Energy Landscape Statistics of the Random Orthogonal Model

July 31, 2002

M. Degli Esposti, C. Giardinà, S. Graffi

Dipartimento di Matematica

Università di Bologna,

Piazza di Porta S. Donato 5, 40127 Bologna, Italy

Abstract

The Random Orthogonal Model (ROM) of Marinari-Parisi-Ritort [13, 14] is a model of statistical mechanics where the couplings among the spins are defined by a matrix chosen randomly within the orthogonal ensemble. It reproduces the most relevant properties of the Parisi solution of the Sherrington-Kirkpatrick model. Here we compute the energy distribution, and work out an extimate for the two-point correlation function. Moreover, we show exponential increase of the number of metastable states also for non zero magnetic field.

1 Introduction: review of the model and outlook

Random (symmetric) matrices out of a given ensemble can be taken as interaction matrices for Ising spin models. The most famous example is the Sherrington-Kirkpatrick (SK) model of spin glasses, where the elements are i.i.d. Gaussian variables with properly normalized variance. Aim of this paper is to discuss a very specific example of these spin glass models, which also share some interesting connections with number theory, and show how random matrix theory could be useful to investigate its properties.

For the sake of simplicity, let us start with a very concrete question: let $N \geq 1$ be a positive integer and denote Σ_N the space of all possible configurations of N spin variables

$$\Sigma_N = \{\sigma = (\sigma_1, \ldots, \sigma_N)\ ,\ \sigma_j = \pm 1\}, \quad |\Sigma_N| = 2^N.$$

Given $k = 1, \ldots, N-1$, denote C_k the correlation function:

$$C_k(\sigma) \ = \sum_{j=1}^{N} \sigma_j \sigma_{j+k}, \quad \text{where } j+k := (j+k-1 \bmod N) + 1,$$

and define the Hamiltonian function

$$H(\sigma) \ = \ \frac{1}{N-1} \sum_{k=1}^{N-1} C_k^2$$

For each N the ground state of the Hamiltonian H can be looked at as the binary sequence with lowest autocorrelation and finding it will have some relevant practical applications in efficient communication (see [4] and references in [13]).

It is remarkable that no concrete procedure for reproducing the ground state for generic N is known, but *ad hoc* constructions based on number theory exist for very specific values of N: if N is prime number with $N = 3 \mod 4$, then the sequence of the Legendre symbols[1] ($\sigma_N = 1$)

$$\sigma_j := \left(\frac{j}{N}\right) = j^{\frac{1}{2}(N-1)} \mod N\,, \quad j = 1, \ldots, N-1$$

gives the ground state of the system [9, 13].

Through the use of the discrete Fourier transform, it is not difficult to see [13, 16] that the previous problem is in fact equivalent to finding the ground state for the so called *Sine model*, which represents our starting point:

$$H(\sigma) \;=\; -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j.$$

Here J is the following $N \times N$ real symmetric orthogonal matrix with almost full connectivity:

$$J_{ij} \;=\; \frac{2}{\sqrt{1+2N}}\,\sin\left(\frac{2\pi\, ij}{2N+1}\right), \quad i,j = 1,\ldots,N$$

Here again, if $2N+1$ is prime and N odd, the Legendre symbols $\sigma_j = j^N \mod 2N+1$ give the ground state of the system for these very specific values of N.

A natural approach is to extend the study of the ground state to the more general thermo-dynamical behavior of the model in terms of

[1] $\left(\frac{j}{N}\right) = 1$, if $j = x^2 \mod N$ and -1 otherwise.

the inverse temperature $\beta = \frac{1}{T}$. As usual, the two basic objects are:

$$\textit{the partition function} \quad Z_J(\beta) := \sum_{\sigma\in\Sigma_N} e^{-\beta H(\sigma)},$$

and *the free energy density (at the thermodynamical limit)*

$$f_J(\beta) = \lim_{N\to\infty} -\frac{1}{\beta N}\log Z_J(\beta)$$

It is important to remark now that even if *there is no randomness in the system*, the ground state of the model *looks like* an output of a random number generator and the numeric of its thermo-dynamical properties resemble the one of disordered systems. This observation was in fact the starting point of an approach developed in [13, 14, 16] where this model is seen as a particular realization of a disordered model where the coupling matrix is chosen at random out of a suitable set of matrices:

Definition 1 *The Random Orthogonal Model (ROM) with magnetic field $h \geq 0$ is the disordered system with energy*

$$H_J(\sigma) = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_j\sigma_i + h\sum_j \sigma_j, \tag{1}$$

where the coupling matrix J is chosen randomly in the set of orthogonal symmetric matrices:[2]

$$J = ODO^{-1},$$

Here O is a generic orthogonal matrix and D is diagonal with entries ± 1. The numbers ± 1 are the eigenvalues of J.

[2] In the ROM model generic matrices have non zero diagonal elements. Often these terms will be set to zero and orthogonality will be reconstructed in the large N limit.

The natural probability measure μ on this set is the product of the canonical Haar measure on the orthogonal group by the discrete measure on the diagonal terms.

