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TESI

Changes in meaning of mathematical objects due to semiotic transformations: a comparison between semiotic perspectives.

George Santi

Settore scientifico disciplinare **MAT/04**

Tutor :

Prof. Bruno D'Amore

Coordinatore:

Prof. Filippo Spagnolo

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INTRODUCTION

AMBIT OF THE RESEARCH

The present study develops within a field of research that in the past years has gained more and more importance and relevance in the progress of Mathematics Education; the study of signs in the teaching and learning of mathematics that in its most general acceptation has been termed as *semiotics*. Conferences, special issues of prestigious journals, books etc. constantly dedicated to semiotics testify the interest in this fascinating topic and confirm its effectiveness both in educational research and teaching design.

Historically semiotics has not developed into a monolithic system of thought but we can recognize different semiotic traditions, Vygotsky's, Peirce's, De Saussure's just to recall the most important (Radford, 2006a). Also within Mathematics Education, semiotic investigation has developed along a plurality of routes that, in turn, confirms what momentum the study of sign has acquired in our discipline. Part of the present study is devoted to such a plurality of perspectives in order to understand semiotics both in its specific acceptations and as global theoretical tool.

Through the semiotic lens, the present research addresses a vast and, we believe, intrinsically open issue that interweaves epistemological, cognitive, sociocultural and educational aspects: *the meaning* of mathematical objects. We will look at meaning through a specific and unexpected semiotic phenomenon, without the pretension to give an exhaustive and conclusive treatment of the topic. The present work can be considered satisfactory if it brings further insights on the role of signs in teaching-learning processes and shows how meaning is an elusive notion, and yet consubstantial to our psychological, social and cognitive growth.

We declare from the beginning that our investigation is informed by an anthropological and sociocultural (D'Amore, 2003; Godino, 2002; Radford, 2006) stand that focuses on the human being intended as an individual that reflecting against a social and cultural reality encounters himself. One could claim that understanding the role of signs in mathematics entails an investigation of their structural organization and discursive functions. This is certainly true, but it oversimplifies the picture:

> «Obviously, mathematics is an intrinsic symbolic activity. […] Semiotics, with its arsenal of concepts, appears well suited to help us understand the mathematical processes of thinking, symbolizing and communicating.
> At the same time, the answer is complex, for processes of thinking, symbolizing, and communicating are – as sociologists, anthropologists and literary critics found out several years ago-subsumed in more general encompassing symbolic systems. […] The inevitable embedded nature of our ways of thinking and doing into these ever-changing symbolic systems makes mathematical thinking and discourse not a mere personal affair, but something entangled with the cultural, historical, political, dimensions of life. Semiotics, as a reflective step backward, offers an advantageous viewpoint – a fissure of the symbolic, disturbance of the familiar, bracketing of the quotidian- whence to investigate, resist and transform the sign and sign systems through which we breath and live» (Radford, 2008a, p. vii-viii).

ORIGIN OF THE PROBLEM

The study we developed in this thesis stemmed from D'Amore and Fandiño's (2006) researches highlighting that at all school levels we witness student's difficulties in dealing with the meaning of mathematical objects in relation to their semiotic representations. This research highlights unexpected behaviours on the part of the students that defy Duval's claim that only conversion is the most important cognitive function which ensures correct conceptualization of mathematical objects and is the main cause of students' difficulties and learning failures.

D'Amore and Fandiño's results seem to defy Duval's claim that focuses the problem of the conceptualization of mathematics only on conversion. Such researches clearly show that subjects exposed to semiotic treatment transformations encounter severe difficulties in dealing with the meaning of mathematical objects. We synthetically recall here some paradigmatic examples of such unexpected behaviour taken from the results of the aforementioned scholars:

- *Primary School*. Students working on probability recognise that 1/2, 3/6, 50%, represent the probability of rolling an even number when throwing a 6 face die, but after a treatment transformation from 1/2 to 4/8 students and teacher don't recognise in 4/8 the same probability.
- *High school*. y=ax+b is a straight line, but x-y/a+b/a=0 obtained after a treatment transformation is no longer recognized as a straight line and assumes another meaning.
- *University level*. (n-1)+n+(n+1) and 3n two semiotic representations linked through a treatment transformation are interpreted as 3 consecutive numbers and the triple of a number respectively but in no way the triple of a number is the sum of three consecutive numbers.
- *University level*. Students working on the famous Gauss sum of the first 100 natural numbers after semiotic transformations arrive at the correct solution of the problem: 101x50. This semiotic representation is not recognised as a representation of the original object. They were looking for objects whose sense

The didactical phenomenon we investigated in the present research has been encountered and interpreted within a structural and functional view of semiotics that rests on a basically realistic view of mathematical objects, semiotic representations and meaning: one object with many representations. In other words, according to a realistic viewpoint meaning is a relation between the *signifier* and the *signified*, i.e. between the representation and the entity the representation refers to. Since a mathematical object has more semiotic representations there are more signifiers for the same signified, that are equisignificant.

The unexpected semiotic phenomenon we described above has been termed as *change of meaning due treatment transformations* to express the idea that students break the equisignificant relation that bind the different representations. In the following chapters we will argue this terminology in order to formulate with more precision our research questions. Without dropping this realistic viewpoint of mathematical objects and the basic relation signifier-signified we will highlight a more complicated and comprehensive structure of meaning.

OBJECTIVES

From a theoretical and experimental point of view the aim of the present investigation is two show how the approach to meaning based on the idea that there are many representations for the same object is inadequate to frame students' learning behaviour. We will move from the realistic ontological stand that considers mathematical objects as ideal *a priori* entities and we will go beyond the epistemology that conceives meaning within the structure of semiotic systems, assuming that the meaning of a semiotic representation is the object it refers to. Our investigation we will take into account the role of activity embedded in sociocultural and historical dimensions. The analysis will rest not only on Duval's approach but we will take into account also Radford's cultural semiotic and Godino's ontosemiotic approaches.

Our investigation develops in two directions. The mainstream of our research is to address the issue of changes of meaning by taking into account more semiotic perspectives. We will also devote part of our investigation to find out the kind of connections that can be established between the semiotic perspectives we are dealing with, thereby inserting our research on meaning in a more comprehensive frame. We believe that it is advantageous to outline the issue of changes meaning through Duval's and Radford theoretical constructs and arrive at a possible solution taking into account also the ontosemiotic approach.

Our objective is to understand why a semiotic transformation causes changes of mathematical objects' meaning and what is the specific role of conversion, treatment, and the combination of the two triggering this semiotic behaviour. We will give another formulation of the problem in the cultural semiotic approach shifting the problem from a mere transformation of semiotic representations to the coordination of local meanings objectified by the individual through semiotically mediated reflexive activities. Our intention is also to understand the interplay between semiotics and activity, and the relationship between signs in semiotic systems and signs as semiotic means of objectification that in their broader understanding mediate activity.

Our hypothesis is that basically student's difficulties in dealing with signs are ascribable to the intrinsic inaccessibility of mathematical objects. The meaning of mathematical objects cannot be reduced to the reference to an ideal entity but it is grounded in the practices students culturally and socially share in the classroom. We believe that the semiotic function is an effective tool to understand how the intrinsic local meanings emerging from practices is synthesized in a general mathematical meaning. We conjecture that there is no contradiction between Duval's and Radford's interpretations of meaning. In fact, our claim is that they are two faces of the same medal; the referential use of semiotic representations transforms entities emerging from activity through semiotic means of objectification into a cultural and general mathematical object, thereby establishing a coordination between semiotic systems and semiotic means of objectification.

The aim of this study is also to identify the boundaries between the structural and functional, cultural semiotic and ontosemiotic approaches, in terms of common features, differences, complementarity, classes of problems that are described by only one of the three theories and those that are described by all of them, etc. Our question is if it is possible to synthesize them into a unitary frame successful in describing the changes of meaning.

Connecting theoretical perspectives is a forefront research topic developed in the recent years to overcome the proliferation of unrelated theories, that makes Mathematics Education a conceptually and methodologically disjoint field of study without a coherent scientific character. We will resort to the results of these studies to compare semiotic perspectives through coherent connecting strategies and methodologies.

Our first conjecture was that the three semiotic perspectives we are advocating had very frail boundaries; mislead by the belief that having at their core semiotics, the three theories would have been easily integrated. We will discuss this claim according to specific theoretical "networking tools" to identify to what degree the three theories can be integrated.

STRUCTURE OF THE DISSERTATION

Chapter 1 deals in a general form the nature of mathematical objects and their meaning. We will describe two basic philosophical viewpoints, the so called realistic theories and pragmatic theories. The first believes that mathematical objects have an independent ideal existence and meaning is a semiotic relation to such entities. According to pragmatic theories, mathematical objects, their representations and meanings are entangled through a social and cultural praxis.

In chapters 2, 3 and 4 we will analyse the three semiotic perspectives we will use to frame theoretically the issue of changes of meaning; Duval's structural and functional approach, Radford cultural semiotic approach and Godino's ontosemiotic approach. For each approach we have singled out the essential notions that are at the core of the theory to frame the issue of change of meaning in each frame. As regards Duval's theory we will address the inaccessibility of mathematical objects, the cognitive paradox, semiotic systems and semiotic registers, the cognitive functions specific of mathematics accomplished through the coordination of semiotic systems. As regards Radford's perspective we will describe mathematical objects, thinking and learning through the notion objectification, reflexive activity and semiotic means of objectification. As regards Godino's approach we will analyse systems of practices, primary entities emerging from the practices and organised in configuration of objects and related through cognitive dualities. We will focus on the semiotic function that is an effective tool that provides an effective access to meaning.

In chapter 5 we will face the problem of networking theories. We will try to understand what characterises a theory in Mathematics Education. Amongst the different possible acceptations of a theory, during our work we will focus on Raford's (2008) view of a theory, a triplet (P,M,Q) formed by a system of principles, a methodology and a template of research questions. Referring to Prediger Bikner-Ahsbahs and Arzarello (2008) we will describe the "landscape" of strategies to connect theories. Chapter 5 is also devoted to the construction of a theoretical framework to address our research questions on the change of meaning. We begin by describing the theoretical elements that constitute the framework. By "comparing and contrasting" the three perspectives, we will outline their compatibility and complementarity. Through the connecting strategy of "coordinating and combining" we will arrive at a possible theoretical framework in which pose our questions, formulate hypothesis and carry out experiments.

In chapter 6 we will describe the experimental phase. In the first part of the chapter we will analyse an experiment conducted in scientific high school of Bologna dealing with the meaning of the tangent to a curve in Euclidean geometry, analytic geometry and calculus. In the second part we will examine and experiment carried out in a primary school of Bologna where students worked with sequences represented by different geometrical figures. The aim was to verify if a change in representation changed the meaning of the mathematical object.

In chapter 7 we propose concluding remarks on the issue of change meaning and the degree of integration of the three semiotic perspectives we considered for the research. We will then propose some open questions that need further and specific investigation and possible educational prospective the present work.

1

Mathematical objects and meaning

1.1 Introduction

Before we enter into semiotic perspectives to tackle the issue of changes of meaning due to semiotic transformations, it necessary to reflect on the nature of mathematical objects and meaning in mathematics. One recognizes this need not only when focusing on the relation between mathematical objects, meaning and signs but also on teaching learning processes in general; educational research and theories of knowledge develop along different lines depending on their philosophical view point on the nature of mathematical objects. Furthermore, many beliefs and conceptions (D'Amore, Fandiño, 2004) both of students and teachers are rooted in an often implicit and unaware ontological position regarding mathematical objects.

The problem of meaning is recognized as a primary issue by many scholars in Mathematics Education. Radford claims that

> «One can very well survive doing mathematics without adopting an explicit ontology, that is, a theory dealing with the nature of mathematical objects. This is why it is almost impossible to infer from a technical paper in mathematics its author's ontological stand. The situation has become very different when we talk about mathematical knowledge. Probably this has to do with the evolution of mathematics education as an academic discipline» (Radford, 2004, p.3).

Balacheff considers meaning as a keyword in mathematics education

«A problem belongs to a research issue on mathematics teaching if it is specifically in relation with the mathematical meaning of the students behaviour during mathematics lessons» (Balacheff, 1990, p. 258)

And Sierpinska

«Understanding a concept will be therefore conceived as the act of acquiring its meaning. Such an act will probably be an act of generalization and synthesis of meanings, in relation with particular elements of the "structure" of the concept (the structure of the concept is the net of meanings of the of statements we have considered)» (Sierpinska, 1990, p.35)

Godino and Batanero highlight the connection between ontology and meaning.

«The theoretical study of meaning from a mathematical, psychological and didactical point of view, that is the formulation of an explicit theory of mathematical objects, can be useful to establish connections between different approaches to the issues that are object of Mathematics Education research» (Godino, Batanero, 1994, p.7).

Of course, mathematical objects, meaning and the use of signs are strictly interwoven, but we will carry out a more detailed analysis that fully encompasses the role of representations both in the ontology and meaning of mathematical objects in the next chapter.

Following Kutshera (1979) we will look at the ontology and meaning in mathematics comparing two philosophical perspectives *realist theories* and *pragmatic theories.* Although clearly distinct and to a certain extent opposed perspectives, in the next chapter we will show that, in different ways, both in the cultural semiotic approach and ontosemiotic approach they cam be seen as complementary view points to understand the nature of mathematical objects and their meaning.

For a detailed treatment of the topic, we refer the reader to (D'Amore, 1999, 2001a, 2001b, 2003; Ernst, 1991; Godino and Batanero 1994), that for sake of brevity we quote only once here.

1.2 Realistic theories

For a detailed treatment of the topic we refer the reader to D'amore (2001). Realistic theories correspond to a Platonic view of mathematical objects: concepts, propositions, theories, structures, contexts etc. have a real and a priori existence in an ideal domain, independent of human beings. Mathematical knowledge consists in discovering pre-existing relations between these objects.

Radford (2004, 2005, 2007) highlights the basic differences in perspectives within different realistic theories. The point is the nature of the ideal domain where the mathematical entities exist: it can be outside or inside the human mind.

For Plato objects of knowledge are objects that do not change, they belong to the realm of essences or forms, the mathematical procedures move solely through forms to forms and finish with form. Mathematical objects cannot belong to the human mind because they would be subjected to the transformations that characterize human being and lose their absolute character.

In other realistic perspectives – rationalism, instrumental rationalism etc. – ideal objects are in the mind and knowledge of such ideal entities is acquired by reason applying the rules of knowledge.

Meaning of mathematical objects is strictly related to this ontological viewpoint. Realistic theories consider meaning as a conventional relationship between signs and the aforementioned ideal entities that exist independently from linguistic signs.

> «According to this conception the meaning of a linguistic expression doesn't depend on its use in concrete situations, but it happens that use rests on meaning, since it is possible a clear distinction between semantics and pragmatic» (Kutschera, 1979, p. 34) .

In this perspective meaning is given through the relation between a signifier and the signified, a linguistic expression and the object it refers to. In realistic theories linguistic expressions have a semantic relation with their entities, therefore the semantic function of linguistic expressions simply consists in a conventional relation to the object, i.e. a nominal relation. A word acquires meaning when it is in relation to an object, a concept or a proposition that are not necessarily concrete entities but are always given in an objective way and determine the meaning of a linguistic term.

We will turn back to this point when discussing Duval's cognitive paradox; this relational and referential idea of meaning is certainly very effective but it is deficient when dealing with the inaccessibility of mathematical objects. Resnik rightly argues that

> « since platonic mathematical objects do not exist in space or time the very possibility of our acquiring knowledge and beliefs about them comes into question ... Thus the Platonist seems to be in the paradoxical position of claiming that a given mathematical theory is about certain things and yet be unable to make any definitive statement of what these things are» (Resnik, 1981, p. 529).

Just to give some examples, the meaning of the proper noun "Born" refers to the German physicist, the atomic predicate like the sky is blue describes reality and the binary predicate the "pen is on the table" designates an attribute. The arithmetic expression 2+2=4 resembles the relation real existing numbers. In a realistic perspective the

We list below the basic features that characterise meaning and mathematical objects in realistic theories:

- **Meaning** is a conventional relation between signs and concrete or ideal entities that exist independently from their representations.
- There is a clear distinction between **semantics and pragmatics**, therefore the use of mathematical entities rests on meaning.
- Linguistic expressions have purely **semantic functions**, in terms of the nominal relation to the object they refer to. Each linguist term posses a specific meaning given by the concrete or ideal reality it describes.
- It is possible to carry out an objective and universal **analysis** on the objects of knowledge as for example logic.
- From a **epistemological** analysis one derives that realistic theories share a platonic viewpoint of objects, concepts, structure etc. that have a real and a priori existence, independent of time and culture, and accessible through the logos or applying the rules that govern the rational functioning of the mind.
- **To know** is to discover a priori entities and relations between such entities and verify the correspondence between the ideal structures and the functioning of the world. **Knowledge** is considered objective and absolute, formed by a system of certain and stable truths.

- Typical examples of realistic theories in mathematics are those of the first Wittgenstein, Frege**,** Carnap, Cantor, Goedel…

Realistic theories are very effective when dealing with the meaning of mathematical objects. Then reduction of meaning to a nominal designation of an ideal entity or ideal structures establishes a precise definition of meaning, but it rests on two assumptions that are difficult to justify and sustain.

The fact that we assign language a designating function doesn't imply that the ideal object actually exist. The problem of where and how mathematical entities exist is left unanswered. Furthermore nothing is said on what an ideal mathematical entity is. What mathematics does is defining its objects through language but how can one establish a relationship between an object and its representation? It is as if it were something intrinsic to semiotic representations. Duval (2006) remarks that the inaccessibility of mathematical objects raises the problem of how mathematical objects are designated and furthermore how do we recognise the same reference when changing representation through a semiotic transformation if there is nothing to compare the two representations with.

The second assumption regards the distinction between linguistic terms and mathematical objects. We are assuming that we can distinguish an entity called sign and an entity called object and create a representational relationship between the two. Such distinction doesn't reveal an a priori matter of fact but it's a conventional relation established by human beings. The claim that the meaning of representation is the content it refers to requires to clarify on what basis we can establish a distinction between something called object and something called sign. Godino remarks

> «"Mathematical object" is a metaphor that consists in bringing one of the characteristics of physical things (the possibility to separate from other "things") to mathematics. Therefore, for a start, everything that can be "individualized" in mathematics will be considered an object (a concept, a property, a representation, a procedure, etc.). We are constantly involved in decomposing reality in some way in a multiplicity of identifiable and differentiable objects that we refer to as through singular or general terms (this chair, the table, the letter x on the blackboard, the function f(x)=3x+2 etc.). […] Given that both a "sign" and an "object" are something, we must

consider them both as objects. To be an object or a sign is something relative. Therefore it is convenient to distinguish between objects and signs. It is an important distinction; however it is a circumstantial and not substantial difference since what in one moment is a sign in another can become an object and vice versa. According to what is more convenient, subjects can identify or distinguish the sign from the object. To be in place of, that is, being in such a relation in respect to the other, that for certain purposes it can be considered as in some way as if it was something else» (Godino, 2002, p. 287).

The act of designating itself is subjected to the same kind of critics, it is not something a priori that can be taken as a foundation of meaning. The relativity of the distinction between sign and objects makes designation something that has to be revisited under another framework. In chapter three we will recover designation but subsumed into a specific cultural and social practice.

We have presented the realistic ontology in mathematics that overcomes the delicate issue of meaning by seeing meaning as a referential relation between a priori entities and linguistic terms. The realistic theories are extremely effective if we don't question their basic assumptions. If we precisely look into the problem we conclude that they do not provide an untainted view of mathematical meaning but they rather have important pitfalls at their bases.

The limits of realistic theories we have highlighted are not meant to decry a view point that has been very effective in the development both of mathematics and hard sciences. The perspective that there is an ideal reality made of objects, concepts, logical structures, regularities that are mirrored by an appropriate use of signs on which our mind can rely, has inspired and still inspires today many mathematicians. We don't want to discard completely this ontological and epistemological stand. This investigation is an attempt to show that meaning intended as a designation of objects is not something a priori that human beings discover but it is a point of arrival of a long and deep endeavour of individuals organized in communities of practice and moved according to problems, sensibilities, philosophical stands that characterize the culture and society they belong to. Of course, this requires revisiting our notion of object, sign, designation and cognition.

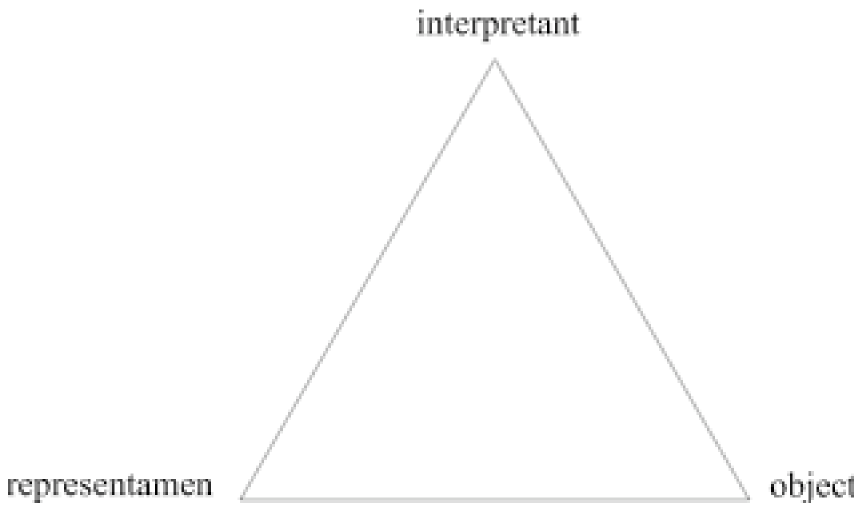
1.2.1 Semiotic triangles.

In this paragraph we introduce a more structured way of describing the relation between a mathematical object, its representation and its meaning that historically has been schematized through the following *semiotic triangles*. This semiotic study of content, conveniently modified, will be useful to the analysis we will carry out in the next chapter.

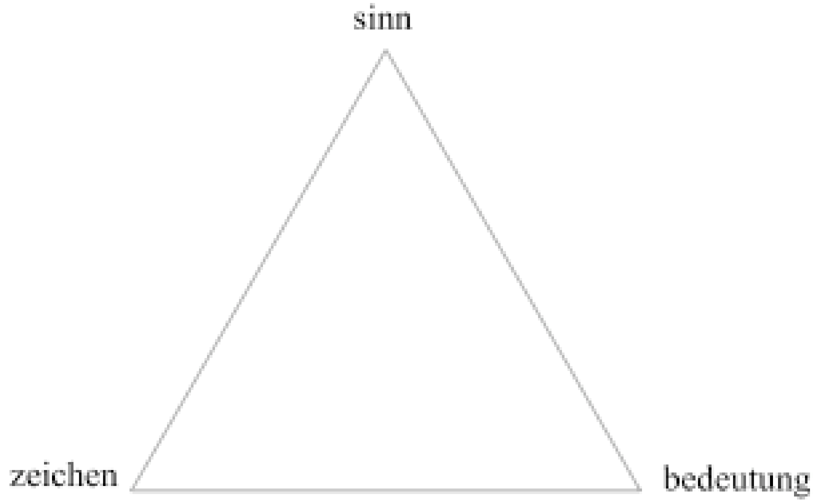
With different acceptations the semiotic triangles we propose relate an object, its representation and the relation that can be established between the object and its representation. Usually the term “sign” is attributed to the vertexes of the semiotic triangle and the relations between them and the term “sign-object” or “sign-vehicle” is used for representations.

The first form of the semiotic triangle was introduced by *Peirce* in (1883) whose vertexes are the *representamen*, the *interpretant* and the *object*. The philosophy that gave rise to triangles is explained by Peirce as follows:

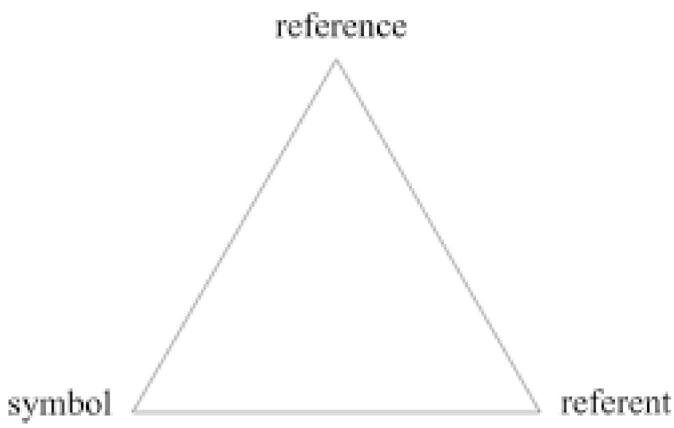
> «A sign... [in the form of a representamen] is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for that object, not in all respects, but in reference to a sort of idea, which I have sometimes called the ground of the representamen» (Peirce 1931-58, 2.228).



In 1892 *Frege* proposed the following triangle whose vertexes are *Sinn* (sense), *Bedeutung* (meaning/denotation) and *Zeichen* (expression). This triangle will be useful for our further analyses on meaning. We will concentrate on it below.



Richards' triangle published in 1923 whose vertexes are the *symbol*, the *reference* and the *referent* was a tentative to summarize the other two. The idea being that the relation between a sing and its referent is mediated by the meaning of the object, termed as reference.



The use of semiotic triangles oversimplifies the representation of meaning and has to face the problem of the referent, that Eco calls *extensional fallacy*:

> «Finally we face the problem which is mainly the concern of a theory of signs production and in particular a theory of mentions; but it is important to consider it at this point because its shadowy presence could disturb the proper development of a theory of codes. The problem in question is that of the referent, in other words of the possible states of the world supposedly corresponding to the content of a sign-function. Although of considerable importance within its proper domain, the notion of 'referent' has most unfortunate results within the framework of a theory of codes, and to underestimate its malignant influence leads to a *referential fallacy»* (Eco, 1979, p. 58) .

> «Since a theory of codes does not consider extension as one of its categories (and similarly doesn't take referents into account) it is able to consider, for instance, the so-called "eternal propositions" while disregarding their extensional value. If it does not disregard this factor, it falls, when dealing with code theory, in the extensional fallacy» (Eco, 1979, p. 63)

Eco focuses his critics to Frege's triangle because it is applied to objects belonging to a real and concrete dimension.

> «If one assumes that the *Bedeutung* is an actual state of the world, whose verification validates the sign, one must ask oneself how this state of the world is grasped or analyzed, how its existence is defined or demonstrated when the sign-function is decoded. It will be quickly be seen that, in order to know something about the *Bedeutung*, one must indicate it through another expression, and so on. […] Thus the *Bedeutung* is grasped through a series of its *Sinn*, and in this sense it can be very imprudent to assume that the *Sinn* can be recognized as appertaining to the same *Bedeutung*, since it is the *Bedeutung* which is defined by the *Sinn* and not vice versa» (Eco, 1979, p. 61).

We can partially overcome the extensional fallacy and use Frege's semiotic triangle to analyse meaning in mathematic because as D'Amore suggests

> «Now, the most naïve and immediate interpretation is that signified of the signifier is the object itself to which it refers. This stand leads to a fallacy ("extensional fallacy"); although it throws into crisis every code theory that needs objectual extensions to a real state of the world, it doesn't disturb mathematics whose objects can be defined in an extensional form, but without the need of any reference to an empiric objective state of the world. (It is not by chance that that the mathematical logician Frege can allow himself to consider *Bedeutung* in a strictly extensional sense, since above all he thought of mathematics and not of natural language».(D'Amore, 1999, p. 241)

Of course, the problem of what *Bedeutung* refers to and how we can distinguish different *Sinn* belonging to the same referent remains an open problem. We now detail what *Sinn* and *Bedeutung* are and try to apply them to frame the meaning of mathematical objects. Frege's concern was to show that the identities $a=a$ and $a=b$ are not the same. The second identity conveys information in a way that the first cannot give. For example $n+n+1=n+n+1$ doesn't give the same information as $n+n+1=2n$. The second identity tells me that the sum of two consecutive numbers is always an even number.

The *Bedeutung* is the is the object the semiotic Zeichen (expression) refers to that offers such object with a particular *Sinn*, the way in which it is given through an expression. Frege's (1993) classical example is that of Venus, considered as the *bedeutung* that can be given with two different *Sinn*, Hesperus, the morning star and Phosphorus, the evening star. For instance in arithmetic the expression $5*7$ and $30+5$ are two different *Sinn* for the same *Bedeutung,* the natural number 35. A good mathematical application of this model is in algebra. In general algebraic expressions are different *Sinn* through which the *Bedeutung* belonging to a particular numerical universe is given. For example the expressions $2n+1$ and $(n+n+1)$ are two ways of expressing the set of odd numbers, if n is a natural number. If we consider the two real functions $f(x)=x(x+1)$ and $g(x)=x^2+x$, dropping considerations regarding domain and codomain, the two expressions give two *Sinn* of the same *bedeutung,* the graph of the function $G \subseteq RxR$.

As regards signs, within this model, we can basically assign two semiotic functions; the *symbolic function* and the *transformational function* (Arzarello, 2006; Arzarello et. al, 1994). The symbolic function gives two *Sinn* of the same *Bedeutung* through change of frame (D'Amore, 1999) or context in which the object is used. For example the expression y=x^2 +1 can be interpreted as a *function* or as an *equation* with two unknowns.

The transformational function is a common reference obtained both through a change of *Sinn* and a change of expression. Algebraic transformations are typical examples of the transformational function but it can also involve an equation and a graph. For example $(x^2+1)' = 2x$ is an example of a transformational function.

The problem of changes of meaning can be framed within Frege's semiotic triangle. We can say that in the examples we proposed in the introduction, students have to deal with different *Sinn* of the same *Bedeutung.* This is a basic route to model the changes of meaning due to semiotic transformations and, of course, with a lot of pitfalls but very effective to give a first frame to the problem. Within a realistic stand there is no insight regarding the nature of the supposed existing ideal mathematical object. Furthermore, Frege's example of Venus that has different senses given by the expressions morning star and the evening star is viable because we can someway access the denotation, and carry out a comparison and a distinction between the object and the two expressions. This kind of operation is not possible in mathematics, so we need to scrutinize the relationship that it is established between mathematical objects and signs and between signs themselves. The interpretation of changes of meaning in terms of *Sinn* and *Bedeutung* is our starting point that we will develop and modify into a completely different model of the relation between meaning, mathematical objects and signs.

