1. QUASILINEAR EQUATIONS

1.1. **an explicit example of non regular solution.** A first order PDE is called quasilinear if its coefficients depend on the variable u. In particular let us consider the class of problems

$$u_t + c(u)u_x = 0 \quad u(x,0) = f \tag{1.1}$$

A simple example could be enough to ensure that in this case the characteristics can cross.

Example 1.1. *We call Burger equation the equation*

$$u_t + uu_x = 0 \quad u(x,0) = f,$$

Let us find solutions with initial conditions

$$f(x_0) = \begin{cases} 1 & x_0 < 0 \\ 1-x & 0 < x_0 < 1 \\ 0 & x > 1 \end{cases}$$

The characteristics take the form:

$$\begin{cases} t' = & 1 \\ x' = & v \\ v' = & 0 \end{cases}$$

Hence they can be evaluated as

$$\begin{cases} t = & s \\ x' = & v(0)s + x_0 \\ v(s) = & v(0) \quad constant \end{cases}$$

The constant value of v depend on the value of x_0.

Indeed if $x_0 < 0$ $v(0) = 1$, hence inserting the value of $v(0)$ in (1.1) we obtain

$$\begin{cases} t = & s \\ x' = & s + x_0 \\ v(s) = & 1 \quad constant \end{cases}$$

In the region reached by these characteristic lines $u = 1$ identically

If $x_0 > 1$ $v(0) = 0$, hence inserting the value of $v(0)$ in (1.1) we obtain

$$\begin{cases} t = & s \\ x' = & x_0 \\ v(s) = & 0 \quad constant \end{cases}$$

In the region reached by these characteristic lines $u = 0$ identically

The characteristics are represented in figure 1. The function u is constantly equal to 1 *on the characteristics starting on the negative axis. It is constantly equal to* 0 *along the characteristics starting on the positive axis. But there lines cross, and it is not clear if the solution will be defined at points there there is the crossing. And if we can define it, it is necessarily discontinuous (see also figure 2).*

We have now to define a notion of non regular solutions, which seems necessary in this case, but it is difficult to give, since the equation contains derivatives.

1.2. **Integral solutions of differential equations.** We will now introduce a weak notion of solution. In order to do this, we start with the notion of test function:

Definition 1.2. *a function ϕ is called a test function, or a function with compact support if it is of class C^∞ and it is identically* 0 *out of a bounded set. The set of test functions is denoted C_0^∞*

(the definition can be given in R or in R^n for every n.)

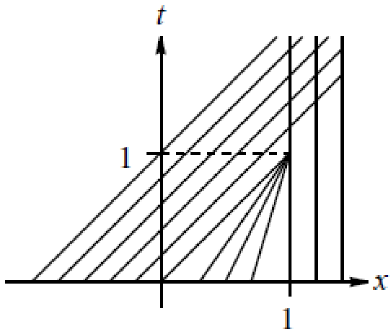


FIGURE 1. the characteristics start crossing at the point $(1,1)$

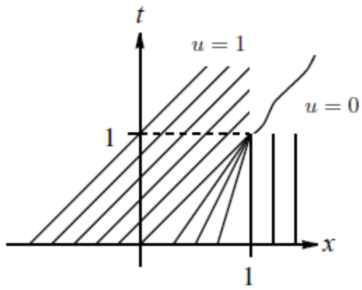


FIGURE 2. starting at the point $(1,1)$ there will be a curve such that on the left $u = 1$, and on the right the function is identically 0. Hence it is discontinuous

Proposition 1.3. *Assume that u is continuous. Then u is identically* 0 *if and only if*

$$\int u\phi dx = 0 \tag{1.2}$$

for every $\phi \in C_0^\infty$.

Proof. We prove the assertion in $\mathbb{R}$. The same assertion holds in $\mathbb{R}^n$. Assume by contradiction that u is not identically 0. Then u is different from 0 at a point x_0. We can assume that $u(x_0) > 0$. Then it is positive in a neighborhood of the point x_0:

$$u(x) > 0 \text{ if } x \in]x_0 - a, x_0 + a[$$

If we choose a test function ϕ which is 0 out of this same interval, but positive on it, we have that both ϕ and u are both positive on $]x_0 - a, x_0 + a[$, and their product is 0 outside this set. Then

$$\int_{\mathbb{R}} u\phi = \int_{x_0-a}^{x_0+a} u\phi > 0$$

which contradicts the assumption (1.2).

□

Remark 1.4. *If c is continuous and G is a primitive of c, the*

$$u_t + c(u)u_x = 0$$

can be equivalently rewritten as

$$u_t + (G(u))_x = 0.$$

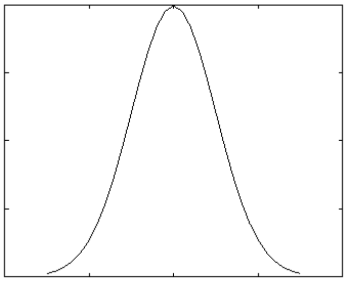


FIGURE 3. a test function ϕ

This is a simple computation, since

$$(G(u))_x = G'(u)u_x = c(u)u_x$$

Proposition 1.5. *Assume that u is of class C^1. Then and it is a solution of the system (1.1) if and only if*

$$\int_0^\infty \int_{\mathbb{R}} u\phi_t dxdt + \int_0^\infty \int_{\mathbb{R}} G(u)\phi_x dxdt + \int_{\mathbb{R}} f(x)\phi(x,0)dx = 0 \textit{ for all } \phi \in C_0^\infty \tag{1.3}$$

