GRAPH THEORY

LECTURE 3

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DEFINITION OF A GRAPH

A graph **G** is an ordered triple (**V**(**G**), **E**(**G**), $\boldsymbol{\psi_G}$) consisting of :

- a non-empty set of vertices **V**(**G**)
- a set **E**(**G**) of edges
- an incidence function $\boldsymbol{\psi_G}$: E(G)→ $[V(G)]^2$

that associates an edge to an unordered pair of vertices which are the ends of the edge.

LECTURE PLAN

- Equivalence Relations and Partitions
- Implications and Contrapositives
- Proof by Induction
- Distances in Graphs

Relate these concepts to what they already know

- How equivalence relations connect to graph isomorphisms, components, and bipartiteness.
- Using contrapositives in proving graph properties.
- Proof by induction applied to graph structures .
- Brainstorming on Graph Distances.

EQUIVALENCE RELATIONS & PARTITIONS

Sometimes the notion of “equals ” is less evident

Example: Is $0 = 2\pi$?

When we consider 0 and 2 as numbers, the answer is clearly: “No!”

After all, $2\pi \approx 6.28 > 0$. On the other hand, when we consider them

as angles measured in radians, the answer is a definite: “Yes!” How can it be that

sometimes 0 does "equal" 2π and sometimes it does not?

Two graphs H, G are identical if and only if

$$\Psi_G = \Psi_H,$$
$$V(G) = V(H),$$
$$E(G) = E(H)$$

EQUIVALENCE RELATIONS

What is a relation?

A relation ~ is a predicate in two free variables, **a** and **b.**

If the predicate ~ is true for a particular choice of **a** and **b**, we write **a~ b**, which we read as "**a** is related to **b**".

DEFINITION

Suppose that ~ is a relation. It is:

- **reflexive** if, for all x , x ~ x .
- **symmetric** if, for all x, y , if x ~ y , then y ~ x.
- **antisymmetric** if whenever x ~ y and y ~ x, then x = y .
- **Transitive** if, for all x, y, z, if x ~ y and y ~ z , then x~ z .

If ~ is reflexive, symmetric, and transitive, it is an equivalence relation.

THINK OF AN EQUIVALENCE RELATION FROM GRAPH THEORY

Graph Isomorphism:

Two graphs G and H are isomorphic- written $G \simeq H$

if there are bijections

- $\theta : V(G) \rightarrow V(H)$
- $\phi : E(G) \rightarrow E(H)$

such that $\psi\_G(e) = uv$ if and only if $\psi\_H(\phi(e)) = \theta(u)\theta(v)$

EXERCISE:

Show that graph isomorphism is an equivalence relation

EXERCISE:

An automorphism of a graph G is an equivalence relation on the vertices.

Recall:

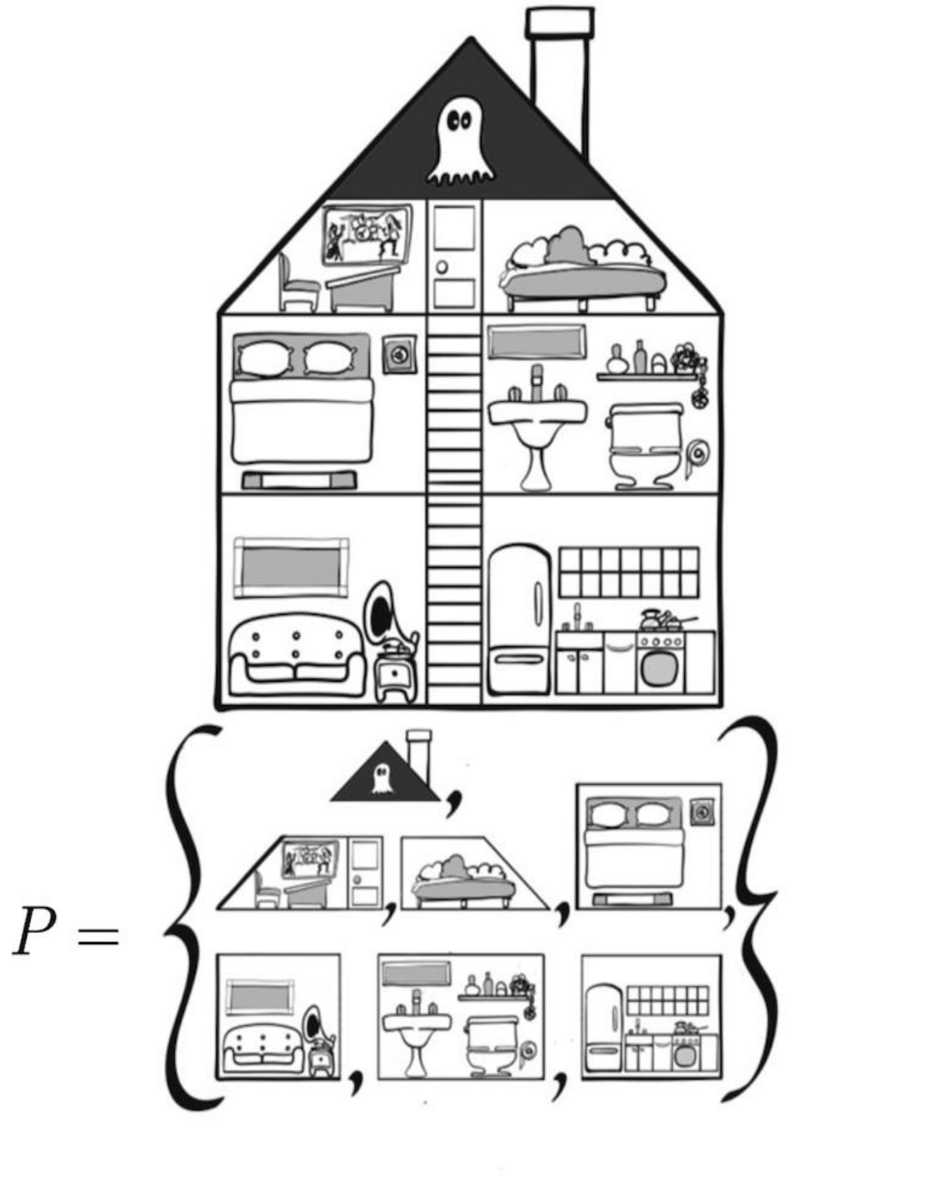
A graph automorphism is an isomorphism onto itself.

Hint: First write down the automorphism using the definition of a graph isomorphism.

THE FLOOR PLAN

Consider the floor plant of a house. The house is separated into rooms. Each room must cover some area of the house. Each two rooms cannot cover the same area or part of the same area. Finally putting all the rooms together should give you back the whole house.

This list of rooms is a partition of the house.



DEFINITION (PARTITION)

Let X be a set. A a family of sets $P \subset P(X)$ is a partition of X if the following hold:

- **Does not contain the Empty Set** $\emptyset \notin P$
- **Covering** $X = \bigcup_{A \in P} A$
- **Pairwise disjoint** If $A, B \in P$, then either $A = B$ or $A \cap B = \emptyset$

An element A in P is called a *room*.
If x is an element of a room A, then we say x inhabits A.

OBSERVE THAT A PARTITION IS A SET WHOSE ELEMENTS ARE SETS.

In the floor plan analogy:

- The non empty criterion ensures that each room covers some area
- The covering criterion ensures that putting together all the rooms will give you the whole house
- The pairwise disjoint criterion ensures that the kitchen with the bathroom (or any other two rooms) cannot share even a single tile

EXERCISE

Show how connected components partition a graph's vertex set.

Hint:
- First, try to express connectedness as an equivalence relation
- What is a connected component in the "language" of equivalence relations?

Recall

- Two vertices u,v of a graph G are connected if and only if there is a (u,v)-path in G.
- A (u,v)-path is an alternating sequence of vertices and edges that starts from u and ends at v, without a repetition of vertices, such that each edge in the sequence is preceded and followed by its ends.

