# Quantum maps: why and why not

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Abstract. We review here some results concerning semiclassical properties of quantum maps. We concentrate mostly on the mathematical aspects of the eigenfunctions distribution problem and few other related aspects in the semiclassical theory of chaotic dynamical systems (quantum chaos). Results for the "arithmetic" case (hyperbolic automorphisms) and more recent non hyperbolic discontinuous dynamics (triangle maps) are presented and discussed.

## 1. Introduction

The aim of understanding the properties of quantum systems with an underlying *chaotic* classical dynamics has an old origin.

In particular, understanding the properties of eigenfunctions and the (related) statistical features of eigenvalues distribution have represented two formidable problems, both from the mathematical and also from the physical point of view.

Actually, for the aim of these notes, we will interpret the term *chaotic dynamical system* as a synonymous of a classical phase flow (continuous or discrete) which is at least ergodic (with respect some *natural* measure: Liouville/Lebesgue) and sometimes (but not always) with positive topological entropy and mixing, i.e. with (e.g. exponential) decay of correlations.

Discrete chaotic dynamics over compact phase spaces has been proposed twenty years ago as testing ground for some of these general semiclassical questions [**BH**]. It is the purpose of these notes to discuss some of these aspects for symplectic maps over $\mathbb{T}^2$.

Some simplified aspects concerning the classical and quantum formalisms, as such as an almost complete classifications of periodic orbits in the case of linear hyperbolic automorphism over $\mathbb{T}^2$, or the finite dimensionality of the corresponding quantum Hilbert space, make these models suitable for a complete or almost complete characterization of important semiclassical properties.

On the other hand, these specific features of the model do not compromise the occurrence of some characteristic behaviours which are typical of more general chaotic systems, as the ones obtained from more realistic Schrödinger operators or from geodesic flow on negative curvature manifold, with the corresponding Laplace-Beltrami operator.

In a very broad sense, we can depict two classes of different mathematical approaches and questions concerning quantum maps over $\mathbb{T}^2$. The first one is represented by the study of quantized linear automorphism. Here, most of the questions concern semiclassical properties of eigenfunctions and some specific non trivial issues about the statistic of eigenvalues, such as the asymptotic degeneracies and the relations with classical periodic orbits. The "arithmetic" nature of the model makes natural the application of mathematical tools out of analytic and algebraic number theory, both at the classical and quantum level. [**PV1, PV2, DEGI, KR**]. The strong connections with other arithmetic model (at least conceptual), as such as the class of arithmetic surfaces with constant negative curvature [**RS**] does not in fact come as a surprise.

It is also important to say that this arithmetic situation is very delicate and the structure is lost (almost completely) under any infinitesimal perturbation. In dealing with semiclassical estimates sometimes one would like to deal with results and properties which are more robust as such as the $C^0$ stability of the classical Anosov property. This is also why most people prefer to look at the second class of examples as the one described below. It is our opinion that despite its high non genericity, the case of linear automorphism can be seen as a appealing mathematical model with interesting connection with number theory. Moreover, some properties of the system, as such as the distribution values of the eigenfunctions, could in fact be more generic than expected.

A second kind of approaches, on the other hand, consists in exploring the semiclassical properties of maps for which any arithmetic structure is fully or partially lost, as in the case of perturbed automorphism or discontinuous maps, such as the Baker map [**SV**] or the more recent *triangle map*, introduced by Casati and Prosen [**CP1, CP2**].

Here the approach is clearly more closely related with genuine dynamical systems and semiclassical techniques. Questions as equidistribution of eigenfunctions, asymptotic (in time and $\hbar$) properties of evolved choerent states, trace formula and diffraction effects induced by the singularities of the maps (in relation with billiards flow) are in fact some of the more explored issues.

It is also worth mentioning that for most of these cases, the numeric clearly shows that these toy models can probably be useful to test some of the basic conjectures regarding the relations between Random Matrix Theory and distribution of quantum eigenvalues.

## 2. Quantization and Dynamics

In order to define the fundamental spectral problem we are interested in, it is essential to introduce the formalism which allows to construct the quantum counterpart of a given discrete symplectic map of $\mathbb{T}^2$.

