where the $J_{n,n'}$ are independent normal Gaussian variables and the sum runs over all the couples of nearest neighbors sites $|n - n'| = 1$. Identifying the space of nearest neighbors with the d-dimensional *bond*-lattice $b \in \mathbb{B}^d$ by $b = (n, n')$, calling $B(\Lambda)$ the d-bond-cube associated to Λ and defining $S_b = \sigma_n\sigma_{n'}$ and $T_b = \tau_n\tau_{n'}$ we introduce the local bond-overlap between σ and τ

$$p_b(\sigma,\tau) = S_bT_b \tag{3}$$

and the global one

$$p(\sigma,\tau) = \frac{1}{|B(\Lambda)|}\sum_{b\in B(\Lambda)} S_bT_b \ . \tag{4}$$

For a discussion of the relevance of bond-overlap and for its use in numerical experiments to study the low temperature phase of finite dimensional Ising spin glasses one may see [MPRLZ] and [MPRL2]. The main advantage of the quantity (4) with respect to the (1) is that while a spin flip inside a bond-connected region $C \subset \Lambda$ for τ changes $q(\sigma,\tau)|\Lambda|$ of an amount proportional to the volume $|C|$ it only changes $p(\sigma,\tau)|B(\Lambda)|$ of an amount proportional to the surface $|\partial C|$. We want to notice that the usual overlap and the bond-overlap are, in the mean field case like the Sherrington-Kirkpatrick model, related by the algebraic formula $q^2 = 2p + 1/N$ so that it is totally irrelevant which one of the two is studied. On the contrary in finite dimension it does not exist such a simple relation among the two quantities even if from a bond configuration $\{S_b\}$ one may reconstruct the spin configuration $\{\sigma_n\}$ (up to a global sign) and viceversa, see for this purpose the treatment of Gauge invariance in [BF] and [N].

But definitely the deeper reason to introduce the bond overlap is related to the mathematical structure of the Hamiltonian (2): being a sum of Gaussian variables it is, for each configuration, a Gaussian variable itself and, for the Wick theorem, completely identified by its covariance matrix which turns out to be proportional to the bond-overlap $p(\sigma,\tau)$. Indicating by Av the Gaussian average we have:

$$\begin{aligned} Av(H_\Lambda(J,\sigma)H_\Lambda(J,\tau)) &= \sum_{(n,n'),(l,l')} Av(J_{n,n'}J_{m,m'})\sigma_n\sigma'_n\tau_m\tau'_m = \\ &= \sum_{(n,n'),(l,l')} \delta_{(n,n')(m,m')}\sigma_n\sigma'_n\tau_m\tau'_m \\ &= |B(\Lambda)|p(\sigma,\tau)\;. \end{aligned} \tag{5}$$

From the previous observation we expect then that all the observables of the system which are derived from the Hamiltonian, like the internal energy, the specific heat and all the upper order derivatives (in β) of the free energy are described by the bond-overlap moments with respect to the quenched measure. The same situation occurs for instance for the mean-field case with respect to the relative covariance. The structure of the computations that lead to a precise mathematical formulation of our statement is indeed independent of the model as far as one considers Gaussian interactions. Let for completeness show how in the Edwards-Anderson case the internal energy is related to the quenched average of the matrix p. For this purpose we recall that the quenched measure over an arbitrary number r of copies of the system is defined assigning to the l-th copy its own Hamiltonian $H^{(l)}_\Lambda(J,\sigma) = H_\Lambda(J,\sigma^{(l)})$

where we stress the same disorder J in each copy. For each $H_\Lambda^{(l)}(J,\sigma)$ we build the random-Gibbs-Bolztmann state:

$$< - >^{(l)} = \frac{\sum_\sigma -e^{-\beta H_\Lambda^{(l)}(J,\sigma)}}{\sum_\sigma e^{-\beta H_\Lambda^{(l)}(J,\sigma)}} \,, \tag{6}$$

$$\ll - \gg = \otimes_{l=1}^{r} < - >^{(l)} \ . \tag{7}$$

The quenched measure $E(-)$ is then defined as

$$E^{(\Lambda,\beta)}(-) \ = \ Av(\ll - \gg) \,, \tag{8}$$

and the matrix P, whose elements are defined by:

$$P_{l,m} \ = \ p(\sigma^{(l)}, \sigma^{(m)}) \tag{9}$$

is then a symmetric random matrix with respect to the measure $E^{(\Lambda,\beta)}(-)$ and its distribution turns out to be invariant under the action of the permutation group over the r copies. Example: for $r = 3$ we have

$$P = \begin{pmatrix} 1 & P_{1,2} & P_{1,3} \\ P_{2,1} & 1 & P_{2,3} \\ P_{3,1} & P_{3,2} & 1 \end{pmatrix} \tag{10}$$

and for instance the expectation of the monomial $P_{1,2}P_{2,3}$ is

$$\begin{aligned} E^{(\Lambda,\beta)}(P_{1,2}P_{2,3}) \ &= \\ = \ Av &\left[\frac{\sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} p(\sigma^{(1)},\sigma^{(2)}) p(\sigma^{(2)},\sigma^{(3)}) e^{-\beta(H_\Lambda(J,\sigma^{(1)}) + H_\Lambda(J,\sigma^{(2)}) + H_\Lambda(J,\sigma^{(3)}))}}{(\sum_\sigma e^{-\beta H_\Lambda(J,\sigma)})^3} \right] \end{aligned}$$