EXERCISE

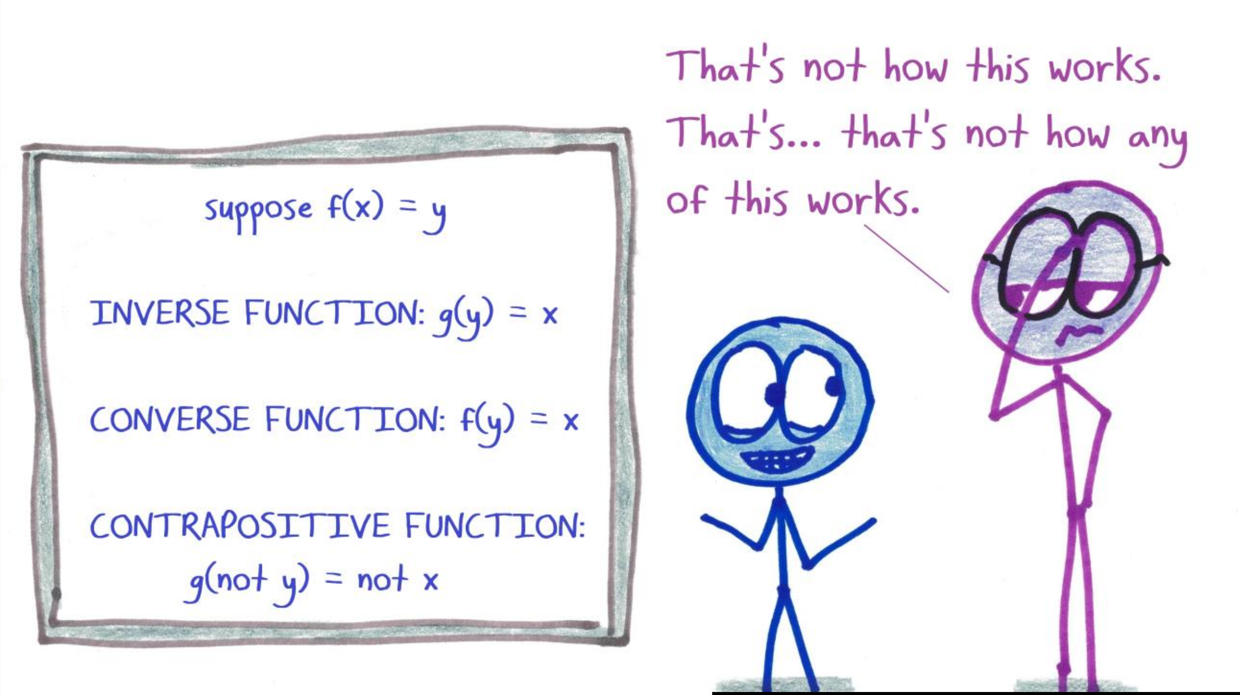
Given a bipartite graph, find a natural equivalence relation on its vertices.

Which is the induced partition?

Recall

A bipartite graph is one that its vertices can be organized in two sets X and Y such that each edge has one vertex in X and one in Y

IMPLICATIONS AND CONTRAPOSITIVES



IMPLICATION

An **implication** is a logical statement in the form of **"If A, then B"**.

- **A** is called the **hypothesis** (or antecedent), and **B** is called the **conclusion** (or consequent).
- It is written as: **A ⇒ B** (A implies B).

Little Timmy's Mom tells him, "if you don't eat all your broccoli, then you will not get any ice cream." Of course, Timmy loves his ice cream, so he quickly eats all his broccoli (which actually tastes pretty good).

After dinner, when Timmy asks for his ice cream, he is told no! Does Timmy have a right to be upset? Why or why not?

CONTRAPOSITIVE

The **contrapositive** is the logical reversal and negation of the implication. It is **"If not B, then not A"**.

- This is written as **¬B ⇒ ¬A** (where ¬ means "not").

What does Timmy know for sure?

EXERCISE

Prove that if a graph has exacty one connected component, then any pair of vertices is connected by a path.

Hint:
Use contrapositive

EXERCISE

Think of implications regarding graph isomorphisms:

e.g. If two graphs are isomorphic, then …..

Make a list, it will come in handy!

PROOF BY INDUCTION

Quite often in mathematics we find ourselves wanting to prove a statement that we *think* is true for every natural number $n \in \mathbb{N}$.

For example:

Show that, in any group of two or more people, there are at least two with the same number of friends.

This is a statement on the number of people in the group which can be any natural number larger than 1.

TEST

- For n=3, there are two options: either having two friends or one.

Since there are 3 people and only two options, at least two of them will have to pick the same option.

- For n=4, there are three options: having three friends, two or 1. Again since there are 4 people but only 3 options at least two of them will have to take the same option.

- For n=5, there are four options: having 4 friends, 3 friends, 2 friends or only 1.

Again, since there are more people than options, at least two will have to take the same option.

We notice a pattern...

But we want to *prove* that this is true for *all* positive integers , and it's going to be impossible if we try to do this by putting in all possible values! Instead, we're going to have to be a bit more cunning

THE STATEMENT

Suppose we have a statement $P(n)$ that we want to show that it is true for all n.

For example:

*P(n)={If the group has **n** people, then at least two of them have the same number of friends}*

Think of every $P(n)$ as a domino.
If we can show:

- that $P(1)$ is true
- that if $P(n)$ is true then so is $P(n+1)$

then we have shown that $P(n)$ is true for every n.