1.2.2 Vergnaud's triplet C(S,I,*S*)

To conclude our overview on realistic theories in mathematics, we present what can be a considered (D'Amore, 2001) as a possible conclusion of the classic strand of semiotic triangles: Vergnaud's "definition" of a concept. Our intention is not facing this fundamental topic but to show a more comprehensive interpretation of the semiotic triangles. According to Vergnaud (1990) a concept is a triplet C=(S,I,*S*) where :

- S is the set of situations that give sense to the concept, the *referent*
- I is the set of invariants that support the operativity of the schemas, the *signified*
- *S* is the set of linguist and non linguistic forms that represent the concept in its different instantiations, the *significant.*

To understand the nature of a concept we must analyse the sets (S,I,*S*) and their mutual relationships. Conceptualization entails an important the passage from *concepts-as-instruments* and *concepts-as-objects* in which signs play a fundamental role through *nominalization*. Behind this approach to concepts and meaning there is the notion of *schema* and *operational invariant* that Vergnaud borrowed from his master, Jean Piajet. Through nominalization process, the invariants that sustain the schemas are objectified. We can recognize here a correspondence with the dual nature of mathematical conceptions, processes and objects, and reification introduced by Sfard (1991).
Vergnaud's triadic approach gives a more comprehensive description of its elements and the relation between them. We don't have an ideal and a priori object but situations in which the invariance that sustain the operativity of the schemata emerge and are nominalized. We are still within a realistic approach that, however, to a certain extent opens the way to a pragmatic view of mathematical objects and meaning that we will treat in the next paragraph. Godino and Batanero recognize in Vergnaud's triad the dialectic relationship between praxes and concept that characterizes their pragmatic approach.

> «We agree with him when he says that situations and representations are strictly bounded to the activities from which the mathematical objects culturally defined emerge; i.e. we believe it is necessary to single out the dialectic relationship between activity (praxis) and concept» (Godino, Batanero, 1994, p. 34)

We believe, anyway, that Piaget's stand that underlies Vergnaud notion of schemata and operational invariants, doesn't allow to place his triadic vision of concepts in to a fully pragmatic conception of mathematical objects and meaning.
Radford suggests that

> «Nevertheless, both Kant and Piaget were wrong in seeing knowledge as a process that ascends from the concrete to the abstract, from the tangible

world to the world of the intangible, leading, in the case of Kant, to an embedding theory in which the sensual object is subsumed into the concepts of reason and, in the case of Piaget, to an emphasis in activity with concrete objects in the sensori-motor stage which vanishes into thin air in later 'development stages'» (Radford, 2004, p. 17)

«Indeed, following Kant –who attempted to achieve reconciliation between empiricist and rationalist trends– Piaget emphasized the role of sensorial-motor actions. If, however, body and artefacts played an epistemological role in his genetic epistemology, it was only to highlight the logical structures that supposedly underlay all acts of knowledge. The semiotic function, as Piaget called it (which includes representation, i.e situations in which one object can stand for another; imitation where sounds are imitated, evocation, etc.) was the bridge between the sensual and the conceptual, between concrete schemas and their intellectualized versions. This is why "operations [i.e. reflective abstracted actions] can sooner or later be carried out symbolically without any further attention being paid to the objects [of the actions] which were in any case 'any whatever' from the start." (Radford, 2005b, p.114).

1.3 Pragmatic theories

Recalling Kutschera distinction between realistic and pragmatic theories of meaning, in this paragraph we will address the latter approach to meaning. Analysing pragmatic theories require to take into account and privilege the role of the human being in cognition and learning and teaching processes. This entails a high complexity and a variety of perspectives within this same strand. We will have to take into account activity, human consciousness, signs and sociocultural features just to quote only some of them.

We can trace pragmatic theories back to Wittegnstein's notion of "linguistic game" that he introduces in his masterpiece *Philosophical Investigations,* first published in 1953. According to Wittgenstein, it is impossible to identify an a priori ideal mathematical object, but we can only refer to the context in which linguistic terms are *used* according to a specific "language game". There is no object per se but only linguist terms that play a specific function within a language game that endows it with a specific use. The linguist term has no a priori meaning but it is meaningful within the language game in which it is used.

Wittgenstein withdraws from the view point that the meaning of a linguistic term is the object the linguistic term refers to, because there is no such a relation between the representation and the object. The Philosophical Investigations begin with a critics to Augustine's (*Confessions*, I.8.) explanation of a plain functioning of language in terms of words and their referents, or sense and denotation to use Frege's terminology. This is how Wittgenstein comments Augustine's passage from the *Confessions*:

> « These words, it seems to me, give us a particular picture of the essence of human language. It is this: the individual words in language name objects--sentences are combinations of such names. In this picture of language we find the roots of the following idea: Every word has a meaning. The meaning is correlated with the word. It is the object for which the word stands.
> Augustine does not speak of there being any difference between kinds of word. If you describe the learning of language in this way you are, I believe, thinking primarily of nouns like 'table', 'chair', 'bread', and of people's names, and only secondarily of the names of certain actions and properties; and of the remaining kinds of word as something that will take care of itself» (Wittgenstein, 1953, #1).

In fact, outside a language game there is no object and there is no linguistic term. Lets take for example the term "chair"; it apparently looks very simple and natural to say that the meaning of the word is the object I am sitting on while writing with my laptop. There is no chair per se outside a net that interweaves symbol, words and actions. There is no meaning to the term "chair" and a so-called object to which the term refers independent of an act of sitting, of myself actually sitting on the chair, of the table that makes sense of me using the chair etc. Wittgenstein suggests the following example:

> «Now think of the following use of language: I send someone shopping. I give him a slip marked 'five red apples'. He takes the slip to the

> shopkeeper, who opens the drawer marked 'apples', then he looks up the word 'red' in a table and finds a colour sample opposite it; then he says the series of cardinal numbers. I assume that he knows them by heart up to the word 'five' and for each number he takes an apple of the same colour as the sample out of the drawer. It is in this and similar ways that one operates with words. "But how does he know where and how he is to look up the word 'red' and what he is to do with the word 'five'?" Well, I assume that he 'acts' as I have described. Explanations come to an end somewhere. But what is the meaning of the word 'five'? No such thing was in question here, only how the word 'five' is used» (Wittgenstein, 1952, #1).

After introducing (aphorism #2) an example of a language game in which a builder and an assistant use building stones as blocks, pillars, slabs and beams, in the following aphorism Wittgenstein describes the nature of a language game.

> «In the practice of the use of language (2) one party calls out the words, the other acts on them. In instruction in the language the following process will occur: the learner names the objects; that is, he utters the word when the teacher points to the stone. And there will be this still simpler exercise: the pupil repeats the words after the teacher, both of these being processes resembling language.
>
> We can also think of the whole process of using words in (2) as one of those games by means of which children learn their native language. I will call these games "language-games" and will sometimes speak of a primitive language as a language-game. And the processes of naming the stones and of repeating words after someone might also be called language-games. Think of much of the use words in games like ring-a-ring-a-roses.
>
> I shall also call the whole, consisting of language and the actions into which it is woven, the "language-game"» (Wittgenstein, 1952, #7).

In Wittgenstein's pragmatic perspective a linguistic term is meaningful as long as it is used in a specific linguistic game with a precise objective. There is no objective reality organized in more or less complicated structures that can be traced back to elementary entities with an intrinsic essence. Semiotic representations are not a privileged access to this pre-existent reality but what we call the “world” is an emergence from human practices underlain by a specific network of linguistic terms and “objects” interwoven through the language game.

In aphorism #50 Wittgenstein claims that it is doesn’t make sense to say that a specific element exists separated by its connection with other elements of the linguist game. He gives the insightful example of the “meter”, there is no existing “metre” outside a practice based on comparison between the standard metre placed in Paris at Sèrves museum and the object we are measuring. There is no “metre” as an independent unit, nor can we ascribe the platinum-iridium rod placed in Paris a privileged role in the metre unit, in fact we can say nothing about that metal bar outside the linguistic game of measurement.

> « There is one thing of which one can say neither that it is one metre long, nor that it is not one metre long, and that is the standard metre in Paris. But this is, of course, not to ascribe any extraordinary property to it, but only to mark its peculiar role in the language-game of measuring with a metre-rule» (Wittgenstein, 1953, #50).

An argument that it is often brought in favour of realistic theories, mainly by my physicist friends is the amazing correspondence between the results obtained through mathematical models to describe reality and the experimental result. They are right when they say that the correspondence is amazing but they fail when they claim that such correspondence is the proof of the existence of mathematical reality. They don’t take into account that the notions of correspondence and measurement make sense within the language game of physics and cannot imply an objective existence of the mathematical entities physics uses.

Linguistic terms acquire meaning because language is part of an activity, also called by Wittgenstein a form of life; there are as many meanings and uses of language as possible linguistic games:

> «There are countless kinds: countless different kinds of use of what we call "symbols", "words", "sentences". And this multiplicity is not something fixed, given once for all; but new types of language, new language-games, as we may say, come into existence, and others become obsolete and get

forgotten. (We can get a rough picture of this from the changes in mathematics.)

Here the term "language-game" is meant to bring into prominence the fact that the speaking of language is part of an activity, or of a form of life» (Wittgenstein, 1953, #23).

We have shown how in the pragmatic stands, typically from Wittgenstein's Philosophical Investigations, the notion of *use* and *activity* is what endows meaning to linguistic terms and semiotic representations in general. Meaning is no longer given by a reference to a not well defined objective entity that is considered independent from the linguist terms. Kutshera claims that pragmatic theories overcome the problem of coping with supposed existing ideal entities that we cannot access and whose nature remains unknown:

«The vanishing of concepts and propositions as elements independent from language dissolves the problem of how these entities are known, and we bring near the phenomena that justify the dependence of thought and experience from language» (Kutshera 1979, p. 148)

We have focused our attention on activity and linguistic terms, but what can we say of mathematical terms in a pragmatic theory. Of course, we cannot refer to an objective, monolithic and fixed entity, but within the notion of practice it still makes sense to speak about a mathematical object. The complexity and variety of pragmatic approaches don't allow to give a single definition of a mathematical object. We present three different definitions that, although with different acceptations, at their core are characterized by activity and the use of signs.

We begin with the following definition proposed by Chevallard,

«An emergent from a system of practice where material objects, that break down in different material semiotic registers, are manipulated; oral register, register of words or pronounced expressions; register of gestures, domain of inscriptions, i.e. that which is written or drawn (graphs, formulae, calculations, …), that is to say, the register of writing» (Chevallard, 1991, p.8).

And continue with the definition proposed in the Ontosemiotic approach:

«Mathematical objects are therefore symbols of cultural unities that emerge from a system of uses that characterizes human pragmatics (or at least of an homogeneous group of individuals); and they continuously change in time, also according to needs. In fact, mathematical objects and their meaning depend on the problems that are faced in mathematics and by their resolution processes. In sum they depend on human praxes» (D'Amore, Godino, 2006, p. 14).

Both definitions pivot around the notion of practice, but while the former sees a passage from objects to semiotic representations the latter focuses directly on the use of symbols, characterizing the mathematical practice basically as a semiotic practice. This definition condenses the richness and complexity of the ontosemiotic approach that we will face in the next chapter. This approach precisely highlights the fact that it is rather naïve to think of "the" mathematical object. What we think of as a mathematical object in fact is a configuration of entities that accomplish a system of practices.

Below the definition of a mathematical object introduced in the cultural semiotic approach:

> «The theory of knowledge objectification suggests that mathematical objects are historically generated during the course of the mathematical activity of individuals. More precisely, mathematical objects are fixed patterns of reflexive human activity incrusted in the everchanging world of social practice mediated by artifacts» (Radford, 2008, p. 222).

We will analyze in thorough what a reflexive mediated activity really is in the next chapter. Again, practice plays a central role in specifying the nature of a mathematical object that it is identified with the emerging of a pattern that resembles the notion of operational invariants of the schema used by Vergnaud to define a concept. In a truly pragmatic perspective, signs completely lose their representational function, and become the mediators that accomplish the reflexive activity. We will see how the role of representing, generally attributed to signs, is substituted by a more comprehensive process that Radford calls *objectification*.

In the next chapter we will carry out a through analysis of the theoretical background behind the aforementioned definitions and the implications they have, within different pragmatic perspectives, on meaning and changes of meaning of mathematical objects.

We list below the basic features that characterise meaning and mathematical objects in pragmatic theories:

- **Meaning** is no longer a relation of reference between a linguistic expression and an ideal entity. The meaning of a linguistic expression is the determine by the use of such expression in a specific linguistic game.
- There is no clear distinction between **semantics and pragmatics**, since the meaning of as linguistic term depends on its use in a specific context. It is therefore senseless to base pragmatics on a pre-existing meaning that is drawn from a contextual use.
- Linguistic expressions lose the purely **semantic functions** they accomplished in realistic stances. Linguistic expressions have personal meanings in the contexts in which they are used and we cannot recognise any absolute and per se meaning. There is no correspondence between the structures of language and supposedly existing ontological structures.
- It is impossible to carry out a scientific, absolute and inter-subjective analysis of abstract entities. It is possible to carry out only a personal and subjective **analysis** on the contextual use of language.
- From an **epistemological** point of view we have a problematic conception of mathematical objects. On the one hand there is no fixed entity that we can refer to as an object, and the other hand the notion of a mathematical makes sense and we have shown different possible definitions from a pragmatic perspective. We cannot say what a mathematical object is, unless we are involved in an activity that makes sense of it.
- **To know** is to become involved in a practice that is carried out in a specific cultural and social context. In this perspective, to know is not a form of apprehension, that etymologically means to “grasp” something. In fact, there is nothing there that we can cognitively “grasp”, learning is an endeavour that requires to become part of a social and cultural dimension where students, while sharing activities, share also meanings. **Knowledge** is no longer an absolute and objective system of facts that correspond to an a priory mathematical reality. Knowledge is relative to a specific practice and field of problems that accomplish a specific language game.

- We introduced pragmatic theories taking into account Wittgenstein's notion of language game as a paradigmatic example. For our analysis we will consider the ontosemiotic and cultural semiotic approaches. The former is based on the notion of language game, in the latter we cannot fully identify activity with a language game.

We believe that from an educational point of view the pragmatic perspective is the most appropriate to face the complexity of teaching-learning processes in mathematics. This is even more true when facing the issue of changes of meaning. A purely referential relation between a signified and a signifier, advocated by realistic theories, doesn't fully account for the true nature of mathematical objects, representations and their meaning. If we look at how mathematical knowledge develops both at a phylogenetic and ontogenetic level we cannot identify a binary structure consisting of mathematical objects existing per se and their corresponding representations. What we will actually see is individuals involved in communities of practices using linguistic terms in a culturally and socially defined language game.

If we look for numbers, straight lines, points etc. we will find no such "objects" but specific language games in which the *use* of linguistic terms give sense to what we call a number, a straight line, a point etc. As Wittgenstein points out in aphorism #23 of his Philosophical Investigations, language games are not fixed once for all but they continuously evolve as the cultural and social needs of individuals develop. From a realistic point of view we can think of the straight line as an independent existing concept. But can we really identify "the" straight line? What are we thinking of? The straight line used in Euclidean geometry, the straight line used in analytic geometry or the straight line used in Riemannian geometry? There is

Mathematics is generally meaningless to students because we hide the practices and field of problems that are the core of the concepts and we present only the final product in its axiomatic-deductive form. Lakatos (1979) claims that mathematics is a living science that produces dead theories. In the pragmatic viewpoint we are advocating here,

Mathematical objects, concepts and meanings are indissolubly interwoven through a shared practice that entails a personal and social relationship to the object.

Such practices, in mathematics, are intrinsically mediated by signs that are consubstantial to thinking and mathematical objects.

We will face the issue of changing meaning scrutinizing the entanglement between, mathematical objects and meaning focussing on the systems of practices mediated by signs. We will not completely discard the signified-signifier structure that is effective to understand both mathematics as a field of knowledge and its teaching-learning processes, if we don't miss the practices that are beneath the use of signs in a referential way. The referential relation between signs and objects will be interpreted, in turn, as a language game, that in the cultural semiotic approach is called *objectification* and in the ontosemiotic approach *semiotic function.* Objectification processes and semiotic functions are two different notions that conveniently combined can help to frame the problem of changes of meaning.

Our pragmatic choice requires a thorough analysis of the role of signs in mathematical cognition and learning and of the relation between signs and their use in linguistic games accomplished in cultural and social activity. This analysis will be carried out in the next chapter.

1.4 Synthesis of the chapter

In this chapter we faced the issue regarding the nature of mathematical objects signs and meaning. Following Kutschera distinction we examined two basic epistemological approaches to the problem: *realistic theories* and *pragmatic theories*.

Realistic theories hold a platonic vision of mathematical objects as a priori entities existing in an ideal domain independent of human experience. The meaning of a linguistic expression is the object that the expression stands for. Meaning is therefore seen as a referential relation between a signifier and a signified. In a realistic approach it is useful to represent the relation between object and representation through three semiotic triangles that in their different versions have as vertexes, representamen-interpretant-object (Peirce), expression-denotation-sense (Frege) and symbol-referent and reference (Richards). We focused on Frege's relation between sense and denotation to frame the problem of meaning and of changes of meaning. Different representations of the same object give different ways (senses) of accessing the same object (denotation). From a didactical point of view, students' difficulties can be interpreted in terms of the way they handle the side sense-denotation of the semiotic triangle. As regards our problem of changes of meaning we can say that students do not recognize different sense of the same denotation given by different semiotic representations. The realistic viewpoint has the pitfall of leaving unanswered the problem of what a mathematical object as an a priori entity really is and the problem of their inaccessibility.

Pragmatic theories deny the existence of objective and absolute ideal entities. It is therefore senseless to look for the meaning of a linguistic term in an object it refers to. Taking as paradigmatic example Wittgenstein's notion of language game, the meaning of a linguistic term is the use of that term in social and cultural shared practices. The focus is shifted on individuals and the different mathematical contexts in which they engage in activities to pursue their cognitive, cultural and social needs. We have seen how with different acceptations we mathematical objects are considered symbols of cultural unities that emerge from a system of uses that characterizes human pragmatics. Therefore, there is no fixed and objective knowledge that, instead, continuously changes in time, also according to needs. The signified-signifier model is baseless in a pragmatic approach since mathematical objects, concepts and meaning are entangled through social and cultural practices accomplished through signs. The object –representation model can be recovered as a possible practice that interweaves signs objects and meanings. In pragmatic approaches the absence of entities that are independent of language and human practices allows to shift our focus from the apprehension of such entities to the practices that entangle cognition, mathematical objects and signs.

2

The structural and functional semiotic approach

2.1 Introduction

In following three chapters we will give the basic elements of the three semiotic perspectives that we will use to frame the problem of changes of meaning. We will address Duval's structural and functional approach, Radford's cultural semiotic approach and Godino's Ontosemiotic approach.

Our objective is to outline the basic elements that characterise each approach in view of giving an interpretation and an explanation of changes of meaning. In each perspective we will analyse the role of signs in mathematical cognition and learning and provide a specific frame for the meaning of mathematical objects; we will focus our attention on the relation that can be established between signs and mathematical objects in the three perspectives.

The idea of using the semiotic lens to analyse mathematical cognition and learning processes can be traced back to the forefront researches carried out by Raymond Duval (1988a,b,c, 1993, 1995, 1996, 1998) from the beginning of 90's. Duval's concern was to characterise the specific cognitive functioning of mathematics that he identifies with a complex coordination of semiotic system that we will describe in paragraph 2.2.3. For Duval, mathematical conceptualization, meaning, and thinking are strictly bound to such complex semiotic functioning. He arrives to this conclusion because of the special epistemological condition that characterises mathematical objects with respect to other fields of knowledge. We begin our treatment of Duval's approach looking at such peculiar condition.

2.2 Inaccessibility of mathematical objects

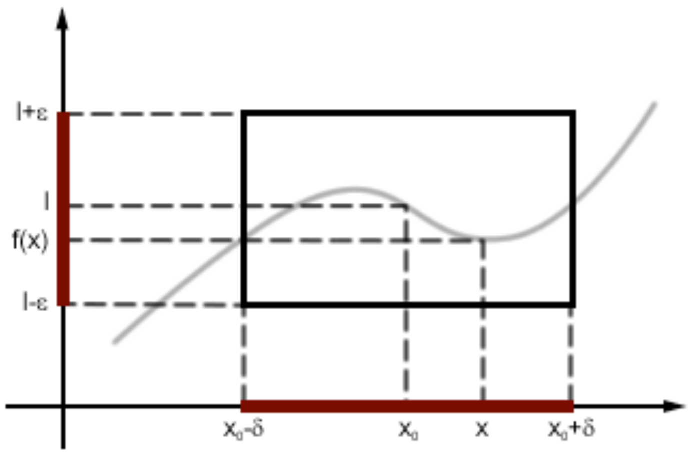
The starting point of Duval's argumentation that justifies the priority role bestowed on semiotics in mathematical thinking and learning lies on a realistic ontology. Meaning is conceived as a the reference to a mathematical object through a semiotic representation.

In D'Amore (2005) we find the following renowned work of Magritte entitled *Ceci n'est pas un pipe*, that he realized in different versions between 1929 and 1946.

> « Beyond the embarrassment it created when it was first exposed, seen with the critic and sharp eyes of today, the sense of this intentionally popular work is fully evident: in fact, the observer DOESN'T see a pipe but one of its representations that alludes to a pipe; what we see, then, is a representation an allusion, an evocation, not the object per se» (D'Amore, 2005, pp. 418-419).

Duval (2006) proposes the following work of Kosuth entitled *One or three chairs* thatb he realized in 1965.

Duval (2006, p. 592) suggests that this artwork realizes a double juxtaposition: the juxtaposition (O, R(O)) and the juxtaposition (RA(O), RB(O)). We can think of an object and a series of its possible representations but we can also think of a series of representations alone independent of the object. Kosuth's art work can be interpreted as the first juxtaposition, the physical chair and the two representation, the photograph and the definition pinned to the wall. But we can also interpret it as a juxtaposition of representations of the chair; the photograph, the photograph of the photograph and the verbal definition.
But what can we say as regards the following representations:



$\lim_{x \to x_0} f(x) = l$ and

"For every open neighbourhood V of L, there exists an open neighbourhood U of x_0 such that $f(U \cap \Omega - \{x_0\}) \subseteq V$"

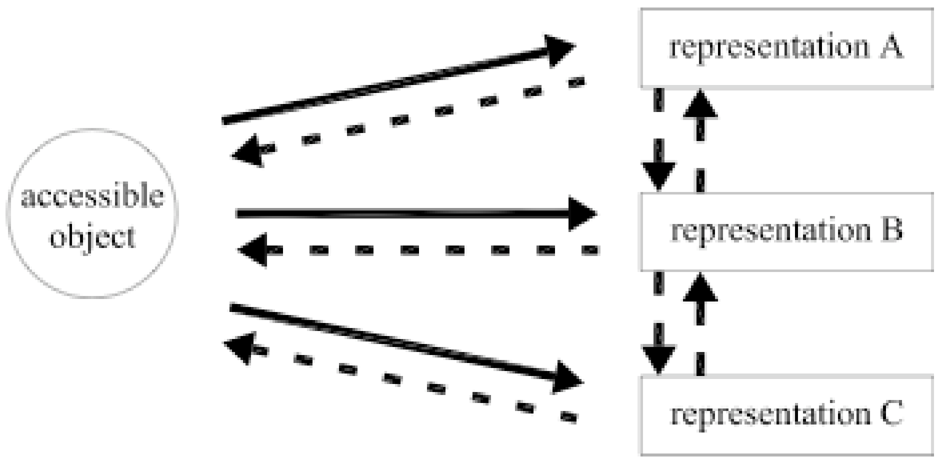
How many limits are there? "One or three limits"? Can we recognize a double juxtaposition? Duval's answer is that there is no double juxtaposition. In the case of Magritte's pipe it is evident that the photograph of the pipe is not a pipe and the double juxtaposition is immediate; but in the case of the limit of a function how can we establish a distinction between the mathematical object and the representation if we have no access to "something" like a function but only to semiotic representations? In the aforementioned article Duval claims that:

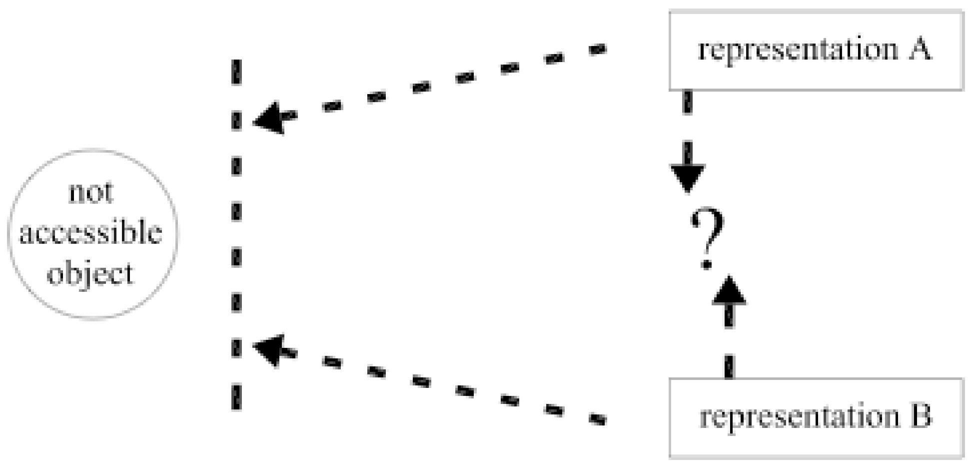
> «the special epistemological situation of mathematics compared to other fields of knowledge leads to endow semiotic representations a fundamental role. In the first place they are the only way to access mathematical objects which raises the cognitive issue of the passage form one representation of the object to another of the same object» (Duval, 2006, p. 586).

In fact, mathematical concepts refer to "non objects" (D'Amore, 2001), that is mathematical knowledge doesn't allow an ostensive relation to a reality of concrete objects. Conceptualization intrinsically requires the introduction of signs organized in systems of signs. In Duval's analysis, mathematical objects play a priority role with respect to mathematical concepts:

> «The notion of object is a notion that we cannot use when questioning the nature, the conditions of validity and the laue of knowledge» (Duval, 1988, p. 139)

The impossibility to establish a double juxtaposition, on the one hand confers to semiotics a central role in mathematics, on the other it entails an intrinsic difficulty in using the variety of representations that are intrinsic to the growth of mathematical knowledge and its learning. In fact, Duval (1995, 2006) claims that the strength of semiotic representations lies in the fact that they can be transformed one into another. From a didactical point of view, the question is how the pupil can correctly handle such transformations if he has no accessible mathematical object with which he can compare the different representations. Furthermore, how can he recognize that the different representations refer to the same mathematical object? The inaccessibility of its objects of knowledge endows mathematical thinking with a specific cognitive functioning that Duval identifies with the coordination of semiotic systems that we will outline in paragraph 2.2.5.





Duval's approach rests on the strong assumption that a mathematical object really exists as an ideal entity. What makes his analysis interesting and original compared to other realistic approaches is that he focus on the issue of inaccessibility of mathematical objects which grounds the constitutive role played by semiotics in mathematical thinking and learning. The issue of inaccessibility doesn't regard only realistic theories, but, to an even greater degree, plays a central role also in pragmatic theories. In Chapter 1 we have seen that it is senseless to search for a fixed mathematical object and that we are left only with a social practice in a linguistic game. The structural and functional semiotic analysis conducted by Duval can be an important resource to understand the nature of linguistic games in mathematics.

2.3 From the ternary to the binary structure of meaning: signs and systems of signs

In chapter 1 we presented Vergnaud's triadic representation of a concept, C(S,I,*S*) that can be interpreted as a possible conclusion of the classical strand of semiotic triangles. The interplay of the elements that form the tern requires the passage from concepts-as-instruments to concepts-as-objects obtained by *nominalization* processes. The view point behind this kind of approach –shared by others distinguished authors like, Piaget, Vygotsky, Borousseau, Chevallard – is that the notion of concept plays a priority role with respect to the relationship between mathematical objects and signs, the idea being that the constructions of concepts in mathematics is independent of the semiotic activity; *noesis* is independent from *semiosis* In piagetian theories, for example, through actions, adaptation processes and the recognition of operational schemata the pupils conceptualizes logical structures that underlay mathematical objects.

Furthermore, the priority function played by concepts with respect to signs is associated with the distinction between external semiotic representations and internal non semiotic mental models where concepts "live". Signs are used only for appropriation and communication of the concept, *after* it has been obtained by other means. In mathematics, both when dealing with the production of new knowledge and with teaching-learning processes, this position is untenable, due to the ontological and epistemological nature of its objects. In fact, we witness a reverse phenomenon:

> «Of course, we can always have the "feeling" that we perform treatments at the level of mental representations without explicitly mobilising semiotic representations. This introspective illusion is related to the lack of knowledge of a fundamental cultural and genetic fact: the development of mental representations is bound to the acquisition and interiorisation of semiotic systems and representations, starting with natural language» (Duval, 1995, p. 29).

According to Duval we cannot separate conceptualization from the use signs. The special nature of mathematical objects that we described in the previous section intrinsically require the use of signs to conceptually apprehend mathematical objects. Mathematics entails a specific cognitive functioning that is characterised by the mobilisation of semiotic representations:

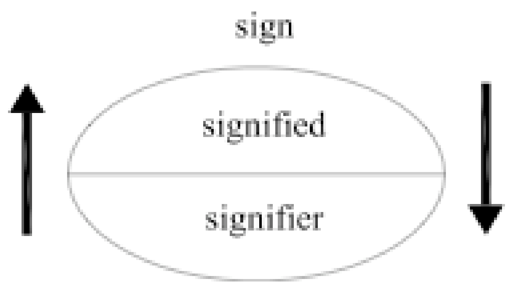
> «The analysis of problems regarding the learning of mathematics […] leads to recognise a fundamental law for the functioning of thought : there is no *noetics* **without *semiotics***, that is without resorting to a potential multiplicity of semiotic systems that entails their coordination on the part of the subject himself» (Duval, 1995, p.5).

Because of the special epistemological situation of mathematical objects, Duval shifts his attention from the semiotic triangles to the binary structure (Sign, Object) to describe the relation between signs, objects and meaning. Duval considers a mathematical object as

> « the invariant of multiple possible representations and that such invariant is discovered recognizing that two representations are representations of the **"same thing",** even when they have nothing in common. Because, also when one representation is privileged, an object of knowledge is never opposed to "its" representation but to the ensemble of its possible multiple representations. The variety, or variation, of representation is essential to become aware of the epistemological gap inside each representation» (Duval, 2009, p. 85).

The aforementioned quotation entails that a sign cannot be reduced to a conventional and independent symbol that directly refers to an object; there is no sign outside a system of signs that Duval calls a *semiotic system*. This is a reasonable condition when supposing the existence of an object that is not directly accessible. Meaning cannot be in the reference to the object itself but in the structure of the system of signs and in the relation between different systems of signs. An isolated mathematical sign for instance an "x" is meaningful because it belongs, for example, to algebraic, system of signs in which it can be interpreted as variable or an unknown.

This structural approach back to the Swiss linguist Ferdinand de Saussure. Saussure (1983) argued that signs only make sense as part of a formal, generalized and abstract system. His conception of meaning was purely structural and relational rather than referential: primacy is given to relationships rather than to things. The meaning of signs was seen as lying in their systematic relation to each other rather than deriving from any reference to material things. For Saussure, signs refer primarily to each other. Within the language system, everything depends on relations (Saussure 1983, p.121). No sign makes sense on its own but only in relation to other signs. Both signifier and signified are purely relational entities (Saussure 1983, 118). For example the meaning of an individual word such as “book” is not a concrete object to which it refers, but its meaning depends on its context in relation to the other words with which it is used. In mathematics the meaning of y=3x+2 is not the straight line itself but the relation between signs in the algebraic system and the relation between the algebraic system and the Cartesian one. Saussure (1983, 101) called the relation signifier-signified within the system a *sign.*



According to Duval (2006) a semiotic system is formed by:

1) Organizing rules to combine or group together elements in significative unities, that is in elementary expressions of figural units
2) Elements that assume sense when compared to other signs: by a distinctive difference or opposition between two signs of the system.