We will use the notation $\langle \cdot \rangle$ to denote the average with respect the measure μ. In particular, we are interested in the *quenched* (i.e., the average is performed after taking the logarithm) free energy density:

$$\langle f_J(\beta) \rangle = - \lim_{N \to \infty} \frac{1}{\beta N} \langle \log Z_J(\beta) \rangle \tag{2}$$

Average over the ROM disorder is performed by the following fundamental formula, which has been obtained by adapting the results in [12] (see also [2]) valid for the unitary case to the orthogonal one [14]. For any $N \times N$ symmetric matrix A:

$$\begin{aligned} \langle \exp \left\{ tr \left(\frac{JA}{2} \right) \right\} \rangle &= \exp \left\{ N tr \left(G \left(\frac{A}{N} \right) \right) \right\} + R_N(A), \\ &\cong \exp \left\{ N \sum_{j=1}^{N} G(\lambda_j) \right\} \end{aligned} \tag{3}$$

where $R_N \to 0$ in the thermodynamical limit $N \to \infty$, the λ_j's are the (real) eigenvalues of $\frac{1}{N} \cdot A$ and $G(x)$ is given by

$$G(x) = \frac{1}{4} \left[\sqrt{1 + 4x^2} - \ln \left(\frac{1 + \sqrt{1 + 4x^2}}{2} \right) - 1 \right]$$

The same formula is exact for the SK model, i.e Gaussian independent symmetric couplings, with

$$G_{SK}(x) = \frac{x^2}{4}$$

Note that $G(x) = G_{SK}(x) + o(x)$. For example, up to the 10-th order

$$G(x) \; = \; \frac{x^2}{4} - \frac{x^4}{8} + \frac{x^6}{6} - \frac{5\,x^8}{16} + \frac{7\,x^{10}}{10} + O(x^{11})$$

The ROM model has been chosen in a such a way that, at least for not too small temperature, the *deterministic* Sine model and the one with quenched disorder share a common behavior. More precisely, the couplings are always of order $N^{-\frac{1}{2}}$; the diagrams contributing to the thermodynamical limit of the high temperature expansion for the free energy density have the same topology and they can all be expressed in terms of positive powers of the trace of the couplings. By construction, the high temperature expansion of the free energy density $f_J(\beta)$ in powers of β is then independent of the particular choice of the symmetric orthogonal matrix J and it does coincide with the annealed average w.r.t. μ. In particular [16]:

$$-\beta\langle f_J(\beta)\rangle \; = \; \log 2 + G(\beta).$$

Besides SK and in general the large class of p-spin models, where couplings have a gaussian distribution, the ROM model provide another interesting class of disordered mean-field spin glass. This model has received considerable interest in recent years, especially in the contest of structural glass transition. Indeed it can been seen as the random version of a wide class of models (for example the fully frustrated Ising model on a hypercube or the above mentioned sine model) which despite having a non-random Hamiltonian display a strong glassy behavior [3, 13, 9]. This model has been studied in the framework of replica theory [14], where it was shown that replica symmetry is broken and there are many equilibrium states available to the system. Mean field (TAP) equations have been derived for this model by resumming the high temperature expansion and the average number of solutions of these equations has been studied in ref. [16].

It is a well established fact that the observed properties of mean-field spin glass models are due to the large number of *metastable states* the system possesses. Despite being not fully justified from a mathematical point of view, the Parisi scheme of breaking replica symmetry furnishes a clear picture of equilibrium statistical properties: states with similar macroscopic behavior have vastly different spin configurations, and have large relaxation times for transition between them. As a consequence, the ground state is accessible only on very long time scales. It is worth mentioning that rigorous results validating the Parisi solution have been accumulating in recent times.

For example, Guerra and Toninelli[10] have proved the existence of thermodynamical limit, i.e. the existence of the limit for quenched average of the free energy (eq. 2). See also [6] where the result has been extended to general correlated gaussian random energy models. Finally, more recently [11], Guerra showed that the Parisi Ansatz represents at least a lower bound for the quenched average of the free energy.

However there is not yet an unambiguous way to identify those metastable states which are relevant for thermodynamics in the infinite volume limit. At zero temperature the metastable states can be defined as the states *locally stable* to single spin flips (definition recalled in Section 3 below) and the calculations are relatively straightforward. Complete analysis of the typical energy of metastable states and of the effects of the external field have been undertaken both for the SK model [19, 18, 7] and for general p-spin model [15]. The zero temperature dynamics for the deterministic *Sine model* has been instead studied in [9].

At non-zero temperature the identification is less obvious and most studies [5, 17] rely on the counting of the number of solutions to the celebrated TAP equations [20]. According to the general belief, one can associate to each metastable state a solution of the TAP equation, but the inverse is not true: a TAP solution corresponds to a metastable state only if it is separated from other solution by a barrier whose height diverges with the volume.

It appears, however, that even the calculations at zero temperature in a presence of external field have not been carried out. One expects, in analogy with SK model, the existence of an AT line [1] indicating the onset of replica-symmetry breaking. In this paper we study at length the effects of the magnetic field on the structure of local optima of the energy landscape. We are able to use these results to shed further light on the nature of the AT instability at zero temperature.