Proof. We prove the implication $\Rightarrow$, the other being similar.

$$u_t + c(u)u_x = 0 \quad u(x,0) = f$$

we can multiply both members by ϕ and get

$$\int u_t\phi + \int (G(u))_x\phi = 0 \quad \text{for all } \phi \tag{1.4}$$

and

$$u(x,0) = f$$

now we integrate by parts. We have to be very careful: in the first term we integrate by parts in the variable t:

$$\int_{\mathbb{R}} \Big(\int_0^\infty u_t\phi dt \Big) dx = \int_{\mathbb{R}} \Big[u\phi \Big]_0^\infty dx - \int_{\mathbb{R}} \Big(\int_0^\infty u\phi_t dt \Big) dx = \tag{1.5}$$

(since $\phi = 0$ at infinity):

$$= -\int_{\mathbb{R}} u(x,0)\phi(x,0)dx - \int_{\mathbb{R}} \Big(\int_0^\infty u\phi_t dt \Big) dx$$

(since $u(x,0) = f(x)$)

$$= -\int_{\mathbb{R}} f(x)\phi(x,0)dx - \int_{\mathbb{R}} \Big(\int_0^\infty u\phi_t dt \Big) dx$$

In the second term we integrate by parts in the variable x:

$$\int_0^\infty \Big(\int_{\mathbb{R}} (G(u))_x\phi dx \Big) dt = \int_0^\infty \Big[u\phi \Big]_{x=-\infty}^{x=\infty} dt - \int_0^\infty \Big(\int_{\mathbb{R}} G(u)\phi_x dx \Big) dt = \tag{1.6}$$

(since ϕ is 0 at infinity)

$$-\int_0^\infty \Big(\int_{\mathbb{R}} G(u)\phi_x dx \Big) dt.$$

Now we insert both (1.5) and (1.6) into equation (1.4), and we get:

$$-\int_{\mathbb{R}} f(x)\phi(x,0)dx - \int_{\mathbb{R}} \Big(\int_0^\infty u\phi_t dt \Big) dx - \int_0^\infty \Big(\int_{\mathbb{R}} G(u)\phi_x dx \Big) dt = 0$$

which is the thesis.

□

Since expression (1.3) can be defined also for non regular functions u, we will adopt it as a definition of a non regular solution of the quasilinear equation

Definition 1.6. *We will say that u is a weak (or integral) solution of the system (1.1) if and only if the following condition is satisfied*

$$\int_0^\infty \int_{\mathbb{R}} u\phi_t dx dt + \int_0^\infty \int_{\mathbb{R}} G(u)\phi_x dx dt + \int_{\mathbb{R}} f(x)\phi(x,0)dx = 0 \text{ for all } \phi \in C_0^\infty \tag{1.7}$$

We will see that this definition does not only allow to define a solution, but will tell us which is the shape of the solution itself.

Let us go back to the solution of the Burger equation with the initial conditions of Example 1. We have mentioned the fact that the solution can be discontinuous. Precisely there will be a curve $x = \xi(t)$ such that u is constantly be equal to 1 on the left of the curve, and 0 on the right.

We will now to study the shape of the jump interface for weak solutions of the general quasilinear equation (1.7). We will assume that u is a solution on an open set $V \subset \mathbb{R} \times [0, \infty[$, it is discontinuous along an interface $x = \xi(t)$ and that it is regular out of this curve. We will then call V_+ the set on the right of the interface:

$$V_+ = \{(x,t) : x > \xi(t)\}$$

and we will call $u_+(\xi(t), t)$ the limit value attained by u approaching the point $(\xi(t), t)$ from the right. Analogously V_- is the set on the left of the interface:

$$V_- = \{(x,t) : x < \xi(t)\}$$

and $u_-(\xi(t), t)$ the limit value attained by u approaching from the left.

Theorem 1.7. *Rankine-Hugoniot jump condition. Assume that u is a weak solution of (1.7) on a set $V \subset \mathbb{R} \times [0, \infty[$ and that it is discontinuous across the curve $x = \xi(t)$ but u is smooth out of this curve.*

Then u satisfies the condition

$$\xi'(t) = \frac{G(u_+) - G(u_-)}{u_+ - u_-}$$

We note that this condition allows to find the solution of the Burger equation, with the condition of example 1.1

The Burger equation can be written in the form (1.1) choosing $c(u) = u$, hence its primitive is

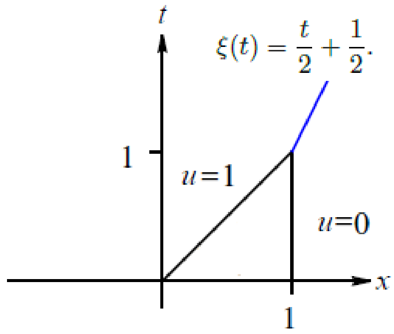
$$G(u) = \frac{u^2}{2}$$

Indeed we already know that the first point in which characteristics cross is $(1, 1)$, hence the curve $(\xi(t), t)$ starts from this point and satisfies:

$$\xi'(t) = \frac{G(u_+) - G(u_-)}{u_+ - u_-},$$

We also know that

$$u_+ = 0, \quad u_- = 1$$



Hence we obtain

$$\xi'(t) = \frac{1}{2},$$
$$\xi(1) = 1$$

this means that

$$\xi(t) = \frac{t}{2} + C,$$

and imposing the initial condition,

$$\xi(t) = \frac{t}{2} + \frac{1}{2}.$$

Finally we get the expression the explicit expression of the solution as in figure 4.