By definition of internal energy and using integration by parts for normal Gaussian variables

$$Av(Jf(J)) = Av\left(\frac{df}{dJ}(J)\right) \tag{11}$$

we have

$$\begin{aligned} U_\Lambda(\beta) &= Av(< H_\Lambda(J,\sigma) >) = \sum_{b \in B(\Lambda)} Av(J_b < S_b >) = \\ &= -\beta \sum_{b \in B(\Lambda)} (1 - Av(< S_b >^2)) = -\beta \sum_{b \in B(\Lambda)} (1 - E^{(\Lambda,\beta)}(p_b)) \ , \quad (12) \end{aligned}$$

where we have used the identities

$$Av(< S_b >^2) = Av(< S_b >< T_b >) = Av(<< S_b T_b >>) = E^{(\Lambda,\beta)}(p_b) \, , \quad (13)$$

which may be immediately derived from the definitions (6-8). From (5) and (12) we get

$$\frac{U_\Lambda(\beta)}{|B(\Lambda)|} = -\beta(1 - E^{(\Lambda,\beta)}(p)) \, , \quad (14)$$

which clearly shows the role of bond-overlap in the finite dimensional case and shows that the couple $\{E, P\}$ completely specify the model in the sense that the quantities $E\left(\prod_{l,m} P_{l,m}^{n_{l,m}}\right)$ for positive integers $n_{l,m}$ represent the physical observables of the theory.

Stochastic stability in finite dimension

In this section we develop an approach to stochastic stability for finite dimensional systems which runs parallel to the one introduced in [AC] for the mean field case. The starting point is the observation that the addition to the Hamiltonian of an independent Gaussian term of finite size:

$$h_\Lambda(\tilde{J},\sigma) = \frac{1}{\sqrt{|B(\Lambda)|}} \sum_{b \in B(\Lambda)} \tilde{J}_b S_b \quad (15)$$

amounts to a slight change in the temperature. In fact for the sum law of independent Gaussian variables one has that in distribution

$$\beta H_\Lambda(J,\sigma) + \lambda h_\Lambda(\tilde{J},\sigma) \ \stackrel{\mathcal{D}}{=}\ \beta'(\beta,\lambda) H_\Lambda(J',\sigma)\ , \tag{16}$$

with

$$\beta'(\beta,\lambda) \ = \ \sqrt{\beta^2 + \frac{\lambda^2}{|B(\Lambda)|}}\ , \tag{17}$$

so that, indicating by $E_\lambda^{(\Lambda,\beta)}$ the quenched state of Hamiltonian $\beta H_\Lambda(J,\sigma) + \lambda h_\Lambda(\tilde{J},\sigma)$ the (16) implies

$$E_\lambda^{(\Lambda,\beta)}(-) \ = \ E^{(\Lambda,\beta')}(-)\ \ , \tag{18}$$

when the expectations are computed on bond-overlap polynomials. Since $\beta' = \beta + \frac{\lambda^2}{2|B(\Lambda)|\beta} + o\left(\frac{\lambda^2}{|B(\Lambda)|\beta}\right)$ the (18) immediately entails that if, over a certain temperature range, the state $E^{(\Lambda,\beta)}$ is uniformly continuous in β, as $\Lambda \to \mathbb{Z}^d$, then its thermodynamic limit E (see [CG] for the proof of monotonic convergence) is stable under deformations in λK, in the sense that for every overlap monomial M:

$$E_\lambda(M) \ = \ E(M). \tag{19}$$

where E_λ is the $\Lambda \to \mathbb{Z}^d$ limit of the measure defined on the left hand side of (18). Such a property, introduced in the mean field case in [AC], is called *stochastic stability* [P2] and was later investigated in [FMPP1] and [FMPP2] to determine a relation between the off-equilibrium dynamics and the static properties. Stochastic stability has important consequences for the quenched state. In particular it says that all the even derivatives of the λ-deformed state have to be zero (the odd being zero for antisymmetry). An elementary computation that only uses the Wick theorem gives (following the method of [AC]) for instance