In the next Section 2 we study the statistics of energy levels over the whole configuration space. We compute energy distribution of a generic spin configuration and the pair correlations for a given couple of spin configurations with a fixed overlap. In Section 3 we analyze metastable states at zero temperature, also in a presence of an external field.

2 Statistics of energy levels

We start by analyzing the statistical features of the landscape generated by the energy function (1). In this section we will always consider zero magnetic field $h = 0$. Let us begin with the energy distribution for a single fixed configuration.

2.1 Distribution of energy

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ denote a given configuration with energy $H_J(\sigma)$. The probability $P_\sigma(E)$ is then given by:

$$P_\sigma(E) := \langle \delta(E - H(J, \sigma)) \rangle$$

Due to gauge invariance, the probability $P_\sigma(E)$ does not depend on the spin configuration σ and it will be denoted just by $P(E)$, in fact: $H(J, \sigma) = H(J', \sigma')$ and $P(J) = P(J')$ where $J'_{ij} = J_{ij}\sigma_i\sigma'_i\sigma_j\sigma'_j$. Introducing the integral representation for δ function

$$\delta(x - x_0) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dk \; e^{k\,(x - x_0)}$$

we get

$$P(E) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dk \; e^{kE} \; \langle e^{\frac{1}{2}\sum_{i,j=1}^{N} k\, J_{ij}\sigma_i\sigma_j} \rangle$$

and we can apply formula (3) to average over disorder considering the matrix $A_{ij} = k\sigma_i\sigma_j$.

It is easy to prove that A admits only one non-zero, simple eigenvalue $\lambda = kN$, so that

$$P(E) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dk\ \exp\left[N\left(\frac{kE}{N} + G(k)\right)\right]$$

In the large-N limit the integral can be evaluated using the saddle-point method. Clearly, the equation

$$\frac{E}{N} + G'(k) \ = \ \frac{E}{N} + \frac{k}{1+\sqrt{1+4\,k^2}} = 0$$

admits the solution $\bar{k} = \frac{2\,E\,N}{4\,E^2 - N^2}$.

This gives:

$$\begin{aligned} P_{ROM}(E) \ &\sim \ \exp\left[N\left(\frac{\bar{k}E}{N} + G(\bar{k})\right)\right] \qquad (4) \\ &= \ \left(1 - \left(\frac{2E}{N}\right)^2\right)^{N/4} \\ &\sim \ \exp\left[-\frac{E^2}{N} - 2\frac{E^4}{N^3} - \frac{16}{3}\frac{E^6}{N^5} + \cdots\right] \qquad (5) \end{aligned}$$

apart for an unimportant constant, not predicted by the saddle-point. As a comparison, in the case of SK model one finds exactly the gaussian distribution:

$$P_{SK}(E) \sim \exp\left(-\frac{E^2}{N}\right)$$

To check the validity of formula (3) which has been used to average over disorder, we computed $P_{ROM}(E)$ for a relative small ROM

(N=100) numerically. For a given spin configuration, random disorder realizations $J = ODO^{-1}$ were generated by using an orthogonal matrix O obtained from a gaussian matrix by applying Gram-Schmidt orthogonalization algorithm and coin tossing for the diagonal D. The resulting distribution of energies was binned and is shown as the data points in Fig. (1).

