Moebius: a film and an invitation to topology

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Introduction

The film *Moebius* (see Figure 1) offers various points of interest. First, it was conceived as a complex film exercise within the activity of the Universidad del Cine in Buenos Aires; as such, it was produced on a rather tight budget: about 250,000 US dollars. All the same, it achieved a fair audience success, and an international distribution. Moreover, it makes a courageous use of concepts and terms which might turn difficult for the general public: those of *topology*. Finally, it finds its place in a hardly definable genre, by suggesting several reading levels.

In this article, I'll deal almost only with the mathematical aspects of the movie, and I shall try to expand, in nontechnical terms, the elements which may have stimulated the fantasy of the authors.

The film

Directed by Prof. Gustavo Mosquera in 1996, *Moebius* takes inspiration from a short story by an American astronomer, A. J. Deutsch [3]. The plot is built around a mathematician, more precisely a topologist, who is engaged in solving the mystery of a disappeared underground train. Already the story has the features of fantastic tale, rather than of a science fiction one; this trend grows more marked, of course, in the hands of a South American! The film production has the vague and dreamlike tones of García Marquez rather than the lucid reveries of Borges (to whom the movie explicitly pays homage). Beyond the mere literary tradition, a strictly Argentinian element is added in the film transcription of the story. In the words of Mosquera: "I started to re–write the text, replacing the names of the underground stations with the names of the stations that already existed in Buenos Aires [...]. In this way, however, everything began to fill with a special significance [...]. Changing the places was quite simple, when compared with the deep changes in the meaning that arose when one even only imagined the possible dialogues about the disappearance of a train filled with people... specially in a country in which so many people had disappeared for political reasons" [9].

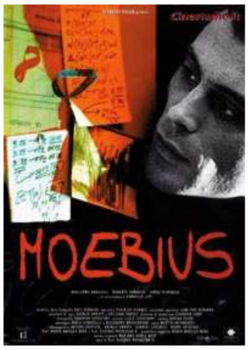

Fig. 1. The poster of *Moebius*

The reference is clearly to the disconcerting phenomenon of "desaparecidos": at least 12,000 people were kidnapped and never returned by the Argentinian dictatorship between 1976 and 1983, beside several thousands more whose corpses were given back. Some explicit hints to this reading level are present in the film.

However, the most evident level is the fantastic one. The film indeed refers to the topology of the underground network, and in particular (importing a mistake already present in the tale) to a "singularity" of a Moebius strip. Still, there is never the claim of (pseudo)scientificity of science fiction. Topology is present almost as a sort of launching pad for the mind to jump beyond everyday experience, identified with the geometry of material experience.

But is the Moebius strip really an "out of our world" object? First of all: what is a Moebius strip? What are its mathematical aspects which can turn out to be so unusual as to border on the fantastic? Is it an isolated phenomenon, or does it find its place within a family of topologically interesting objects? These are the questions which I try to answer in the next sections. I state from the beginning that I won't give precise definitions and statements; the terms to which to refer for a proper study will be listed in the section "Adequate tools". The present article is exclusively an invitation to topology, surely not a paper *of* topology.

The Moebius strip

It is very easy to build a Moebius strip; but first let us start with something more usual. Let's take a rectangular cloth strip; there are several ways of