$$\frac{d^2}{d\lambda^2} E_\lambda^{(\Lambda,\beta)}(p)|_{\lambda=0} = 2E^{(\Lambda,\beta)}(p_{1,2}^2 - 4p_{1,2}p_{2,3} + 3p_{1,2}p_{3,4}) \ , \tag{20}$$

so that from (19) we have in the infinite volume limit

$$E(p_{1,2}^2 - 4p_{1,2}p_{2,3} + 3p_{1,2}p_{3,4}) \ = \ 0 \ . \tag{21}$$

Remark 1: the (21) is one of the possible polynomial expressions in the matrix P entries that can be shown to have zero average [AC]. It was shown in [C] that the family of those identities coincides with the one obtained in the framework of the replica equivalent ansatz introduced in [P1]. Those polynomials play in spin glass models a role which is reminiscent of the Hermite polynomials in classical Gaussian field theories; in the same way the identities that can be found in [GG] allow a computation of the moments in terms of sum of products of lower order ones and are, in parallel, reminiscent of the Wick formula for the Gaussian case.

The assumption of almost everywhere continuity in beta of the quenched quantities is based on very solid physical grounds: in fact even if it is reported that for a given realization of the disorder the equilibrium state has a very erratic dependence on the temperature (temperature chaos) it seems reasonable to expect that with the average over the disorder the quenched state might vary continuously with β except possibly on isolated singularities. To understand this last point it is useful to clarify the physical meaning of the identity that we just derived. Using (16) we obtain

$$\frac{1}{\beta}\frac{d}{d\beta} E^{(\Lambda,\beta)}(p) = |B(\Lambda)| \frac{d^2}{d\lambda^2} E_\lambda^{(\Lambda,\beta)}(p)|_{\lambda=0} \ . \tag{22}$$

From the definition of specific heat per particle and from (14), (20) and (22)

$$
\begin{aligned}
c_\Lambda(\beta) &= \frac{d}{d\beta}\frac{U_\Lambda(\beta)}{|B(\Lambda)|} = -\,(1 - E^{(\Lambda,\beta)}(p)) \,+\, \frac{d}{d\beta}E^{(\Lambda,\beta)}(p) \,= \\
&= -\,(1 - E^{(\Lambda,\beta)}(p)) + 2\beta|B(\Lambda)|E^{(\Lambda,\beta)}(p_{1,2}^2 - 4p_{1,2}p_{2,3} + 3p_{1,2}p_{3,4})\,.
\end{aligned} \tag{23}
$$

The identity (20) can be then seen as a simple consequence of the boundedness (almost everywhere in beta) of the specific heat per particle which is a classical result in statistical mechanics based on convexity [Ru, G]. For a classification of the set of identities that can be derived from stochastic stability and for the control of higher order derivatives see [C].

Remark 2: in [CG] it was introduced a general finite-dimensional spin-glass multibody interaction of the type

$$
H_\Lambda(J,\sigma) = -\sum_{X\subset\Lambda} J_X\sigma_X\ , \tag{24}
$$

where

$$
\sigma_X = \prod_{i\in X}\sigma_i\ , \tag{25}
$$

($\sigma_\emptyset = 0$) and the J_X's are independent Gaussian variables with zero mean and variance Δ_X^2. A simple calculation like the one in (5) gives

$$
\begin{aligned}
Av(H_\Lambda(J,\sigma)H_\Lambda(J,\tau)) &= \sum_{X,Y\subset\Lambda} Av(J_XJ_Y)\sigma_X\tau_Y = \\
&= \sum_{X\subset\Lambda}\Delta_X^2\sigma_X\tau_X\ .
\end{aligned} \tag{26}
$$

The previous observation shows that the argument we have used for the Edwards-Anderson model can be reproduced exactly for the interactions (24) with the result that in such a general case the model is stochastically stable and replica equivalent with respect to the generalized overlap:

$$
\widetilde{P}(\sigma,\tau) \,=\, \frac{1}{|\mathcal{P}_\Lambda|}\sum_{X\subset\Lambda}\Delta_X^2\sigma_X\tau_X\ , \tag{27}
$$

where we have indicated by $\mathcal{P}_\Lambda$ the set of interacting subsets of Λ.

Remark 3: we want to stress the fact that our result is different from the one mentioned in [G]. The author there (at the end of section 4) suggests a method to prove some factorization identities for the standard overlap (1) for a general spin model. The result is achieved with the addition to the Hamiltonian of a term proportional to a mean-field spin-glass interaction and sending the interaction strength to zero after the thermodynamic limit is considered. The resulting state turns out to be replica equivalent but the addition of a mean field perturbation could, in principle, select a subphase of the whole equilibrium state. Our result instead obtains replica equivalence with respect to the bond-overlap for the whole quenched state.

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