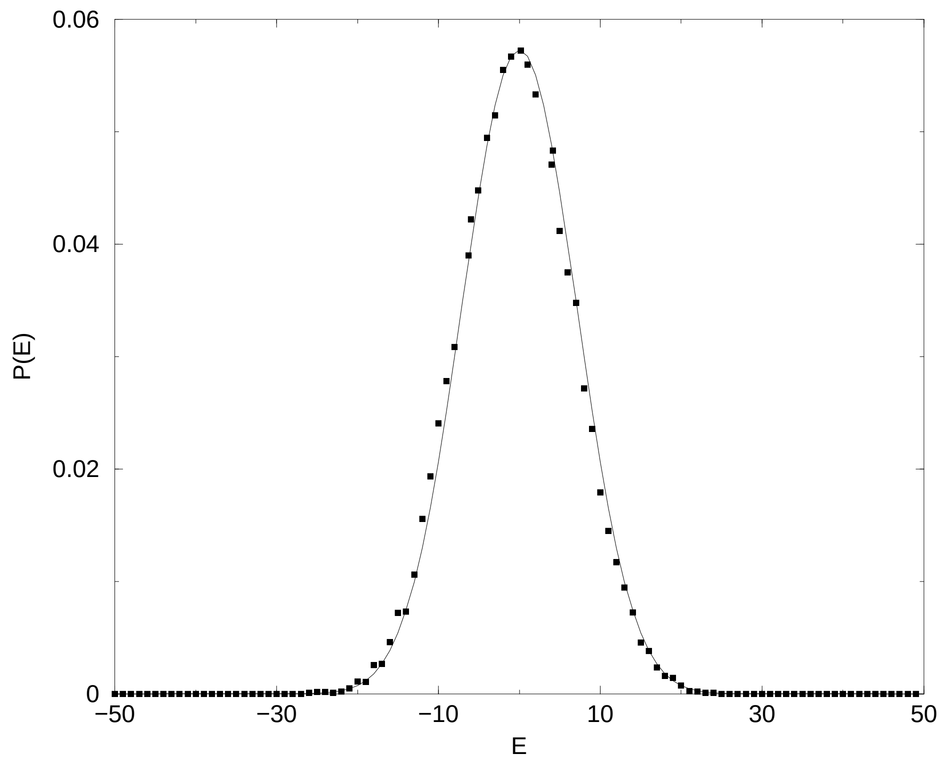


Figure 1: Probability distribution function $P_{ROM}(E)$ for ROM model (full curve). Simulation for a $N = 100$ ROM (data points). For a fixed spin configuration, 10^6 realizations of disorder were generated.

As it should be, the support of $P_{ROM}(E)$ is almost all in the interval $[-N/2, N/2]$. Indeed, the orthogonality of J imposes simple bound on the energy of any spin configuration: the lower bound $-N/2$ (resp.

upper bound $N/2$) is reached if and only if σ is an eigenvector of J relative to eigenvalue $+1$ (resp. -1).

2.2 Two-point Energy Correlation

We consider now the probability $P_{\sigma,\tau}(E_1, E_2)$ that two configurations $\sigma, \tau \in \Sigma_N$ have energies E_1 and E_2 respectively. Gauge invariance implies that this probability can only depend on the overlap between the two configurations:

$$q(\sigma, \tau) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tau_i$$

Proceeding as before, we get:

$$\begin{aligned} P_{\sigma,\tau}(E_1, E_2) &= \langle \delta(E_1 - H(J, \sigma))\, \delta(E_2 - H(J, \tau)) \rangle \\ &= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dk_1 \int_{-i\infty}^{+i\infty} dk_2 \; \exp(k_1 E_1 + k_2 E_2) \\ &\quad \langle \exp\left(\frac{1}{2} \sum_{i,j=1}^{N} J_{ij}(k_1 \sigma_i \sigma_j + k_2 \tau_i \tau_j) \right) \rangle \end{aligned} \tag{6}$$

Consider now the matrix $A_{ij} = k_1 \sigma_i \sigma_j + k_2 \tau_i \tau_j$ which has two non-zero simple eigenvalues

$$\lambda_\pm = \frac{N}{2} \left[(k_1 + k_2) \pm \sqrt{(k_1 - k_2)^2 + 4 k_1 k_2 q^2} \right]$$

Applying formula (3) we obtain:

$$\begin{aligned} P_{\sigma,\tau}(E_1, E_2) &= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dk_1 \int_{-i\infty}^{+i\infty} dk_2 \\ &\quad \exp\left[N \left(\frac{k_1 E_1}{N} + \frac{k_2 E_2}{N} + G(\lambda_+/N) + G(\lambda_-/N) \right) \right] \end{aligned}$$

The saddle-point method yields the equations:

$$\frac{E_j}{N} + \frac{1}{N}\, G'(\frac{\lambda_+}{N})\, \frac{\partial \lambda_+}{\partial k_j} \;+\; \frac{1}{N}\, G'(\frac{\lambda_-}{N})\, \frac{\partial \lambda_-}{\partial k_j} \;=\; 0, \qquad j = 1, 2$$

For the SK model, one immediately find:

$$\frac{E_1}{N} + \frac{1}{2}(k_1 + k_2 q^2) = 0, \quad \frac{E_2}{N} + \frac{1}{2}(k_2 + k_1 q^2) = 0$$

with solutions:

$$k_1 = \frac{2(E_1 - E_2 q^2)}{N(-1+q^4)}, \quad k_2 = \frac{2(E_2 - E_1 q^2)}{N(-1+q^4)}.$$

This yields the well known [8] ($\sigma, \tau \in \Sigma_N$ fixed, with overlap q):

$$\begin{aligned} P_{SK}(E_1, E_2) \;&=\; \left(\frac{\sqrt{1-q^4}}{N\pi}\right) \exp\left[-\frac{(E_1+E_2)^2}{2N(1+q^2)}\right] \exp\left[-\frac{(E_1-E_2)^2}{2N(1-q^2)}\right] \\ &=\; P_{SK}\left(\frac{E_1+E_2}{\sqrt{2(1+q^2)}}\right) \cdot P_{SK}\left(\frac{E_1-E_2}{\sqrt{2(1-q^2)}}\right). \end{aligned} \tag{7}$$

For asymptotically uncorrelated configurations $q = 0$, one clearly get a product measure, whereas one recover complete degeneracy when $q = 1$:

$$P_{SK}(E_1, E_2) \;=\; P_{SK}(E_1) \cdot P_{SK}(E_2), \quad q = 0, \tag{8}$$

and

$$P_{SK}(E_1, E_2) \;=\; P_{SK}(E_1) \cdot \delta\,(E_2 - E_1)\,, \quad q = 1. \tag{9}$$