In the literature there are various ways of associating a suitable quantum Hilbert space and a quantum evolution operator to a given map over the torus [**BH, DE1, DDG**]. Essentially all these quantization procedures yield the same result, even if they rely on different mathematical tools. Which one to use is mostly a matter of personal taste.

For example one can choose a direct approach, even if not completely general, as the one suggested originally by Berry and Hannay [**BH**], based essentially on requiring the periodicity of the wavefunction in both configuration and momentum representation, and on the generating function technique (suitably adapted to the $\mathbb{T}^2$ topology) for defining the quantum dynamics.

If this does not fulfill the personal mathematical taste, then one can choose other methods; from a geometric quantization approach (real or complex), essentially based on the classification of sections of line bundles over the torus ([**DDG**] and references therein) and on the description of the lift to the space of sections of maps on phase space, to a more group theoretical approach, based on the classification of irreducible representations of discrete Heisenberg groups and on the description of the corresponding metaplectic representations ([**DE1**] and references therein). All of these approaches yield the same picture. For the purposes of these notes, we choose here to recall only the main definitions and notations, referring to the above papers for further details. One of the main consequence of implementing quantum mechanics over $\mathbb{T}^2$ is clearly the constrain on the possible value of the Planck constant $\hbar$, which can only take a countable set of values: $2\pi N\hbar = 1$, $N \in \mathbb{N}$. In particular the associated Hilbert space is just an $N$-dimensional Hilbert space $\mathcal{H}_N \approx \mathbb{C}^N$. This fact, which sometimes contributes to a skeptical attitude towards quantum mechanics over the torus (in particular from physicists), does in fact constitute a main advantage in developing semiclassical asymptotics and it allows furthermore to relate semiclassical estimates to results from algebraic number theory and finite field theory.

Roughly speaking, the situation is the following:

- *The Planck's constant* $\hbar$ can only takes values $2\pi\hbar N = 1$, $N \in \mathbb{N}$
- Given $h = 1/N$, $N \in \mathbb{N}$, the *space of states:* $\mathcal{H}_N$ is finite dimensional, of dimension $N$, and consists of periodic point-masses at the coordinates $q = k/N$, $k \in \mathbb{Z}$. We identify $\mathcal{H}_N$ with the $N$-dimensional vector space $L^2(\mathbb{Z}/N\mathbb{Z})$, with inner product
$$< \Psi\, , \Phi >= \frac{1}{N}\sum_{k=0} \bar{\Psi}(k)\Phi(k)$$
- *Quantization of observables:* there is a well defined map[1]
$$f(q,p) \in C^\infty(\mathbb{T}^2) \;\to\; Op_\hbar^W(f)$$
which allows to associate a suitable operator on $\mathcal{H}_N$ to any smooth observable on phase space. This map does in fact implement the usual commutation relations and it can be seen as the discrete analog of the usual Weyl quantization which associate zero-order pseudodifferential operator to any given symbol.
More precisely, if
$$f(q,p) = \sum_{n,m\in\mathbb{Z}} f_{n,m}\, e^{2\pi i(nq-mp)} = \sum_{n,m\in\mathbb{Z}} f_{n,m}\, \chi(n,m)$$
then
$$Op_\hbar^W(f) \;=\; \sum_{n,m\in\mathbb{Z}} f_{n,m} T(n,m), \qquad T(n,m) = e^{i\pi\frac{nm}{N}}\, t_2^m t_1^n$$

[1]To be more precise, there are a two parameter family of this possible quantization procedures. This is in fact related with the classification of sections of line bundles over the torus with respect their quasi-periodic boundary conditions , or equivalently with the existence of a two parameter family of irreducible, $N$-dimensional, non equivalent representation of the discrete Heisenberg algebra. We refer to [**DE1**] and references therein for a more complete treatment.

where the generators $t_1$ and $t_2$ are defined by $(t_2\Psi)(q) = \Psi(q+1)$ and $(t_1\Psi)(q) = e^{\frac{2\pi i q}{N}}\,\Psi(q)$ respectively.