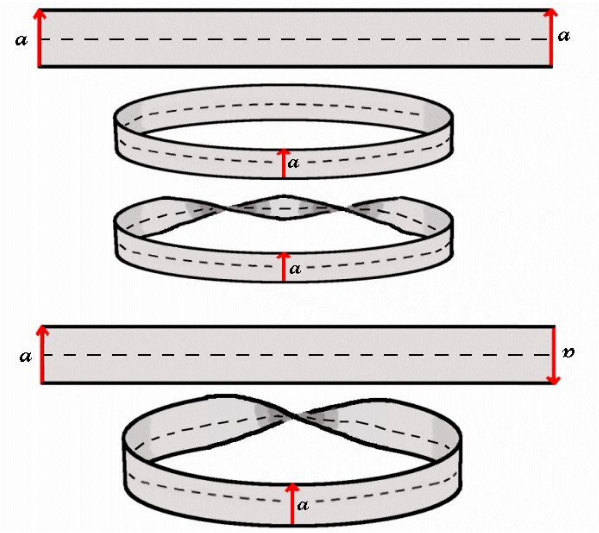

Fig. 2. Construction of cylinder frusta and of the Moebius strip

sewing two of its sides together. The simplest one (which we would use for making a belt with the strip) gives rise to a familiar object: a cylinder. (More properly, it is what is called a cylinder *frustum* in analytic geometry; in solid geometry it is called the *lateral surface* of a cylinder). This construction is shown in the upper part of Figure 2 by the two vertical arrows: they have to be identified (sewn together) tip to tip, tail to tail. Note that, as expected, the boundary of the resulting cylinder is formed by two separate curves. Moreover, just as in a normal belt, it has two faces: one internal and one external. We note at once an essential difference with respect to the geometry we are used to: here we don't speak of a rigid cylinder, but of an object that we can subject to modification by bending segments, modifying areas etc. In Section "Development and embedding" we shall even iron it flat on a plane!

But we are interested in another construction. Instead of identifying the two sides "directly" as before, we first carry out a half-turn twist, by reversing one of the sides, as in the lower part of Figure 2; this is indicated also this time by the arrows (note the right hand arrow, upside down with respect to before). We have built a concrete model of the Moebius strip. The first noticeable difference resides in the boundary: now it consists of a single curve. The second relevant difference with respect to the cylinder is that the Moebius strip has a single face: one can pass from one side to the other of the strip without crossing the boundary. This property is wonderfully depicted in a print by M.C. Escher: *Band van Moebius II* amusingly parodied in a logo of the University of Maryland, which we reproduce in Figure 3; we shall talk about it in Section "Orientability". Note also another characteristic, which is distorted in the movie: it is true that a turtle going through the strip once happens to be at the starting point, but on the other side of the surface, so "it is there but not there" like the disappeared train of the film. But after a further tour (in general, after an even number of tours) it reaches the starting point and on the same side!

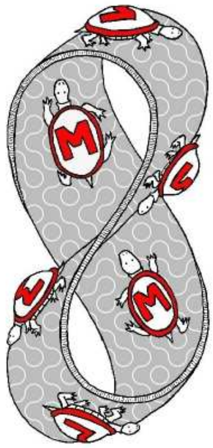

Fig. 3. Turtles going along a Moebius strip

A historical note: the name traditionally attached to this object, derives from the German astronomer and mathematician August Ferdinand Moebius (Schulpforta 1790 - Leipzig 1868; equivalent spelling: Möbius), who conceived it in 1858 and described it in a paper in 1865 [8]. However, the same object had been previously studied by another German mathematician: Johann Benedikt Listing (Frankfurt am Main 1808 – Göttingen 1882) [7].

Development and embedding

Figure 2 shows the two main contrivances with which topologists usually imagine a surface: development and embedding. Development consists in representing an object by starting from an easily conceivable figure, and imagining identification of some of its parts: e.g. by starting from a polygon with gluing instructions (as in most cardboard packets), but also from 3D objects (see Section "Beyond the Moebius strip"). This procedure is very handy and still is based on rigorous mathematics. Embedding is perhaps more natural, but more inconvenient: it consists in imagining the considered object in a "larger" space: e.g. in our 3D Euclidean space.

Development permits us to appreciate the intrinsic properties of the object under study, without confusing them with the ways in which we can place the object in an ambient space. When working at the development of a surface, a topologist thinks a bit like a citizen of the *Flatland* imagined by Abbott [1, 2], i.e. as though a third dimension did not exist. We can imagine the cylinder to be cut open and rolled out into a rectangle as in the upper part of Figure 2; then we can imagine that we crawl along the rectangle up to the arrow delimiting it at the left hand; at that point, we can cross the arrow, but then - due to the identification - we would re–enter at once the rectangle through the arrow on the right hand side.

Embedding (the "belt" which we find immediately below, in Figure 2) is undoubtedly more natural but, for one thing, it does not allow us to see the whole surface, a part of it being hidden by another part. Moreover, there is another problem: there can exist several embeddings of the same object (in this case a cylinder), not equivalent to one another. So far, actually, we have not talked about the third object from the top in Figure 2. We can build a model of it by starting, as before, from a cloth strip and by twisting it, this time, a whole turn before sewing together the two sides marked by the arrows; just to make things clearer, the Flatland citizen would not notice the difference: the topology is the same as the one we would get by not twisting the strip.

