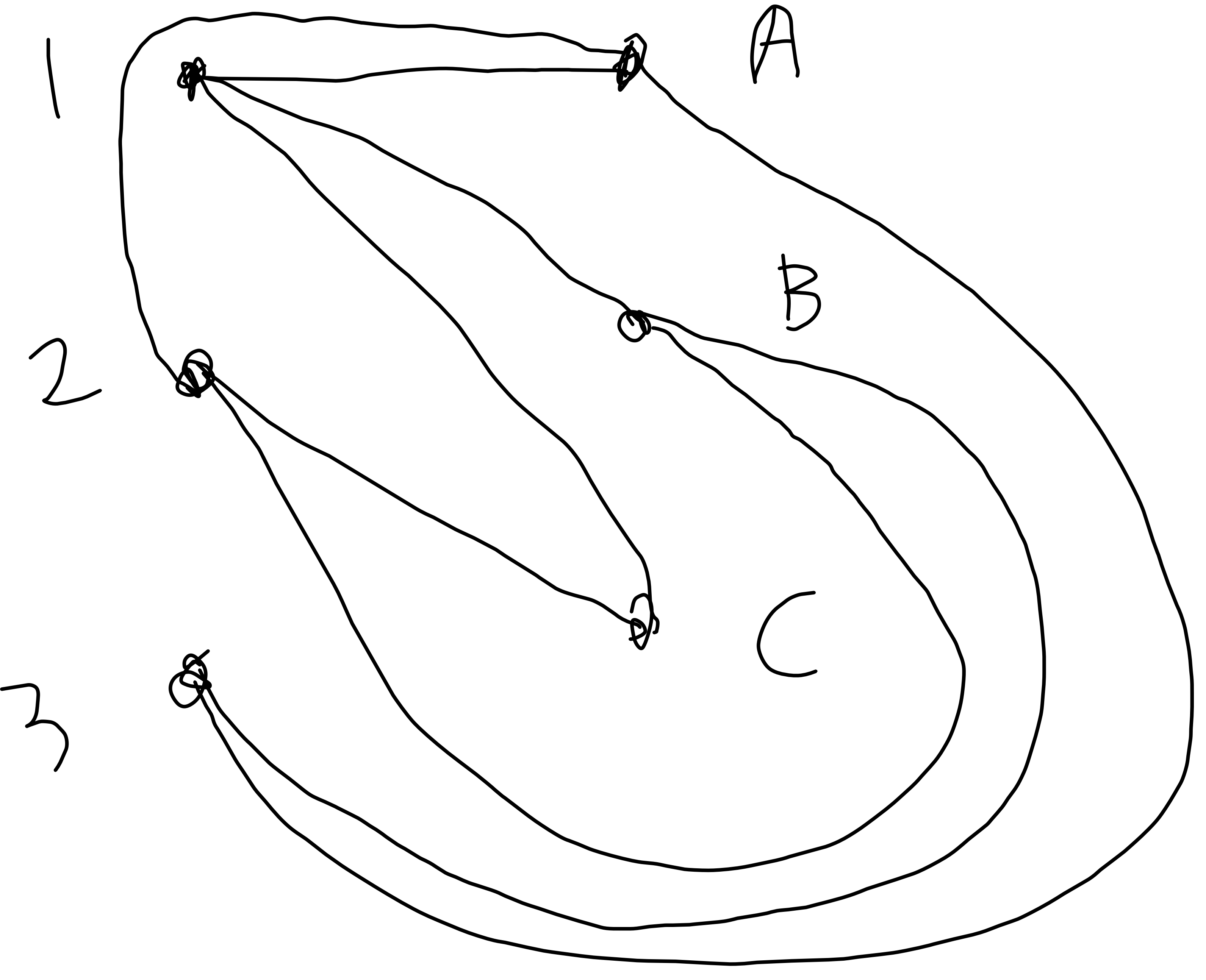
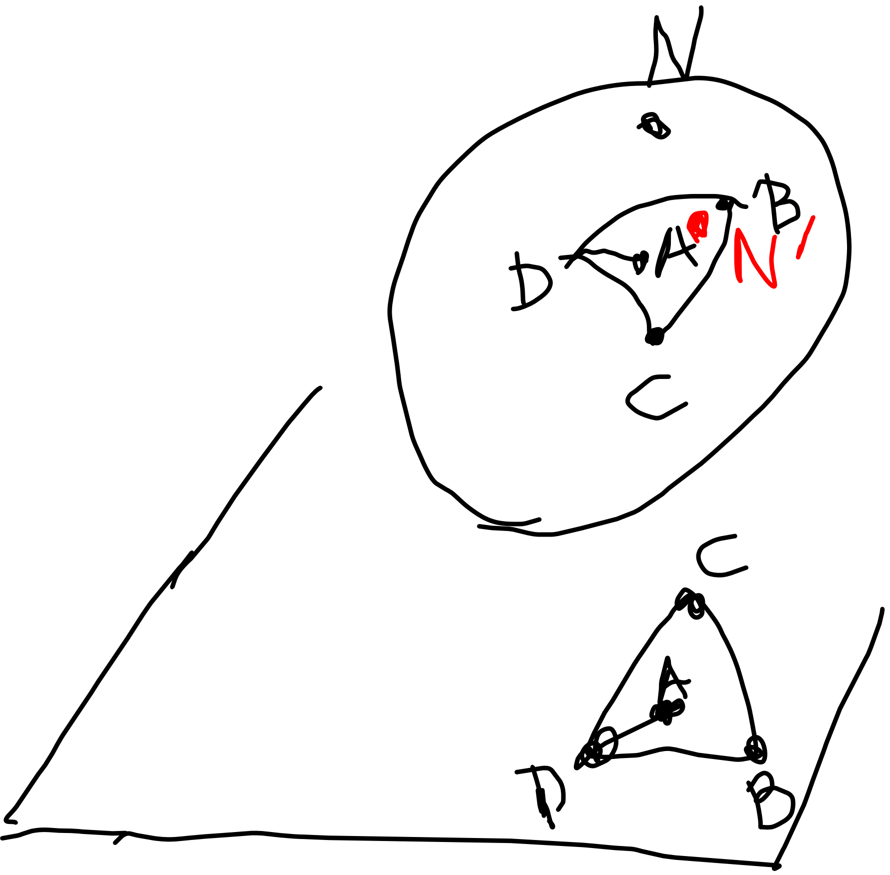