- *Quantum Dynamics:* if $F$ is an area preserving maps over the torus, which is either a linear automorphisms, a fixed time map of a global Hamiltonian flow, or a skew translation, then there exists a well defined unitary operator $U_F$ on $\mathcal{H}_N$, which implements the dynamics at the quantum level. We will not enter here in the details of this construction, but we just recall that a decomposition theorem allows to quantize basically any regular Anosov maps $F$ over $\mathbb{T}^2$, basically just writing it as a composition of a skew translation, an hyperbolic automorphism and a time one map of a global hamiltonian flow. Moreover, we also stress that this quantization scheme can also be extended to a broad class of discontinuous maps (see [**DD**]).

## 3. Equidistribution of eigenfunctions and quantum ergodicity

Assume here that a given symplectic dynamics $F$ of $\mathbb{T}^2$ is given, together with the corresponding family (in $N$) of quantum operators $U_F^{(N)} = U_F$. We denote by $\sigma_N(U_F) = \{\lambda_1, \ldots, \lambda_m\} \subset S^1$ the corresponding spectrum, where $\lambda_j$ is an eigenvalue with multiplicity $d_j$ ( $\sum_{j=1}^m d_j = N$).

Avoiding the discussion about the distribution of the degeneracies, we assume that a given orthonormal set of eigenfunctions $\{\phi_1^{(N)}, \ldots, \phi_N^{(N)}\}$ is given.

Let $\mathcal{M}_F$ being the (convex) set of invariant measures for $F$. In particular we are interested in the case when $\mathcal{M}_F \neq \{m\}$ ( $m$: the Lebesgue measure), as such as for an hyperbolic map or a map that besides preserving Lebesgue it also admits an eventually infinite number of periodic orbits.

Actually for a genuine symplectic Anosov map, $\mathcal{M}_F$ does in fact have a rich structure and it does contain far more invariant ergodic measures, besides $m$ and the atomic ones concentrated on periodic orbits.

From a semiclassical point of view, one of the main problem is to describe the possible limits of the expected values of the observable when $N \to \infty$.

More precisely, we give the following [**Sa**]

Definition 3.1. We denote by $\hat{\mathcal{M}}_F$ the set of probability measures of $\mathbb{T}^2$ which can be obtained as $*$-weak limit of expected values.
More precisely, we say that $\nu \in \hat{\mathcal{M}}_F$ if and only if there exists a sequence of eigenfunctions $\{\phi_{j_k}^{(N_k)}\}_{k\geq 1}$, $N_{k+1} > N_k$ such that for each $f \in C^\infty(\mathbb{T}^2)$

$$\lim_{k\to\infty} < \phi_{j_k}^{(N_k)}, Op_\hbar^W(f)\,\phi_{j_k}^{(N_k)} > = \int_{\mathbb{T}^2} f\,d\nu$$

A main subject in the theory of quantum chaos is to understand the structure of $\hat{\mathcal{M}}_F$ and its relations with the set $\mathcal{M}_F$.

In order to explore this problem, note first of all that it is possible to see that for all the case we are interested, our quantum propagators satisfy an "Egorov property"[2]; namely for any suitable nice observable $f$, say $f \in C^\infty(\mathbb{T}^2)$:

$$\| E_F(f) \| := \| U_F^{-1} Op_\hbar^W(f)\, U_F - Op_\hbar^W(f\circ F) \| \to_{N\to\infty} 0$$

[2]Actually, for smooth dynamics this is one the basic requirement for a well defined quantization scheme $F \to U_F$.

REMARK 3.2. For discontinous maps the above uniform strong convergence is in general not true and we should rely on a weaker condition, which correspond (roughly speaking) in assuming convergence to zero of the operator $E_F(f)$ on asymptotically large (density 1) subspaces of $\mathcal{H}_N$ ([**DD**] and section below)

In both cases, assuming the "Egorov property" implies that $\hat{\mathcal{M}}_F$ only contains classical invariant measures:

$$\hat{\mathcal{M}}_F \subset \mathcal{M}_F$$

So now the basic question, which we would like to stress is important from both the mathematical and physical point of view, becomes the following:

> Which invariant classical measures can we reproduce in the semiclassical limit ?
>
> In particular, if $\mathcal{M}_F$ contains atomic invariant measures concentrated on periodic orbits, can we reconstruct some of them starting from the eigenfunctions (in the limit $N \to \infty$) ?
> In other words, do *scars* exists ?