On the contrary we, living in 3D Euclidean space, see a remarkable difference between the two embeddings. In fact, the two boundary curves are linked together like the rings of a chain if we perform the one turn twist, whereas they are not if we don't twist, and we can realize it by cutting the strip along the dashed line. (By the way, can the reader imagine what happens if one cuts a Moebius strip along the dashed line?). From the "intrinsic" viewpoint of the Flatland citizen, there are just two possibilities of identification of the vertical sides: either with the arrows pointing the same way (both upwards, or equivalently both downwards), or with the arrows pointing in opposite ways (one upwards, the other downwards). In our 3D space, on the contrary, the same object (in this case a cylinder) can be embedded in infinitely many ways (by identifying the edges after a twist of one, two, …, n whole turns or none at all).

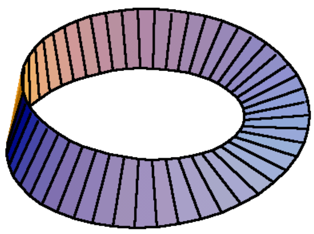

Fig. 4. The Moebius strip as a segment locus

By the way, note that the cylinder can be embedded (by "ironing" it) even into a plane, in form of an annulus. This is not possible with a Moebius strip:

however we try to do it, there will always be that sort of self–crossing of the boundary, which we see in the pictures; this is probably the singularity to which both the tale and the movie refer; it is not, anyway, a singularity of the surface, but of its projection on a plane.

One might wonder: besides the homemade construction with a cloth strip, does the embedding of the Moebius strip into our space have a secure mathematical foundation? Yes indeed! There is a formal construction, illustrated in Figure 4: the strip is seen as the locus drawn by a segment which is dragged along a circle (horizontal in the picture) and which at the same time rotates a half twist around its middle point.

Orientability

The property of having two sides like the cylinder, or just one like the Moebius strip is strictly related with an embedding. One can remark that also the "twisted" versions of the cylinder show two faces separated by the boundary. In the same way, the other possible embeddings of the Moebius strip (obtained by twisting the strip one and a half, or two and a half turns) also have only one face. This is the manifestation of an important feature of the surface: the cylinder is orientable, while the Moebius strip is not.

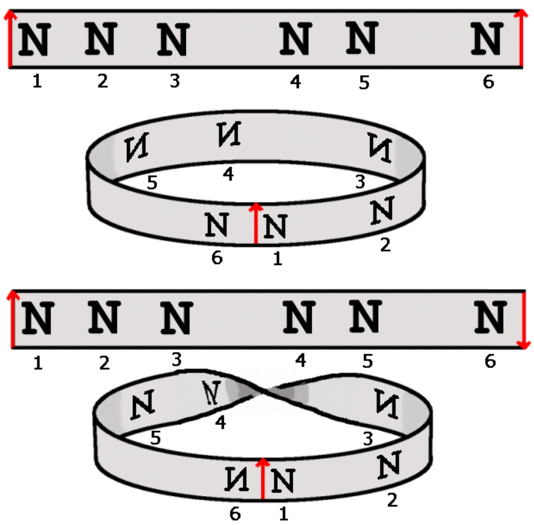

Fig. 5. A letter "N" glides along a particular path

Is then the orientability character associated with the embedding, rather than to the intrinsic topology of the surface? No! Orientability can be defined as follows, without considering the surface as a part of the ambient space.

Note that cylinder and Moebius strip are locally very similar: for each point of either space, there is a neighborhood[1] of the point in the surface which is essentially a patch of plane (better still: of a half–plane, if we consider the presence of the boundary). At every point, then, we can consider a Cartesian frame of the plane patch forming the neighborhood. If we move along a curve on the surface, we can drag the frame with continuity, i.e. without jerks. We define the surface to be *orientable* if, whatever the closed path starting and arriving at the same point, the initial reference frame and the final one coincide or - at most - can be superimposed on each other by a rotation (not a reflection). If however there is a closed path along which this does not happen, the surface is said to be *nonorientable*. The concept can be effectively illustrated by dragging, along a closed path, a figure having no axial symmetry, e.g. a letter "N". See Figure 5, where we can follow the letter along a particular loop in the cylinder and in the Moebius strip; for greater convenience, the same phenomenon is reproduced also in their embeddings.