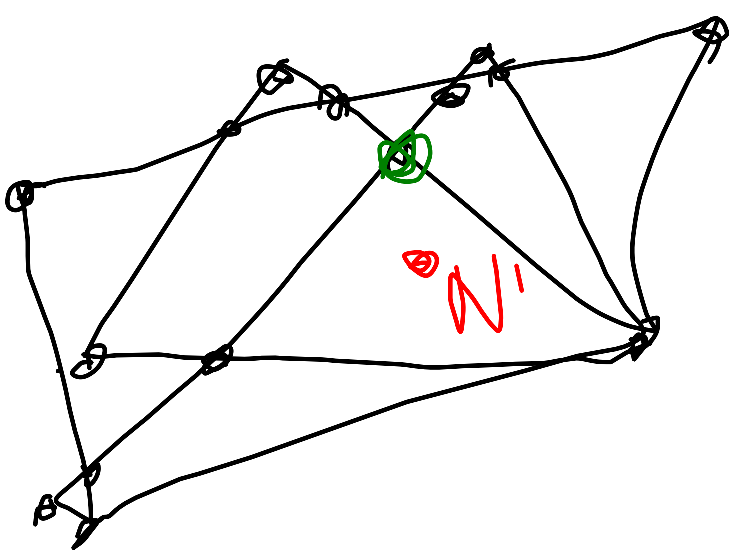