A typical example of a semiotic system is the decimal system; there are precise rules to combine elements and the meaning of signs is determined by comparison to other elements of the system. The structure of a semiotic system *structure* allows both the production and transformations of signs and characterizes the system according to the discursive *functions* (Duval, 1995, pp. 88-98). A semiotic system in general accomplishes a communication function.

We speak of a *semiotic register* when three meta-discursive functions are accomplished by the structure of the system:

1) Objectification
2) Communication
3) Treatment

A semiotic register is considered a *language* when it accomplishes the following discursive functions:

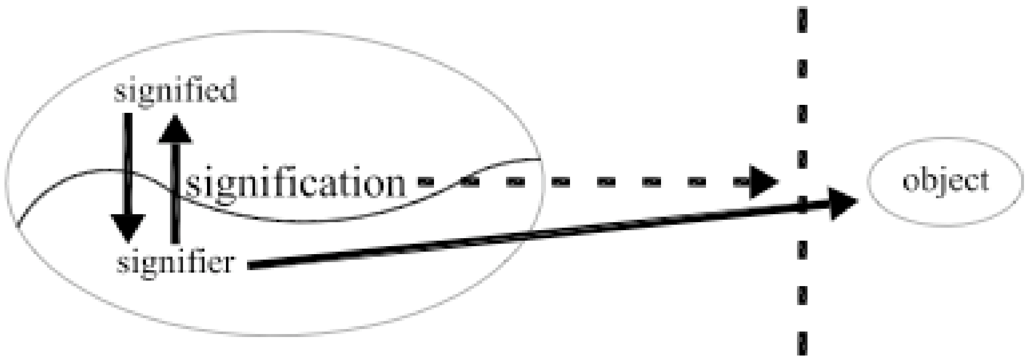
1) Referential: Designation of objects
2) Apophantic: constitution of complete statements
3) Discursive expansion: articulation of complete statements in a coherent unity.

If the systems accomplishes also *discursive reflexivity* used to support reasoning, we speak of a *natural language*.

In the couple (sign,object) the sign is related to other signs within a semiotic system. In Duval's approach, semiotic representations refer to an object, even though it is inaccessible. So, on the one hand semiotic representations exist independently within the sign system on the other they refer to a "real" inaccessible object. Duval analysis originates from Frege's notions of Sinn and Bedeutung but he believes that Frege's account doesn't address the following issues (Duval, 2008):

- Inaccessibility of mathematical objects
- The relation between the sense of a semiotic representation and the features of the system used for producing it.
- The coordination of different semiotic systems

Duval condenses in the sign, conceived as a complex structure, both sense (Sinn) and Bedeutung (denotation); Sinn is understood in the aforementioned Saussurian sense and denotation, that cannot be a direct relation between the object and the representation, is accomplished as invariant from the use of representations within the semiotic system or the connection of more than a single semiotic system. Denotation requires recognizing a differentiation between signs and mathematical objects; such differentiation is difficult to achieve because of the cognitive paradox that leads to an overlap of the representation with the mathematical object.



In the (sign,object) couple, Duval synthesizes both the referential relation that characterises semiotic triangles and the structural notion of sign belonging to a semiotic system.

2.4 Mathematical cognitive operations: choice of the distinctive features, treatment and conversion

We have seen that inaccessibility of mathematical objects implies that there is no noetics without semiotics. Duval claims that the conceptual apprehension of mathematical objects *is* a complex connection of significative units and representations within a semiotic system and between different semiotic systems. In this respect, we can identify a specific cognitive functioning for mathematics that is different from the cognitive functioning of other fields of knowledge that allow the double juxtaposition we introduced in paragraph 2.2.1. Such cognitive functioning is carried out by three basic cognitive operations (Duval, 1993, 1995, 2006): choice of the distinctive features, treatment and conversion.

Before we detail these operations it is helpful to introduce the following notation taken from D'Amore (2001, 2006):

- r^m is a generic semiotic register (m=1,2,3…)
- $R^m_i(A)$ is the i-th semiotic representation of the object A in the register r^m (m= 1, 2, 3, …; i = 1, 2, 3, …)

The choice of the distinctive factors is the operation that selects the representation of the object A within the semiotic register. The inaccessibility of the mathematical object makes this operation a purely semiotic operations that depends on the structure of the semiotic systems that determines the production of semipotic representations and their transformation. However, a purely structural and functional approach doesn't recognize that the choice of distinctive features cannot only be carried out at a semiotic level. It requires to take into account also cultural and social aspects that characterise mathematical practices in the classroom. It requires an interplay between the strictly semiotic functioning displayed by the structure of the semiotic system and the sociocultural interaction in the classroom

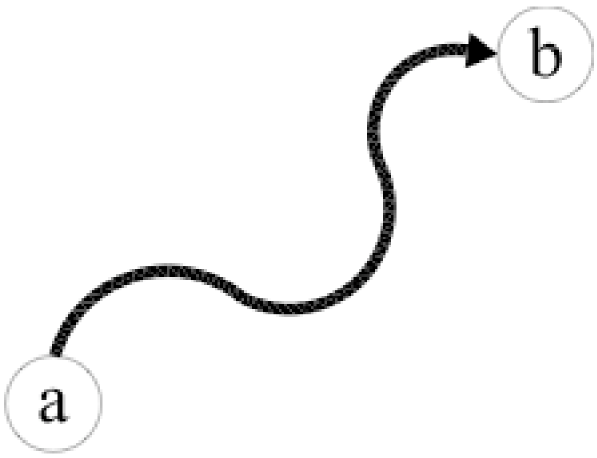
Treatment is a semiotic transformation from a representation $R^m{}_i(A)$ into another representation $R^m{}_k(A)$ in the same register r^m. From a mathematical point of view, treatments are considered the most important kind of semiotic transformation, because it is only through treatment that mathematical reasoning develops to carry out algorithms explanations and proofs. So, the most powerful semiotic systems are those highly structured for treatments and mathematical algorithms, such as symbolic systems. needed are those that have the greatest potential for treatment, and above all, those whose treatment procedures can be made into algorithms, such as within symbolic systems. A typical example of treatment is algebraic calculus, but also geometric transformations and constructions, arithmetical operations.

Conversion is a semiotic transformation from a representation $R^m{}_i(A)$ belonging to a register r^m into another representation $R^n{}_k(A)$ belonging to another register r^n. We believe that the term transformation is not the most appropriate for conversion since it is more an abrupt leap from one semiotic register to another with no semiotic connection between the two. A typical example of conversion is Cartesian geometry that coordinates the algebraic and the geometric register, but also problem solving usually requires conversions at least between natural language, a figural register a symbolic semiotic system.

Duval considers conversion the most important cognitive operation that on the one hand is the main cause of students' learning difficulties and on the other is a cognitive threshold to reach conceptual apprehension of mathematical objects. Duval (1993, 1995, 2006, 2008) highlights the following features that confer conversion a priority role:

- Conversion is not a form of encoding between two semiotic registers. Unlike treatment transformations there are no rules available to perform the transformation nor the reference to the object as a mediator between the representations.
- Non congruence between the two registers (Duval, 1995, pp. 48-49) that don't have corresponding significant units, one to one correspondence between significant units and corresponding ordering between significant units in the two registers.
- Conversion is not symmetric, i.e. direct and inverse transformations can be cognitively very different and raise obstacles and difficulties that have nothing in common.

The solution of the following problem is a significative example of the peculiarity of conversion as a cognitive operation, especially in terms of non-congruence and asymmetry between the semiotic registers involved. Consider the tremendous semiotic leap in passing from the figural to the tabular representations and the differences when going in the opposite "direction".
Consider the following relation of a set of only two elements, a and b. Is it a transitive relation?

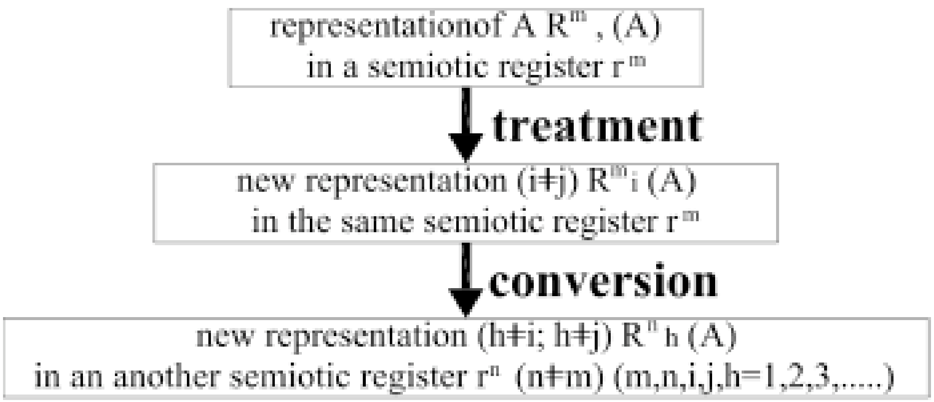


We must verify that $\forall x \forall y \forall z (xRy.\text{and}.yRz \Rightarrow xRz)$

A conversion to the following truth tables, taking into account the truth table of material implication, solves the problem effectively. For all the truth values of a and b it is easy to verify the following table.

a	a	a	V
a	a	b	V
a	b	a	V
a	b	b	V
b	a	a	V
b	a	b	V
b	b	a	V
b	b	b	V

The following schema synthesizes the interplay of cognitive operations that according to Duval's approach characterise mathematical thinking and learning.



In this paragraph we have outlined mathematical cognitive processes in terms of three semiotic operations. Of these operations, Duval privileges conversion that has a totally different nature from conversion and is considered a cognitive threshold towards adequate mathematical thinking and learning. We stress the fact that what has been described in this paragraph is a specific cognitive functioning that is transversal to all of mathematics but it must not be confused with its contents and its logical-deductive structure (Duval, 1995b). Fandiño Pinilla (2008) considers this specific cognitive functioning one of the five basic learnings that characterises mathematics and she calls it semiotic learning. Around this specific semiotic cognitive functioning pivots mathematical activity in all its complexity: objects, noetics, thinking, learning and meaning. Duval claims it is not spontaneous and it has to be enhanced through a specific didactical action on the part of teachers.

Duval's analysis sharply focuses on this cognitive functioning that he considers determined a priori by the structural and functional features of semiotic registers and it doesn't take into account other aspects that are important to mathematics teaching learning processes; nevertheless, this framework is an extremely precise and powerful theoretical lens to understand mathematical cognition. When facing the issue of meaning we must take into account the specificity of Duval's analysis.

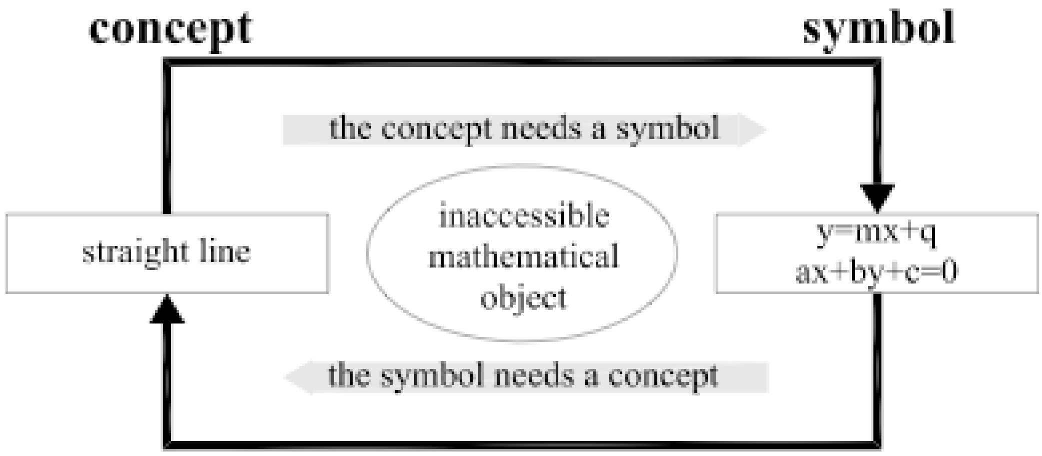
2.5 The cognitive paradox

The cognitive functioning we have described in the previous paragraph clashes with the problem of inaccessibility of mathematical objects. Students unavoidably identify the semiotic representations with the mathematical objects. This hinders the coordination of semiotic registers and gives rise to the phenomenon of compartmentalization (Duval, 1995, p. 52). Duval describes the cognitive paradox as follows:

> «(...) on the one hand the learning of mathematical objects cannot be but a conceptual learning, on the other an activity on mathematical objects is possible only through semiotic representations. This paradox can be for learning a true vicious circle. How could learners not confuse mathematical objects with their representations if they cannot have

relationships but with semiotic representations? The impossibility of a direct access to mathematical objects, which can only take place through a semiotic representation, leads to an unavoidable confusion. And, on the contrary, how can learners master mathematical treatments, necessarily bound to semiotic representations, if they do not already possess a conceptual learning of the represented objects?» (Duval, 1993, p.38).

The following schema synthesizes the functioning of this paradox.



There are many examples of the effects of this paradox on students' behaviour; the following amusing episode occurred with a high school student at his last year of a scientific high school of Bologna.

Teacher: «Imagine you had to explain a classmate what a straight line is.»

Student: «I would say three dots, a segment, three dots. »

In this example we can see how the student identifies the mathematical object with one of its possible representations.

> «the student is unaware that he is learning signs that stand for concepts and that he should instead learn concepts; if the teacher has never thought over this issue, he will believe that the student is learning concepts, while in fact he is only "learning" to use signs» (D'Amore, 2003, p. 43).

In general, the effect of the paradox is to fix a particular representation, that is confused with the mathematical object and the semiotic activity is confined to the register the representation belongs to. This leads to compartmentalization of semiotic registers. For example, equations are always represented in the algebraic register, and geometrical figures always in a figural register.

The importance given by Duval to conversion is connected also to the cognitive paradox. According to Duval, the true differentiation between the mathematical object and its semiotic representation is achieved through conversion. Conversion is a cognitive threshold to overcome such confusion and recognise in the sign both the reference to the object and its meaning within the structure of the semiotic systems.

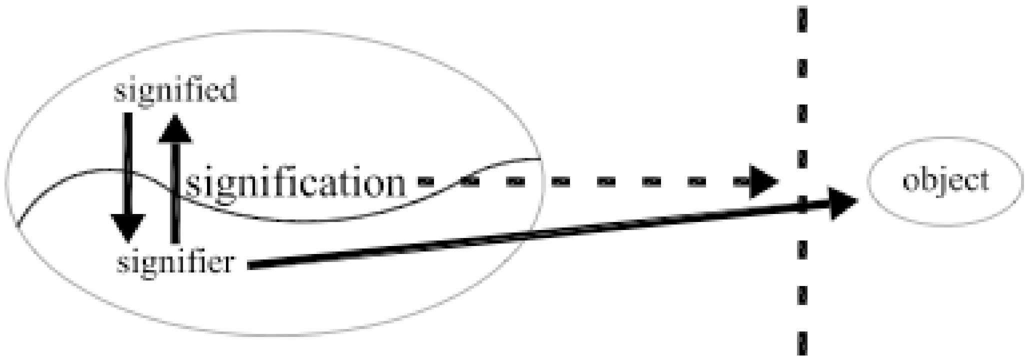
> «(…) coordination of registers is the condition to master understanding since it is the condition for a real differentiation between mathematical objects and their representation. It is a threshold that changes the attitude towards an activity or a domain when it is overcome. (…) Now, in this coordination there is nothing spontaneous» (Duval, 1995b, p.259).

There is an issue we need to raise at this point. Conversion is certainly a cognitive threshold to overcome the cognitive paradox and plays a special role in mathematics both as regards cognition and learning difficulties. We would expect treatment to overcome the difficulties tied to inaccessibility through the structure of the semiotic system that allows to pass from one representation to the other; the reference to the mathematical object should be in terms of invariance of treatment transformations within a same register. The change of meaning due to treatment transformations seems to defy this interpretation. Lets go back to one of the episodes we described in the introduction: $(n-1)+n+(n+1)$ transformed by treatment into $3n$. The subjects perform the treatment correctly but they don't recognize the same object. Each semiotic representation is identified with a different mathematical object, the sum of three consecutive numbers and the triple of a number, although subjects write a correct equality: $(n-1)+n+(n+1)=3n$. Treatment raises an unexpected situation as regards inaccessibility and the cognitive paradox. We would have expected this behaviour tied only to the representational leap required by conversion when there is no direct reference to the object and no connecting rule between representations.

The cognitive paradox rests on the assumption that there is a fixed and inaccessible mathematical object and that meaning can be traced back to the structure of the semiotic system. The unexpected outcome of treatment transformations, we mentioned above, shows that this point of view is insufficient to grasp the complexity of the problem. A more thorough analysis, that we will carry out in the following paragraphs, requires to take into account also the practices behind the semiotic representations and a different ontology.

2.6 Meaning and changes of meaning

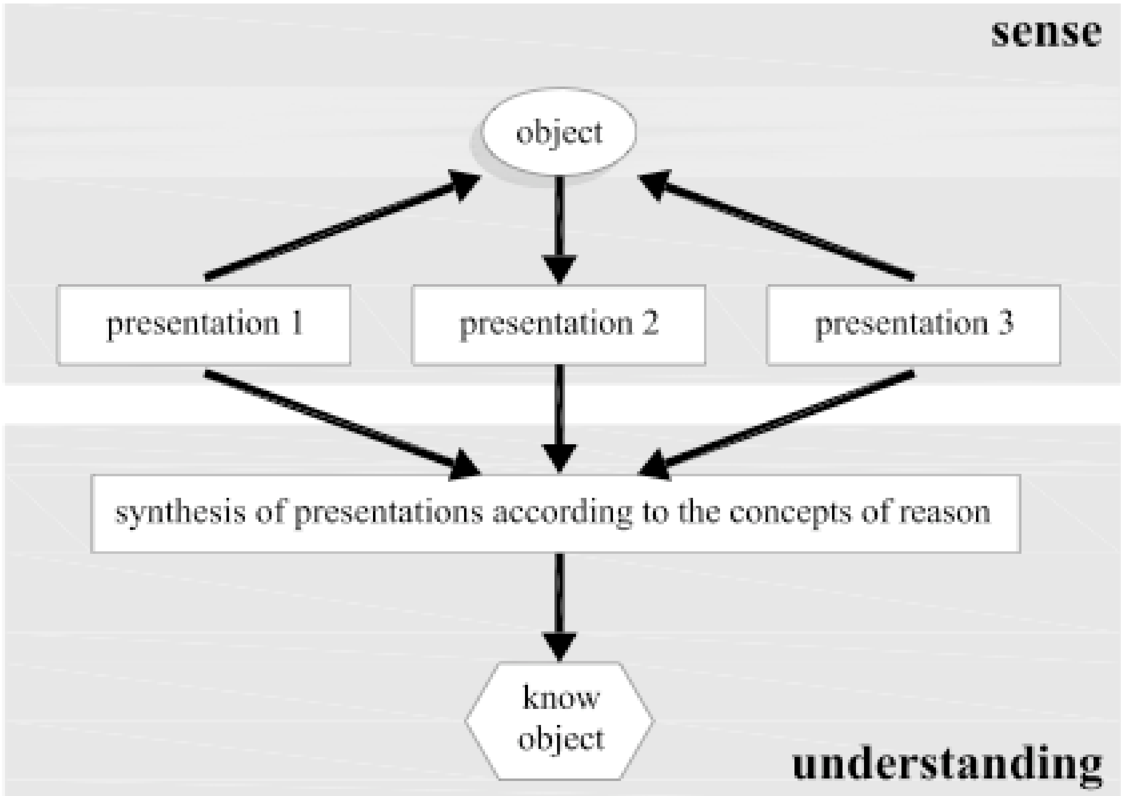
In paragraph 2.2.2 we introduced the couple (sign-object) and we have shown how the sign condenses both Sinn and Bedeutung. Sinn is the sense that a semiotic representation assumes within the structure of the semiotic system by opposition and comparison of its elements and Bedeutung is the reference to the inaccessible mathematical object as an invariant of relations between elements that form representations or coordination of representations themselves. We recall the schema of the couple (sign,object).



For example, if we consider the expression $f(x)=2x^2+3x+c$ its Sinn derives from the opposition and comparison between the symbolic elements that makeup the algebraic register (numbers, letters, operational symbols, etc.) and its reference is the mathematical object we call parabola.

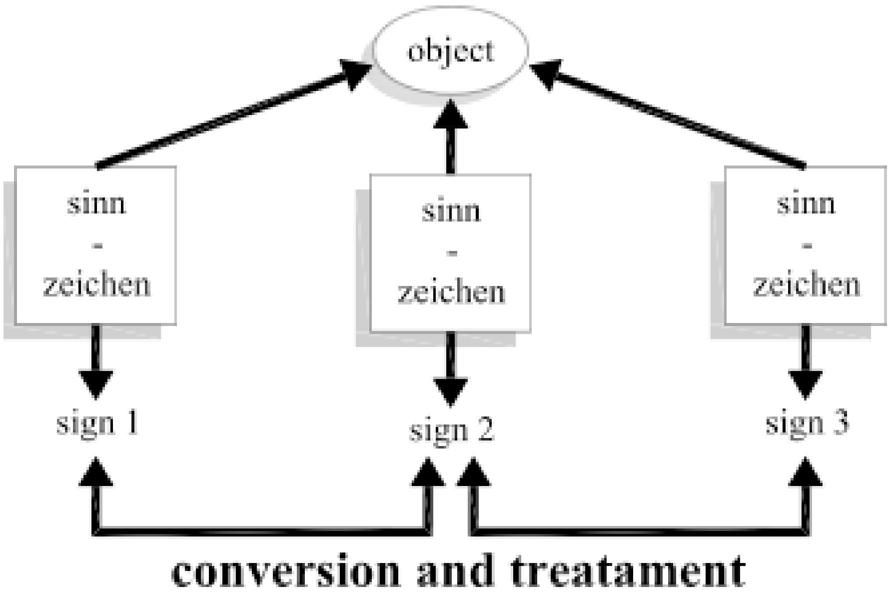
According to Duval, meaning has therefore a double nature in terms of sense and denotation. A more comprehensive understanding is obtained by involving in this Sinn and Bedeutung interplay other semiotic representations and other semiotic registers; in our example it could be a graph of the parabola in the Cartesian register or the function expressed in an implicit form. In chapter 1 we introduced the transformational function that, in Frege's triangle, brings to a change of Sinn through a change of expression but without changing bedeutung. The problem of changes of meaning, referring to Frege's triangle, can be seen as a change of Sinn with a common Beduetung. We need a more complete description of the phenomenon to take into account both the dyadic structure (sign,object) and the coordination of semiotic representations and semiotic registers.

Radford (2004, p.15) to describe Kant's endeavour to harmonize the sensible and intelligible proposes the following schema:



> «These intuitions are the brute material of knowledge. They still need to be regrouped by the mind thanks to the concepts of the pure intellect, which Kant called the a priori forms of knowledge. Without these forms, our perceptions and intuitions would remain dispersed. These concepts of the pure intellect are not concepts of objects; they are logical skeletons without content; their function is to make possible a regrouping or synthesis of intuitions. The synthesis is the responsibility of what Kant identified as the cognitive faculty of Understanding» (Radford, 2004, p.15)

In Duval's perspective there is no sensible relation to the mathematical object, but the strength of this schema is to frame effectively the fact there is a coordination of several representations. To adapt Radford's schema to Duval's viewpoint, we must translate presentation into representation and synthesis of presentation into coordination of representations through treatment and conversion.



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In this schema the arrows point to the mathematical object without "touching" it to convey the fact that it is an inaccessible entity. There is no passage from an- object to a known object, because the cognitive activity from the beginning consists in constructing a network of semiotic representations; the only possible access to the mathematical object is through semiotics, there isn't *noetics without semiotics*. The Kantian idea of synthesis of presentations is an associative synthesis of the diversity of intuitions and it doesn't require the distinction between Sinn and Bedeutung. Duval, instead, considers synthesis as invariance deriving from the coordination of different semiotic representations and it therefore requires the creativity provided by the semiotic registers and a clear distinction between Sinn and Bedeutung.

Duval's articulate structure of meaning allows to embed changes of meaning into a broader frame. In terms of Sinn a change of meaning is an unavoidable event. Both treatment and conversion entail a change of representation therefore a change of the way in which linguistic terms are connected to form a significative unit. In this perspective the research issue is the relation between Sinn and Bedeutung, that is how can student deal with different senses given by semiotic representations without loosing reference to a common inaccessible mathematical object. As we pointed out in the previous paragraph, the relation to the object as invariance of semiotic transformations, is lost even with treatment, when the rules of the semiotic register should maintain the common Bedeutung.

We need to look deeper into meaning and the mathematical object, in particular the model one object-many representations is insufficient to conceive meaning in a comprehensive way. In the next paragraphs we will show, within a pragmatic perspective, different routes that allow to broaden the way to conceive mathematical objects and their meaning.

2.7 Synthesis of the chapter

Duval's structural and functional analysis bears a basically realist approach to meaning and its aim is to single out the specific cognitive functioning that characterizes mathematical thinking with respect to other fields of knowledge. The analysis focuses on the structure of systems of signs and the discursive and meta-discursive functions they accomplish. Mathematical cognition and learning is identified with a coordination of semiotic systems. Meaning is seen as the referential relation between one object and its possible representations, modelled in terms of sense and denotation but in a dyadic structure that is a development of Frege's semiotic triangle.

3

The cultural semiotic approach

3.1 Introduction

In this chapter, presenting Radford's (2000, 2002, 2003) cultural semiotic approach we will shift to a pragmatic viewpoint of mathematical objects and meaning. The cultural semiotic approach draws from Vygotsky's historical-cultural school, Husserl and Merlau- Ponty's phenomenology, Ilyenkov's philosophy and Leont'ev activity theory. Radford's approach ascribes to semiotics a chief role but it broadens the notion of sign whose nature is interwoven with activity, individual consciousness and social and cultural elements.

> «We take signs here not as mere accessories of the mind but as concrete components of 'mentation'. […]instead of seeing signs as the reflecting mirrors of internal cognitive processes, we consider them as tools or prostromanoheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage. As a result, there is a theoretical shift from what signs *represent* to what they *enable* us to do. […]the signs *with* which the individual acts and *in* which the individual thinks belong to cultural symbolic systems which transcend the individual *qua* individual. Signs hence have a double life. On the one hand, they function as tools allowing the individuals to engage in cognitive praxis. On the other hand, they are part of those systems transcending the individual and through which a social reality is objectified» (Radford, 2000, p. 240-241).

The central element that characterizes the cultural semiotic approach is the notion of objectification, a powerful theoretical tool to understand learning and meaning of mathematical objects. The cultural semiotic approach is also called theory of knowledge objectification; we will use both terminologies when referring to Radford's approach.

3.2 Mathematical thinking and mathematical objects

Before analyzing the objectification of knowledge it is necessary to characterize thinking and the nature of mathematical objects according to the cultural semiotic approach.

Mathematical thinking

In the previous paragraph we described Duval's approach that analyses the specific cognitive functioning that characterises mathematical thinking. Cognitive functioning was identified with a network of semiotic representations and semiotic registers. Many educational theories conceive thinking and learning as an isolated activity, that takes place inside the individual's mind, aiming at discovering an a priori reality. Social interaction is often taken into account but its role is mainly to trigger the cognitive activity in terms of adaptation to an external environment. The cultural semiotic approach takes into account anthropological, historical and cultural elements as constitutive of thinking.

> «The theory of knowledge objectification adopts a non-mentalist position on thinking and intellectual activity. This theory suggests that thinking is a type of a social practice (Wartofsky, 1979), praxis cogitans. To be more precise thinking is considered to be a *mediated reflection in accordance with the form or mode of activity of individuals*» (Radford, 2008, p.218).

Thinking is not something immaterial that happens in the mind of the individual but it is embedded in social activity, it is a *praxis cogitans*. Nevertheless, in this perspective mathematical thinking has a form of ideality that

> «is rather like a stamp impressed on the substance of nature by social human life activity, a form of the functioning of the physical thing in the process of this activity. So all the things involved in the social process

> acquire a new 'form of existence' that is not included in their physical nature and differs from it completely – [this is] their ideal form» (Ilyenkov, 1977, p. 86).

To think, therefore, is to take part in sense giving activities, subsumed in a social and cultural context, form which forms of rationality, problems and needs emerge.

Mediation means that, in a pragmatic view and continuing Vygotsky's (1986) path, signs are *constituents* of thinking because they carry out the social activity and bind the individual, historical and cultural dimensions. Such mediators include sign systems, objects, instruments, gestures etc.

> «thinking is not something occurring merely in the students' mental plane. Thinking also occurs along the social plane, in a region that, paraphrasing Vološinov (1973), I want to call the territory of artifactual thought. It is within this territory that subjectivity and cultural objectivity mutually overlap and where the mind extends itself beyond the skin (Wertsch, 1991)» (Radford, 2008, p. 219)

The reflexivity of thinking regards the role of subjective consciousness in thinking. Activity is carried out as an intentional act directed on the one hand *towards* a historical and cultural reality on the other *through* that same cultural and historical reality. Thinking is not an isolated activity in which the individual assimilates knowledge, but it is a reflection on the part of the subject, accomplished in a socially shared activity, of a cultural and historical reality; the term reflection refers to the manner in which the individual intends reality according to cultural and social criteria.

The form and mode of activity refer to cultural and historical factors that direct the individuals' intentional acts into what we call thinking and knowledge. Radford (2008) calls these cultural factors *Semiotic Systems of Cultural Signification.*

> «In their interaction with activities (their objects, actions, division of labour, etc.) and with the territory of artifactual thought, the *Semiotic Systems of Cultural Signification* give rise, on the one hand, to forms or *modes of activities*, and, on the other hand, to specific *modes of knowing or epistemes* (Foucault, 1966). While the first interaction gives rise to the particular ways in which activities are carried out at a certain historical

> moment, the second interaction gives rise to specific modes of knowing» (Radford, 2008, p. 219)

Mathematical objects

In the pragmatic viewpoint advocated by Cultural Semiotic approach mathematical objects are strictly related to the reflexive mediated activity.

> «The theory of knowledge objectification suggests that mathematical objects are historically generated during the course of the mathematical activity of individuals. More precisely mathematical objects *are fixed patterns of reflexive activity incrusted in the ever changing world of social practice mediated by artefacts*» (Radford, 2008, p. 222).

In Radford's pragmatic view, mathematical objects are strongly embedded in a pragmatic view in which both the individual and social activity play a prominent role, and lose any character of a-priori identities. This is a key point when discussing the relation between meaning and semiotic representations of mathematical objects. We cannot confine the issue of meaning to the relation between signs in a semiotic system and the coordination of different semiotic representations, referring to a common somehow a priori object, through treatment and conversion. Each representation is imbued with personal and social practices that oblige to broaden meaning beyond the symbolic structure. In the cultural semiotic trajectory we are following, we recognize a duality between the structure of signs and social activity that doesn't allow to endow a priority role to practice in respect to semiotics and vice versa. Is it the semiotic structure that determines activity or the needs that emerge from a field of problems that inform the structure of the semiotic systems that allow to carry out the activity linked to such problems?