It is our opinion that this problem is appealing from the mathematical point of view, via its relations to problems and conjectures in number theory and in the theory of L-functions as for the "arithmetic" case of hyperbolic automorphisms ; but the same problem is also very important from a physical point of view, being intimately related to the conceptual meaning of any semiclassical theory:

As it is well known, this question about the structure of $\hat{\mathcal{M}}_F$ does represent a formidable mathematical problem also in the case of the geodesic flow on compact surfaces with constant negative curvature.

DEFINITION 3.3. For a given Lebesgue preserving dynamics $F$ over $\mathbb{T}^2$, we say that *quantum unique ergodicity* (QUE) holds if and only if

$$\hat{\mathcal{M}}_F = \{m\}$$

It has been conjectured that in presence of strong chaoticity for the classical dynamics, e.g. uniform hyperbolicity and/or strongly mixing, the so called *quantum unique ergodicity conjecture* should hold true. This has been proposed by Sarnak for geodesic flows of surfaces of constant negative curvature [**Sa**]

In particular there should be no localization around periodic orbits, i.e. no *scars*. We will see some (partial) confirmation of this conjecture for the case of hyperbolic maps on $\mathbb{T}^2$

In the case of the geodesic flow, as for a broad class of toral maps, a first partial answer can be given by basically only using classical ergodicity (with respect to Lioville/Lebesgue measure) and some minimal information about eigenfunctions in the average.

In fact, a first result is the *Schnirelmann's theorem* which (in this case of the torus) states that the Lioville/Lebesgue measure is achieved with probability 1, i.e. for *almost all* sequences of eigenfunctions. This results has been proven first for ergodic geodesic flows [**CdV, Ze**] and also for suitable Schrödinger operators corresponding to classical hamiltonian over $T\mathbb{R}^n$ [**HMR**]. Equidistribution of almost all eigenfunctions for linear maps on $\mathbb{T}^2$ and their smooth perturbations has been proven in [**BD**], and in [**DD**] for a class of discontinous maps. We would like to make the following comments.

REMARK 3.4.

- *Schnirelmann* type results are a combinations of quite general properties of the systems: classical *ergodicity*, the *Egorov* property and an average results concerning the eigenfunctions. In particular only ergodicity for the classical flow is required, and one does not need to get into the fine structure of individual eigenstates in order to prove equidistribution with probability one.
- When the classical map is uniquely ergodic, i.e. it admits only the Lebesgue measure as invariant measure, the *quantum unique ergodicity* is clearly almost trivial. This is for example the case of parabolic linear maps as studied in [**MR**]. Here, of course, the important issue is the one regarding "ergodicity rates" , namely the speed of convergence to zero of $c_j(f) = | < \Psi_j \, Op_\hbar^W \, (f) \, \Psi_j > - \int_{\mathbb{T}^2} f \, dm \, |$, or to say more precisely the asymptotic statistics of the (eventually normalized) $\{c_j\}_{j=0,\ldots,N-1}$ and the relations with both classical periodic orbits and quantum degeneracies. See [**MR**] for some insight about this highly non-trivial problem.
- In all the main example considered in the literature, namely linear maps over the torus or geodesic flow over surfaces of *constant* negative curvature. The Lebesgue (or in general Liouville) measure plays a very special rule. In the case of hyperbolic automorphisms, for example, not only $m$ is the unique absolutely continuous invariant measure and contains the time asymptotic of almost all orbits (i.e. $m$ is the so called SRB measure), *but it also* contains the spatial ,i.e. topological, structure of the periodic orbits of the systems, namely in all of this cases $m$ is also the *measure of maximal entropy*. It is then clearly important to explore situations where the SRB measure differ from the maximal entropy one, as for some perturbations of the linear automorphisms. This problem is currently under investigation.

Violation to QUE can then happen only on very *exceptional sequences* of eigenfunctions. For the linear case one rely on number theory in order to gain extra information about individual eigenfunctions and try to prove (or disprove) quantum unique ergodicity.

**3.1. The Quantum Cat.** Let $A \in SL(2, \mathbb{Z})$. We let $\omega = \mathrm{Tr} A$ and assume $|\omega| > 2$ (hyperbolicity). Moreover we denote by $\lambda$ its greater eigenvalue and by $v = (v_1, v_2)$ the corresponding eigenvector. It is very well known that this does constitutes the easiest example of Anosov maps.

