1. Conservation laws

We consider a general system evolving in time. In the simplest situations we can assume that all significant changes happen in a specific direction. For example, while modelling convection in a tube or a bar with constant cross-section. Hence we can assume that our model is one-dimensional.

Hence we describe the monodimensional tube (or bar) as segment $[a, b]$, and we study the concentration of a state variable in the tube (which can be temperature, charge, concentration of a substance. A conservation law impose that any state variable is not created, and that the rate of change of quantity of the state variable in the spatial segment $[a, b]$ equals the total flow at the point $x = a$ minus the flow at the point b plus the total contribution of sources in the segment.

We will denote u the concentration of the state variable, and its rate of change in time will be computed as

$$\frac{d}{dt}\int_a^b u(x,t)dx$$

The concentration of sources will be denoted with f, so that the total contribution of sources will be

$$\int_a^b f(x,t)dx$$

The flow through the endpoints can be evaluated as

$$\phi(a,t) - \phi(b,t) = -\int_a^b \phi_x(x,t)dx$$

Hence the conservation law becomes

$$\frac{d}{dt}\int_a^b u(x,t)dx = -\int_a^b \phi_x(x,t)dx + \int_a^b f(x,t)dx$$

Or in other words

$$\int_a^b (u_t(x,t) + \phi_x(x,t) - f(x,t))dx = 0$$

Since this happens on each interval $[a, b]$ we finally obtain

$$u_t(x,t) + \phi_x(x,t) - f(x,t) = 0$$

2. First order differential equation

2.1. **Transport equation.** We assume that a liquid is flowing in the tube and we assume that a solid polluting substance is is transported in the channel. The concentration of the contaminant will be denoted by u. In this case, since the contaminant is moving together with the liquid we can assume that the concentration is proportional to the flux:

$$u = c\phi,$$

where c can be interpreted as the speed.

In particular the equation becomes:

$$u_t(x,t) + cu_x(x,t) - f(x,t) = 0 \tag{2.1}$$

In higher dimension we can assume that the contaminant is transported within a liquid surface, so that the problem becomes 2-dimensional:

$$u_t(x,y,t) + au_x(x,y,t) + bu_y(x,y,t) - f(x,y,t) = 0$$

We can **stationary** the associated problem independent of t:

$$au_x(x,y) + bu_y(x,y) = f(x,y) \tag{2.2}$$

Clearly equations (2.1) and (2.2) have a different meaning, but formally (2.2) is more general than (2.1), since it has an extra coefficient. Note that equation (2.1) depends on space and on time, hence it is natural that the role of the time-variable t is different from the role of the space variable x. On the contrary equation (2.2) only involves two spatial variables and they roles are completely symmetric.

However from a purely mathematical point of view we can consider equation (2.2) more general than equation (2.1), since it has an extra coefficient.

2.2. **constant coefficients equation.** Let us consider the general linear PDE of the first order with constant coefficient:

$$u_t + cu_x = 0 \tag{2.3}$$

Her $u = u(x,t)$ is the unknown function c is a constant. We can interpret this equation as the directional derivative of the function u in the direction of the vector $(c, 1)$.

Definition 2.1. *Let $(c, 1)$ be a vector in R^2. Assume that there exists*

$$lim_{s\to 0}\frac{u((x,t)+s(c,1))-u(x,t)}{s|(c,1)|}.$$

The we call this limit directional derivative in the direction of this vector $(c, 1)$, and we denote it $D_{(c,1)}u$.

Remark 2.2. *We note that if there exists the directional derivative of u in the direction of the vector $(c, 1)$, then*

$$D_{(c,1)}u = \frac{d}{ds}u((x+sc, t+s)) = cu_x + u_t.$$

Hence equation (2.3) expresses the derivative of u along a line parallel to the vector $(c, 1)$.

More generally we can try to solve equation (2.3) along a curve $\gamma(t) = (x(t), t)$ The function u restricted to this curve will be denoted v:

$$v(t) = u(x(t), t).$$

we now check that v satisfies a suitable equation:

$$v'(t) = u_x(x(t), t)x'(t) + u_t(x(t), t). \tag{2.4}$$

Hence in order to have solution we can require that

$$x'(t) = c,$$

so that, from (2.4) we get

$$v'(t) = u_t(x(t), t) + cu_x(x(t), t) = 0$$

We have now transformed the first order PDE in a system of ordinary differential equations:

$$\begin{cases} x'(t) &= c \\ v'(t) &= 0 \end{cases} \tag{2.5}$$

The method we have applied is called the characteristic method, and the curve $\gamma(t) = (x(t), t)$ is called characteristic.

Remark 2.3. *If the equation has constant coefficients, then the characteristic are lines*

We already know that a system of ordinary differential equation has unique solution only if we impose initial condition. It is natural to impose initial conditions also to our transport equation

Proposition 2.4. *If c is a constant, and $f : R \to R$ a continuous function, then the problem*

$$u_t + cu_x = 0 \quad u(x,0) = f(x) \tag{2.6}$$

has a solution of the form

$$u(x,t) = f(x - ct)$$

Proof

We apply the method of characteristics, and look for a solution of the system (2.5) which clearly becomes

$$\begin{cases} x(t) &= ct + x_0 \\ v(t) &= costant \end{cases} \tag{2.7}$$

In particular we get

$$v(t) = v(0) = u(x(0), 0) = f(x_0)$$

Now we have obtained both the values of x and v in terms of (x_0, t):

$$\begin{cases} x(t) &= ct + x_0 \\ v(t) &= f(x_0) \end{cases} \tag{2.8}$$

Since we want to find the solution u in terms of (x,t) we have to invert:

$$x_0 = x - ct$$

$$u(x(t), t) = v(t) = f(x_0) = f(x - ct)$$

Definition 2.5. *it is natural to call travelling wave any function of the form*

$$f(x - ct)$$

Indeed the graph of the function $f(x - ct)$ at the time t is the same of the graph of the function $f(x)$, but it is translated to the right (if c is positive) by the value ct. Thus with growing time, we see the graph of f translating to the right with a speed c.