In chapter 1 we highlighted that realistic and pragmatic theories are not intrinsically conflicting theories, but the realistic idea of referring to an existing object can be recovered as the final outcome of the practices from which mathematical objects emerge. In the cultural semiotic approach we cannot ascribe to mathematical objects an ideal and a priori existence as they are strictly bound to the reflexive activity they emerge from. Nevertheless, within the Semiotic Systems of Cultural Signification we can ascribe a form of existence to the fixed patterns that emerge from the praxes cogitans:

> «Plato was absolutely right in affirming, in the Parmenides, that ideas are not in the mind. And that he was equally right in conceiving of ideas as "fixed patterns". But instead of seeing these patterns "as fixed in nature" we should say "*as fixed in social practice*". In doing so, the wall that divided the seen and the unseen worlds that Plato mentioned in the Phaedo falls into pieces and mathematical objects lose their eternal aura and their atemporality. They become part of the always changing world of the individuals. […]In the anthropological epistemology that I am considering, mathematical objects retain an aspect of their platonic ideality. But what this ideality is about is an ideality resulting from a reflection that the individuals carry out of their world in the forms of their actions and activities» (Radford, 2004, p. 20).

The cultural and historical ideal existence we ascribe to mathematical object through activity in Ilyenkov's sense allows a form of reference to the mathematical object that we cannot identify with a strict designation obtained through the relation object-sign or with a form of construction and reconstruction of knowledge. It has to do with the depth and subtlety of reflexive activity and the meaning making processes we are scrutinizing in this work. Radford calls the "reference" to the cultural object *objectification* that we describe in the next paragraph.

3.3 Learning: objectification and semiotic means of objectification

Learning is considered a mediated reflexive activity but addressed to the mathematical objects that bare a cultural and historical dimension. The cognitive and epistemological situation is very different when we consider learning in respect to the historical and cultural construction of the mathematical objects. In the historical development of mathematics, mathematicians' reflexive activity aims at creating new object, while learners' reflexive activity addresses an object the already exists, not in a realistic sense, but as a culturally and socially recognized entity.

> «Students' acquisition of a mathematical concept is a process of becoming aware of something that is already there, in the culture, but that the students still find difficult to notice. The awareness of the object is not a passive process. The students have to actively engage in mathematical

activities not to “construct” the object (for the object is already there, in the culture) but to make sense of it. This process of meaning-making is an active process based on understandings and interpretations where individual biographies and conceptual cultural categories encounter each other – a process that, resorting to the etymology of the word, I call *objectification*. To learn, then, is to objectify something» (Radford, 2005a, p. 111).

Learning is an intentional act in which the subject encounters and puts in “front” of his consciousness the mathematical object through a mediated activity that gives sense to the learned object.

In this perspective signs are not e reduced to their representational function but they culturally mediate the reflexive activity that brings to the objectification of the mathematical objects. The way learners intend the mathematical object through their intentional acts is not a neutral subject-object relationship, but it is intrinsically “tainted” by culture, history and social structures through the semiotic mediators that direct our intention:

> «Sense-giving acts and all that makes them possible are essentially cultural. […]What appears in front of us in our intentional experience is consequently ubiquitously framed by the cultural history of the means that we use to apprehend it. Sense-giving acts and all that makes them possible are essentially cultural. […] In giving meaning to something, we have recourse to language, to gestures, signs or concrete objects through which we make our intentions apparent […]. Language, signs, and objects are bearers of an embodied intelligence (Pea, 1993) and carry in themselves, in a compressed way, cultural-historical experiences of cognitive activity. […] I termed the whole arsenal of signs and objects that we use to make our intentions apparent *semiotic means of objectification*. Regardless of whether or not what we intend is personal or impersonal, what we convey in the experience of meaning can only be achieved in and through them» (Radford, 2006, p. 52).

Semiotics has an instrumental role but this could not fully convey their strength in accomplishing reflexive activity both in thinking and learning. Semiotics means of objectification are not technical means but they culturally and socially direct the individual's intentional sense-giving acts , differently there would be no thinking, no learning and no meaning; is an interiorization of the historical social practices that are condensed in the semiotic means of objectification.

We must also take into account that learning is also a social activity, by definition an activity is always social. Social interaction is not a facilitator nor it provides an adaptive environment but it is consubstantial to learning. It is through social interaction that Semiotic Systems of Cultural Signification orient and select the reflexive activity and recognize the cultural dimension embedded in the semiotic means of objectification.

> «Objects cannot make clear the historical intelligence that is imbedded in them. This requires that they be used in activities as well as in contact with other people who know how to "read" this intelligence and help us to acquire it. Symbolic-algebraic language would otherwise be reduced to a group of hieroglyphics. The intelligence that symbolic-algebraic language carries would not be noticed without the social activity that takes place in the school» (Radford, 2008, p. 224).

To carry out the reflexive mathematical activity, mediation cannot be accomplished only through the semiotic registers that characterises mathematics in its axiomatic and deductive form . If we observe students, but not only students, immersed in mathematical activity they resort to forms of mediation that cannot be traced back only to Duval's semiotic registers, typical of mathematics. They also use gestures, artefacts, objects, kinaesthetic activity, bodily movements that along with semiotic registers form semiotic means of objectification. We need to broaden our notion of sign and semiotics to include elements that we wouldn't consider strictly mathematical nor include in Duval's structural and functional framework.

The enlargement of the notion of sign that semiotic means of objectification entail is not a need to draw on more powerful representative systems. What really distinguishes semiotic means of objectification from the signs we usually recognize as belonging to mathematics is the teleology behind their use. In the pragmatic route we are following, the role of signs is not to represent "something"; in mathematics there is nothing fixed and stable that we can somehow grasp with a sign. What we undoubtedly recognize is a reflexive activity. In their broadened understanding, signs are deeply interwoven with the reflexive activity they accomplish, we look at sign not to draw on something but to do something.

Semiotic means of objectification are consubstantial to the mathematical activity; even when using objects and artefacts the cultural and social mathematical practice endows them with a semiotic nature. For example, consider primary school students working with bingo chips or toothpicks to learn natural numbers. In terms of the reflexive activity they are accomplishing, those objects are no longer concrete objects but they are signs that bare an embodied cultural experience and at an institutional level we recognize them as mathematics. Radford (2003) recognizes in novice students in algebra, that objectify sequences expressed with figures, the use of rhythmic use of bodily movements as a semiotic mean of objectification to generalize the terms of the sequence. In the social and cultural context of the mathematics classroom such bodily movements embody a cultural and historical mathematical intelligence.

We have referred to the notion of activity and practice several times, without specifying what they really are. In the cultural semiotic perspective, activity is a synthesis of sensual and intellectual aspects that characterize thinking and learning. In its personal and social dimension activity expresses the richness and complexity of human reflexive experience. To understand activity we must focus on consciousness's intentional acts when individuals make sense of their social and cultural reality; human experience is characterised by a space-time dimension, movement, perception, feelings, emotions procedures and at an intellectual level by abstraction, schemas, generalization processes, structures etc. The interplay of such sensual and intellectual experience immersed in a social and cultural account for a variety kinds of activities; we will show how the ontosemiotic approach has outlined six kinds of objects that emerge from different practices that individuals can accomplish.

Considering activity in terms of a strong interweaving of sensual and intellectual features on the on hands explains the need for an enlargement of semiotics, when it mediates mathematical practices, on the other provides insights to understand how semiotic means of objectification are used in thinking and learning.

It is senseless to look for transformational operations with semiotic means of objectification in analogy with the operations that characterize Duval's semiotic registers. Duval's semiotic analysis considers the passage from one semiotic representation to another; semiotic operations are carried out in a diachronic timeline. The theory of knowledge objectification is based on a synchronic analysis, because reflexive activity requires the use of more semiotic means of objectification at the same time. Intentional acts are pursued in a network of intellectual and sensual activities, we cannot separate such dimensions nor outline a passage from a sensual level to so called rational structures as for example in Piagetian theories. Semiotic means of objectification are not used on at a time but they are rather organized in *semiotic nodes*

> «pieces of the students' semiotic activity where action, gesture, and word work together to achieve knowledge objectification» (Radford et al., 2003).

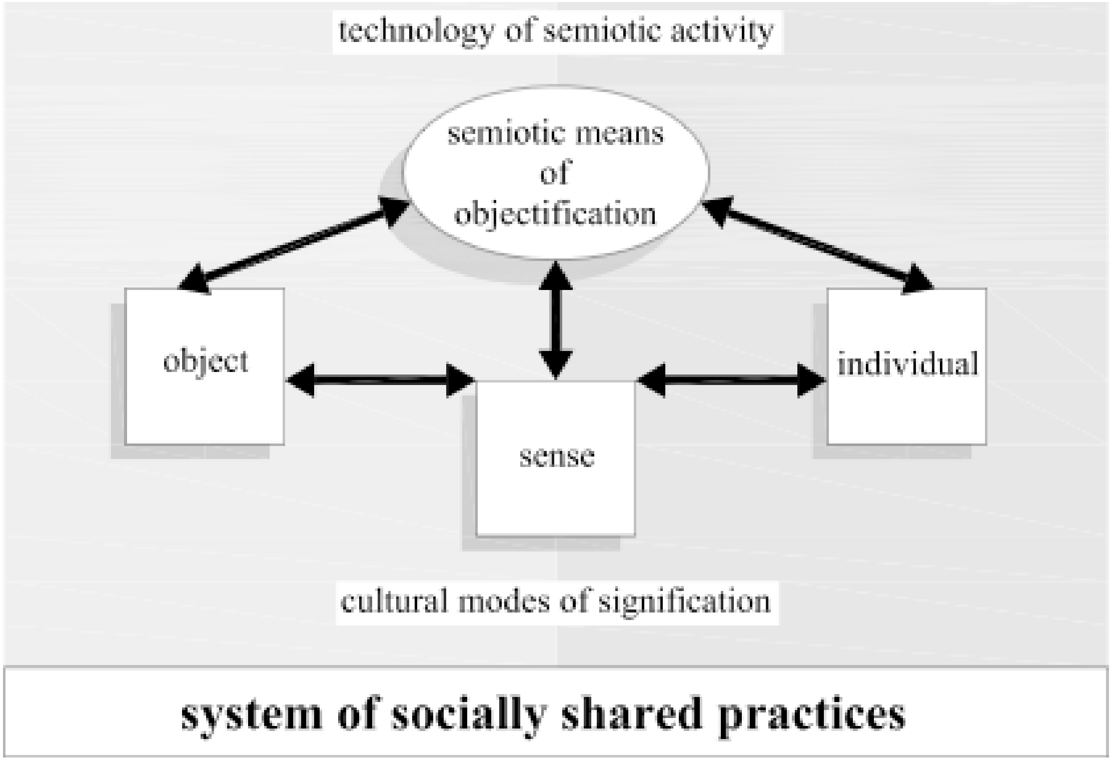
In general, in their intentional sense giving acts , students resort at the same time to gestures, bodily actions, and signs and we testify an interplay of sensual and intellectual features. Students' mathematical experience develops diachronically but the semiotic means of objectification are used synchronically through semiotic nodes. For example, if we observe students working in algebra, besides symbolic language they also resort to gestures, actions, artefacts, deictic use of natural language. Without an appropriate blend of semiotic means of objectification, within a socially shared reflexive activity, algebraic symbolism and mathematical symbols in general are meaningless.

The synchronic use of semiotic means of objectification is not in contradiction with the diachronic use of semiotic representations; we cannot discard conversion and treatment transformations of semiotic representations, thereby losing an important part of the picture. When analyzing students' behaviour we cannot skip the complicated network of semiotic systems they have to handle; mathematical activity and therefore also mathematical learning is interwoven with semiotics; without resorting to the transformation of signs within semiotic systems, mathematics wouldn't have developed into the refined form of rationality we know today. Duval's coordination of semiotic registers is the tip of the iceberg of the encompassing reflexive activity. If our analysis focuses on a symbolic expression within the semiotic node as the practice evolves, we can single out Duval's semiotic transformations. The evolution of semiotic nodes and the objectification process they accomplish, provide an effective tool to understand *how* semiotic transformations are carried out. Duval's semiotic transformations are not the final outcome of reflexive activity but they are an effective way to look at reflexive activity to scrutinize mathematics learning process.

We have described activity in terms of consciousness intentional acts that are pursued in sensual and intellectual forms. The theory of knowledge objectification calls the sensuous spatial and temporal aspect of activity as embodied experience and the intellectual one as disembodied meaning (Radford, 2000, 2002, 2005). The term embodiment is not used with a strictly neuroscientific acceptation as other scholars usually do in Mathematics Education (Lakoff and Nuñez, 2000). The embodied experience is an intrinsically cultural and social experience and the individual consciousness is acquires its identity within reflexive social practice (Leont'ev, 1977) ; in fact, Radford (2008) claims that through objectification processes the individual "*finds her self"* through the counterpart of objectification called subjectification.

According to the cultural semiotic approach, learning entails a cognitive rupture to disembody mathematical meaning (Radford, 2003, 2005) when reflexive activity is mediated by symbolic semiotic means of objectification. Our experimental results confirm students difficulty as the mathematical practice resorts to symbolic language. We believe that it is misleading to consider learning characterised by a threshold between an embodied and disembodied dimension. On the one hand mathematics is disembodied by definition, in fact its cultural objects have no concrete nature and are accessible only through a mediated practice; on the other its language game, "its form of life" as Wittgenstein (1953) would say, require an interplay of embodied and disembodied activity, of sensual and intellectual dimensions. The problem is that when symbolic language enters in the semiotic node, there is no interplay between the disembodied and embodied dimension. Students usually break the unity of the semiotic node and they hold on to representations that allow an embodied experience, typically iconic or figural representations, or they use symbolic language as an empty manipulation of sign unrelated to the cultural and social practice they mediate.

The following schema (Radford, 2005) synthesizes the complexity of learning as an objectification process of a cultural object that is accessed through semiotic means of objectification within social and cultural modes of signification.



3.4 Levels of generality

In the cultural semiotic approach mathematical objects are conceived as fixed pattern of reflexive mediated activity. We have seen that it is possible to ascribe a form of ideal existence, in Ilyenkov sense, to mathematical objects in the cultural and historical dimension from which they originated. Their cultural ideal existence allows to consider learning an objectification process in which pupils make sense of the cultural object.

The mathematical object is not a fixed, compact and homogeneous entity but it changes as the reflexive activity evolves both at an ontogenetic and phylogenetic level. For example, we can think of the historical development of the concept of limit, from Euclid’s and Archimedes’ exhaustion methods to Weirstrass’ ε-δ formal definition. Bagni (2004) proposes an historical and socio-cultural analysis which shows that, at a cultural level, we cannot recognize a unitary object but the concept of limit evolved as the reflexive activity resorted to more powerful semiotic means of objectification. Bagni remarks that from a realistic viewpoint, we would consider Wallis’ and Euler’s definitions incorrect and lacking mathematical rigour the we ascribe to Weirstrass’ definition that grasps the “true” concept of limit. From a pragmatic standpoint this position is untenable. Referring :

> «“His wording is loose”: what do we mean by that? If we investigate Wallis’ correctness against our contemporary standards we must conclude that his expression is not rigorous. But such investigation would be historically weak: obviously Wallis’ wording would not be correct, *nowadays*; but Wallis was rigorous, in his own way» (Bagni, 2004, p. 100).

This historical argument has an educational correspondent. In their cognitive history, students are exposed to several practices that decompose the mathematical object according to several reflexive activities. Teachers usually direct their educational action towards a supposedly ideal and unchanging correct and rigorous mathematical reasoning, without taking into account the net of activities that characterize both the cultural phylogenetic and the individual ontogenetic “elusiveness” of the mathematical object.

In their objectification processes, students have to coordinate a network of reflexive activities in which they have to recognize a unitary but stratified mathematical object. Their reflexive activities are characterized by a blend of cultural, historical and social elements. In our experimentation, students working with the tangent had to synthesize several strata of the concept that can be traced back to different activities they encountered in their educational path: the classroom activity during our experiment, the Euclidean reflexive activity as a single point of contact between the straight line and the circumference; the activity in analytic geometry in which they solved second degree systems; the activity that resorts to infinitesimal techniques through the derivative. The students’ difficulties express a lack of coordination between these activities. If we analyse their behaviour it is coherent with the way they coordinate their activities. The educational challenge is to lead them to an objectification of the mathematical object that doesn’t compartmentalise the different activities and layers that make the mathematical object, thereby missing the cultural objectivity of mathematical concepts.

The aforementioned coordination of activities has a direction towards the higher layers of generality (Radford, 2003, 2005, 2008) of the cultural object.

> «This movement has three essential characteristics. First, the conceptual object is not a monolithic or homogenous object. It is an object made up of layers of generality. Second, from the epistemological point of view, these layers will be more or less general depending on the characteristics of the cultural meanings of the fixed pattern of activity in question (for example, the kinaesthetic movement that forms a circle; the symbolic formula that expresses it as a group of points at an equal distance from its centre, etc.). Third, from the cognitive point of view, the layers of generality are noticed in a progressive way by the student» (Radford, 2008, p. 226).

The layers of generality are strictly connected to semiotic means that mediate the reflexive practice. When the semiotic node is centred on gestures, bodily movements, kinaesthetic activity, deictic use of natural language we have a strong embodied experience and a low level of generality. When the semiotic node contains symbolic language, abstract structures, the object is objectified with a higher level of generality. In section 2.3.2 we mentioned the issue of disembodiment of experience. The need of a sensual interpersonal space-time experience hinders the access to higher layers of generality.

As an example of the stratification of the object in layers of generality we propose the concept of distance. Individuals, in their first interaction with space, recognize as a fixed the pattern the Euclidean notion of distance, expressed by the length of the segment between two points. At a higher level of generality, in vector spaces the distance is expressed by the norm of the vector connecting the two points, and we can further generalise the notion of distance introducing metric spaces. Dropping any relation to the embodied notion of Euclidean space, in functional spaces we can objectify the notion of distance between functions.

We believe students have to face two kinds of generalizations. The first generalization is bound to the semiotic means of objectifications students use to mediate reflexive activity. The other regards the representation of a class of objects with a single representation. In the first case, we can objectify the natural number five with an iconic representation, the symbol 5, x-5=0, etc . The second form of generalization is typical of algebra. For example, to express the set of odd numbers, we can use the sequence $a_n=2n+1$. The ontosemiotic (Contreras, Font, Luque, Ordoñez, 2005) approach describes this generalization in terms of the duality intensive-extensive within a linguistic game.

There is an interaction between the two. The set of odd numbers can be objectified using an algebraic expression, rhythm, deictic and generative use of natural language as it is clearly shown in Radford (2003).

Radford (2003, 2005) recognizes three forms of generalization:

- A factual generalization bound to operational schemas within the students' space-time embodied experience, using rhythm, bodily movements, gestures, the generative and deictic use of natural language and working on specific objects
- A contextual generalization bound to an invariant operational schema that keeps memory of space-time contextual experience without referring to a particular representation of the object that is objectified as conceptual object using linguistic deictic and generative terms. They are objectified through
- A symbolic generalization that requires to drop the relation with contextual space-time elements using symbolic semiotic means of objectification.

3.5 Meaning and changes of meaning

In the previous paragraphs, we have seen how thinking, mathematical objects, signs and meaning are indissolubly entangled through the reflexive activity mediated by semiotic means of objectification. In the cultural semiotic approach meaning is conceived as an intentional act on the part of the individual, a way of intending what he encounters in his personal experience.

The idea that meaning is an intentional act draws back to Husserl's phenomenology that conceives knowledge as relation between the individual consciousness and the object of its intentional act. From the realistic viewpoint he adopted, he had to overcome the opposition between the relativity of the subjective experience and the objectivity of absolute and eternal ideal mathematical objects. Husserl (1913-1959) overcomes this dichotomy introducing an interplay of *noesis* and *noema*: noesis refers to the way we attend objects through intentional acts and noema is the conceptual content, one of the meanings that the individual addresses through his personal experience. The meaning of the object, according to Husserl, is stratified in partial meanings each of one corresponding to a specific noetic intentional act. The global meaning is the synthesis of the local noemas attained through the corresponding noeses; the transcendental and ideal object coordinate the intentional acts into a single unitary meaning.

As it usual happens with realistic approaches, this viewpoint is very effective in describing the construction of meaning but it rests on assumptions that contradict the actual state of affairs. In fact, there is no isolated individual that directs his intentional act to a fixed object. Both individuals' intention and the mathematical object are interwoven with social and cultural activities.

The way learners intend the mathematical object through their intentional acts is not a neutral subject-object relationship, but it is intrinsically "tainted" by culture, history and social structures through the semiotic means of objectification and the semiotic systems of cultural signification that direct our intention. Meaning is a reflexive act on the part of the individual, but his intentions are culturally and socially directed by shared activities.

When we focus our attention to learning, the objectification process obliges to broaden our notion of meaning of a mathematical object. Indeed, in the objectification process, meaning entails a relationship between a cultural dimension and a personal dimension, between a cultural meaning and a personal meaning. On the one hand the student is the protagonist of learning through his sense-giving intentional acts on the other hand such intentional acts, through social activity, are directed to an interpersonal and general cultural object.

> «I want to suggest that it is advantageous to think of meaning as a double-sided construct, as two sides of the same coin. On one side, meaning is a subjective construct: it is the subjective content as intended by the individual's intentions. On the other side and at the same time, meaning is also a cultural construct in that, prior to the subjective experience, the intended object of the individual's intention (l'object visé) has been endowed with cultural values and theoretical content that are reflected and refracted in the semiotic means to attend to it» Radford (2006, p. 53).

The sense giving activity students are involved in can be seen as a convergence of the cultural meaning with the personal meaning. At an ontogenetic level the personal activity mediated by the semiotic means of objectification traces out the phylogenetic activity culturally condensed in the mathematical object.

«I believe that the mathematical learning of an object O on the part of an individual I within a society S is nothing else but the adhesion of I to the practices that other members of S develop around the object O. How do we express such adhesion? Accepting the practices that are mainly linguistic» (D'Amore, in: Bagni, D'Amore, Radford, 2006, p. 22)

The objectification process entails mainly two difficulties on the part of the student:

1. The mathematical object is an entity stratified in layers of generality. Each layer of generality is associated with a particular reflexive activity determined by the characteristics of the semiotic means of objectification that mediate it. The diversity of the student's reflexive activities splits his intentional acts towards objects that he considers disconnected but at an interpersonal level are recognized as belonging to the same cultural entity. The objectification process therefore doesn't require a coordination of semiotic representations as such but of the different activities mediated by those representations.
2. Meaning has a strongly embodied nature related to the personal space-time sensual and emotional experience of the student, but at higher levels of generality the student has to include within the semiotic nodes formal and abstract symbols that brake the relationship with his spatial and temporal experience. Students have to experience a disembodiment of meaning that hinders the objectification of the interpersonal and general aspects of the mathematical object.

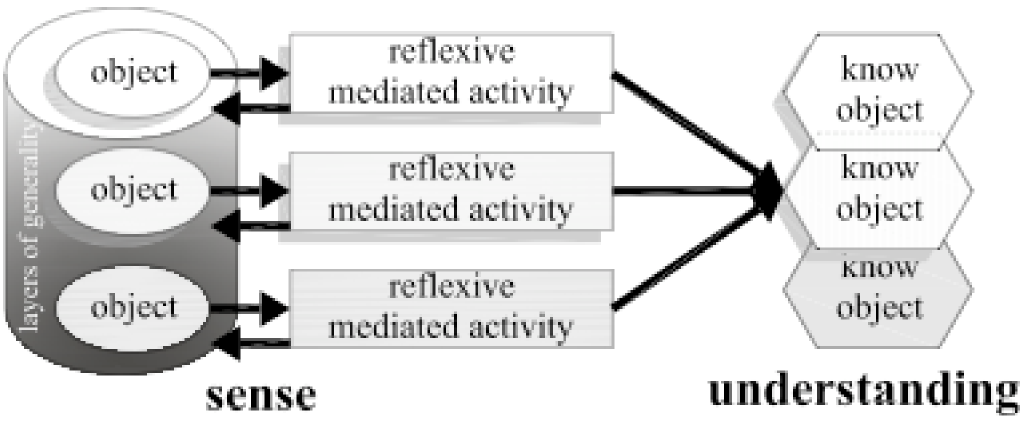
In the cultural semiotic approach meaning is not conceived as the reference to an ideal object of a linguistic term, nor as a pure intentional act of a subject directed towards such ideal object. Meaning is the net of mathematical activities condensed in the semiotic means of objectification, but in the cultural semiotic approach meaning is also meaning for someone immersed in reflexive activity. The educational challenge is to transform the cultural meaning condensed in semiotic means of objectification into a meaning for someone; the noetic-noematic layers of the individual intentional acts achieve a unity of meaning as they reflect the cultural object.

«Meaning also has a cultural-historical dimension which pulls the interaction up in a certain direction – more precisely, in the direction of the cultural conceptual object. […] Through the design of the lesson and the teacher's continuous interpretation of the students' learning, classroom interaction and the students' subjective meaning are pushed towards specific directions of conceptual development. Cultural conceptual objects are like lighthouses that orient navigators' sailing boats. They impress classroom interaction with a specific teleology» (Radford, 2006, p. 58).

The problem is that the mathematical "lighthouses" are stratified and changing entities as the reflexive activities develops, that often conflict with the students' individual embodied personal experience. The difficult path that the student has to follow in his learning experience towards a flexible and adaptive coordination of local meanings incorporate in each layer that forms the mathematical object, is an effective frame of the problem of changing of meaning. We cannot stick to structural and functional analysis of the semiotic transformation but we need to broaden our analysis to the reflexive mediated practices.

As we have already mentioned above, when considering learning processes in the cultural semiotic perspective, thinking, mathematical objects, signs and meaning are indissolubly entangled through the reflexive activity. Analyzing students' sense-giving acts therefore requires to shift our focus from the duality object-representation to the network of reflexive activities that entangles objects, signs and meaning.

We propose the following schema- that adapts the aforementioned Radford's schema representing Kant's interpretation of learning- to frame meaning and learning of mathematical objects in the Semiotic Cultural approach



The double arrows express the alignment between the cultural meaning of the object an individual meaning acquired through the mediated reflexive activity. Understanding is no more a synthesis of sensual intuitions attained through a priori forms of knowledge. Both the sensual intuitions and the synthesizing process, in the cultural semiotic approach, are reflexive mediated activity.

We propose a protocol taken from (Radford, 2003) that can be interpreted as a change of meaning due to treatment transformation. Novice students in algebra had to find the general term of the following sequence:

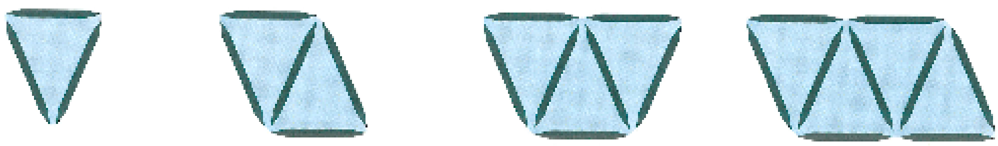


Fig.1 Fig. 2 Fig. 3 Fig. 4

We refer the reader to (Radford, 2003) for a thorough analysis of the students' mathematical activity. It comes out that students resort to a variety of semiotic means to objectify the general term of the sequence: gestures, rhythm, generative and deictic use of natural language etc. After factual and contextual generalizations, students try to reach a symbolic generalization proposing the following expression for the general term: $(n+1)+n$. Anyway students were reluctant in performing treatment transformations to obtain $(n+n)+1$ or $2n+1$ as gneral terms of the sequence. We testify a change of meaning due to treatment transformation, similar to the one encountered by D'Amore and Fañdino with university students: $(n-1)+n+(n+1)=3n$. In the latter example, the origin of the problem is concealed by a contractual use of treatment transformation. In the former example, where the students were novices in algebra, we can clearly trace back the problem to a missed connection of the reflexive praxes mediated by $(n+1)+n$ and $(n+n)+1$. The first expression is strictly connected to students' previous embodied activity mediated mainly by gestures whereas the second entails a cognitive rupture with their previous reflexive activity and, therefore, results meaningless. In fact, the use of symbols is carried out within a contextual generalization.

We remark that from a structural and functional point of view students are able to perform rather complicated conversions between natural language, the figural representation of the sequence, arithmetical representations, and the algebraic register. The geometric figures that represent the terms of the sequence are strucutred representations of natural numbers. The students, in contradiction with Duval's expectation, can easily perform conversions from such figural representations to the algebraic and arithemtical registers but they are stuck when they have to face the treatment from $(n+1)+n$ to $2n+1$. If we focus only on the structure of the semiotic registers, we cannot explain students' difficulties in performing this trivial algebraic transformation. We have to take into account the reflexive practices that are mediated by the semiotic representations and the connections that students are able to establish between them.

The geometric representations of natural numebrs, although highly structured, along with the use of gestures and bodily movements mediate an activity that easily incorporates their space-time experience; when students includein their practice $(n+1)+n$, this representation is still connected with their previous embodied activity with the geometrical representations; the parentheses recall the relation between two successive figures and the number of toothpiks in the n-th figure that lead them to recognize the number of the general term of the sequence. Students couldn't relate $2n+1$ to any significative mediated activity and has been disregarded; $2n+1$ could have been a possible representation of the general term if students had noticed that in the n-th figure there are always n triangles and that two contigous triangles have a common side, so they had to count two sides for each triangle plus 1. In an experiment caried out in a primary school, resorting to gestures and deictic us of natural language ten years old students recognized this general schema. Obviously, they couldn't express in the symbolic language, instead they used natural language and arithmetic representarions.

This example relfects the strucutre of the aforememtioned schema. As the students activity reaches higher levels of generality, they are not able to incorporate more abstract semiotic means of objectificarion to mediate their reflexive activities that remain somehow unconnected. From a structural and functional point of view the required treatment is trivial, but if we focus on the mediated reflexive activity the situation entails a great cognitive comlexity in harmonizing cultural and individual meaning.