We denote by $Q[\sqrt{d}]$ the real quadratic field extension associated to the characteristic polynomial of $A$, where $d$ is the field discriminant. We denote by $\mathcal{D} = \mathbb{Z}[\lambda] \subset Q[\sqrt{d}]$ the order associated to $A$. Both classical (periodic orbits distribution) and quantum properties of the system induced by $A$ on the torus can be studied through multiplicative actions over $\mathcal{D}$. [3] In particular, there exists an ideal $I_A = \mathbb{Z}[v_1, v_2] \approx \mathbb{Z}^2$ such that the action of multiplication by $\lambda$ on $I_A$ has a matrix representation given by $A$ (in the basis given by $v_1, v_2$).

Non uniqueness of prime factorization in Dedekind domain force Ideal theory into the picture and it is possible to reduce the problem of periodic orbit distribution to considerations regarding factorization of ideals into prime ideals. This fact have been exploited in [**PV1, PV2, Ke**] and also used in [**DEGI**].

[3] Actually over quotients of the form $\mathcal{D}/(M)$, $(M)$ a not always principle ideal: $M \in \mathbb{Z}$ and $(M) = M\mathcal{D}$

In particular one can use these techniques to understand the distribution of the maximal period of periodic orbits lying on discrete lattices of points with rational coordinates and fixed prime denominator [4]. In this prime case, thanks also to the fact that basically any such ideal $(p)$ can decompose in only basically two different way (split or inert), these relations are clear and not too difficult to explore. The usual Chinese remainder theorem then allows then to examine the general case and to relate the erratic behaviour of the period of classical orbits to well known conjectures regarding the order of elements in finite fields. We refer to the previous references for a detailed and complete treatment. Also for the quantum model, the algebraic and number theoretical techniques are crucial and have been used in particular in [**DEGI**] and [**KR**]. Here is a very brief account of the main results:

- Given $A \in SL(2,\mathbb{Z})$, the quantum propagator $U_A$ (for a fixed $N \in \mathbb{Z}$) depends only on the reduction of $A \mod 2N \in SL(2,\mathbb{Z}/2N\mathbb{Z})$.
- If we define the subgroup

$$\Gamma(4,2N) = \{A \in SL(2,\mathbb{Z}/2N\mathbb{Z}) : A = I \mod 4(2), \text{if } N \text{ even (odd)}\}$$

  then there exists an overall choice of the phase for $U_A$ such that $A \in \Gamma(4,2N) \to U_A$ is a homomorphism:

$$U_{AB} = U_A \circ U_B$$

- "Egorov is exact": $\forall A \in \Gamma(4,2N)$ and $\forall (n,m) \in \mathbb{Z}^2$

$$U_A^{-1} T(n,m) U_A = T(A^t(n,m))$$

- If one factor $2N = \prod_p p^{k_p}$, then thanks to the Chinese remainder theorem one can reduce most of the analysis to the prime case, through the following isomorphisms:

$$\mathbb{Z}/2N\mathbb{Z} \approx \prod_p \mathbb{Z}/p^{k_p}\mathbb{Z},$$

  which implies the following natural decomposition (see e.g. [**KR**] )

$$U_A = \bigotimes_p U_A^{(p)}, \qquad T(n,m) = \bigotimes_p T^{(p)}(n,m)$$

A very first step toward QUE has been performed in [**DEGI**], where (roughly speaking) one relates eigenfunctions for prime values of $\hbar$ to exponential sums over finite fields (Kloosterman sums). In this way, the problem is reduced to one-variable exponential sums, which can be estimated using Weil's theorem, namely the Riemann hypothesis for a curve over a finite field.

Theorem 3.5. *Let $A$ an hyperbolic automorphism and let $N_k$ any increasing sequence of primes numbers such that the degeneracy of $U_A$ is uniformly bounded [5], then for any given sequence of eigenfunctions $\{\Psi_{j_k}^{(N_k)}\}_{k\geq 1}$, $\Psi_{j_k}^{(N_k)} \in \mathcal{H}_{N_k}$, any*

[4] The set of periodic orbits does in fact coincide with rational points over the torus. Moreover, the distribution of the periods in terms of the lattice denominator turns to be related to quantum degeneracies.