In generale, one has

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_1 E_2\, dP_{SK}(E_1, E_2) \;=\; \frac{N q^2}{2}.$$

For the ROM model, it is immediate to see that the analog of (8) and (9) hold true with the single energy distribution $P_{ROM}(E)$ given by (4). For generic value of $0 < q < 1$, a first crude estimate is achieved by using the stationary points of the gaussian approximation and $G(x) = \frac{x^2}{4} - \frac{x^4}{8}$ to evaluate the exponent. This yields:

$$P_{ROM}(E_1, E_2) \sim P_{SK}(E_1, E_2) \cdot Exp[-\Phi_q(E_1, E_2)],$$

where

$$\begin{aligned}\Phi_q(E_1, E_2) \quad := \quad & -2\frac{-8\,E_1^3\,E_2\,q^4 - 8\,E_1\,E_2^3\,q^4 + E_1^4\,\left(1 + 2\,q^2 - q^4\right)}{N^3\,\left(-1 + q^2\right)^2\,\left(1 + q^2\right)^4} \\ & + \frac{{E_2}^4\,\left(1 + 2\,q^2 - q^4\right) + 2\,{E_1}^2\,{E_2}^2\,q^2\,\left(2 - q^2 + 4\,q^4 + q^6\right)}{N^3\,\left(-1 + q^2\right)^2\,\left(1 + q^2\right)^4}\end{aligned}$$

Further corrections can be now calculated, but we do not known a systematic way of doing it at all orders.

3 Zero temperature metastable states

Metastable states at zero temperature are defined as the configurations whose energy can not be decreased by reversing any of the spins [9]. Since the energy change ΔE_i involved in flipping the spin at site i is given by

$$\Delta E_i = 2\left(\sum_j J_{ij}\sigma_i\sigma_j + h\sigma_i\right)$$

the constraint a configuration σ must satisfy in order to be metastable is

$$\sum_j J_{ij}\sigma_i\sigma_j + h\sigma_i > 0 \qquad \forall i = 1, \ldots, N$$

The average number of metastable configurations $\langle\mathcal{N}(e, h)\rangle$ with a given energy density $e = E/N$ is then

$$
\begin{aligned}
\langle \mathcal{N}(e,h) \rangle &= \langle \sum_{\{\sigma\}} \prod_{i=1}^{N} \left[\int_0^\infty d\lambda_i \; \delta \left(\lambda_i - \sum_j J_{ij}\sigma_i\sigma_j - h\sigma_i \right) \right] \\
& \qquad \delta \left(Ne + \frac{1}{2} \sum_{i,j} J_{ij}\sigma_i\sigma_j + h \sum_i \sigma_i \right) \rangle
\end{aligned} \tag{10}
$$

One should really calculate the average value of the logarithm of the number of metastable states, this being the extensive quantity, and hence introduce replicas; indeed, as pointed out in [5], the introduction of a uniform magnetic field should introduce strong correlations between the metastable states. However, we shall proceed with the direct calculation of $\langle \mathcal{N}(e,h) \rangle$ as it suffices to bring out the most relevant features of the problem.

Introducing integral representations for δ functions we have

$$
\begin{aligned}
\langle \mathcal{N}(e,h) \rangle &= \sum_{\{\sigma\}} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \; e^{zNe} e^{zh \sum_i \sigma_i} \\
& \prod_{i=1}^{N} \left[\int_0^\infty d\lambda_i \; \int_{-i\infty}^{+i\infty} \frac{dk_i}{2\pi i} \right] e^{\sum_i k_i (h\sigma_i - \lambda_i)} \; \langle e^{\sum_{i,j} J_{ij} (\frac{z}{2}\sigma_i\sigma_j + k_i\sigma_i\sigma_j)} \rangle
\end{aligned}
$$

To apply the formula (3) for averaging over disorder we define the matrix $A_{ij} = \left(\frac{z}{2} + k_i\right)\sigma_i\sigma_j + \left(\frac{z}{2} + k_j\right)\sigma_j\sigma_i$. The non-zero eigenvalues of A_{ij} are easily calculated and read

$$
\mu_\pm = \sum_i \left(\frac{z}{2} + k_i \right) \pm \sqrt{ N \sum_i \left(\frac{z}{2} + k_i \right)^2 }
$$

so that we obtain:

$$
\langle \mathcal{N}(e,h) \rangle = \sum_{\{\sigma\}} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \; e^{zNe} e^{zh \sum_i \sigma_i} \prod_{i=1}^{N} \left[\int_0^\infty d\lambda_i \; \int_{-i\infty}^{+i\infty} \frac{dk_i}{2\pi i} \right] \; e^{\sum_i k_i (h\sigma_i - \lambda_i)}
$$

$$\exp\left\{N\left[G\left(\frac{1}{N}\sum_i\left(\frac{z}{2}+k_i\right)+\sqrt{\frac{1}{N}\sum_i\left(\frac{z}{2}+k_i\right)^2}\right)+\right.\right.$$
$$\left.\left.G\left(\frac{1}{N}\sum_i\left(\frac{z}{2}+k_i\right)-\sqrt{\frac{1}{N}\sum_i\left(\frac{z}{2}+k_i\right)^2}\right)\right]\right\} \tag{11}$$