D'Amore and Fandiño Pinilla (2007, pp. 2-3) remark:

> «The process of meanings endowment moves at the same time within various semiotic systems, simultaneously activated; we are not dealing with a pure classical dichotomy: treatment/conversion, that leaves the meaning prisoner of the internal semiotic structure, but with something much more complex. Ideally, from a structural point of view, the meaning should come from within the semiotic system we are immersed in. Therefore, in Example 2, the pure passage from (n-1)+n+(n+1) to 3n

should enter the category: treatment semiotic transformation. But what happens in the classroom practice, and not only with novices in algebra, is different. There is a whole path to cover, that starts from single specific meanings culturally endowed to the signs of the algebraic language (3n is the triple of something; 101×50 is a product, not a sum). Thus, there are sources of meanings relative to the algebraic language that anchor to meanings culturally constructed, previously in time; such meanings often have to do with the arithmetic language. From a, so to speak, "external" point of view, we can trace back to seeing the different algebraic writings as equisignificant, since they are obtainable through semiotic treatment, but from inside this vision is almost impossible, bound as it is to the culture constructed by the individual in time. In other words we can say that students (not only novices) turn out bridled to sources of meaning that cannot be simply governed by the syntax of the algebraic language. Each passage gives rise to forms or symbols to which a specific meaning is recognised because of the cultural processes THROUGH which it has been introduced»

3.6 Synthesis of the chapter

The cultural semiotic approach bears a pragmatic viewpoint of mathematical objects and their meaning. Learning is seen as an objectification process realized through a reflexive praxis mediated by semiotic means. The cultural semiotic approach drops the viewpoint of meaning as referring. Meaning has an intrinsically cultural nature embodied in the semiotic means we use to accomplish reflexive activity.

4

The onto-semiotic approach

4.1 Introduction

We have seen that Duval's approach recognizes an a-priori inaccessible mathematical object to which semiotic representations refer and that in Radford's approach mathematical objects lose such ideal existence, as they are bound to individuals' culturally framed activity.

The ontosemiotic approach (Godino, 2002; D'Amore, Godino, 2006; Font, Godino, 2007) also develops within a pragmatic theory of mathematical objects and generalizes the notion of representation, through the notion of semiotic function which relates an antecedent with a consequent: the role of representation is not played only by language but also emerging objects can be antecedents of a semiotic function; the ontosemiotic approach thus endows mathematics with its essentially relational and general character.

The ontosemiotic approach characterises the notion of meaning as a network of semiotic established between any antecedent and a consequent. Through the semiotic function the ontosemiotic approach welds the opposition between realistic and pragmatic theories.

4.2 Operational and referential phases

In chapter 1 we have analyzed the characteristics of realistic and pragmatic theories and we have presented as two contrasting approaches to mathematical ontology and meaning. We have already seen how the cultural semiotic approach overcomes this opposition considering the ideality and existence of mathematical objects as fixed patterns of mediated reflexive activity. From a cultural and social point of view mathematical objects posses an ideal and independent existence.

We can specify this point of view following Ullman's operational and referential phases that characterises the development of meaning.

> «(…) the meaning of a word can be verified only studying its use. There are no shortcuts towards meaning through introspection or any other method. The researcher first has to organize a suitable sample of contexts and face them with an open mind, allowing meaning or meanings to emerge from such contexts. Once this phase is concluded, we can pass to the "referential" phase and formulate the meaning or meanings that are thus highlighted. The relation between the two methods, or, better, between the two phases of the investigation, ultimately, is the same that we encounter between the language and the spoken: the operational phase deals with meaning in the spoken; the referential phase with meaning in the language. There is absolutely no need to place the two modes of access in opposition, one in front of the other; each of them bears its side of the problem and none of them is complete without the other» (Ullman, 1962, pp. 76-77).

In the ontosemiotic approach there is an interplay between the operational and referential phases. On the one hand mathematical objects and their meaning emerge from a system of practices on the other it is necessary to linguistically refer to that system of practices to apprehend the mathematical object and to construct new objects by addressing new activities that involve the old ones. For example the set of real numbers emerges from a system of practice at the operational level and it is recognized at the referential level by the use of specific language terms. We can construct the object real function by introducing a new system of practices that relates two subsets of real numbers according to the standard definition of a mathematical mapping.

The referential phase can be interpreted as a particular practice within a linguistic game that allows to designate shared activities that become part of the cultural and institutional language. The interplay between the operational and referential phases requires a twofold use of language and semiotics. Semiotics plays an instrumental role when it makes a practice possible and it plays a representational role when it allows to address such practice and the emerging object. We remark that also in the cultural semiotic approach, we need a representational use of semiotics at the referential level to identify the cultural object that is objectified through an instrumental use of semiotic means objectification. The teacher can orient the students' reflexive activity because she can refer to the cultural object to which she directs the students' intentional acts.

In the ontosemiotic approach the interaction between the operational and referential phases is accomplished within the linguist game by the semiotic function that grounds the notion of meaning in this theoretical approach. To carry out a detailed analysis of the semiotic function and meaning we need to introduce the basic theoretical tools that characterise the ontosemiotic approach. In the following paragraphs we will refer mainly to (Godino, Batanero; 1994; Godino, 2002; D'Amore, Godino, 2006; Font, Godino, D'Amore, 2007) and for sake of brevity we will not continuously quote these papers.

4.3 Systems of practices in fields of problems

The ontosemiotic theory develops along two dimensions, the personal and the institutional. The former regards the individual involved in thinking and learning processes, the latter regards group of individuals focused in pursuing and objective. The personal and institutional dimensions are two sides of the same coin that heal the opposition between psychological and anthropological and socio-cultural perspectives in Mathematics Education. The personal dimension is usually termed as cognitive and the institutional as epistemic.

The pragmatic standpoint that characterizes the ontosemiotic approach bestows a central role to the notion of practice. The theory is grounded on the idea that mathematical thinking and learning develops within a language game. A mathematical practice is a institutional realization of specific language game that involves also the personal dimension of individuals:

> «An institution is constituted by the people involved in the same class of problem-situations, whose solution implies the carrying out of certain shared social practices and the common use of particular instruments and tools. Institutions are conceived as communities of practices and they include, for instance, school classes or ability groupings and ethnic groups. Mathematical practices are carried out by persons and institutions in the context of material, biological and cultural backgrounds. Therefore, we assume a socio-epistemic relativity for systems of practices, emergent objects and meanings» (Font, Godino, D'Amore, 2007, p.3).

We can now outline the notion of practice in the ontosemiotic approach, characterized by an operational or discursive activity, a problem that is recognized significant by an individual or an institution and communication and generalizing processes:

> «We define a practice the set of linguistic actions or manifestations carried out by someone to solve mathematical problems, communicate the solution

to others, verify the solution and generalize it to other contexts and problems» (Godini, Batanero, 1994; p. 16)

The ontosemiotic approach usually doesn't refer to a single practice related to a single problem but considers systems of practices related to fields of problems. For example, if we consider the learning of linear equations there is a system of practices that characterize the linguist game: algorithmic processes for the solutions, the use of parameters, recognition of the unknowns, discussion of solutions relative to a set of numbers, to represent straight lines in analytic geometry etc. Equations can be used within a set of problems: to solve geometrical problems, to model situations of daily life, to solve kinematics problems in physics etc.

A mathematical practice can be epistemic or cognitive depending if it is accomplished at an institutional or personal level. Of course, there isn't a definite separation between epistemic and cognitive practices and the two dimensions are strictly interwoven. When we consider teaching-leaning processes that take place in the classroom, the individual experience of the pupil develops within the social practices that involve the class or subgroups of the class.

In the ontosemiotic approach a mathematical object is conceived as an emergent from a system of practices:

> «Mathematical objects are therefore symbols of cultural unities that emerge from a system of uses that characterizes human pragmatics (or at least of an homogeneous group of individuals); and they continuously change in time, also according to needs. In fact, mathematical objects and their meaning depend on the problems that are faced in mathematics and by their resolution processes. In sum they depend on human praxes» (D'Amore, Godino, 2006, p. 14).

The pragmatic nature of the ontosemiotic theory entangles mathematical objects and their meaning through the system of practices. It is difficult to distinguish a mathematical object from its meaning since they are both constitutively related to the system of practices. The meaning of a mathematical object is identified with the cognitive and epistemic practices from which the object emerges. Therefore a mathematical object has a personal and institutional meaning according to the dimension we focus our attention on.

If we ask ourselves what the set of natural number and its meaning are, we cannot but analyse the use we carry out in a particular language game as system of practices in a field of problems. We can focus on cardinality, ordinality, operations, its algebraic structures etc.

Although with differences that we will outline in the following paragraphs, also the ontosemiotic approach bestows a priority role to the notion f activity:

> «[The onto-semiotic approach] assumes a certain socio-epistemic relativity […] for mathematical knowledge, since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the institutions and the social context of which it forms a part» (Font, Godino, Contreras, 2008, p. 160).

4.4 Emerging primary entities and configuration of objects

The ontosemiotic approach performs a very detailed analysis of the type of practices that can be accomplished within a particular language game. While the cultural semiotic approach focuses on the way cultural and social elements, condensed in the semiotic means and the cultural modes of signification, direct intentional act of the individual's consciousness towards the cultural objects, the ontosemiotic proposes a refined characterization of mathematical practices. The denomination "ontosemiotic" refers to the twofold interest of the theory in the semiotic elements that characterise Wittgentsteins' language game and the ontology of mathematical objects in terms of systems of practices. Although signs and language play a prominent role in the onotosemiotic approach, the theory acknowledges a central role to the notion of practice.

> «To Ernest's question if "semiotics potentially offers the base to a unified theory in mathematics education (and mathematics)" we answer affirmatively, under the condition to adopt (and elaborate) an appropriate semiotics and to complement it with other theoretical tools, in particular an ontology that takes into account the variety of objects that are involved in mathematical activity» (Godino, 2002, p. 262)

To fully appreciate the role of semiotics and acknowledge the distinction between a mathematical object and its possible representations, it is necessary to identify the kind of practices that representations make available and refer to in the interplay between the operational and referential phases. The overlap between activity and mathematical objects leads to identify different types objects that emerge from the variety of systems of practices.

The ontosemiotic approach recognizes six types of primary entities emerging from the systems of practices that carry out a mathematical language game:

- Problem situations.
- Procedures.
- Languages.
- Concepts.
- Proprieties
- Arguments

Every mathematical practice develops from *situations* that require to solve a problem, carry out a particular task or pursue an objective. In turn, a problem situation results from the need to conceptualize, to generalize, to interpret and frame mathematically a piece of reality . Problem situations characterize the development of mathematics both at an epistemological and educational level. The learning of mathematics becomes meaningful to the student only when he is exposed to significative problem situations that require the use of mathematical knowledge.

Procedures are typical expressions of a language game in mathematics. Typically they represent mathematical algorithms. In general, procedures can be identified with Vergnaud's schemas and the operational invariants that we described in chapter 1. Every mathematical practice has a process that characterizes it.

Concepts are identified with a particular practice that characterizes mathematics: definition. In a realistic approach, definition is considered as the a definite description of an object that exists a priori. In the ontosemiotic approach the definition is a mathematical practice from which a particular primary entity emerges. To overcome the elusive notion of concept, Godino identifies it with the "rule" that is behind the language game that is accomplished by the system of practices.

Properties emerge from another typical mathematical practice: the use of propositions and predicates. The development of mathematics as an axiomatic and deductive system rests on this type of practice.

Strictly related to properties are *arguments* that refer to another mathematical practice that characterizes mathematics as an axiomatic and deductive form of knowledge. In the learning processes, a typical route that characterises arguments usually starts with a conjecturing activity, goes through argumentations and ends with proving as formal and structured mathematical practice.

Languages are primary entities that are related to the use of semiotic representations and linguistic terms. We believe that the use of semiotics should be considered as a meta-practice that is transversal to the other five systems of practice that embody the mathematical language game. Without a semiotic base, it is impossible to accomplish any mathematical practice; we cannot carry out any procedure, define a mathematical object, express a property, identify a problem situation and develop an argumentative practice. As we have already mentioned, the use of semiotics has twofold role; an instrumental one that can be identified with the mediation role of semiotic means of objectification to accomplish a mathematical activity and a representational role that intervenes in the referential phases when we need to designate a mathematical practice and its primary entities.

Primary entities are not isolated elements that emerge within the mathematical activity but they are rather interwoven in a recursive manner. The network of primary entities forms a *configuration of objects* that can be epistemic or cognitive according to the individual or institutional dimension we consider:

> «The problem – situations promote and contextualise the activity; language (symbols, notations, graphics, …) represent other entities and serve as tools for action; arguments justify the procedures and properties that relate the concepts. These entities have to be considered as functional and relative to the language game (institutional frameworks and use contexts) in which they participate; they have also a recursive character, in the sense that each object might be composed of other entities. Depending on the analysis level for example arguments, these entities might involve, for example, concepts, properties and operations» (Font, Godino, D'Amore, 2007, p.4)

In the ontosemiotic approach we have to fully drop the realistic idea that there is a fixed ideal object. The mathematical object is constitutively connected to the system of practices in a field of problem. Furthermore, the complexity of mathematical activity doesn't allow, at the referential level, to identify a single object but we must consider a configuration of primary entities emerging from as system of practices. In fact the analysis of mathematical thinking and learning develops along the triple of (representations, systems of practices, configurations of objects); the three element don't exist independently, one without the other two, although the investigation requires to describe them separately.

4.5 Cognitive dualities

According to the mathematical language game they belong to, the practices and the emerging primary entities we introduced in the previous paragraph have a double nature that the theory terms as cognitive duality. The onto semiotic approach recognizes five cognitive dualities:

- Personal-Institutional
- Unitary-Systemic
- Expression-Content

- Ostensive-Non Ostensive
- Extensive-Intensive

We have already described the *personal-institutional* duality in paragraph 3.4.1. We recall that the two dimension are strictly connected and that personal dimensions develops within a social and cultural interaction within the classroom. From an educational point of view, the institutional dimension plays a fundamental role in selecting the mathematical knowledge and in the design of the mathematical activity. The didactical transposition (Chevallard, 1985) which is at the core of teaching-learning processes, takes place within the interplay of the institutional and personal dimensions.

The *unitary-systemic* duality describes the relation between the systems of practices at the operational level and the emerging of objects at the referential level. If we analyse the use of semiotic representations, to fully account for the complexity of mathematical activity they accomplish it is necessary to consider the unitary and systemic facets that characterise the linguistic practices. On the one hand semiotics is essential, at the systemic level, to carry out any system of practice on the other, at the unitary level, semiotics allows to recognize from a social and cultural point of view such systems of practices and its emerging objects. For example, if we consider the calculation of the first derivative of a function as a procedure primary entity, we can identify a systemic and unitary facet. The systemic facet is relative to the set of rules we use to calculate the first derivative of a function. At the same time to perform the calculations we resort to algebraic operations, the notion of functions as unitary entities that sustain the practice.

The *ostensive-non ostensive* cognitive duality takes into account the intrinsically abstract nature of mathematical entities. Roughly speaking, the ostensive facet refers to the semiotic means that allow the mathematical practice and the non ostensive refers to mental objects that do not belong to our perception. We cannot consider the non ostensives as an ideal reality that we access through the ostensives, we would fall into a realistic viewpoint that doesn't belong to the ontosemiotic approach. The non ostensive is the social and cultural rule that sustains the language game, the ostensives permit the practices according to such rules. Therefore the non ostensive facet of a primary entity, within a linguistic game, has an ostensive facet that permits its system practices, corresponding to such language game. The ostensive-non ostensive facet has a socio-cultural nature within the language game in which we use something to make something explicit to someone at a personal or institutional level.

The *extensive-intensive* duality deals with the problem of generalizing processes typical of mathematics. We refer to generalization as the process that considers a set or system of elements as a unit. When we define an isosceles triangle we implicitly refer to al possible triangles that respect a fixed characteristics. The generalising practices are accomplished through the extensive-intensive facets, i.e. to grasp the general element we need to go through an intermediate phase in which we use an individual object. To accomplish generalization we have to relate intensive (the whole class of objects) and extensive (an individual object of the class) objects. The use of variable expresses this intensive-extensive duality. The variable is used as an undefined individual element that represents the whole class of elements that is its universe of reference. Generalization processes do not derive from an a priori structure but they are a specific practice in a language game that requires the recognition of the rules that allow to recognise that while we are acting on a specific representation we are interested in the general characteristics of the emerging object disregarding the particular aspects.

The *Expression-Content* duality generalizes the notion of representation and relates an antecedent with a consequent through the semiotic function that we analyse in the following chapter.

4.6 The Semiotic function

The semiotic function is a theoretical tool that sustains the expression-content facet. The semiotic function arises from the need to acknowledge the essentially relational and general nature of mathematical knowledge. Through the semiotic function, it is possible to integrate the operational and referential phases that characterise mathematical development. The advance of mathematics requires firstly to refer to and secondly to connect the primary entities emerging from the system of practices. The configuration of objects that allows a system of practices is not a juxtaposition of primary entities but it has strong relational character obtained thanks to a net of semiotic functions.

The notion was introduced by the Danish linguist Hjemslev (1943) as *sign function* and by Eco (1979) as *semiotic function*. It referred to the dependence between a text and its components and between the components themselves. It is a correspondence in terms of a function or a relation dependence between an antecedent, termed as expression or signifier, and a consequent, termed as content or signified. The semiotic function is established by a personal or institutional subject according to a cultural and social agreement and code. Such codes and agreements play the role of rules in the language game that inform the subjects that relate, through the semiotic function, the antecedent and the consequent in a given system of practices.

> «In the onto-semiotic approach a semiotic function is conceived, interpreting this idea, as the correspondences (relations of dependence or function between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by subject (person or institution) according to a certain criteria or corresponding code. These codes can be rules (habits, agreements) that inform the subjects about the terms that

> should be put in correspondence in the fixed circumstances. In this way, semiotic functions and the associated mathematics ontology take into account the essentially relational nature of mathematics and generalize the notion of representation: the role of representation is not totally undertaken by language (oral, written, graphical, gestures, …)» (Font, Godino, D'Amore, 2007, pp. 3-4)

The role of antecedent and consequent in a semiotic function is accomplished by any of the primary entities. In fact, the distinction between object and sign is not given a priori, but it belongs to the language game and depends on the context and the objective of the mathematical activity. Peirce points out that

> «A sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object» (C.S. Peirce, Collected Papers, 2.228)

In the ontsemiotic approach the notion of object has a metaphoric sense that reflects our sensual experience that allows to access and separate things. In the ontosemiotic approach weaker notion of mathematical object is given by the following definition:

> «All that can be indicated, pointed out, named when we construct, communicate or learn mathematics» (D'Amore, Godino, 2002, p.28)

The relativity of the distinction between objects and their representation is a consequence of the pragmatic standpoint advocated by the ontosemiotic approach. On the one hand in mathematical activity we deal only with representation on the other the emergents from the systems of practices can be indicated and referred to as objects.

The introduction of the semiotic function meets the impossibility to distinguish a priory an object from its representations and the need to establish within a language game representational relation between an "entity" that we call antecedent or signifier and another entity that we call consequent or signified; what plays the role of signifier and signified is decided by the rules of the language game. The semiotic function fully generalizes the notion of representation and it does not bind the function to represent only to language but any object can be a signifier or signified according to its role assigned by the language game.

> «The possibility to distinguish a sign from an object allows that "someone" can establish semiotic function between "two objects" ("something" for

"something"). In this relation ("something" for "something") we interpret one of the two objects as an "expression" that is in relation with a "content" (the other object) (Godino, 2002, p. 290)

There are three types of relation between antecedent and consequent:

- Representational when an object is used in place of another.
- Instrumental when a object uses another object as an instrument;
- Structural when two or more objects are organized into a structure from which new objects can emerge.

Mathematical learning and competence is a net of semiotic functions established by a personal or institutional subject and students' mistakes and difficulties can be explained in terms of semiotic conflicts when they have connect into configurations of objects the primary entities emerging from the systems of practices. In the next paragraph we will frame the meaning in terms of semiotic functions.

The semiotic function justifies the denomination ontosemiotic given to the theory, since it connects both the importance of ontology and semiotics in mathematical thinking and learning.

4.7 Meaning and changes of meaning

In the structural and functional approach meaning is conceived in terms of the relation Sinn and Bedeutung. In the realistic perspective advocated by Duval, we have many representations for one single object and meaning is the result of a complicated network of semiotic representations obtained through treatment and conversion transformation in and between semiotic registers.

The cultural semiotic approach breaks the structure one object-many representation to frame the meaning of mathematical objects. Basically, the meaning of a mathematical objects is identified with a mediated reflexive activity. At a phylogenetic level the mediated reflexive activity condenses in the ideality and "reality" of the cultural object, at an ontogenetic level the mediated activity objectifies the cultural object, i.e. it directs the individual consciousness's acts, allowing the student to make sense of the cultural knowledge by directing the individual consciousness's acts . The sense giving activity students are involved in can be seen as a convergence of the cultural meaning with the personal meaning. At an ontogenetic level the personal activity mediated by the semiotic means of objectification traces out the phylogenetic activity culturally condensed in the mathematical object.

We stressed that the convergence between the personal meaning and the cultural meaning has to face an intrinsic obstacle due to the fact the mathematical object is not a unitary and homogeneous entity; it develops in layers of generality, each of them corresponding to the reflexive activity mediated by semiotic means of objectification. The student is exposed to a variety of reflexive activities that splits his intentional acts towards objects that he considers disconnected but at an interpersonal level are recognized as belonging to the same cultural entity. The ontosemiotic approach, through the semiotic function, provides a theoretical tool that allows to coordinate the local meanings of the different layers of the object into a global cultural meaning.

While the cultural semiotic approach is interested mainly in the relation between consciousness' intentional acts and activity mediated artefacts, the ontosemiotic approach investigates the mathematical objects' ontology, looking at the complex network of the different types of systems of practices, and semiotics in its instrumental and representational uses.

In this paragraph, our objective is to frame the the issue of meaning and changes of meaning resorting to a connection of the basic theoretical tools that make up the ontosemiotic approach:

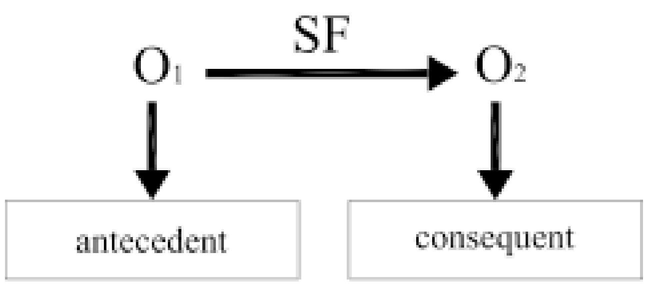
- The triple System of Practices, Configuration of Objects and Representations that we shorten in the (SP, CO, R).
- The cognitive dualities. We will focus mainly on the Personal-Institutional, Extensive-Intensive, Unitary-Systemic, Expression-Contnent.
- The semiotic function that relate the

We start facing the problem of meaning focussing on two opposite aspects, the pure referential and the pure operational.

From a pure referential point of view and in a very general sense we can say the meaning is the consequent of a semiotic function connecting two objects, the antecedent and the consequent:

> «Meaning is the content of any semiotic function, that is to say, the content of the correspondences (relations of dependence) between an antecedent (expression, signifier) and a consequent (content, signifier, or meaning), established by a subject (person or institution, according to a distinct criteria or a corresponding code» (Font, Godino, Contreras, 2008, p.161).

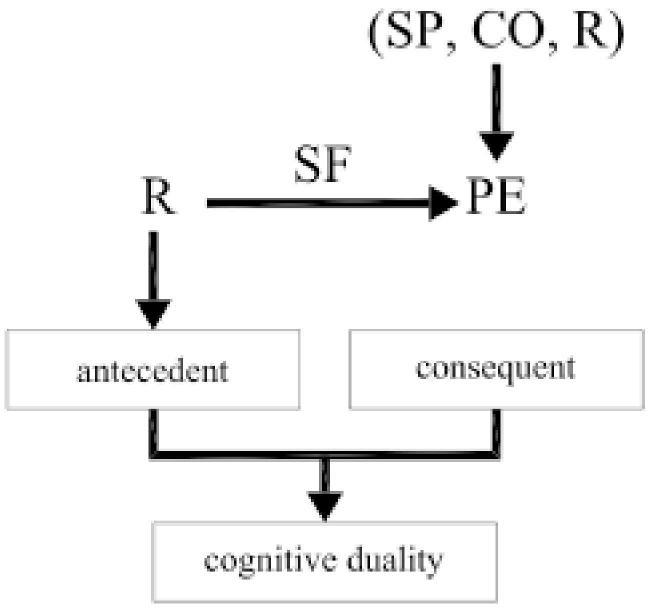
The antecedent and the consequent can be any object and the semiotic function can be used as a very flexible instrument to generalise the notion of meaning. The following schema frames meaning as the consequent of a semiotic function.



At the opposite extreme, at a pure operational level, meaning is the system of practices from which the mathematical objects emerges. In a very general sense, the meaning of a mathematical object is simply the what we do with such object in a mathematical language game while carrying out a system of practices in a field of problems.

Resorting to the unitary-systemic duality, we can give a holistic understanding of meaning connecting the referential and operational dimensions. The referential dimension is related to the unitary facet while the operational one to the systemic dimension. Using the semiotic function and the basic elements that make up mathematical thinking and learning according to the semiotic approach, we propose different modalities in which we can understand meaning, from simple to more complicated structures that relate the systems of practices and the cognitive dualities.

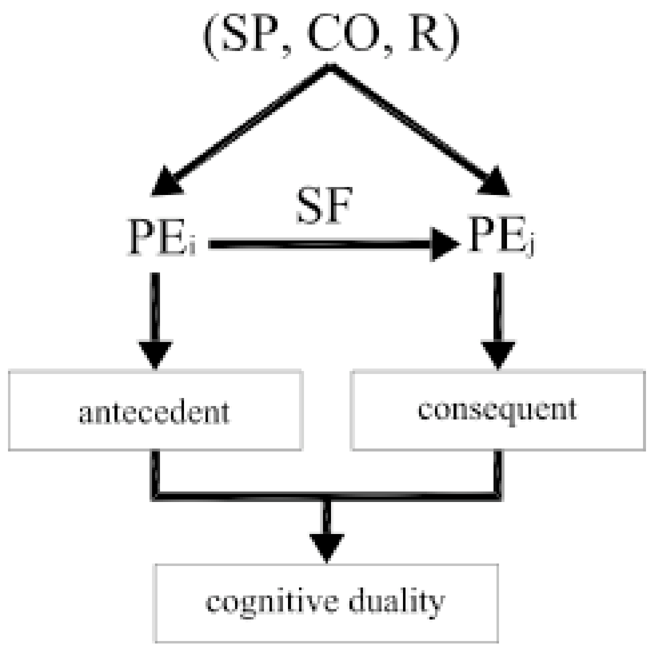
If we focus on one of the primary entities that emerge from a system of practices, the meaning of a mathematical object is the consequent of a semiotic function whose antecedent is a representational object and the consequent is the triple primary entity, system of practices and representations (PE, SP, R)



We remark that we have to intend the representational object within the cognitive duality unitary-systemic; in the antecedent of the semiotic function, the representational object has a unitary facet and the consequent a systemic one. The triple (PE, SO, R) is understood according to the institutional-personal duality.

For example we can consider the meaning of a Riemann integral as the operational primary entity emerging from a system of practices allowed by appropriate representations.

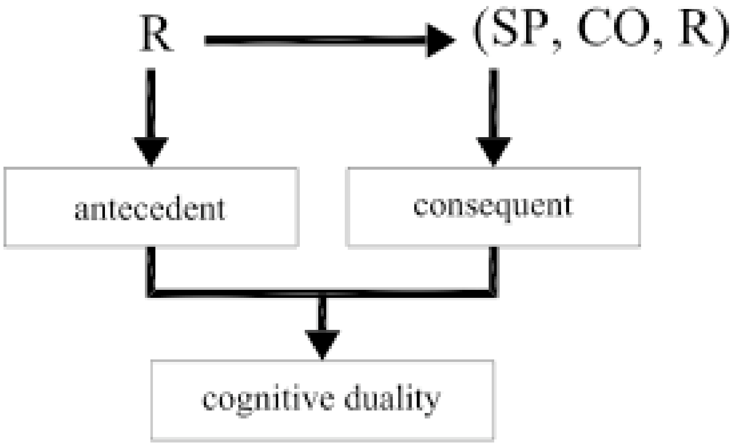
We can further enrich the meaning of a mathematical object connecting through semiotic function two primary entities belonging to a triple (SO, CO, R): the antecedent and the consequent are two primary entities of the configuration of objects.



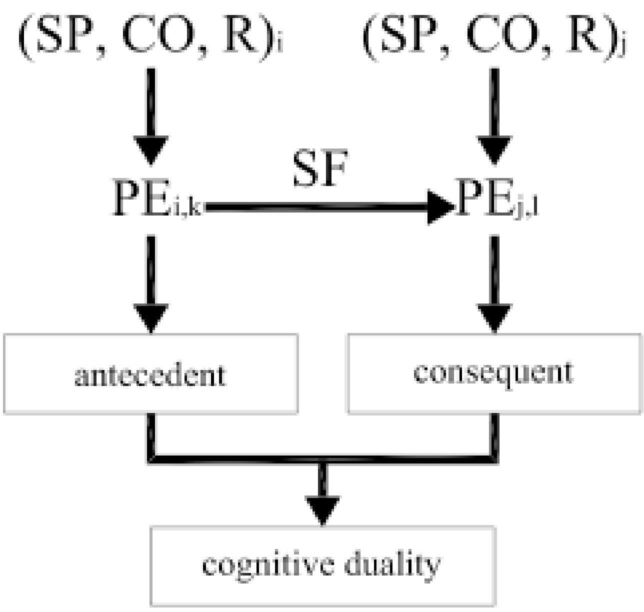
We can look at the relation between the two primary entities from the institutional-personal cognitive duality. If we consider also the extensive-intensive duality we can merge meaning with the generalization of the mathematical concept.

For example, if we consider an integral we can relate a definition primary entity with a procedural primary entity. We can relate the two primary entities using a specific function that through the extensive-intensive duality generalizes the notion of integration to a general function.

Through the semiotic function, we can construct a configuration of objects formed by a network of primary entities. In a more encompassing way, we can consider meaning as the consequent of a semiotic function where the antecedent is a representational object and the consequent is the triple (SP, PE, R).

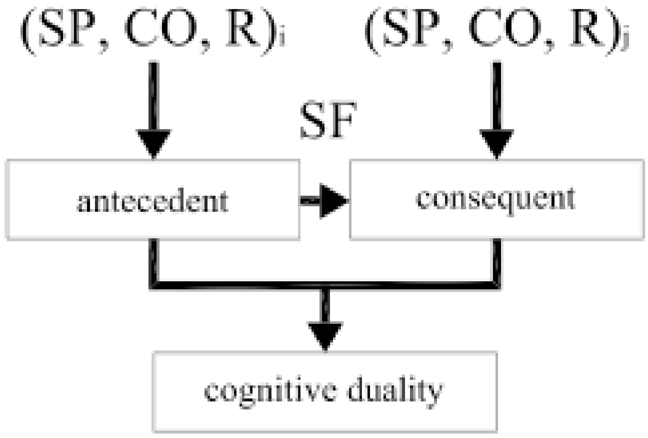


When we construct a semiotic function that connects two primary entities, the emerging objects don't necessarily belong to the same triple (SP, CO, R). The antecedent can belong to a triple $(SP, CO, R)_i$ and the consequent to a triple $(SP, CO, R)_j$.



For example, in analytic geometry to construct the Cartesian plane we need to connect at least operational and conceptual entities belonging to Euclidean geometry and real numbers language games.

We can get a more global meaning if we consider triples (SP, CO, R) as antecedent and consequents of a semiotic function.