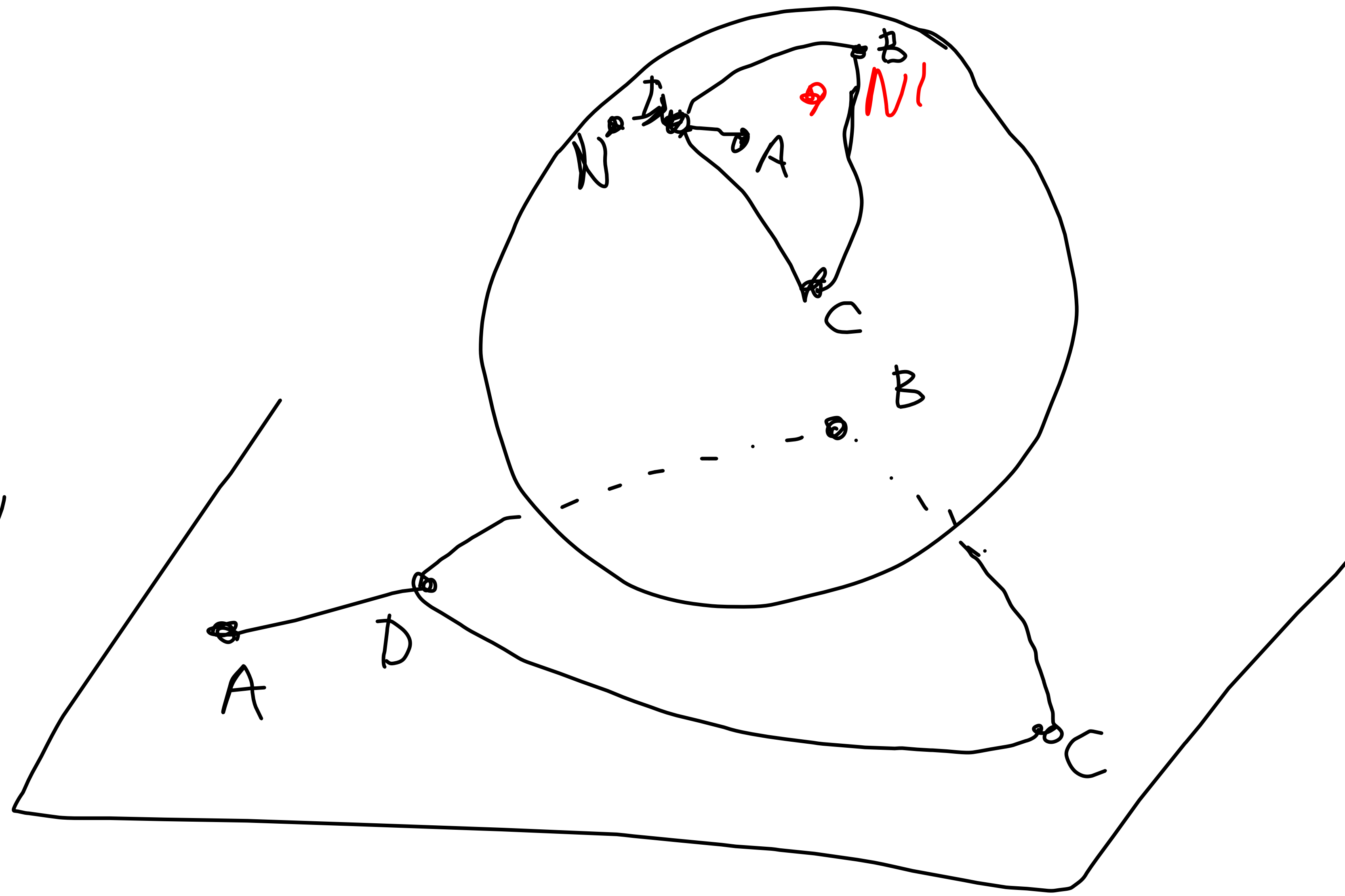
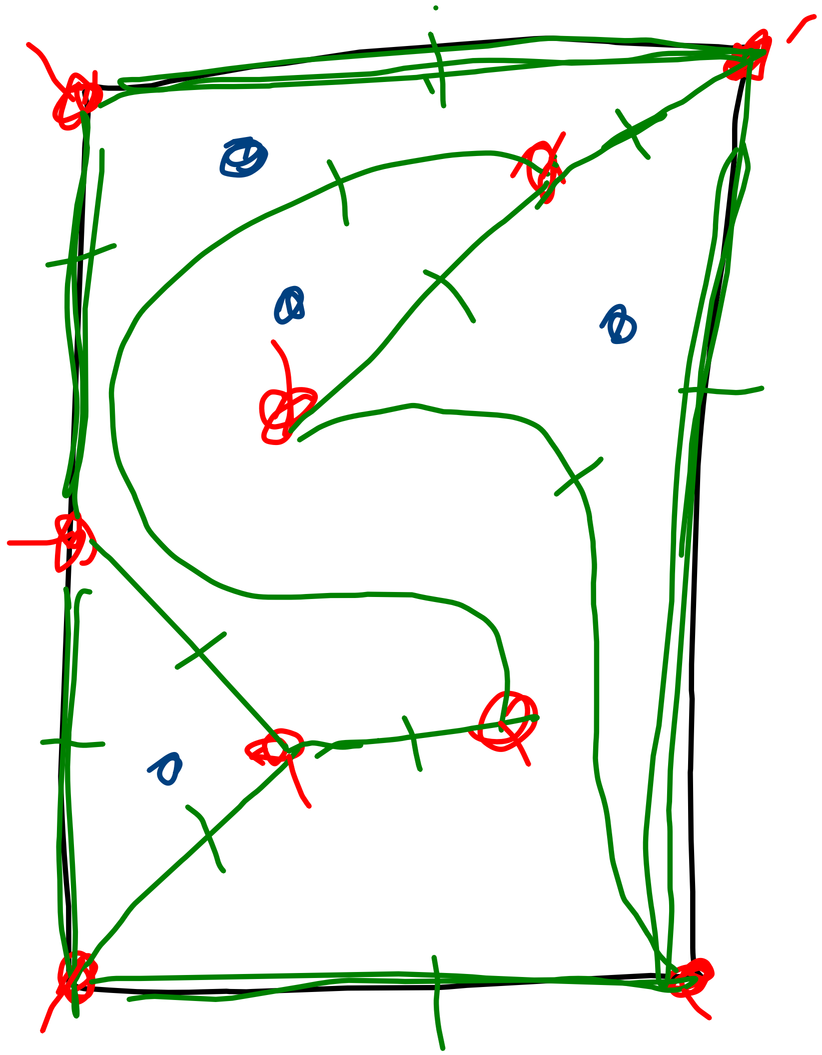