PROOF BY INDUCTION

There are four steps:

1. **Define some statement $P(n)$ that you will prove by induction**
2. **State that you are going to use induction**
3. **State and prove the base case:** You will define what the statement is on **the first admissible** number. Usually, the first admissible number is 0 but not always! What is the first admissible number in the "number of friends problem"?
4. **State and prove the inductive step:** You will say "Suppose $P(n)$ is true for some arbitrary n. I will show that it is then true for $P(n+1)$ as well" and you show it

$$\boldsymbol{P(n) \Rightarrow \mathrm{P}(n+1)}$$

EXERCISE:

Write the proof for the "number of friends problem" using induction.

EXERCISE

Prove that in a bipartite graph, the sum of degrees of vertices in one partition equals the sum of degrees in the other partition.

Recall: The degree of a vertex is the number of edges incident with it.

Hint: Apply induction on the number of edges

EXERCISE

Prove that a complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.

Hint 1: Apply induction on the number of vertices,

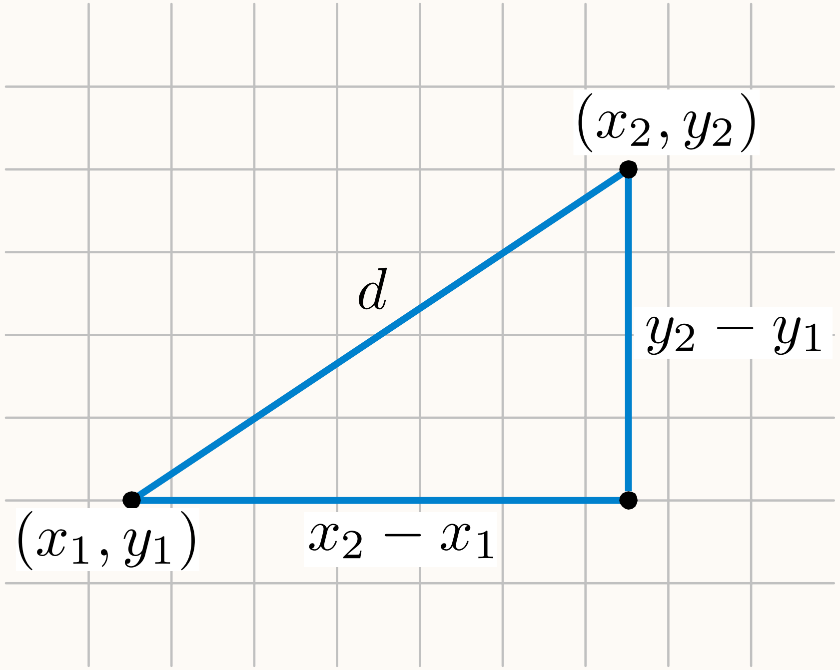
Hint 2: Test some small numbers first to gain some intuition.

Recall: A complete graph is a graph where every vertex is adjacent to every other vertex.