[5] For a given $N$ prime it is not difficult to see that the multiplicity is the same for all eigenvalues . Moreover it is still an open problem to show that there are infinitely many primes where the degeneracy is bounded. Actually, the generalized Riemann hypothesis will guarantees a positive proportion of primes satisfying this assumption.

*$f \in C^\infty(\mathbb{T}^2)$ and for any $\epsilon > 0$*

$$< \Psi_{j_k}^{(N_k)}\, Op_\hbar^W\,(f)\, \Psi_{j_k}^{(N_k)} >= \int_{\mathbb{T}^2} f\, dm \,+\, O_{f,\epsilon}(N^{-1/2+\epsilon})$$

A substantial improvement of the previous result has been recently obtained in [**KR**], where the authors were able to reproduce ( for any given $N$) a subgroup of unit elements in $\mathcal{D}$ (up to some quotients ) which in fact give rise (roughly speaking) to a set of unitary propagators commuting with $U_A$. In analogy with arithmetic surfaces, these operators have been called Hecke operators. They have no classical counterpart and they are in fact strongly $N$-dependent. Since the Hecke operators commute with $U_A$, they act on its eigenspaces, and since they commute with each other, there is a basis of $\mathcal{H}_N$ consisting of joint eigenfunctions of $U_A$ and the Hecke operators (Hecke eigenfunctions). The result is then the following (see [**KR**] for details)

THEOREM 3.6. *For all normalized Heck eigenfunctions $\Phi \in \mathcal{H}_N$ of $U_A$ and for any $f \in C^\infty(\mathbb{T}^2)$, the expectation values $< \Phi, Op_\hbar^W\,(f)\,\Phi >$ converge to the phase-space average of $f$ as $N \to \infty$. Moreover, for all $\epsilon > 0$, we have*

$$< \Phi, Op_\hbar^W\,(f)\,\Phi >= \int_{\mathbb{T}^2} f\, dm \,+\, O_{f,\epsilon}\left(N^{-1/4+\epsilon}\right), \;\; \textit{as } N \to \infty$$

REMARK 3.7.

- If one restricts to sequences of $N$'s with uniformly bounded degeneracy, then clearly one has equidistribution with respect any given basis of eigenfunctions.
- The exponent $1/4$ is probably non optimal and the most likely one should be $1/2$ as for the prime case [**DEGI**]

**3.2. The triangle map.** In [**CP1, CP2**] a two-parameter family of discontinous maps over the torus , the so called "triangle maps" (here denoted by $F_{\alpha,\beta}$, and always $\alpha, \beta \in [0,1] \setminus \mathbb{Q}$) , has been introduced in connection with the mixing properties of flows in certain triangular billiards. Despite its simplicity, very little is proven about the ergodic and mixing properties of such maps. The following is basically the one introduced in [**CP2**]. First of all, let $\theta(q)$ be the discontinuous function over the circle given by $\theta(q) = -1$ if $0 \le q \le 1/2$ and $\theta(q) = 1$ otherwise.

For any $\alpha, \beta \in [0,1]$, we define the map $F_{\alpha,\beta}$ as

$$F_{\alpha,\beta}(q,p) \;=\; (q + p + \beta + \alpha\theta(q)\,,\, p + \beta + \alpha\theta(q))$$

Note that $F_{\alpha,\beta}$ can be written as the composition of three elementary maps,

$$F_{\alpha,\beta} \;=\; B \circ R_\beta \circ G_\alpha,$$

where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a parabolic map, $R_\beta(q,p) = (q, p+\beta)$ is a translation in the $p$ direction and $G_\alpha$ is the "discontinous part" of the dynamics $G_\alpha(q,p) = (q, p + \alpha\theta(q))$.

It is surprising (at least for us) that there are basically no results about the dynamical properties of such map. As far as we know, even ergodicity for $\alpha \ne 0$ is still not proven (even if probably true for irrational $\alpha$'s). One can state rigorous and well known results only when $\alpha = 0$, whereas we have to rely on numerical experiments and conjectures when facing the discontinous case ($\alpha \ne 0$) (see

[**CP1, CP2**]). Using the previous decomposition, it is also quite easy to define the corresponding unitary quantum propagator $U_{\alpha,\beta}$, as described for example in [**DE2**]. It is our opinion that this class of maps represents a nice testing ground for some fundamental questions in the semiclassical theory of chaotic dynamical systems. It is also very clear in fact that for these non-hyperbolic maps the only fundamental process which could eventually reproduce non trivial dynamical properties (as such as the decay of correlations) rely on the "cutting" effect produced by the discontinuities.