Many mathematical concepts entail the interplay of more than one triple (SP, CO, R) through the semiotic function. For example to grasp the global and general meaning of the tangent, we need to connect three language games: Euclidean geometry, Cartesian geometry and calculus.

In the ontosemiotic approach, meaning is the combination of the operational and referential dimension through the semiotic function in which the role of expression and content can be played by primary entities or configurations of objects according to the level of complexity we want to grasp.

This approach goes beyond the idea that meaning stems from a referential relation between an independent object and one of its possible representations. According to the ontosemiotic approach we must think of meaning in terms of an object O1 (antecedent), an object O2 (consequent) and the rule that allows to establish the semiotic function between O1 and O2 considered that can play one of the different roles we described above. .
Meaning is a relation established through the semiotic function that involves triples pairs constituted by a system of practices and a configuration of objects and representations.

Duval's Semiotic transformations are the emerging aspect of a semiotic function that relates a representation R (antecedent) in a triple (SP, CO, R) with another representation S in another triple (SP, CO, R). In its global sense, meaning can be conceived as a relationship between a triple $(SP, CO, R)_i$ and a triple $(SP, CO, R)_j$ established by a semiotic function. The triples allow both a macro analysis if we consider relations between the whole configuration and a micro analysis if we consider relations between primary entities of such configurations.

When facing the issue of meaning, considering mathematical knowledge in terms of one object–many representations is insufficient to grasp the whole of its complexity. As we mentioned above, mathematical objects, representations and meaning are entangled through activity. Such a net distinction between the mathematical object and its possible representations is effective when devoted to the cognitive operations in mathematics but when investigating the teaching and learning processes as a whole such distinction is untenable for the following reasons:

- It is very difficult to identify "the" mathematical object. Mathematical objects are stratified in layers of generality and organized in epistemic and cognitive configurations of primary entities.
- As an emergent from a system of practices, it is difficult to recognize a clear boundary between the object and the representations that mediate the practice. Of course, on the one hand we mustn't confuse the object with it's representation, but on the other if we try to separate the object from its representation we exclude the practices it emerged from. The unitary-systemic cognitive duality allows to take into account both the need to refer to the object and the social activity the objects comes from: a representation has a representational value, as something that stands for something else in a unitary sense; a representation has an instrumental value, as it sustains specific practices in a systemic sense (Font, Godino, D'Amore, 2007).

In the ontosemiotic approach, meaning is a complex and holistic construct that in a linguistic game binds systems of practices, configurations of objects, cognitive dualities and the semiotic function. Meaning has a local value when we consider a particular system of practices obtained by a specific representation and it has a global value when we relate through the semiotic function the possible systems of practices involved in the emergence of a mathematical object.

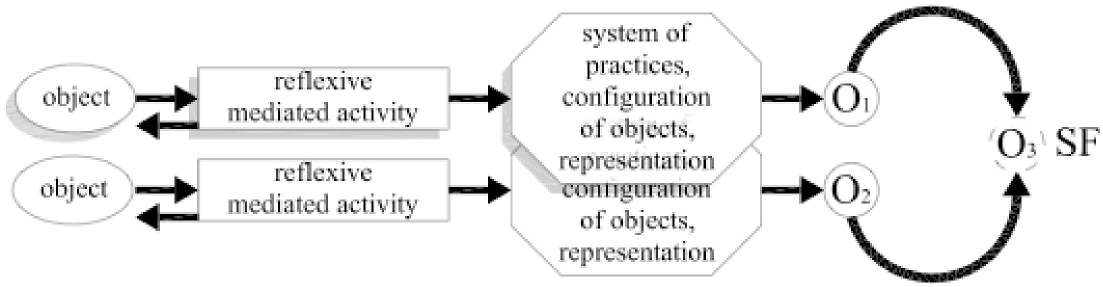
> «We can decide that the meaning of a mathematical concept is the pair "epistemic configuration/practices it entails", where the definition of the concept (explicit or implicit) is one of the components of the epistemic configuration. When the concept has another equivalent definition the concept can be built into another pair "epistemic configuration/practices it entails", different from the pair considered before. In this case, each pair can be considered as a different "sense" of the concept, while the meaning of the concept is the set of all the pairs "epistemic configuration/practices it entails"» (Godino, 2002, p. 5-6, appendix).

The analysis in terms of one object-many representations is extremely effective at a referential level. The coordination of semiotic systems is at the core of mathematical thinking but we cannot disregard the operative phase in terms of systems of practices and configurations of objects in which such coordination is rooted. To understand how and why semiotic transformations occur, we resort to the semiotic function that relates triples "systems of practices-configurations of objects-representations".

We believe that the notion of objectification plays a central role in characterizing systems of practices, configuration of objects and the criteria and rule that connect the antecedent with the consequent in the semiotic function. Looking at the complexity of practices also in terms of reflexive mediated activity provides a thorough understanding of the epistemic and institutional dimension involved in mathematical thinking and learning. Semiotic means of objectification tell us precisely the nature of the practice they accomplish thereby determining both the layers of generality and the configurations of emerging objects involved. The introduction of the semiotic function provides a more refined theoretical tool to analyse the issue of meaning in mathematical thinking and learning, that connects both practices and the relative emerging objects.

We propose the following schema integrating the notion of objectification and semiotic function to frame the issue of meaning in mathematics.

Schema Radford + Godino.



4.8 Synthesis of the chapter

The ontosemiotic approach also conceives mathematical objects and meaning within a pragmatic standpoint. Referring to Wittgenstein's language game, at the core of the theory is the notion of system of practices to face a field of problems. Mathematical objects and systems of practices are overlapped: a mathematical object is the kind of practice it enacts which in turn is its meaning. In the following paragraphs we will detail how the semiotic function connects local meanings emerging from different practice into a global meaning of the mathematical object.

5

Towards a theoretical framework for changes of meaning

5.1 Introduction

In this chapter we want to accomplish a double objective. On the one hand we want to face the issue of connecting theories focussing on the three perspective we are analyzing on the other, through a suitable synthesis of the structural and functional approach, the cultural semiotic approach and the ontosemiotic approach build a specific framework to for the problem of meaning and changes of meaning.

Resorting to forefront research in the field of networking of theories in mathematics education we will briefly present the basic theoretical tools that are necessary to connect theoretical perspectives. We will give the basic criteria to identify the elements that make up a theory in mathematics education and we will present connecting strategies that can be employed when relating different perspectives. Our aim is to arrive at a framework that allows to formulate our research questions and conjecture possible hypotheses we can compare with our experimental result within the same framework to arrive to possible solutions to our research problem.

5.2 Networking theories

Theories in mathematics education

The answer to what a theory in mathematics education is has involved many scholars; Bigalke (1984), Romberg (1988), Mason and Waywood (1996), Maier and Beck (2001), Niss (2007). The analysis of theoretical perspectives in mathematics education basically develops considering theories as static or dynamic constructs. As static constructs, theories have a normative use to organize and systematize empirical data, instead as dynamic constructs, instead, theories are developed to answer research question and the structure of the theory builds up through its use in research activities within an interplay of observation, practice and formulation of theoretical principles.

To analyse theories in mathematics education we will consider Radford's (2008) formulation to frame theoretical perspectives. Within a dynamic perspective, Radford develops his analysis of theories within social-cultural space that referring to Lotman (1990) he calls the semiosphere. A semiosphere is characterized by the following elements:

- A system of practices.
- A meta-language
- Themes and plots that can be developed within this sociocultural space
- Coexistence of multicultural identities.

The semiosphere is extremely effective to study the connection of theories since it is a space that fosters interaction and dialogue between different cultural identities. In the pragmatic stand we are advocating in this study, a theory embodies social, cultural such identities and differences that characterise the objects we want to compare and connect. The meta-language of the semiosphere allows to objectify in a social practice the types of connection and comparison of its objects. In a networking perspective, the semiosphere blends to important plots, integration of its entities and differentiation through dialogue that fosters identity, self-knowledge and extraction of further knowledge and understanding about oneself and others.

Radford (2008) frames a theory in mathematics education as dynamic structure made up of the following ellemts:

- A hierarchical system of principles.
- A methodology.
- A template of research questions.

The system of principles defines the nature of the theory. They are a set of fundamental claims regarding mathematical,epistemological, psychological, educational, social, cultural and philosophical aspects that intervene in mathematical teaching and learning; they define the object of discourse and the research prospective. It is important to note, especially when connecting theories, that the set of systems are not juxtaposition of claims. The principles are organized in an hierarchical system in which we must take into account not only the principles themselves but also their hierarchical position in the system and their relation with other statements that make up the theory. A one or more principles can be common to more than one theory but this doesn't imply that such theories are equivalent, if the principles have a different position in the hierarchical system.

The system of practices informs also the methodology of a theory that defines the operability of a research that must be coherent with the system of principles. According to the object of discourse and the research prospective defined by the principles, the methodology defines the experimental design, the contexts that are more appropriate to perform experiments, the selection of data and the type of data analyses.

A theory is a form of reflection on a cultural practice that emerges due to the rise of cultural and social relevant problems. The templates of research questions are research problems that are generalised through the system of principles. It is rather naïve to think of research question as prior to a theory that develops to answer such questions. Research questions are always theory laden:

> « My argument here is that we cannot answer this question by looking at the theories' research questions alone and that we need to look into the principles as well. For, research questions are not stated in a conceptual vacuum: research questions are stated within a world-view and this world-view is defined by the explicit and implicit principles of any given theory» (Radford, 2008, p. 325).

Thinking in terms of the triple T(P, M, Q) in the semiosphere is very effective when comparing different theoretical perspectives. We can compare theories at the level of the system of principle, the methodology or the research questions. The triple T(P, M, Q) is very effective to outline the boundaries (Radford, 2008) of a theory. When connecting theories, it is important to balance integration and differentiation between theories. The boundary of a theory is a threshold that cannot be overcome without loss of identity. The boundary sets the limit of discourse of a theory, beyond such limit the theory contradicts its systems of principles.

Networking theories

In this section we will refer to Prediger, Bikner-Ahsbahs, Arzarello (2008) framework regarding the connection of theories.

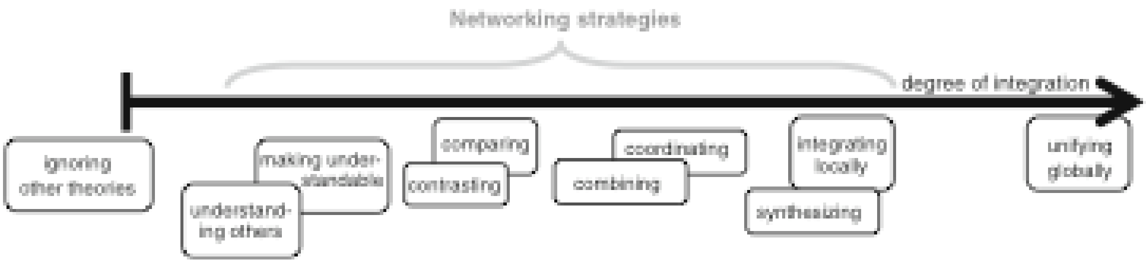
We remark that mathematics education, understood as an epistemology of learning (D'Amore, 1999) is intrinsically the result of a sophisticated connection of theories. The complexity of the teaching-learning processes in mathematics, doesn't allow to study the learning in mathematics looking only at mathematics but we have to resort to other fields of knowledge like, philosophy, pedagogy, sociology, anthropology etc. The present investigation is based on semiotics, that before Duval's forefront studies, one wouldn't have considered so strictly connected to mathematical thinking and learning. Mathematics education is not a juxtaposition of different disciplines, but it is an independent field of research (Romberg, 1988) as an outcome of a *systemic* interaction between different disciplines (D'Amore, 1999). The problem is that the richness and power of the systemic connection of many theories has lead to a proliferation of theoretical perspectives in mathematics education that has to be faced with the plot of identity/differentiation and integration mentioned above:

> «In the present panorama of mathematics education, we observe a sort of theoretical "autism" (closed in its self) and a theoretical and methodological disarticulation. This problem doesn't regard only distant paradigms and schools of thinking (pragmatism, realism, cognitivsm etc.), but also intermediate emerging theories that share the same basic epistemological paradigm» (D'Amore, Godino, 2006, p. 35).

Facing the issue of the proliferation of theory, we need to blend integration and identity/differentiation plots. There are two possible extreme behaviours. On the one hand, if we overweigh the plot of identity, the risk is to have a set of disarticulated theories that ignore themselves on the other if we overweigh the plot of integration theory we "have a theory of everything" that is unable to frame the complexity and variety of teaching learning processes.

The aim of connecting theory is to grasp the richness of the diversity of theoretical perspectives to enhance communication between different viewpoints, integration of empirical results, the recognition of strength and weakness among theories and scientific progress. Connecting theories is important to reduce the inflation of perspectives thereby bestowing mathematics education global coherence, theoretical and methodological unity, effective research design and enhancing a spin off in education to improve teaching and learning.

Prediger, Bikner-Ahsbahs, Arzarello (2008) propose a “landscape” of possible connecting strategies that balance identity and integration. The following schema taken from the aforementioned article shows the networking strategies ordered according to their degree of integration:



At the opposite extremes of the “landscape” are ignoring other theories and unifying globally that for the aforementioned reasons are discarded:

- *Understanding and making understandable* is a precondition for connecting theories that allows to grasp their articulation of research practices and its structure through paradigmatic examples. Understanding and making understandable belongs to the plot identity/differentiation is basically accomplished through the metal-language of the semiosphere that allows the dialogue between different cultural identities and languages.
- *Comparing and contrasting* is a connecting strategy that aims at a better understanding and communication between theories and allows to carry out a choice among two ot more perspectives. Comparing refers to a neutral analysis of two or more theories whereas contrasting aims at highlighting differences to single out specific aspects of theories and make possible connections visible. This connecting strategy is still centre on the plot of identity/differentiation.
- *Coordinating and combining* is a strategy that connects two or more theorie in view of describing an empirical phenomenon or tackling a particular research issue. The outcome of this connecting strategy is not a coherent complete theory, but rather a conceptual framework that allows to coordinate

different theoretical tools for the sake of specific objective. Combining refers to the connection theory through a juxtaposition of theories that give a multifaceted theoretical insight. Coordinating is a strategy that connects theories that share a high level of complementarity and coherence. This connecting strategy entails a balance of the plots of identity/differentiation and integration.

- *Synthesizing and integrating locally* is a connecting strategy that aims at the construction of a new synthesized theory that requires strong preconditions and necessarily rests upon less integrative theories. Synthesizing is a strategy that connects symmetric and equally stable theories that merge into a new theory. Integrating locally is a strategy that networks not symmetric theories, that is when one theory is more dominant with respect to the other. Although this strategy posses a high degree of integration the plot of identity/ differentiation in the semiosphere plays an important role in the construction of the new theory.

5.2.1 Networking research questions

The research problem that informs the present investigation arouse within Duval's structural and functional semiotic approach. Duval's theoretical framework appeared insufficient to fully understand this new research issue. We decided to address also the Cultural-Semiotic and Ontosemiotic approaches gathering other and more inclusive theoretical tools to better frame and understand these unexpected didactical phenomena. The researched project started focusing on Duval's structural and functional approach, eventually taking into account also the Ontosemiotic approach mainly to use the notion of *semiotic function* as a theoretical tool to frame the meaning of mathematical objects and connect systems of practices and emerging objects. Involving more than a single Mathematics Education theory, raised the problem of relating the different perspectives we decided to address for our research.

The research questions regarding the connection of the theoretical approaches we considered were:

1. What are the boundaries of each semiotic perspective we are considering and how can this inform our connecting strategies in terms of the plots of integration and differentiation?
2. What degree of integration is the most appropriate to answer our research questions? At what level is most effective networking: system of principles, methodology or research questions templates?

At the beginning of our research our conjectures were:

1. The fact that the three perspectives are based on semiotics, lead us to believe that at their cores they were very similar theories that could adapt to solve the problems of our research. Our first conjecture was that, in fact, the theories had frail boundaries and that they could be considered as contiguous frameworks.
2. Our hypothesis was that certainly it was necessary to work at the level of understanding and making understandable and comparing and contrasting. Our first naïve conjecture, coherently with what we said above, was that the three perspectives allowed a combination and coordination between them, to reach also a synthesis to understand the role of semiotics in mathematics education and to tackle the issue of meaning. Our conjecture was also that the connection could be carried out at the at the level of the system of principles and research questions.

Taking into account the analysis we conducted in chapters 2,3 and 4 and connecting theoretical tools we described in the present chapter we can answer the research question regarding the connection between the structural and functional approach, the cultural semiotic approach and the ontosemiotic approach.

Answer 1

Our initial hypothesis that the three frameworks were contiguous, with frail boundaries was not correct. Although the three perspectives focus their attention on semiotics, its hierarchical position in the system of principles, is very different. Duval is interested in the specific cognitive functioning of mathematics that he characterises as the coordination of semiotic systems that are the only access to mathematical objects; he claims that there isn't *noesis* without *semiosis*. Radford is interested in learning as objectification-subjectification processes in which individuals become a *aware* through mathematical objects of a *cultural and historical* dimension mediated by semiotic means in social and cultural modes of signification. Godino is interested in the passage from the *operational* phase to the *referential* phase in mathematical learning and generalises the notion of sign as any *antecedent* in a semiotic function.

The nature of the three semiotic perspectives tackles the issue of learning of mathematics in different directions that in the networking semiosphere require to consider both differentiation and integration

Answer 2

In view of the possible level of integration between the three perspectives, chapters 2,3 and 4 express our endeavour in *Understanding* and *comparing/contrasting* to grasp the true identity of each theory and precisely identify their boundaries. Our initial hypothesis to synthesize the three perspectives was obviously too ambitious, overestimating the potentialities and possibilities of a doctoral research, and lacked the necessary theoretical knowledge about networking theories. We cannot exclude that a synthesis of these perspectives is achievable. Integrating at the level of *coordinating /combining* provided effective theoretical tools to tackle our specific issue of "changing of meaning" due to treatment semiotic transformation. It was possible to coordinate because although the boundaries are marked there is strong complementarity between the three perspectives in addressing the following dualities; personal-institutional, referential-operational, cognitive – reflexive activity etc.

We found fruitful to establish connections at the levels of the system of principles and the research questions. We didn't address the issue of integrating at the level of the methodology resorting to qualitative and ecological methods that we reckoned more appropriate to answer our research questions while focussing on the role of reflexive activity.

5.3 A framework for meaning and changes of meaning

In this paragraph we show hoe it is possible to coordinate the structural and functional, cultural semiotic and ontosemiotic approaches to build a framework for meaning and changes of meaning.

Although at the core of their system of principles the three perspectives are very different, this resulted in a resource to face the complex and broad problem of the meaning of mathematical objects. The differences between the three perspectives allows to tackle the problem from different directions and, limitations in one of the theories, are made up for by the others. In the terminology we introduced in the previous paragraph, the aforementioned differences result in a high level of complementarity that accounts for networking by *coordinating* the three perspectives. We have singled out six elements in which the three perspective complete each other:

1. Semiotics in its different acceptations: representational, instrumental, purely relational through the semiotic function. .
2. Cognitive operations specific of mathematics, identifiable with the coordination of semiotic systems and semiotic registers.
3. The role of consciousness in determining sense giving acts.
4. Social and cultural factors
5. Activity in its reflexive understanding and as a practice emerging from a language game in a field of problems.
6. The notion of mathematical object in its operational (pragmatic) and referential (realistic) sense.

Disregarding even one of the aforementioned elements would hinder our analysis and understanding of the issue of meaning and the formulation and answer to the research questions that inform the present investigation. The networking attempt to coordinate the structural-functional, the cultural semiotic and the ontosemiotic approaches, allows to face the problem of meaning in a holistic and open way without encapsulating it in a fixed and definitive frame that would betray its anthropological and socio cultural nature.

1. Semiotics in its different acceptations: representational, instrumental, purely relational through the semiotic function

To understand mathematical meaning the role played is essential access objects through their representational function, to mediate activity and support the variety of practices that characterise mathematics and to construct mathematical relations and structures. Only the coordination of the three semiotic perspectives allows to accomplish the aforementioned functions. We need semiotic registers to analyse representations, semiotic means of objectification to mediate activities, and the semiotic function to construct mathematical relations and structures.

2. Cognitive operations specific of mathematics, identifiable with the coordination of semiotic systems and semiotic registers.

Duval has clearly identified cognitive operations that are specific of mathematical thinking and learning. When analyzing mathematical thinking we cannot disregard the analysis in terms of treatment and conversion transformations. As we pointed out in chapters 3 and 4 they are emerging operations but they give us precious information about the student's behaviour and the instructional and research design.

3. The role of consciousness in determining sense giving acts.

We have described how the cultural semiotic approach considers learning as an objectification process accomplished through the semiotic means of objectification and identifies learning with a meaning making process. Such phenomenological approach to learning is very effective when analysing changes of meaning and it requires the theoretical tools of the cultural semiotic approach.

4. Social and cultural factors

To understand the role of social and cultural factors in determining mathematical activity we need to coordinate both the cultural semiotic and ontosemiotic approaches. The cultural semiotic approach shows the role of social interaction and historical and cultural elements in directing the individual's intentional acts towards the mathematical object, whereas the ontosemiotic, through the social and cultural factors, describes mathematical the complexity of the systems of practices and their relation with the language game.

5. Activity in its reflexive understanding and as a practice emerging from a language game in a field of problems.

When we use the term mathematical activity or mathematical practice, we refer to a very complex object. The coordination of the cultural semiotic approach and the ontosemiotic approach allows to frame tow basic aspects of mathematical activity;

activity that directs the indvidual's intentional acts, in the cultural semiotic approach, and activity as systems of practices that realize a specific language game and its rules.

6 The notion of mathematical object in its operational (pragmatic) and referential (realistic) sense

We have seen how many problems arise when we try give a fixed definition of a mathematical object. During this work we highlighted several dimensions that make up the mathematical object: operational, referential, cultural, invariant, as a fixed pattern, as a primary entity, level of generality, semiotic. Without a coordination of the three semiotic perspectives we are advocating it would be impossible to grasp such complexity and the relation between objects and their meaning.

5.4 Research questions and hypothesis

We now have at our disposal the theoretical tools to formulate our research questions that also show the connection between the cultural semiotic and ontosemiotic approaches at the level of the templates of questions.

An important part of the work we conducted during this doctoral research was devoted to clarify and precisely frame both the research questions and the working hypotheses. We remind the reader that the research questions were formulated within Duval's and Radford's approaches and in terms of a networking of the two theories. Facing the issue of connecting theories, we also formulated questions regarding the networking of the aforementioned theories, which, in the development of our investigation, also included the ontosemiotic approach to take into account the semiotic function as an effective tool to relate and connect different reflexive activities and primary entities and configurations of objects as a whole in the construction of meaning. . Below we briefly recall the research questions followed by the hypothesis.

In Duval's approach we addressed the problem as the loss of meaning understood as a loss of the common reference to the mathematical object when given object given through different semiotic representations:

1. Why do students lose the meaning of the mathematical object when changing semiotic representation and how does this phenomenon occur ?

2. What is the relation between treatment and conversion when students experience a loss of meaning of the mathematical object?

We conjectured that:

1. The loss of meaning is ascribable to the inaccessibility of the mathematical object and it is one of the possible behaviours deriving from Duval's cognitive paradox. The student is unable to disentangle the *sinn* from the *bedeutung* when he carries out a semiotic transformation thereby losing the reference to the common mathematical object or referring to a different one.
2. The phenomenon is independent from the type of transformation, but the phenomenon is more evident in the case of treatment because, from a syntactical point of view, the pupil can carry out the semiotic transformation correctly without necessarily handling appropriately the couple (sign, object).

In the cultural semiotic approach we formulated our research questions focusing on the role of activity in determining the meaning of mathematical objects:

1. How do students *connect* and *synthesize* several contextual meanings into a unitary meaning of the mathematical object?
2. How do students objectify a general cultural and interpersonal meaning of the mathematical object, if they can only access a personal meaning obtained through their embodied space-time contextual reflexive activity mediated through semiotic means of objectification?

We conjectured that:

1. When the semiotic means of objectification mediate contiguous activities there is a strong connection between the different local meanings, each of them representing a step in the path that leads to the general and cultural mathematical object; embodiment is the key element that ensures such contiguity. The transition from one situated experience into another towards higher levels of generality can interpreted as the relation between an antecedent and a consequent of a semiotic function.

2. The semiotic function is a theoretical and practical tool to overcome the dichotomy between the reflexive personal dimension and the general cultural dimension. The disembodied meaning can be seen as a “meaning of meanings” in the sense that each meaning deriving from each couple system of practices-configuration is synthesized in a unitary meaning of object through the semiotic function

In terms of a connection of the two theories we focused our attention on the following issues:

1. What is the relation between the coordination of semiotic systems and activity in the objectification of meaning?
2. The use of semiotic systems is an outcome of the student’s learning process or it is carried out in parallel with the reflexive activity?
3. Is it possible to coordinate the diachronic use of semiotic representations and the synchronic use of semiotic means of objectification?

We conjectured that:

1. The two dimensions coexist and mathematical activity basically is a semiotic practice that allows the emergence of mathematical objects intended as primary entities.
2. The coordination of semiotic systems coexists with the reflexive activity. We cannot disentangle mathematical cognition from the coordination of semiotic systems.
3. The two temporal dimensions are harmoniously coordinated; through the synchronic use of semiotic means of objectification, students diachronically perform semiotic transformations.

6

Classroom experiments

6.1 Introduction

In this chapter we present the results of two experiments we conducted in a high school and in a primary school respectively. The primary school experiment was conducted with a class in its last year of a scientific school working on the first derivative of a real function. The primary school experiment was carried out with ten years old students at the last year of their cycle working on the generalization of sequences.

The high school experiment highlights students' difficulties in dealing with the meaning of the tangent, as they have to synthesize different layers of generality and systems of practices from which the object emerges. The students are locally able to deal with the tangent in specific practices they are not able to coordinate the general meaning of the concepts as they change representations that mediate their reflexive activity.

The primary school experiment was designed on the basis of the sequences used by Radford (2000, 2002, 2003, 2005), whose range is represented by figures. We exposed students to the same sequences represented by different representations always in the figural semiotic register. Most of the students recognized the same mathematical sequence when we changed representation in the figural register. In fact, the situation is similar to that of D'Amore and Fandiño working with university students who didn't recognize the common reference of the two algebraic representations (n-1)+n+(n+1) = 3n.

Besides the above experiments, we conducted other two investigations that we will not describe because they didn't give meaningful and interesting results. Basically because of improper construction of the teaching-learning design in the classroom.

6.2 Experiment 1: The Tangent

Environment and constraints

The experimentation was conducted in a school of Bologna with 19 years old students at the last year of their secondary high school studies. The students followed a five year scientific curriculum that in the last year, besides humanities studies and science, provides mathematics and physics courses, three hours per week each. The syllabus of mathematics focuses on calculus. During our experiment the class was starting the study of derivatives. We worked with a small class, twelve students.

The teacher was very concerned about the final exam that, as regards mathematics, consists of a written test and an oral discussion, that covers the whole mathematics syllabus. Because of the high number of topics and the lack of time it was not possible to interfere with the teaching methodology and the mathematical topics.

Experimental design and methodology

We started videotaping the classroom environment during traditional frontal lessons for five hours on the part of the teacher. We then gave a written text followed by binomial interviews that were also videotaped.

We worked with the class when students started facing derivatives. We recall that during our research the mathematical topic was developed as follows:

- Incremental ratio and its geometrical interpretation.
- Derivative as limit of the incremental ratio and geometric interpretation.
- Non derivability and singular points.
- Derivation and continuity
- Algebra of derivatives.

Below part of the worksheet that was given to the students.

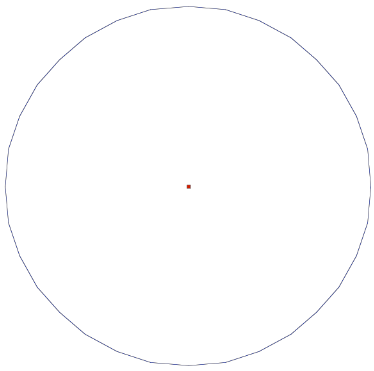
LICEO SCIENTIFICO *"E. Fermi"*
TEST 1
Name______________________________

Surname___________________________

Class______________________________

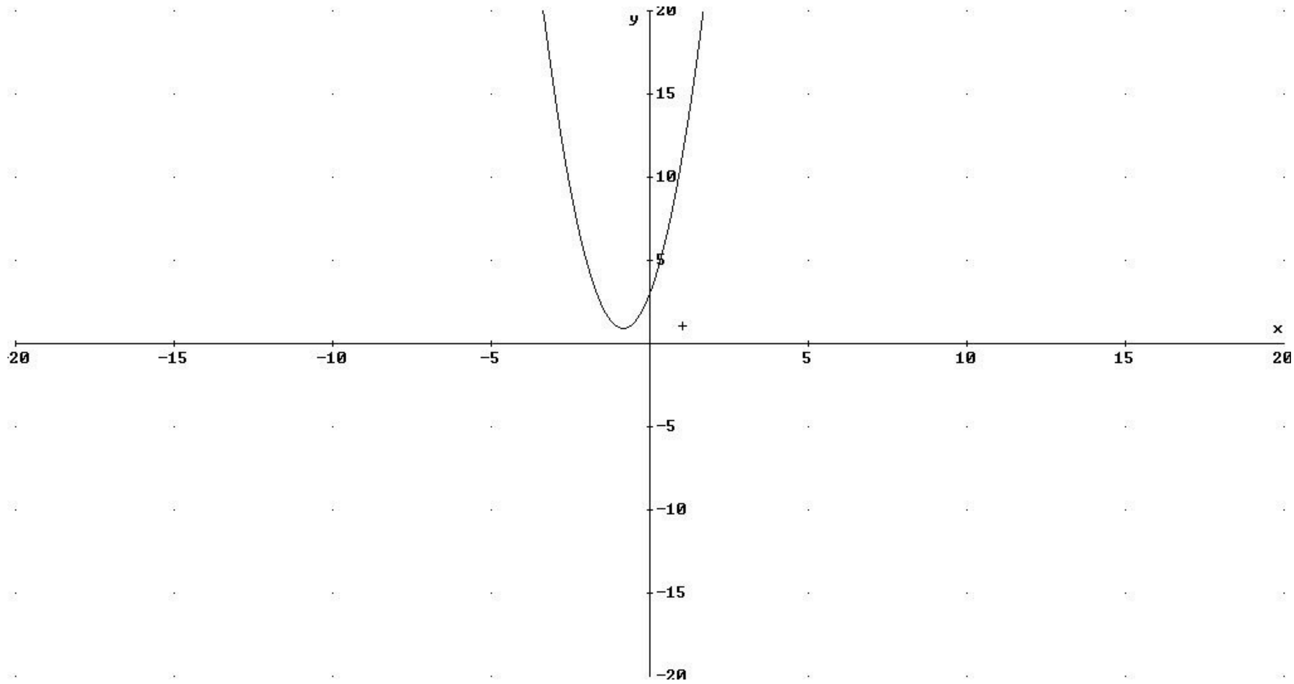
Question 1

1.1 Given the following circumference trace the tangent to a generic point A.



How did you determine the tangent ?
The tangent is unique in A?
Justify your answers.