THE NOTION OF DISTANCE

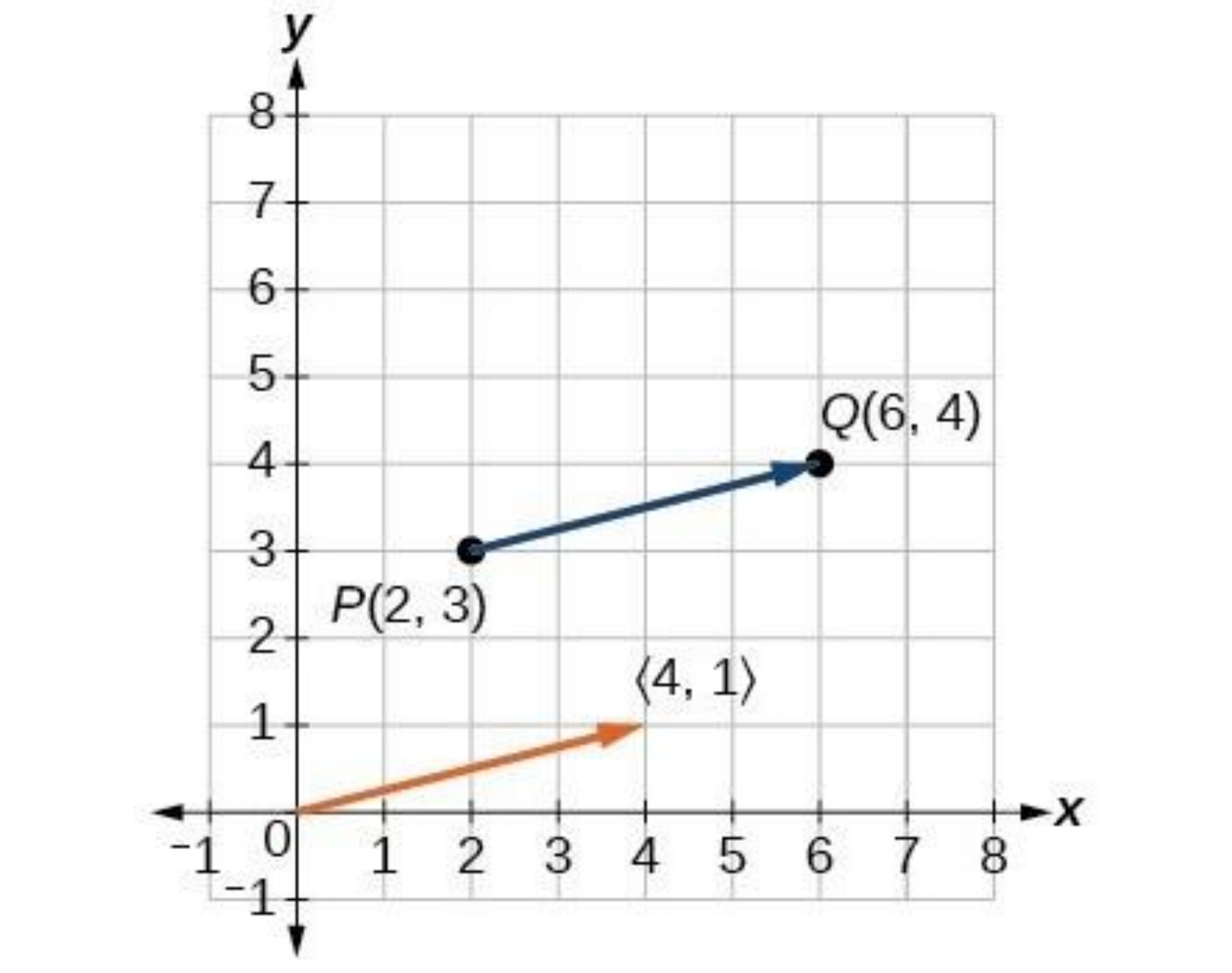
The **Euclidean distance** between two points in Euclidean space is the length of the line segment between them.

It can be calculated from the Cartesian coordinates of the points using the Pythagorean theorem, and therefore is occasionally called the **Pythagorean distance**.



EUCLIDEAN DISTANCE

When viewing Euclidean space as a vector space, its distance is associated with a norm called the Euclidean norm, defined as the distance of each vector from the origin.



BUT WHAT IF WE DON'T HAVE MANY OPTIONS?

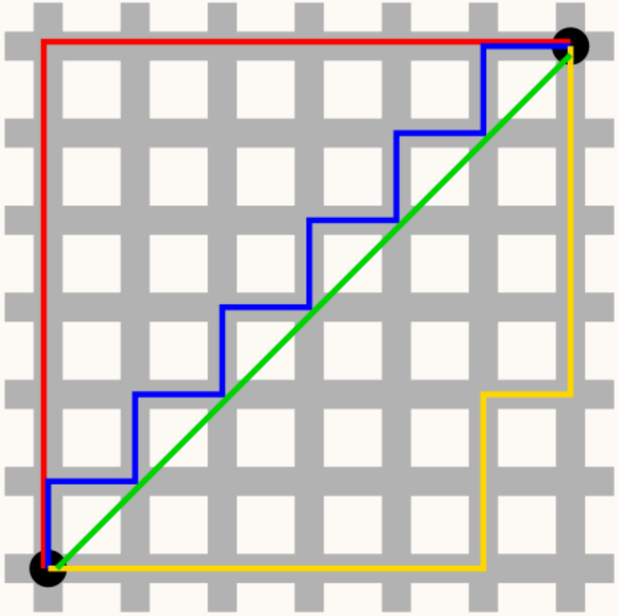
The Manhattan or Taxi Cab distance

THE MANHATTAN DISTANCE

Taxicab distance (L^1 distance), also called Manhattan distance, which measures distance as the sum of the distances in each coordinate.