1
2
3
A
B
C

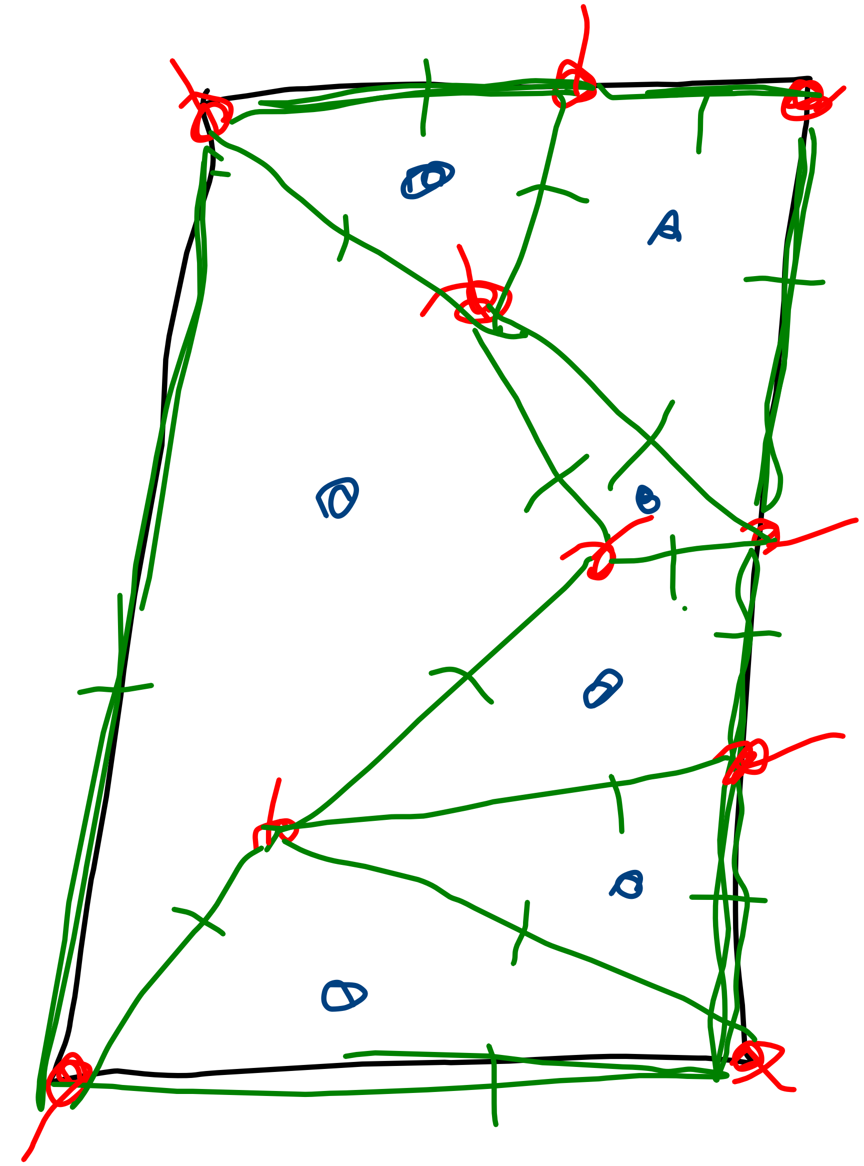

N
D
A
B
N'
C
C
A
D
B


N'