Performing now the trace over spin configuration, defining

$$v=\frac{1}{N}\sum_i\left(\frac{z}{2}+k_i\right) \qquad w=\frac{1}{N}\sum_i\left(\frac{z}{2}+k_i\right)^2$$

and imposing the constraints via two Lagrange multipliers, we have

$$\begin{aligned}\langle\mathcal{N}(e,h)\rangle &= \frac{1}{(2\pi i)^3}\int_{-i\infty}^{+i\infty}dz\int_{-i\infty}^{+i\infty}dv\int_{-i\infty}^{+i\infty}dw\int_{-i\infty}^{+i\infty}dx\int_{-i\infty}^{+i\infty}dy\\ &\exp\left\{N[ze+\frac{zx}{2}+\frac{yz^2}{4}]\right\}\\ &\exp\left\{N[-xv-yw+G(v+\sqrt{w})+G(v-\sqrt{w})]\right\}\\ &\prod_{i=1}^{N}\left[\int_0^\infty d\lambda_i\int_{-i\infty}^{+i\infty}\frac{dk_i}{\pi i}\;e^{yk_i^2+k_i(x-\lambda_i+yz)}\cosh\left(h(z+k_i)\right)\right]\end{aligned}\tag{12}$$

The integrals over the k_i are now gaussian

$$\begin{aligned}\langle\mathcal{N}(e,h)\rangle &= \frac{1}{(2\pi i)^3}\int_{-i\infty}^{+i\infty}dz\int_{-i\infty}^{+i\infty}dv\int_{-i\infty}^{+i\infty}dw\int_{-i\infty}^{+i\infty}dx\int_{-i\infty}^{+i\infty}dy\\ &\exp\left\{N[ze+\frac{zx}{2}+\frac{yz^2}{4}]\right\}\\ &\exp\left\{N[-xv-yw+G(v+\sqrt{w})+G(v-\sqrt{w})]\right\}\\ &\prod_{i=1}^{N}\left[\int_0^\infty d\lambda_i\;\frac{1}{2\sqrt{\pi y}}\left(e^{hz}e^{-\frac{(x+yz-\lambda_i+h)^2}{4y}}+e^{-hz}e^{-\frac{(x+yz-\lambda_i-h)^2}{4y}}\right)\right]\end{aligned}\tag{13}$$

and the integrals over the λ_i can be performed in terms of the complementary error function

$$\mathrm{erfc(x)}=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

so that we find:

$$
\begin{aligned}
\langle \mathcal{N}(e,h)\rangle \quad = \quad & \frac{1}{(2\pi i)^3}\int_{-i\infty}^{+i\infty} dz \int_{-i\infty}^{+i\infty} dv \int_{-i\infty}^{+i\infty} dw \int_{-i\infty}^{+i\infty} dx \int_{-i\infty}^{+i\infty} dy \\
& \exp\left\{ N[ze+\frac{zx}{2}+\frac{yz^2}{4}] \right\} \\
& \exp\left\{ N\left[-xv-yw+G(v+\sqrt{w})+G(v-\sqrt{w}) \right.\right. \qquad (14) \\
& \left.\left. +\ln\left(\frac{1}{2}\left(e^{hz}\mathrm{erfc}\left(-\frac{x+yz+h}{2\sqrt{y}}\right)+e^{-hz}\mathrm{erfc}\left(-\frac{x+yz-h}{2\sqrt{y}}\right)\right)\right)\ \right]\right\}
\end{aligned}
$$

As usual the calculation is concluded by carrying out a saddle-point integration. The r.h.s. of Eq. (14) is to be extremized with respect to the five variables z, v, w, x, y.

3.1 Total number of metastable states

Here we study the total number of metastable states $\langle \mathcal{N}(h)\rangle$ (irrespectively of the energy) as a function of the field. Writing

$$
\log\langle \mathcal{N}(h)\rangle = A(h)N + B(h), \qquad (15)
$$

$A(h)$, in the thermodynamical limit ($N \to \infty$), can be calculated by setting $z=0$ in Eq. (14), which becomes:

$$
\begin{aligned}
\langle \mathcal{N}(h)\rangle \quad = \quad & \frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty} dv \int_{-i\infty}^{+i\infty} dw \int_{-i\infty}^{+i\infty} dx \int_{-i\infty}^{+i\infty} dy \\
& \exp\left\{ N\left[-xv-yw+G(v+\sqrt{w})+G(v-\sqrt{w}) \right.\right. \\
& \left.\left. +\ln\left(\frac{1}{2}\left(\mathrm{erfc}(-\frac{x+h}{2\sqrt{y}})+\mathrm{erfc}(-\frac{x-h}{2\sqrt{y}})\right)\right)\ \right]\right\} \qquad (16)
\end{aligned}
$$