Proposition 2.6. *The solution of equation (2.11) is unique. Indeed we were able to explicitly write its expression.*

Example 2.7. *Solve the problem*

$$u_t - 3u_x = 0, \quad u(x,0) = x^2.$$

and show that the solution is $u(x,t) = (x + 3t)^2$

2.3. **linear equations with non constant coefficients.** The method we applied before can be applied also to equations (2.2) called linear equations.

Remark 2.8. *If a, b, f are functions of class C^1, such that*

$$a(x,0) \neq 0$$

for every x, then we can find characteristic for the equation

$$a(x,y)u_x + b(x,y)u_y = g(x,y) \tag{2.9}$$

The method of characteristic can be applied as before: we try to solve the equation along suitable curve in the plane x, y. However in general the curve will be parametrized by a new variable s: $\gamma(s) = (x(s), y(s))$. As before we set:

$$v(s) = u(x(s), y(s)),$$

and we get:

$$v'(s) = u_x(x(s), y(s))x'(s) + u_t(x(s), y(s))y'(s).$$

If we pretend that this function satisfy the equation, we have to impose that

$$\begin{cases} x'(s) &= a(x(s), y(s)) \\ y'(s) &= b(x(s), y(s)) \end{cases} \tag{2.10}$$

Consequently we have

$$v'(s) = f(x(s), y(s))$$

Proposition 2.9. *If a, b, f are functions of class C^1, and such that*

$$a(x, 0) \neq 0$$

for every x, then the system

$$a(x, y)u_x + b(x, y)u_y = g(x, y) \quad u(x, 0) = f(x) \tag{2.11}$$

has a solution, at least in a neighborhood of the initial condition

We repeat the previous steps:

- we find characteristics We have already noted that the characteristics can be written

$$\begin{cases} x'(s) &= a(x(s), y(s)) \\ y'(s) &= b(x(s), y(s)) \\ v'(s) &= f(x(s), y(s)) \end{cases} \tag{2.12}$$

  with initial conditions

$$\begin{cases} x(0) &= x_0 \\ y(0) &= 0 \\ v(0) &= u(x(0), 0) = f(x_0) \end{cases} \tag{2.13}$$

- we solve the equation, This is a Cauchy Problem, with second member of class C^1. Hence it has a solution unique, but defined only in a neighborhood of 0. The solutions will be dependent on s and on the initial value x_0: The characteristics will be of the form

$$\phi(x_0, s) = (x(x_0, s), y(x_0, s)) \tag{2.14}$$

- we invert and find a solution u depending on the variables (x, y). Condition

$$a \neq 0$$

  is used exactly to ensure invertibility. Indeed for $a = 0$ the characteristics would be

$$x' = a, t' = b$$

  In other words t would be constantly equal to 0, the characteristics will be coincident with the x axis, and it would be impossible to invert.

  More formally we want prove invertibility using the local Invertibility Theorem (see Teo 2.10 below). Hence we have to prove that the Jacobian determinant of ϕ is different from 0, at least when $s = 0$

$$J_\phi = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial x_0} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial x_0} \end{pmatrix}$$

In particular along the characteristics curves in (2.14) for $s = 0$ we have

$$\left\{ \begin{array}{ll} \frac{\partial x}{\partial s} = a, & \frac{\partial x}{\partial x_0} = 1, \\ \frac{\partial y}{\partial s} = b, & \frac{\partial y}{\partial x_0} = 0, \end{array} \right. \tag{2.15}$$

So that the determinant is $-b$, which is different from 0. This implies that ϕ is invertible.

Let us state a general condition for invertibility of a function

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2$$

Theorem 2.10. *(Local invertibility theorem) Let us consider a function*

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2$$

$\phi = (x(s, x_0), y(s, x_0))$*. We call Jacobian determinant*

$$J_\phi = \begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial s} \end{pmatrix}$$

The function ϕ is locally invertible if its Jacobian determinant is different from 0.

Example 2.11. *Solve the problem*

$$u_t + 2tu_x = 1 \quad u(x, 0) = sin(x)$$

- *find charachteristics*

$$\left\{ \begin{array}{lcc} x'(s) & = & 2t(s) \\ t'(s) & = & 1 \\ v'(s) & = & 1 \end{array} \right. \tag{2.16}$$

- *solve the equation. From the second:*

$$t = s$$

$$x' = 2t \Rightarrow x'(s) = 2s \Rightarrow x(s) = s^2 + x_0$$
$$v' = 1 \Rightarrow v(s) = s + v(0) = s + sin(x_0)$$

The we get the solution, expressed in terms of the variables (s, x_0).

$$\left\{ \begin{array}{c} t = s \\ x(s) = s^2 + x_0 \\ v(s) = s + sin(x_0) \end{array} \right.$$

- *We invert:*

$$x_0 = x - s^2 = x - t^2$$
$$v = t + sin(x - t^2)$$

The solution can be expressed as

$$u(x, t) = t + sin(x - t^2)$$

Note that in this example the characteristics are

$$(s, s^2 + x_0),$$

which are not strict lines