1.2 Trace the tangent to the following curves in a generic point A.


y
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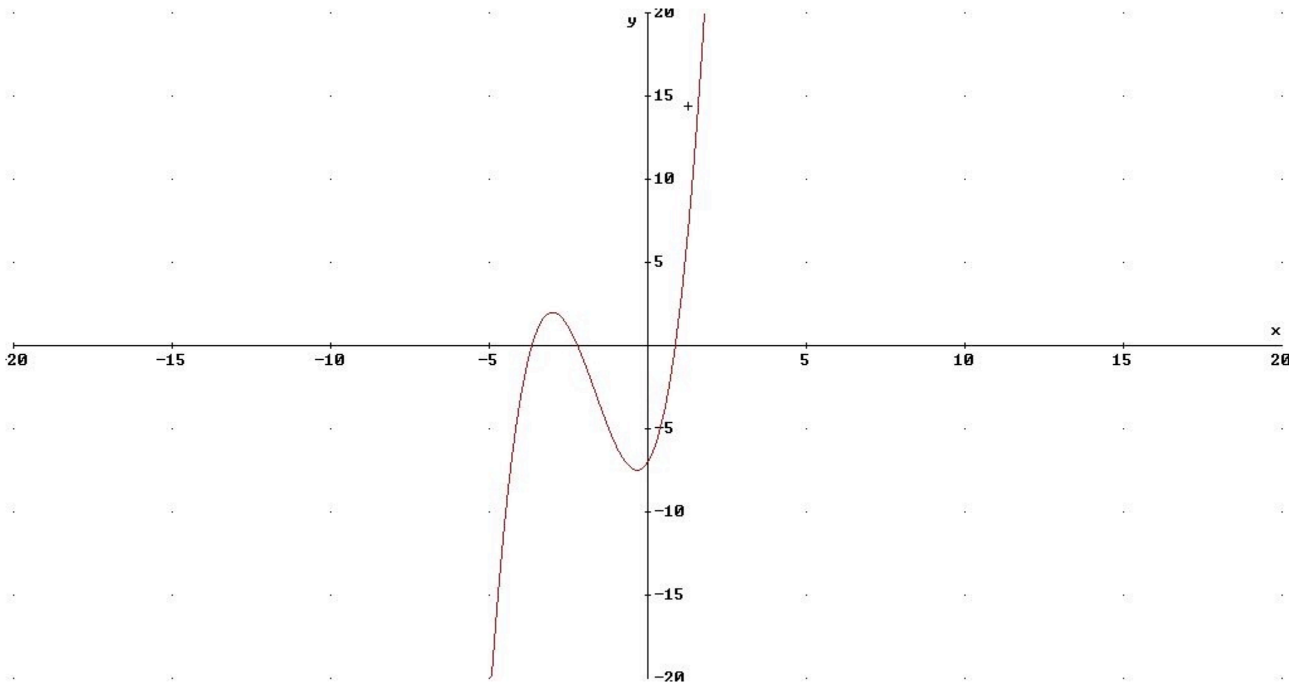
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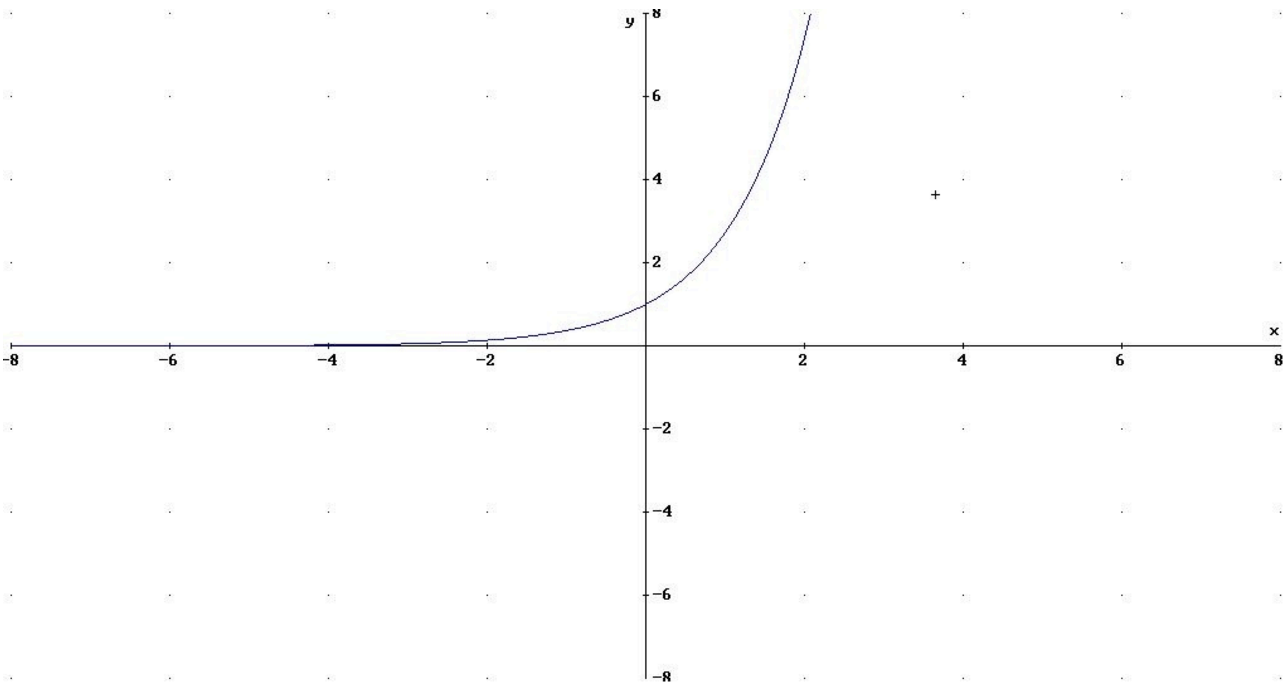
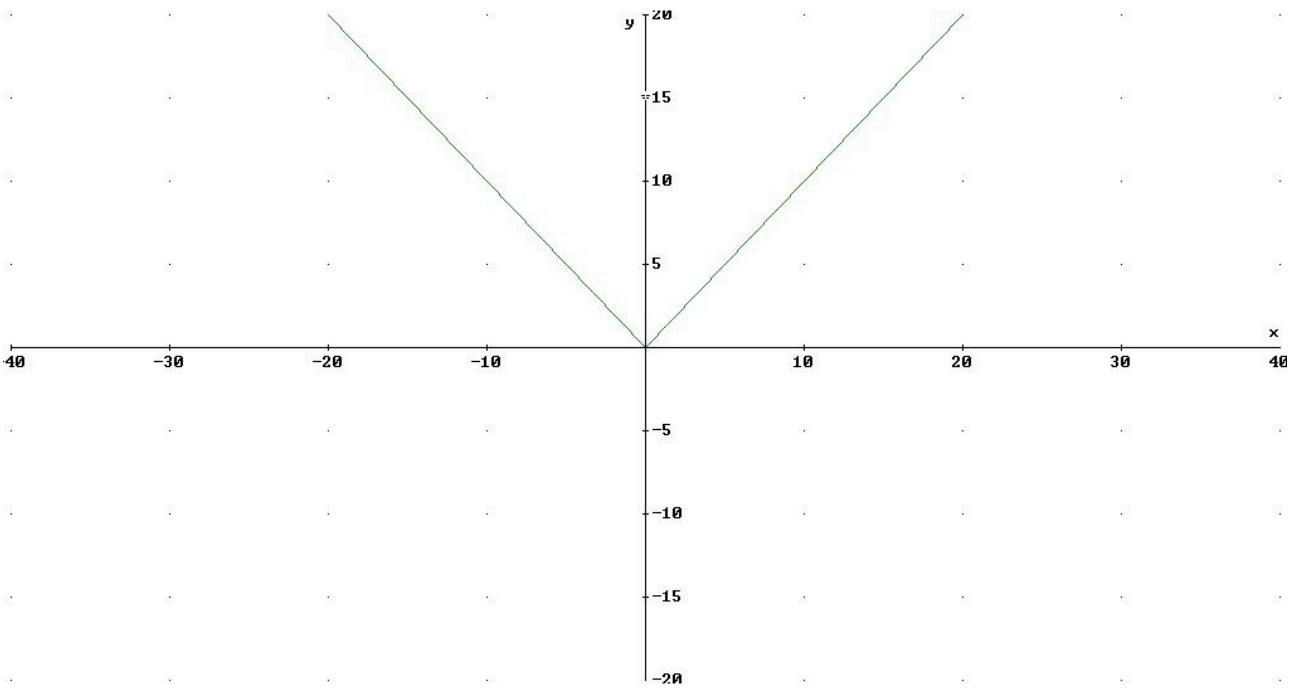


How did you determine the tangent ?
The tangent is unique in A?
Justify your answers.

1.3 Trace the tangent to the following curve in point (0,0).



How did you determine the tangent ?
The tangent is unique in A?
Justify your answers.

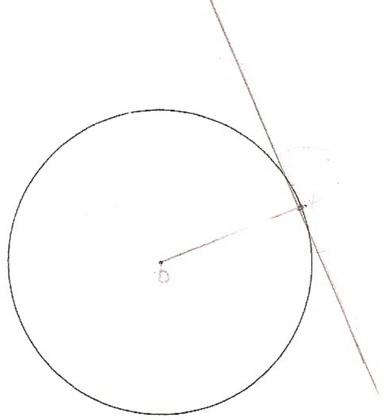
Qualitative analysis

We now analyse the protocol of five students that in different ways highlight their difficulty in dealing with the concept of the tangent as we change curve through a semiotic treatment in the Cartesian register. The experiment was designed to analyse the students' behaviour when dealing with the singular point of the function y=|x|. Most of the students recognize that (0,0) is a singular point and that the function has no first derivative in such point. Anyway, they claim that the graph of the function has two tangents in the origin, the two half lines starting from point (0,0) because "the tangent to a straight line is the straight line itself". Below are the protocols of theese students.

<u>The protocol of Francesca</u>

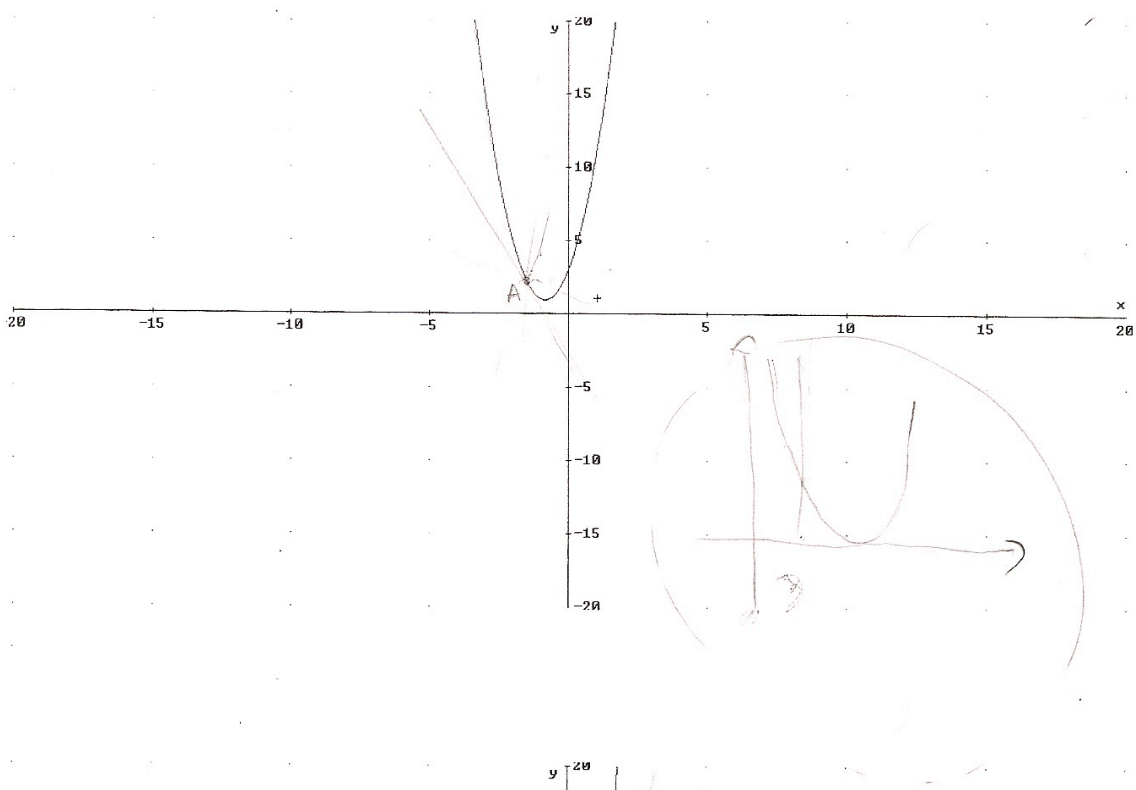
Domanda 1

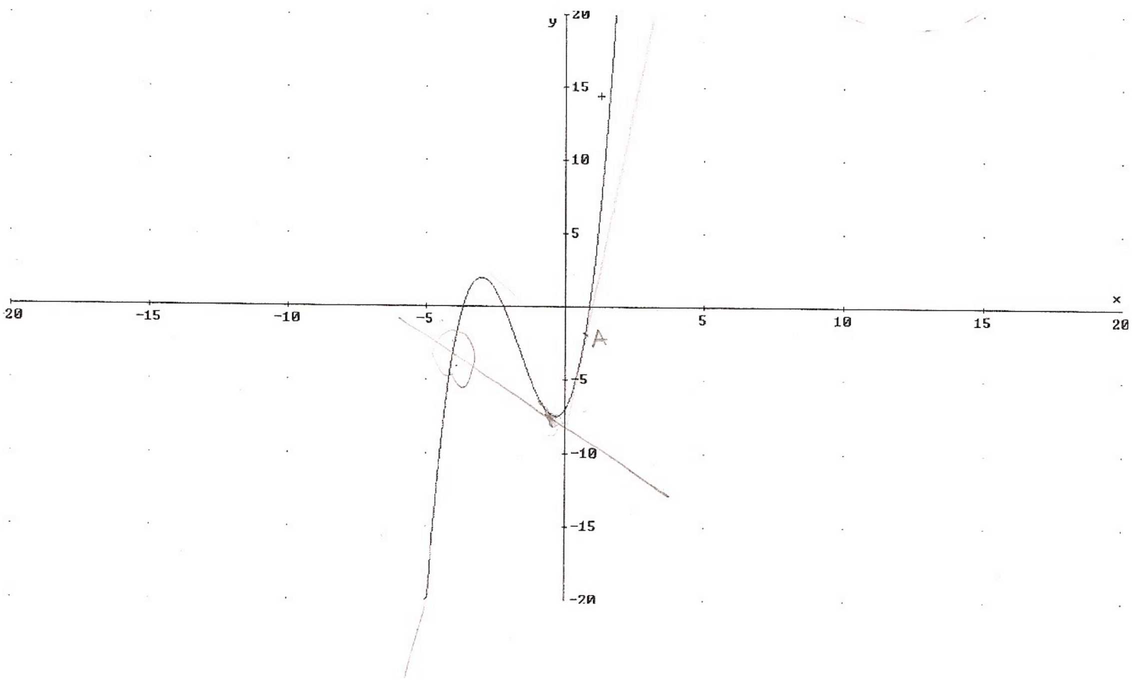
1.1 Data la circonferenza disegnata di seguito, traccia la retta tangente ad un suo punto A.

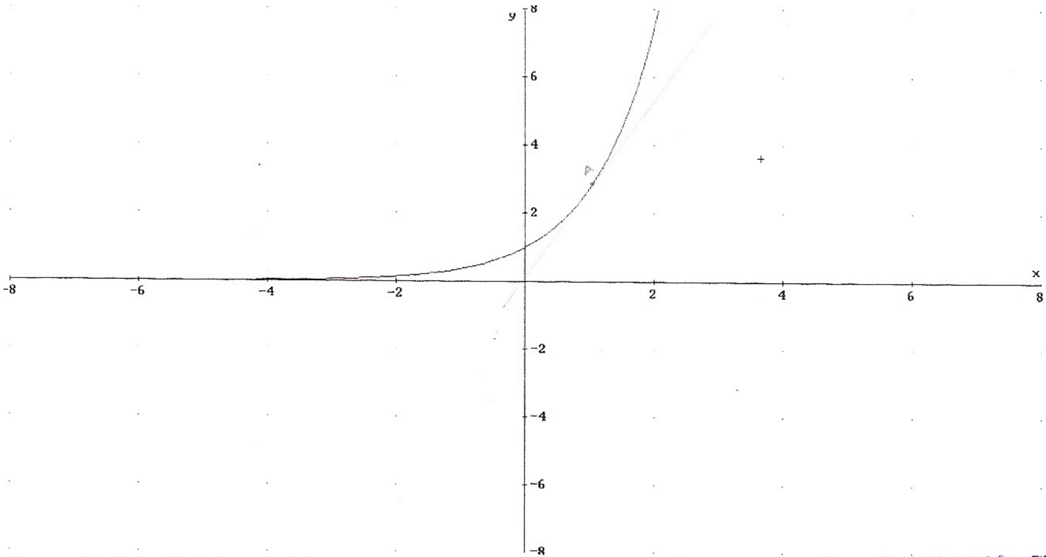


Come hai individuato la tangente? Ho disegnato il raggio. la tangente e' perpendicolare al raggio
La retta tangente è unica nel punto A? Sì; E' unica perchè per due punti passa una sola retta. E perpendicolare ad OA esiste una sola retta che passi per il punto A.
Motiva le tue risposte.

1.2 Considera le curve disegnate di seguito, traccia la retta tangente alle curve in un loro punto A.



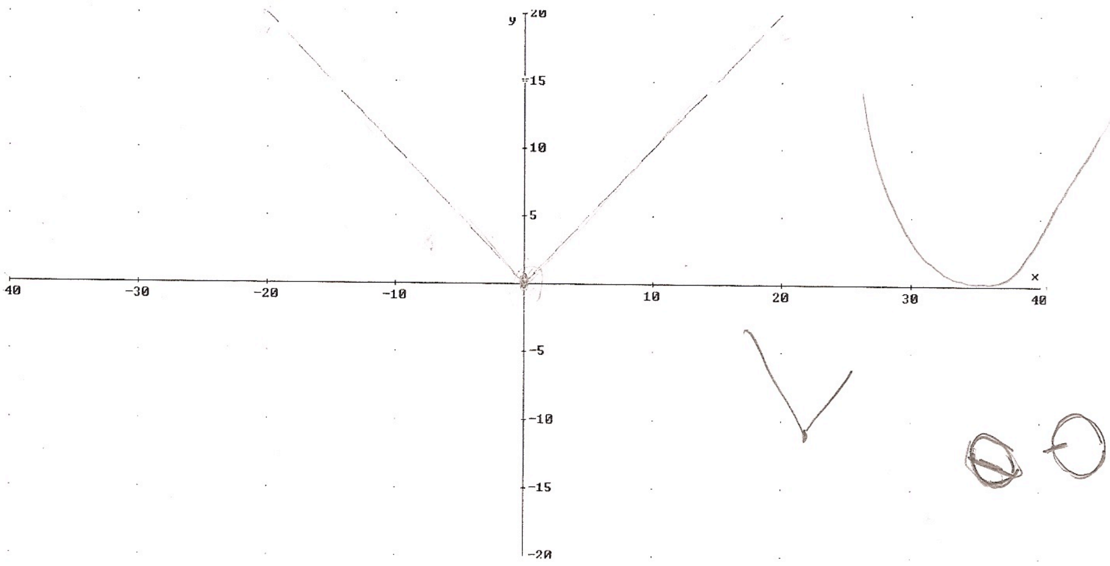




Come hai individuato la tangente?
La tangente è unica nel punto A.
Motiva le tue risposte.

la tangente può toccare la funzione solo in un punto, altrimenti non sarebbe più tangente, ma una retta secante. Quindi fissato un punto A sulla funzione ho cercato, attraverso l'uso del righellino, l'unica retta passante per A non secante.

1.3 Considera la curva disegnata di seguito, traccia la retta tangente alla curva nel punto di coordinate (0,0).



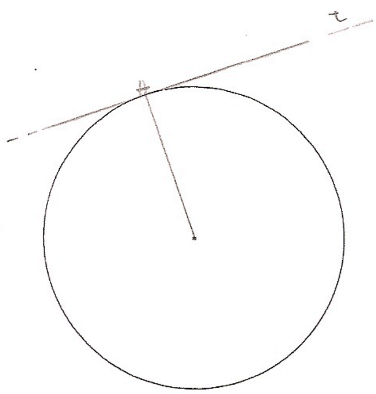
Come hai individuato la tangente ?
La tangente è unica nell'origine?
Motiva le tue risposte.

le tangenti alla funzione nel punto (0,0) sono le rette stesse.
In questo caso quindi le tangenti all'origine sono >1.

The protocol of Chiara

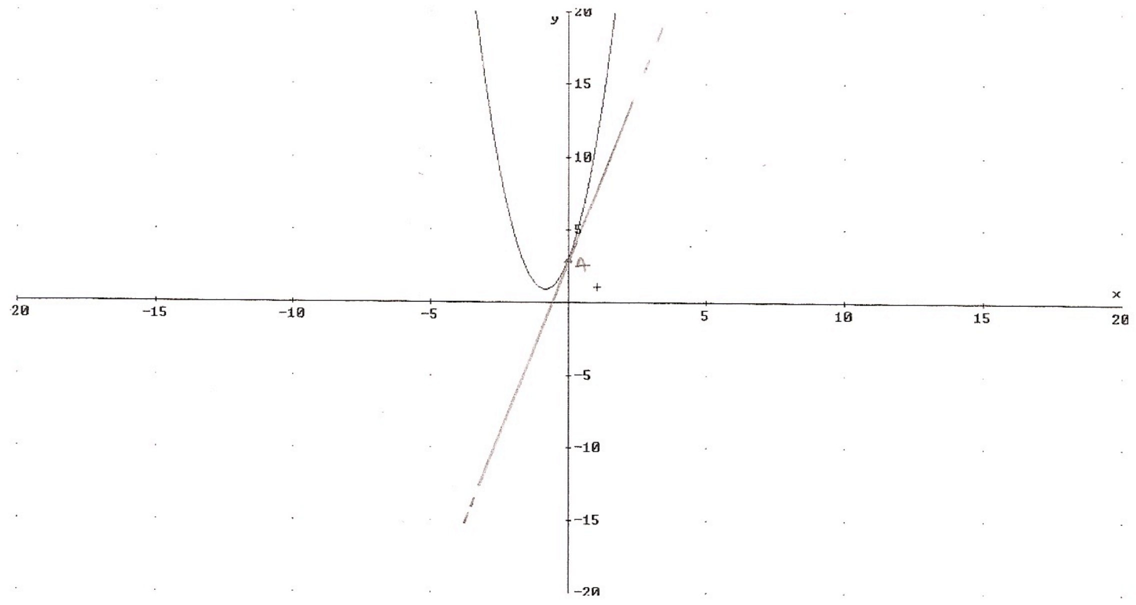
Domanda 1

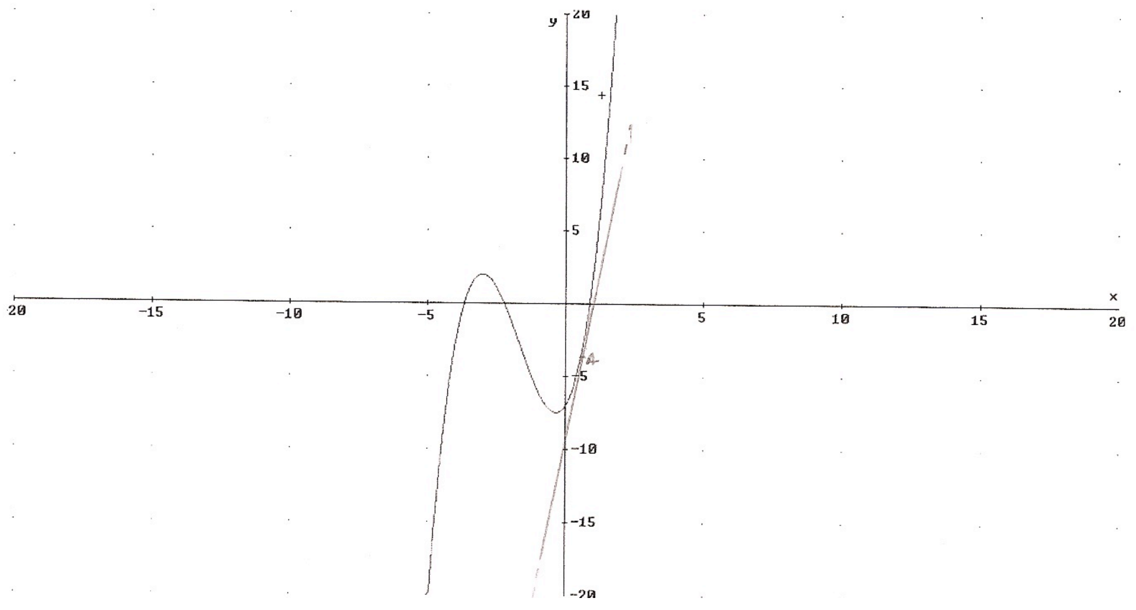
1.1 Data la circonferenza disegnata di seguito, traccia la retta tangente ad un suo punto A.

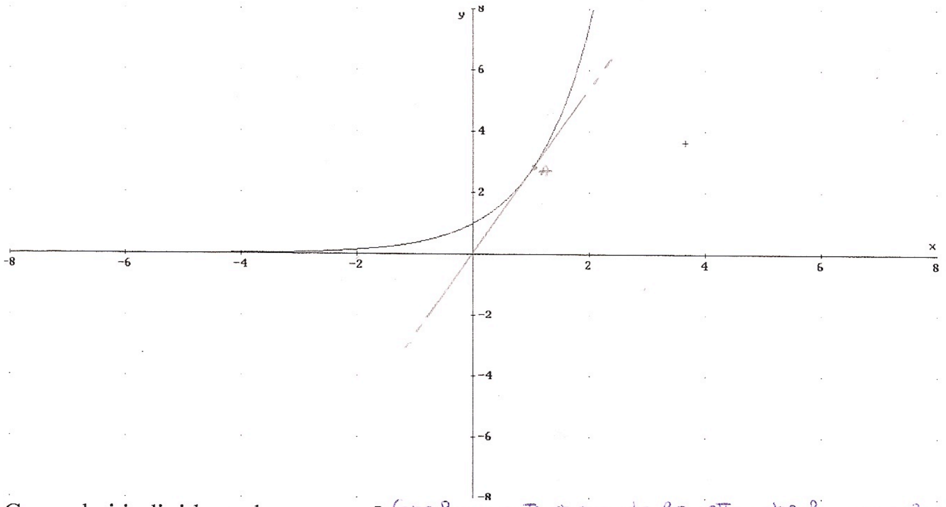


Come hai individuato la tangente? è la perpendicolare al raggio che unisce il centro al punto A
La retta tangente è unica nel punto A? sì
Motiva le tue risposte.

1.2 Considera le curve disegnate di seguito, traccia la retta tangente alle curve in un loro punto A.



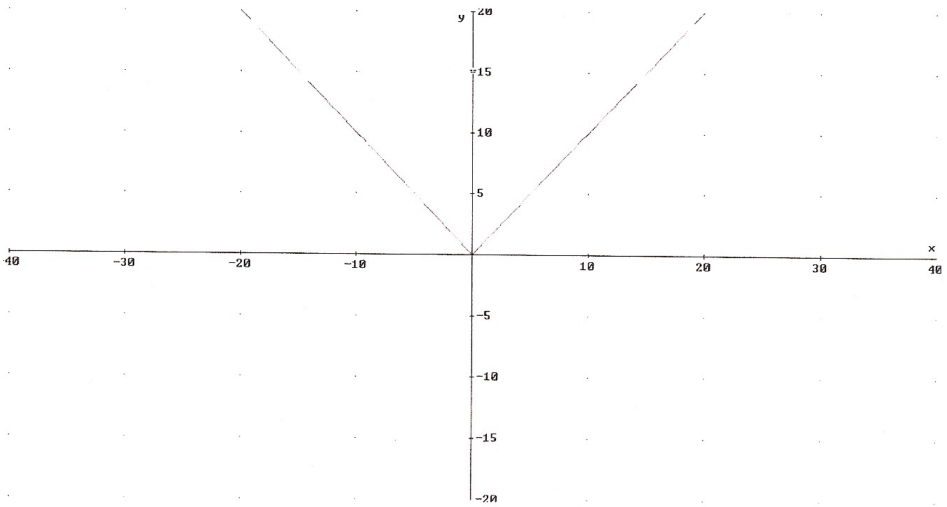




Come hai individuato la tangente? Graficamente: trovando la retta che ha un solo punto in comune con la curva
La tangente è unica nel punto A. Matematicamente: si può porre a sistema l'equazione della curva e della retta (fissato in A) e imporre il $\Delta = 0$. Oppure calcolarla con le regole della $f^I(x)$
Motiva le tue risposte.

La Tg nel punto A è unica perché le curve non presentano punti angolosi

1.3 Considera la curva disegnata di seguito, traccia la retta tangente alla curva nel punto di coordinate (0,0).



Come hai individuato la tangente ? Nell'origine la tangente è doppia e coincide con le due semirette disegnate poiché il grafico individua una $y = |f(x)|$ e crea quindi un punto angoloso.
La tangente è unica nell'origine? Per calcolare la tangente occorre calcolare
Motiva le tue risposte.

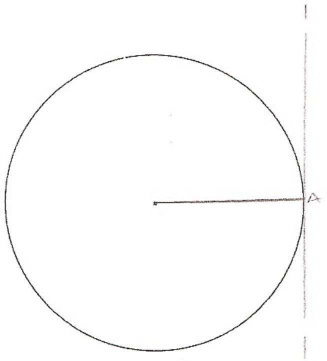
- $f^I_+(x)$ con $f(x) \geq 0$
- $f^I_-(x)$ con $f(x) < 0$

Infatti la tangente a una retta è la retta stessa e il grafico individua una spezzata.