In the case of the SK model one recover the well now one-variable saddle-point equation [7]:

$$
x \;=\; \frac{\exp[-x^2/2]\cosh(hx)}{\int_{-x}^{\infty}\exp[-t^2/2]\cosh(ht)\,dt}
$$

If x_c is the solution to the previous equation:

$$A_{SK}(h) \;=\; \log(2) - \frac{1}{2}(x_c^2+h^2) + \log\left(\frac{1}{(2\pi)^{1/2}}\int_{-x_c}^{\infty}\exp[-t^2/2]\cosh(ht)\,dt\right),$$

in particular $A_{SK}(0) \sim 0.199$, whereas for large h one has $x \sim \left(\frac{2}{\pi}\right)^{1/2} e^{-h^2/2}$ and consequently $A_{SK} \sim \frac{1}{\pi}e^{-h^2}$ (see Fig.(2)).

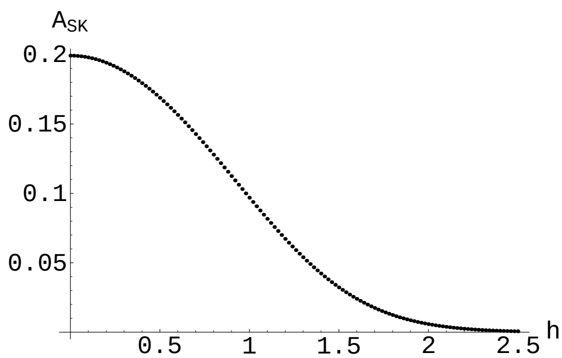
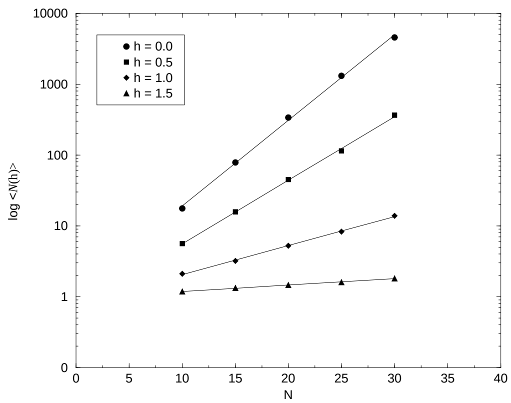


Figure 2: $A_{SK}(h)$

We now turn to the ROM model. We first perform a numerical investigation by doing an exhaustive enumeration of spin configurations and keeping track of metastable states. The system-size dependence of $\log\langle\mathcal{N}(h)\rangle$ is plotted for different values of h in Fig. (3, left). The data are fitted to formula (15), ignoring possible finite size corrections. The resulting $A_{ROM}(h)$ are showed in Fig. (3, right) as data points. Moreover, the saddle point equations corresponding to (16) were solved numerically, and the result is shown by the solid curve in (Fig. (3, right)). The agreement between theory and simulations is very good in spite of the fact that we used admittedly small systems ($N < 30$). As one would expect, metastable states disappear as the magnetic field is increased, since it introduces a tendency towards ferromagnetic



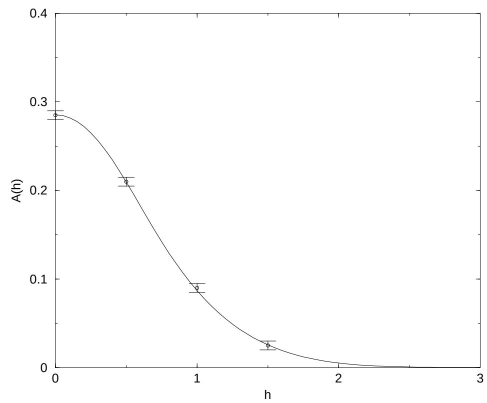


Figure 3: Numbers of metastable states. (Left): Their dependence on system size N at different magnetic field, see legend. (Right): Data points shows the field dependence of $A_{ROM}(h)$ obtained from the fits, while the full curve indicates the analytical results in the thermodynamical limit.

behavior. Most of the processes are the confluence of a metastable state to another with a larger drop of free-energy.

Note that we have $A_{ROM}(0) \sim 0.285$, while the asymptotic behavior for large magnetic field h does coincide with the gaussian case (see Fig.(4,Fig.(5)).

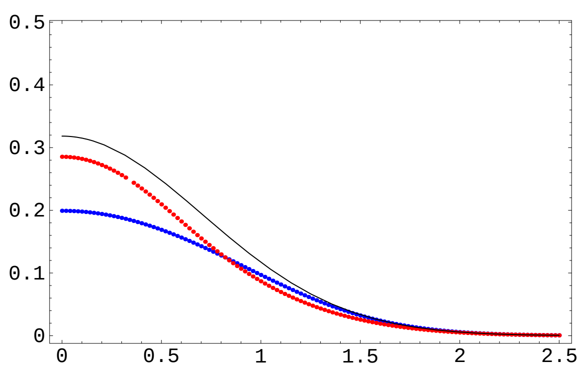


Figure 4: $A_{SK}(h)$ (bottom blue),$A_{ROM}(h)$ (middle red), $\frac{1}{\pi}e^{-h^2}$ (top) .