Manhattan distance between two points (x_1, y_1) and (x_2, y_2)on a grid is:

$$d_{\{Mahattan\}} = |x_1 - x_2| + |y_1 - y\_2|$$

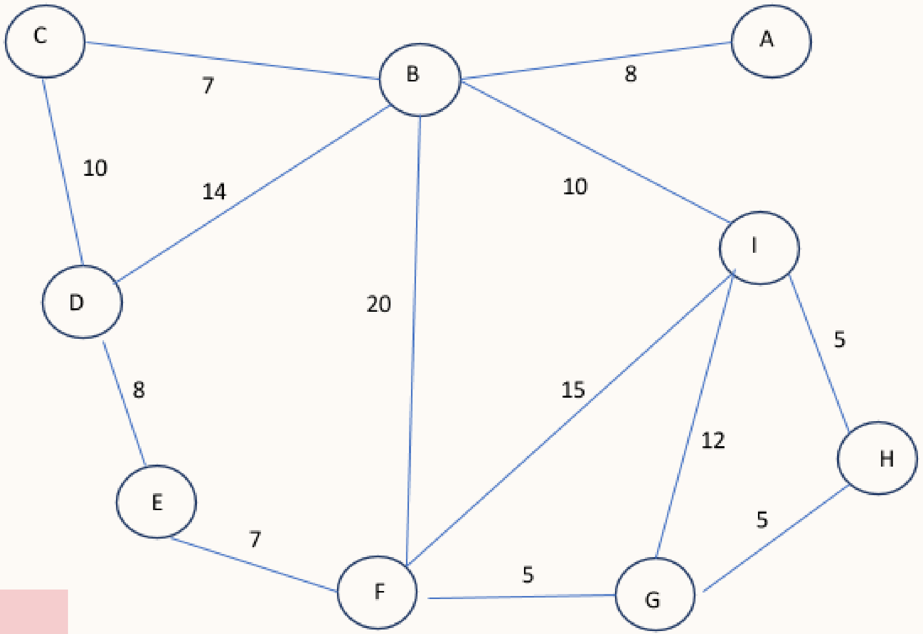


GRAPH DISTANCE

Shortest Path Problem:

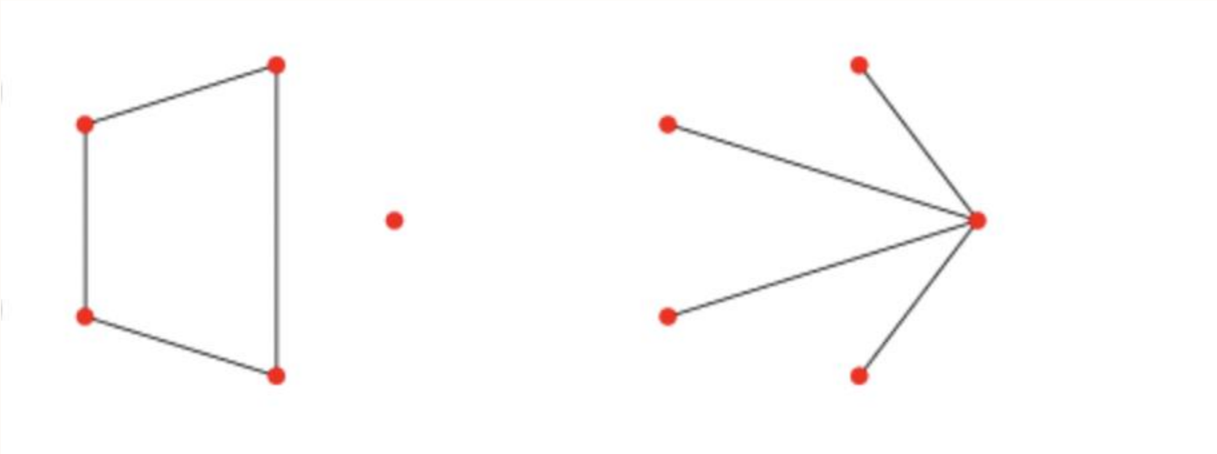
Given a railway network connecting various towns, determine a shortest route between two specified towns in the network.

Is the distance between towns the same?



PROBLEM SOLVING SESSION

1. Show that if there is a (u,v)-walk in G, then there is also a (u,v)-path in G.
2. Use the contrapositive to prove that these two graphs are not isomorphic.
3. Find a bipartite graph that is not isomorphic to a subgraph of any k-cube.



THANK YOU

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And office hours