[**DE2**] is intended to be a first step in developing a semiclassical theory for such kind of maps. In particular, we rely on results and formalism developed in [**DD**] in order to prove some semiclassical Egorov estimates (non completely trivial because of the discontinuities) and equidistribution of almost all eigenfunctions for ergodic triangular maps .

We refer to [**DD**] and [**DE2**] for a more general discussion about the semiclassical properties of discontinuous maps and the rule of diffraction effects close to the singularities.

Given now a function $f \in C^\infty(\mathbb{T}^2)$, in [**DD**] we were interested in the behaviour of the following operator which measure the amount to which quantization and evolution do not commute, $\forall k \geq 1$, and $N \to \infty$:

$$E^{(k)}_{\alpha,\beta}(f) = U^{-k}_{\alpha,\beta} Op^W_\hbar\,(f)\, U^k_{\alpha,\beta} - Op^W_\hbar\,\left(f \circ F^k_{\alpha,\beta}\right), \tag{3.1}$$

where we have assumed that $f \circ F_{\alpha,\beta} \in C^\infty(\mathbb{T}^2)$.

As it is also clear from the discussion in [**DD**], standard semiclassical analysis breaks down in the presence of singularities due to diffraction effects and this does in fact turn into preventing a uniform convergence to zero of the previous operators (for $k$ fixed and $N \to \infty$). This can not be avoid, but fortunately a lot less is necessary for proving equidistribution (for most) of the eigenfunctions. We gave the following definition [**DD**]

Definition 3.8. Let $T$ be a map on the torus and $V_T$ its quantization. We say $(T, V_T)$ satisfies an Egorov estimate up to time $K$ if the following holds:

(1) there exists a closed set $\Sigma_K$ of measure zero so that, if $f \in C^\infty(\mathbb{T}^2)$ is supported away from $\Sigma_K$, then $f \circ T^\ell \in C^\infty(\mathbb{T}^2)$, $\forall \ell \leq K$;
(2) for each family of orthonormal bases $\{\psi^{(N)}_j\}_{j=1,\dots,N}$, there exists a family of index sets $\mathcal{E}^{(K)}(N) \subset \{1, 2, \dots, N\}$ satisfying $\mathcal{E}^{(K)}(N)/N \to 1$ so that $\forall\, 0 \leq \ell \leq K$

$$\sup_{j \in \mathcal{E}^{(K)}(N)} \parallel E^{(\ell)}_{\alpha,\beta}(f)\psi^{(N)}_j \parallel \stackrel{N\to\infty}{\longrightarrow} 0.$$

In [**DD**] we then proved that this is sufficient for having equidistribution of almost all eigenfunctions (see Theorem 2 in [**DD**]). In [**DE2**] we used this fact for proving the second statement of the following Theorem.

Theorem 3.9.

- *For all $\alpha, \beta \in [0, 1]$, $F_{\alpha,\beta}$ satisfy the Egorov estimate up to time $K$, for any finite $K$.*
- *Assume $F_{\alpha,\beta}$ ergodic (w.r.t. $m$) and let $U_{\alpha,\beta} = U$ the corresponding quantum operator.*

*For any integer $N \geq 1$, denote by $\{\phi_1^{(N)}, \ldots, \phi_N^{(N)}\}$ an arbitrary set of normalized eigenvectors of $U$, then there exists a subset $\mathcal{E}(N) \subset \{1, \ldots, N\}$ such that*

(1)
$$\frac{|\mathcal{E}(N)|}{N} \to 1, \qquad N \to \infty$$

(2) $\forall f \in C^\infty\left(\mathbb{T}^2\right)$*:*
$$< \phi_{j_N}, Op_\hbar^W(f)\, \phi_{j_N} > \to \int_{\mathbb{T}^2} f\, d\mu,$$
*when $N \to \infty$ and $j_N \in \mathcal{E}(N)$.*

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