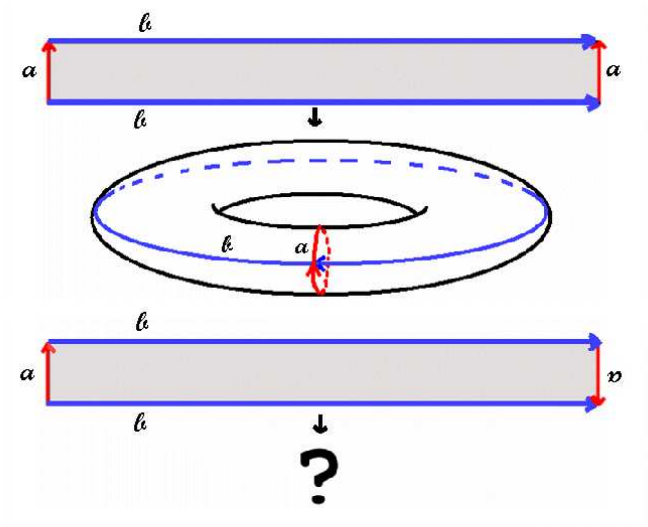

Fig. 6. Other identifications of rectangle sides

[1] Please note: here the notion of *neighborhood*, like many others, is dealt with in a conversational way, but it is actually the object of a rigorous definition. The same holds for *homeomorphism*, which formalizes the idea of similarity which we express in what follows. See Section "Adequate tools".

Beyond the Moebius strip

We can extend the trick of developing surfaces beyond the two objects considered so far. In fact, we can also identify the two horizontal sides of the rectangle as well as the two vertical sides. In the upper part of Figure 6 we see the identification of the two boundary curves of the cylinder. It is not difficult to imagine an embedding of the resulting (boundaryless) surface into our space: it is the *torus*, that sort of life belt drawn just below in Figure 6. Things get complicated if we perform the same identification, but starting from the Moebius strip, as in the lower part of Figure 6: the short sides identified in opposite directions, the long ones in the same direction.

Here the technique of development is really indispensable: we follow rather well what can happen to the Flatland citizen when crossing either the vertical or the horizontal sides. In the case of the torus, embedding is much less practical, as one part of the surface hides another; in the case of the other identification type, embedding into our space is even impossible! The surface one gets was studied by Felix Christian Klein (Düsseldorf 1849 – Göttingen 1925) [5]. It turns out to be a boundaryless, nonorientable surface called the *Klein bottle*. Unfortunately, it is not possible to embed the Klein bottle into our space; but it is possible to embed it in a 4D Euclidean space! In our 3D space we can only see a projection of it (Figure 7).

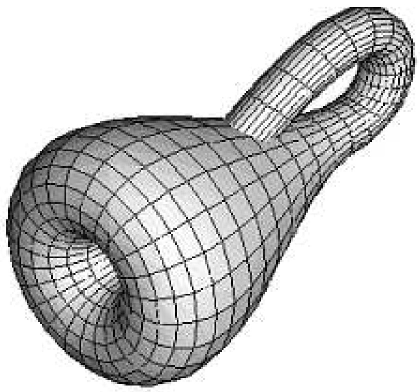

Fig. 7. A 3D projection of the Klein bottle

We can observe an important fact: every 2D projection of the Moebius strip necessarily presents singularities, i.e. points with non-Euclidean neighborhoods (that is to say, not like a disk, possibly continuously deformed). In the same way, also every 3D projection of the Klein bottle has singularities, at which it appears to us as a self–intersecting pipe; it is actually a singularity of the projection and not of the surface itself. The Klein bottle has one more interesting property: if we cut it in a suitable way, it decomposes into two Moebius strips. Can the reader understand how to perform the cut? (Hint: work on the development).