B
N'
N
A
C
B
D
A
C



$9 - 12 + 4 = 1$



$10 - 16 + 7 = 1$

Inductive premise: $\varphi = 1$
G is necessarily a tree, so

$$\nu = \varepsilon + 1$$

$$\nu - \varepsilon + \varphi = \cancel{\varepsilon} + 1 - \cancel{\varepsilon} + 1 = 2$$

Inductive hypothesis:
If G' has less than n faces, then

$$\nu(G') - \varepsilon(G') + \varphi(G') = 2$$

Inductive Thesis:
If G has n faces, then $\nu(G) - \varepsilon(G) + \varphi(G) = 2$

Proof of the inductive step:

Let G be a connected plane graph with $n \geq 2$ faces. Since it has at least 2 faces, there is at least one edge e that is not a cut-edge; let f_1, f_2 be the 2 faces sharing e as a common edge. Now, let G' be the plane graph obtained by deleting e ∴ the faces f_1, f_2 merge in a single face f. So G' has $n-1$ faces, and the inductive hypothesis applies:

$$v(G') - \varepsilon(G') + \varphi(G') = 2$$

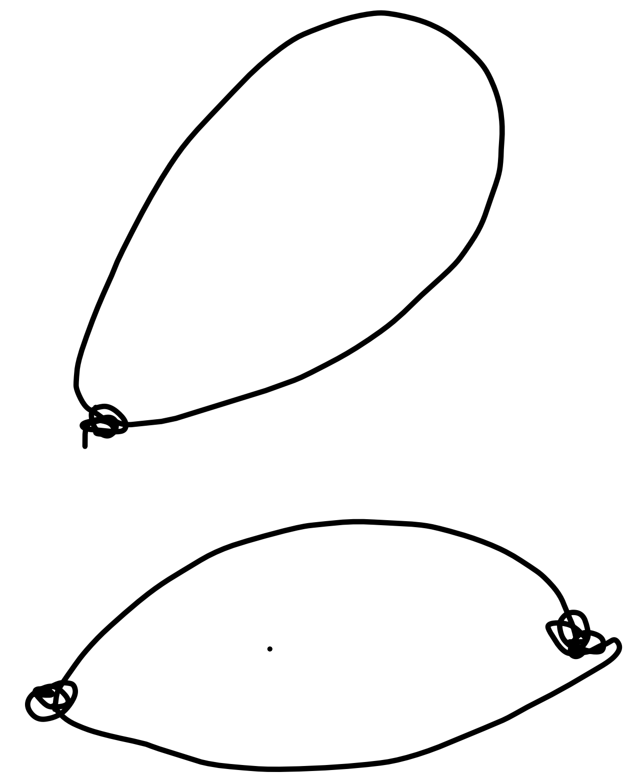
$$\underset{\underset{\nu(G)}{\|}}{\nu(G')} - \underset{\underset{\varepsilon(G)-1}{\|}}{\varepsilon(G')} + \underset{\underset{\varphi(G)-1}{\|}}{\varphi(G')} = 2$$