This indicates that $A_{ROM}(h)$ still remain non-zero for arbitrarily

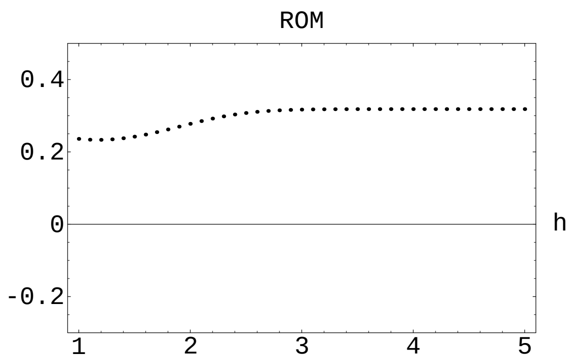


Figure 5: Plot of $e^{h^2} \cdot A_{ROM}(h)$ for values of the magnetic field h between 1 and 5.

large h and hence for any finite value of the external magnetic field the number of metastable states grow exponentially with the system size N. As pointed out by [7] for the SK model, this result is in agreement with the observation that the AT instability occurs for all finite h at zero temperature.

Acknowledgements: This work has been partially supported by the European Commission under the Research Training Network MAQC, n HPRN-CT-2000-00103 of the IHP Programme.

References

[1] J.R.L. de Almeida and D.J. Thouless, "Stability of the Sherrington-Kirkpatrick solution of a spin glass model", *J. Phys. A: Math. Gen.* **11** 983 (1978)

[2] E. Brezin and D.J. Gross, "The External Field Problem in the Large N Limit of QCD", *Phys. Lett.* **97B** 120 (1980)

[3] J.P. Bouchaud and M.Mezard, "Self induced quenched disorder: a model for the glass transition", *J. Physique I* **4** 1109 (1994)

[4] J. Bernasconi,"Low Autocorrelation Binary Sequences: Statistical Mechanics and Configuration Space Analysis", *J. Physique* **48** 559 (1987)

[5] A.J. Bray and M.A. Moore, "Metastable states in spin glasses", *J. Phys C: Solid State Phys.* **13** L469 (1980)

[6] P. Contucci, M. Degli Esposti, C. Giardinà, S. Graffi, "Thermodynamical limit for correlated gaussian random energy models", *cond-mat/0206007*

[7] D.S. Dean, "On the metastable states of the zero-temperature SK mode", *J. Phys. A: Math. Gen.* **27** L889 (1994)

[8] B. Derrida, "Random-energy model: An exactly solvable model of disorder systems", *Phys. Rev. B* **24** 2613 (1981)

[9] M. Degli Esposti, C. Giardina', S. Graffi and S. Isola, "Statistics of energy levels and zero temperature dynamics for deterministic spin models with glassy behaviour", *J. Stat. Phys.* **102** 1285 (2001)

[10] F. Guerra, F.L. Toninelli, "The thermodynamical limit in mean field spin glass model", *cond-mat/0204280*, to appear on22 *Comm. Math. Phys.*

[11] F. Guerra, "Broken replica symmetry bounds in the mean field spin glass model", *cond-mat/0205123*

[12] C. Itzykson and J.B. Zuber, "The Planar Approximation (II)", *J. Math. Phys.* **21** 411 (1980).

[13] E.Marinari, G.Parisi and F.Ritort, "Replica field theory for deterministic models: binary sequences with low autocorrelation", *J. Phys. A: Math. Gen.* **27** 7615 (1994).

[14] E.Marinari, G.Parisi and F.Ritort, "Replica field theory for deterministic models II: A non-random spin glass with glassy behaviour", *J. Phys. A: Math. Gen.* **27** 7647 (1994)

[15] V.M. de Oliveira and J.F. Fontanari, "Landscape statistics of the p-spin Ising model", *J. Phys. A.: Math. Gen.* **30** 8445 (1997)

[16] G. Parisi and M. Potters, "Mean-field equations for spin models with orthogonal interaction matrices", *J. Phys. A.: Math. Gen.* **28** 5267 (1995)

[17] H. Rieger, "The number of solutions of the Thouless-Anderson-Palmer equations for p-spin-interaction spin glasses", *Phys. Rev. B* **46** 14655 (1992)

[18] S.A. Roberts, "Metastable states and innocent replica theory in an Ising spin glass", *J. Phys. C: Solid State Phys.* **14** 3015 (1981)

[19] F. Tanaka and S.F. Edwards, "Analytic theory of the ground state properties of a spin glass: I. Ising spin glass", *J. Phys. F.: Metal Phys.* **10** 2769 (1980)

[20] D.J. Thouless, P.W. Anderson, R.G. Palmer "Solution of a 'solvable model of a pin glass' ", *Phil. Mag.* **35** 593 (1977)

[21] N.J. Vilenkin, *Special functions and the theory of group representation, Translations of Mathematical Monographs*, **22**, American Math. Soc., Providence, Rhode Island, U.S.A.