A remark of combinatorial character: the number of possible pairwise identifications of the rectangle sides is finite. Ignoring the several possible embeddings - and sticking to the intrinsic viewpoint - we can then obtain just a finite number of objects. What are they? We shall find, in several ways, Klein bottles and tori; moreover, we shall find another nonorientable surface which cannot be embedded into 3D Euclidean space: the projective plane. There is also an identification scheme which gives rise to the simplest possible surface: the spherical surface. Of course, we mean the sphere in a topological, not a geometrical sense, i.e. up to continuous deformations: the surface of a ball which can be nicely round, but it can also be deflated and dented. Can the reader find this identification scheme?

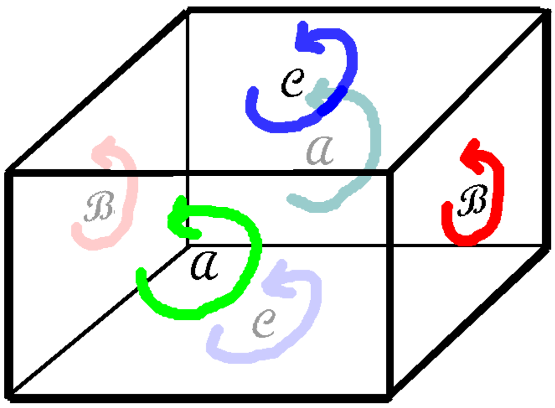

Fig. 8. Development of a 3D torus

There is a quite different direction in which to generalize the described procedures: to increase the dimension. In fact we can consider a parallelepiped, instead of a rectangle, and perform a pairwise identification of the faces. An example is described in Figure 8; the resulting object is locally (but not globally) like our space, and is the 3D analogy of the torus. Of course, we cannot depict it by an embedding and we must be content with its development. The generalization goes actually to dimension four and beyond, but then one really needs the adequate tools for a however vague intuition of what happens.

Adequate tools

The first adequate tool for a mathematician's work is a precisely defined terminology. This is not the place to give definitions, but it may be suitable, at least, to mention the correct terms, with which one describes in a rigorous way the aspects we have skimmed over in the preceding sections. Already the concepts of *relation* and of *equivalence*, which may look so intuitive as not to need a definition, are actually two precisely defined elements within *set theory*. Also the identifications we have seen so many times, have a formalization: the passage to *quotient*. We already enter topology when we speak of *neighborhoods* and *continuity*; also *embedding* and *projection* are rigorously defined terms. The two equivalence relations that we have compared in Section "Development and embedding" are *homeomorphism* - with respect to which two cylinders are considered equivalent (or, as should be said, *homeomorphic*) although they are differently embedded - and *ambient isotopy* which, on the contrary, distinguishes different embeddings. A *surface* is a particular case, of dimension 2, of an *n–dimensional manifold*; the definition of this concept hinges on the property that for each point there should be a neighborhood of it, homeomorphic to a neighborhood of a point in a *Euclidean space* of *dimension n*. For manifolds with *boundary*, like the (frustum of) cylinder and Moebius strip, the model is a *half–space* instead of a space.

The tools with which manifolds are presently studied are those of algebraic topology, i.e. *homotopy groups* - in particular the first one, also called the *fundamental group* - the *homology* and *cohomology modules*, the *cobordism groups* and many other *invariants* in continual development. Apart from the *sphere* and the projective plane, all manifolds named here are *fiber spaces*, to which an important chapter of topology is dedicated. Another extremely vital section is the theory of *knots*, which studies problems of ambient isotopy in depth. There is a very interesting interaction between the topology of a manifold and the *geometrical structures* defined on it. The fact that manifolds are the most typical *configuration* and *phase spaces* offers to topology the possibility of interacting with mathematical physics, mainly through *differential topology*, in particular through the study of *vector fields* and *Morse functions*.

The basic notions of topology and algebraic topology are accessible in a fairly easy way; there is a wide literature of various technical levels [10, 6, 4].

Conclusions

What is so interesting in a Moebius strip, as to inspire a story and a film? Maybe the fact that, although it's easy to make a concrete model of it, it shows several features which can be appreciated even by a superficial investigation - but surely much more by suitable mathematical tools. Moreover, it offers a glimpse of a part of mathematics, which is as rigorous, but less "rigid" than typical school mathematics: topology.

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