$$= \nu(G) - \varepsilon(G) + \cancel{1} + \varphi(G) - \cancel{1} = 2$$

$$\nu(G) - \varepsilon(G) + \varphi(G)$$

$$3 \leq d(f_1)$$
$$3 \leq d(f_2)$$
$$\vdots$$
$$3 \leq d(f_\varphi)$$

$$3\varphi \leq \sum d(f_i)$$



$$2\varepsilon \geq 3\varphi \qquad \varphi \leq \frac{2}{3}\varepsilon$$

$$v - \varepsilon + \frac{2}{3}\varepsilon \geq v - \varepsilon + \varphi = 2$$

$$\frac{3v - \varepsilon}{3} \geq 2$$

$$3v - \varepsilon \geq 6$$

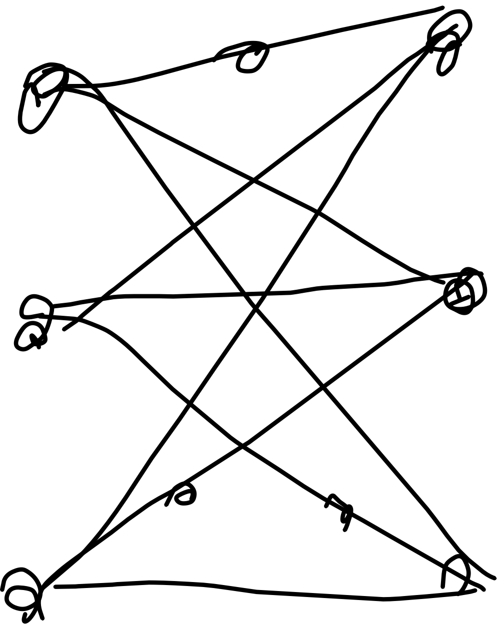
$$\varepsilon \leq 3v - 6$$

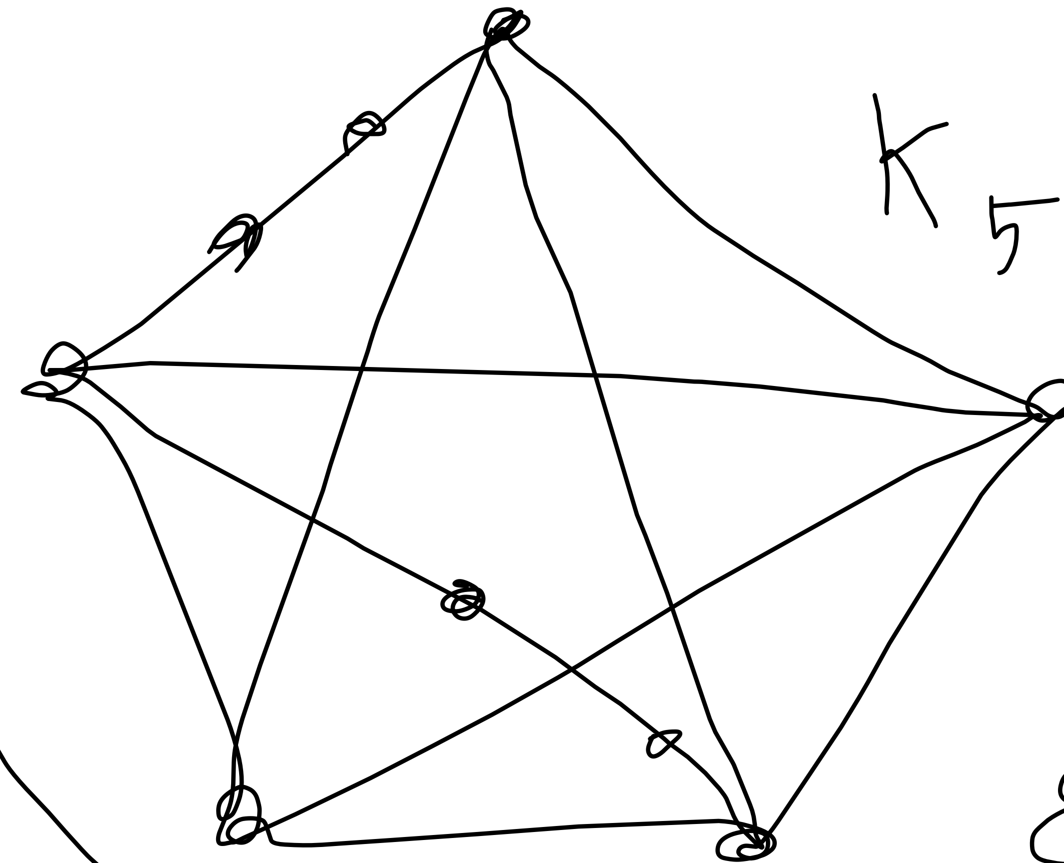
$$\delta v \leq 6v - 12$$

$$\delta \leq 6 - \frac{12}{v}$$

$$< 6$$

$$\leq 5$$





K_5

$\nu = 5$

$\varepsilon = 10$

$K_{3,3}$

$\nu = 6 \quad \varepsilon = 9 \qquad 4\varphi \leq 18 \qquad \varphi \leq \frac{18}{4} = \frac{9}{2} \qquad \varphi \leq 4$