The protocol of Caterina

Domanda 1

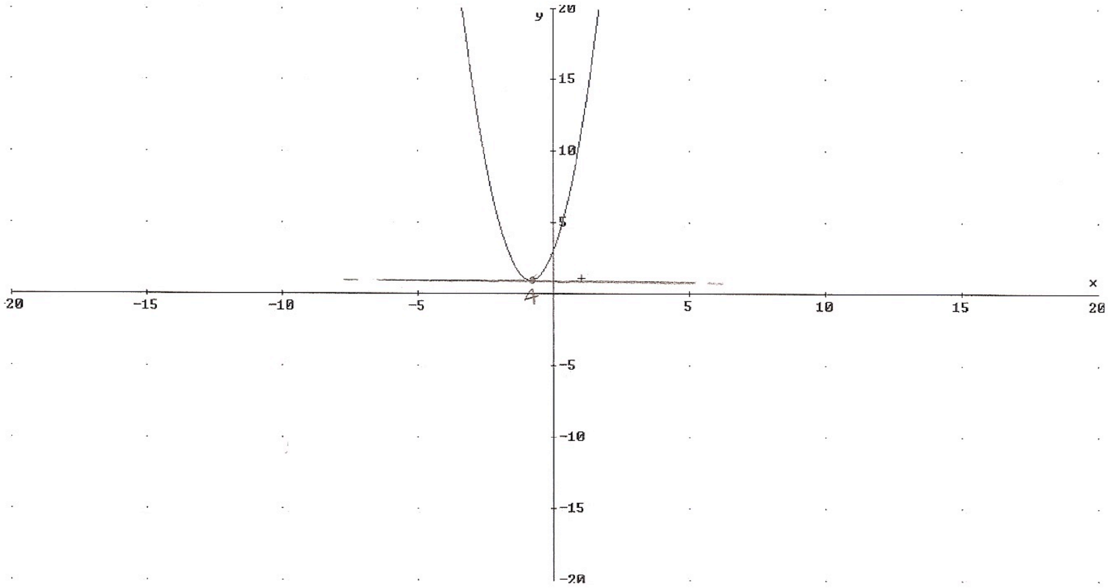
1.1 Data la circonferenza disegnata di seguito, traccia la retta tangente ad un suo punto A.

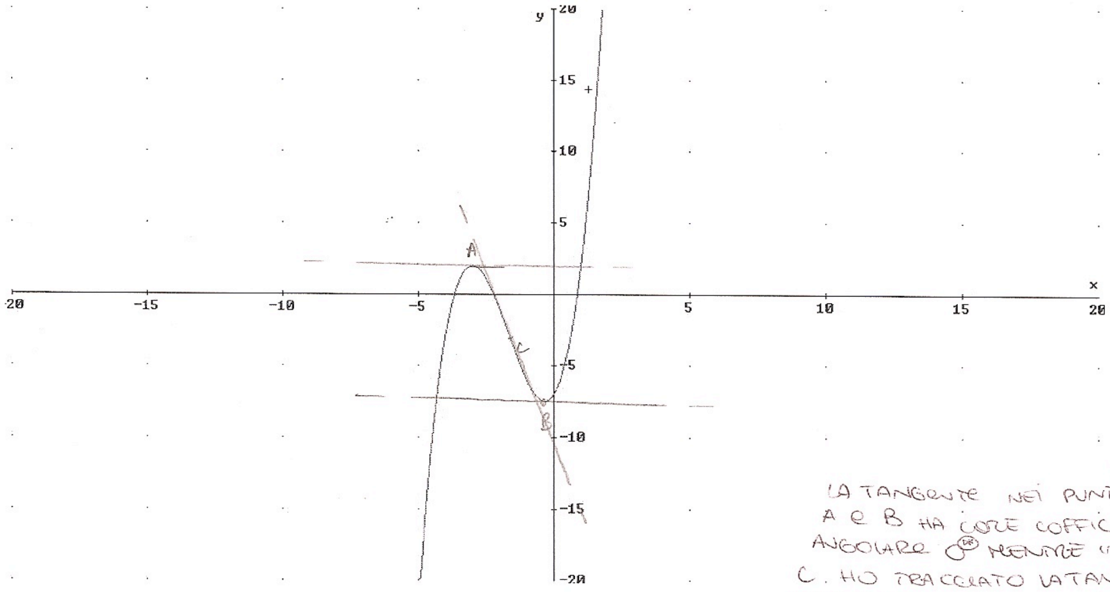


Come hai individuato la tangente?
La retta tangente è unica nel punto A?
Motiva le tue risposte.

PER CREARE LA TANGENTE HO ~~MANDATO~~ TRACCIATO DAL UN RAGGIO E HO TRACCIATO LA PERPENDICOLARE A QUESTO ~~[illegible]~~ OTTENENDO COSÌ LA TANGENTE. NEL PUNTO A ESISTE UNA SOLA TANGENTE POICHÉ TRACCIANDO ALTRE RETTE PASSANTI PER QUEL PUNTO OTTENGO DELLE SECANTI

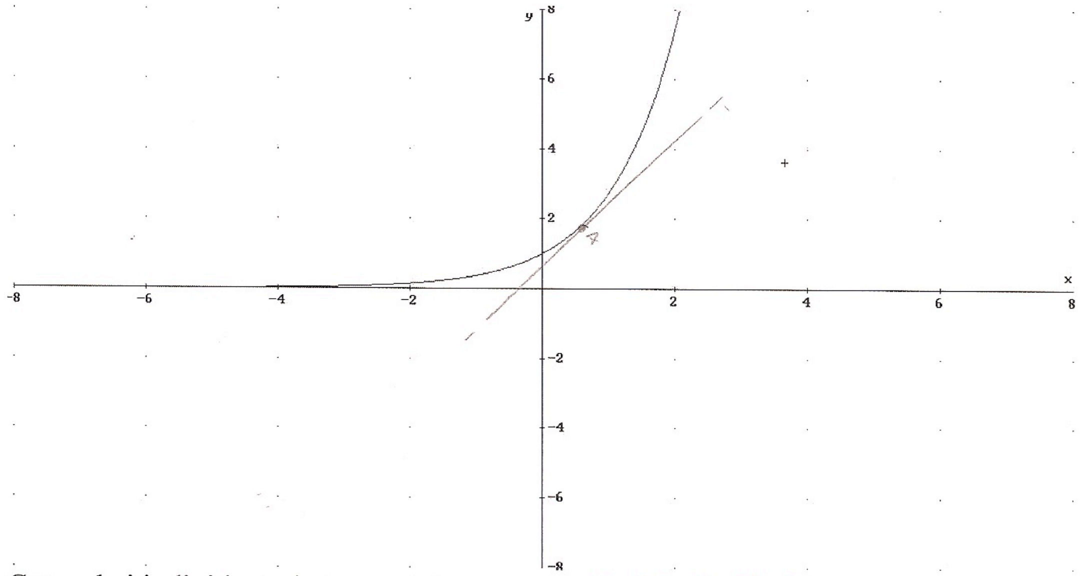
1.2 Considera le curve disegnate di seguito, traccia la retta tangente alle curve in un loro punto A.





LA TANGENTE NEI PUNTI A e B HA COME COFFICENTE ANGOLARE 0 (*) MENTRE IN C. HO TRACCIATO LA TANGENTE INFLESSIONALE.

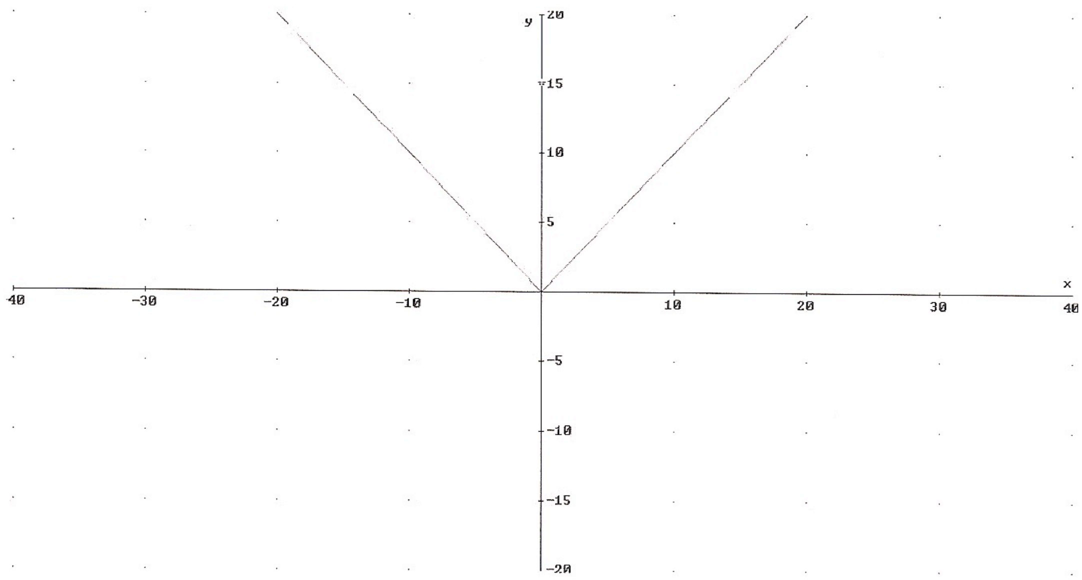
(*) POICHÈ LE RETTE TRACCIATE SONO // ALL'ASSE X



Come hai individuato la tangente?
La tangente è unica nel punto A.
Motiva le tue risposte.

NEL 1° E NEL 3° GRAFICO
HO PRESO UN PUNTO NELLA CURVA È HO
TRACCIATO LA TANGENTE CHE IN A È UNICA

1.3 Considera la curva disegnata di seguito, traccia la retta tangente alla curva nel punto di coordinate (0,0).



Come hai individuato la tangente ?
La tangente è unica nell'origine?
Motiva le tue risposte.

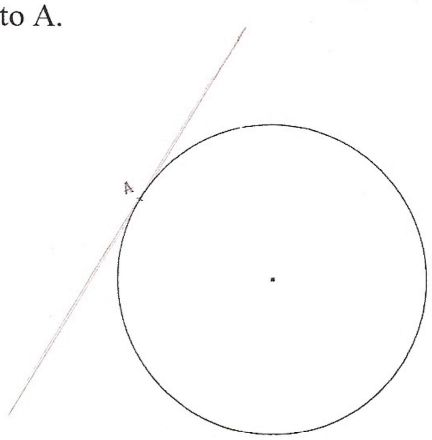
L'ORIGINE È UN PUNTO ANGOLOSO PERCIÒ
NON C'È UNA TANGENTE IN (0,0) MA OGNI
SEMIRETTA HA COME TANGENTE UNA RETTA
COINCIDENTE CON SE STESSA

The following protocol_is very interesting because the student identifies a singular point as a point having of a curve with more than one tangent.

<u>The protocol of Giada</u>

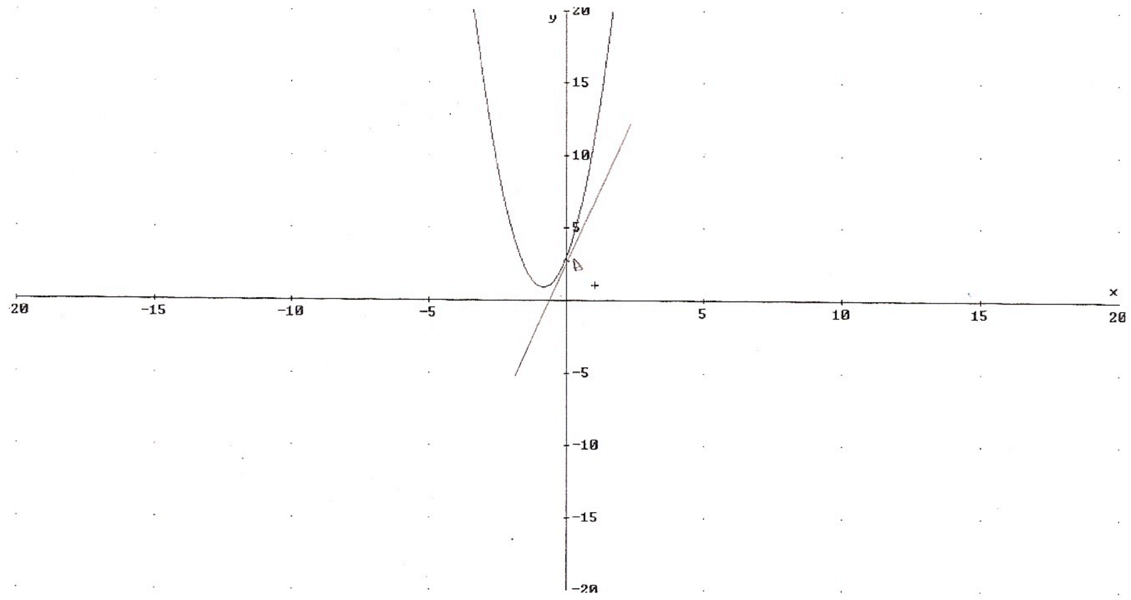
Domanda 1

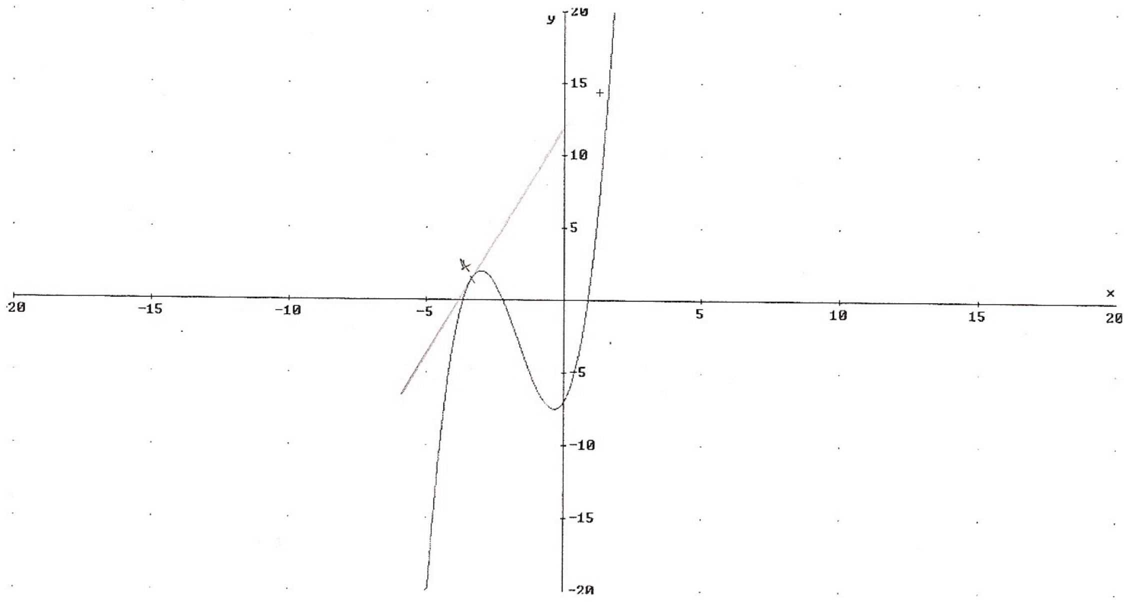
1.1 Data la circonferenza disegnata di seguito, traccia la retta tangente ad un suo punto A.

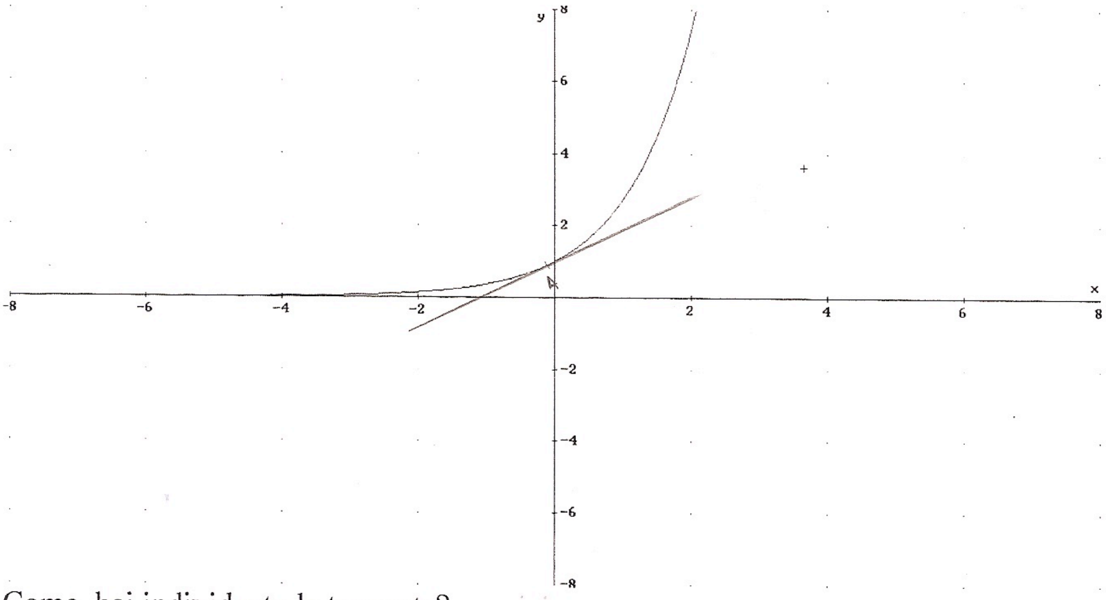


Come hai individuato la tangente? Ho ~~tracc~~ disegnato la retta che passa per il punto A della C e nessun altro
La retta tangente è unica nel punto A? Sì. Per la definizione di tangente ad ~~a~~ una curva.
Motiva le tue risposte.

1.2 Considera le curve disegnate di seguito, traccia la retta tangente alle curve in un loro punto A.

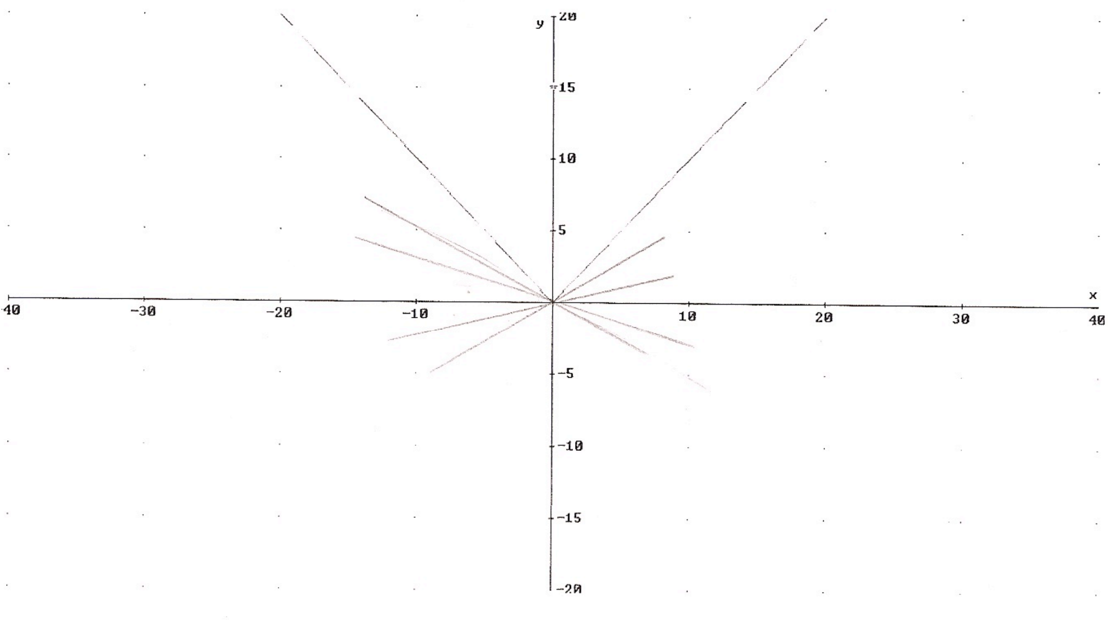






Come hai individuato la tangente?
La tangente è unica nel punto A. Sì perché
Motiva le tue risposte.

1.3 Considera la curva disegnata di seguito, traccia la retta tangente alla curva nel punto di coordinate (0,0).

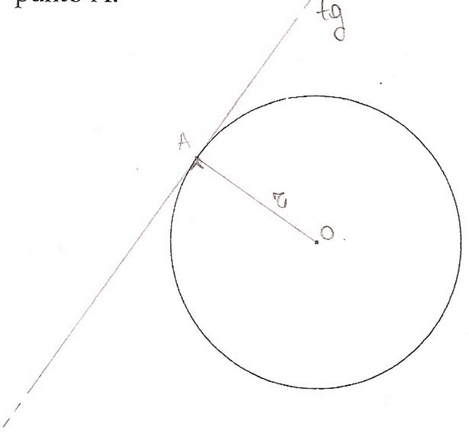


Come hai individuato la tangente ?
La tangente è unica nell'origine? No perché (0.0) è punto angoloso.
Motiva le tue risposte.

The following protocol is very interesting and the student's behaviour is effectively framed by the coordination of the structural and functional approach, the cultural semiotic approach and the ontosemiotic approach. We will analyse the answers of Laura to the questions of the test. Below, Laura's answer to question 1. The scan of the protocol is in Italian, the reader finds the questions and the student's answers translated in English.

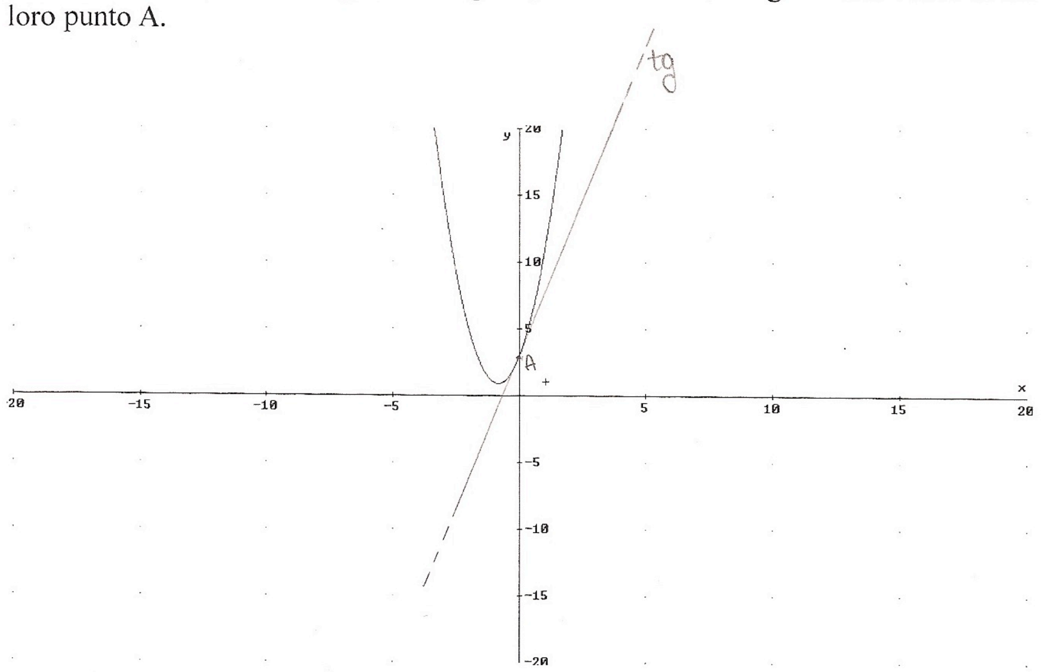
Domanda 1

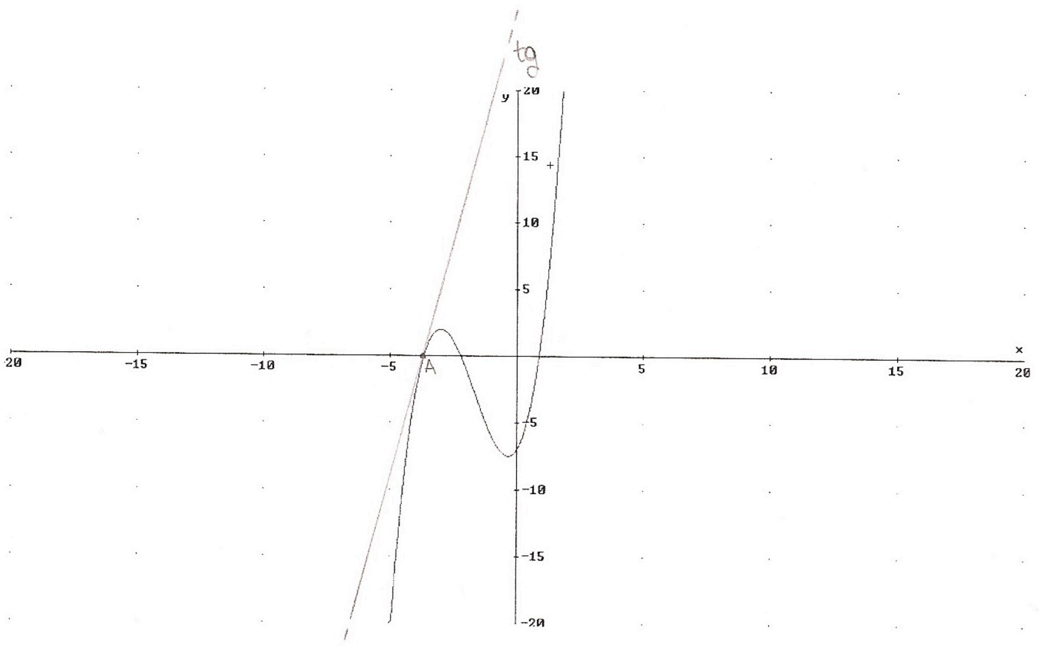
1.1 Data la circonferenza disegnata di seguito, traccia la retta tangente ad un suo punto A.

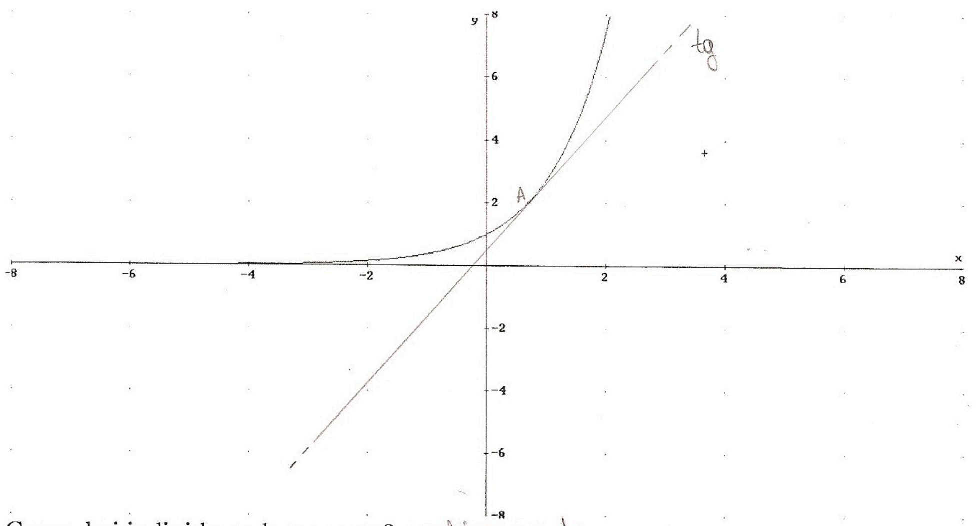


Come hai individuato la tangente? geometricamente, la tangente è la retta ⊥ al raggio nel punto A.
La retta tangente è unica nel punto A? teoricamente sì, anche se il disegno è impreciso.
Motiva le tue risposte.

1.2 Considera le curve disegnate di seguito, traccia la retta tangente alle curve in un loro punto A.



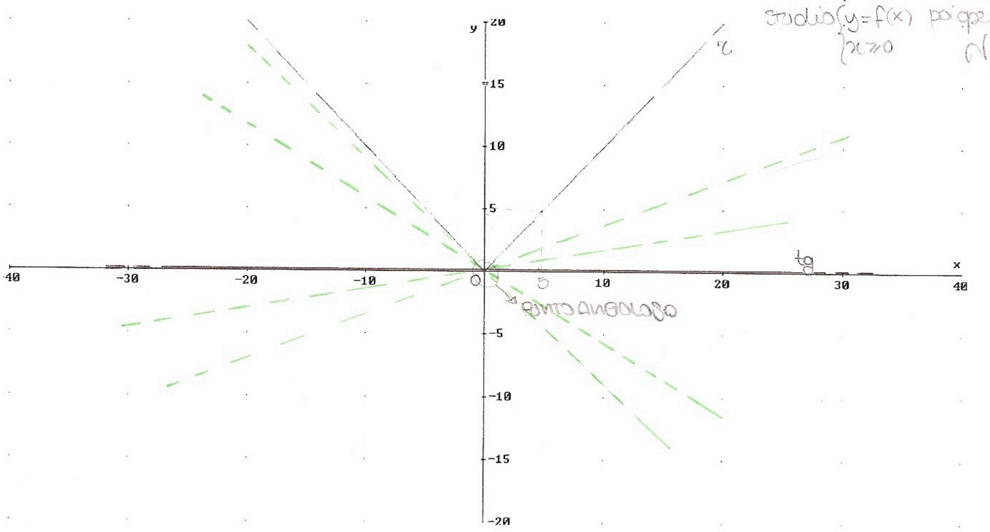




Come hai individuato la tangente? graficamente
La tangente è unica nel punto A. sì, ~~..~~ avendo l'equazione della curva e calcolan=
Motiva le tue risposte. do la $f^{I}(x_0)$ darei ottenere l'equazione di una retta (che è la retta tg).

1.3 Considera la curva disegnata di seguito, traccia la retta tangente alla curva nel punto di coordinate (0,0).

$r: y=|x|$
è una f(x) del tipo $y=f|x|$.
studio $\begin{cases} y=f(x) \\ x\geq 0 \end{cases}$ poi opero S_y.



Come hai individuato la tangente ? ~~[illegible]~~
La tangente è unica nell'origine? non lo so, graficamente mi sembra che ci possano
Motiva le tue risposte. essere più tg* nel punto O ~~...~~ (tutte le rette passanti per (0;0) fino a quella con inclinazione appena inferiore alla retta r). Tuttavia credo che la tg debba essere unica, e che sia quella che coincide con l'asse x.

Question 1

1.1 Trace the straight line tangent to the circumference through a point A.

How did you determine the tangent?

L: *From a geometric point of view the tangent is the straight line perpendicular to the ray in point A*.

The tangent is unique in point A?

L: *Theoretically yes, even if the figure is imprecise*.

1.2 Trace the straight line tangent to the following curves in a point A.

How did you determine the tangent?

L: *Graphically*.

The tangent is unique in point A?

L: *Yes, with the equation of the curve and calculating the derivative in A, I should obtain the equation of a straight line, the tangent.*

1.3 Trace the straight line tangent to the curve through the point (0,0).

How did you determine the tangent?

The tangent is unique in point A?

L: *I don't know, it looks as if there could be more than a tangent in point O (all the straight lines passing through (0,0) with a slope slightly smaller than the one to the straight line r*). *Nevertheless I believe that the tangent must be unique, the one passing through the x axis.*

Laura's protocol highlights the difficulty in handling the meaning of the tangent, a difficulty shared with the other students of the classroom whose behaviour was contradictory when facing the singular point. We will analyse this protocol from the Structural and Functional approach, the Cultural Semiotic approach and the Onto-semiotic one.

Structural and functional analysis

The students in general and Laura in particular are conversant with treatment and conversion semiotic transformations. As regards the function y=|x| almost all the students spontaneously carry out a conversion from the Cartesian register to the algebraic register, and some of them also calculate the left and right first derivative of the function. The students were also exposed to several exercises that required conversions from the algebraic register to the Cartesian register and vice versa. Nevertheless, students have difficulties in grasping the manifold meaning of the tangent. Although they are able to coordinate semiotic registers, they are not able to disembody meaning to reach higher levels of generality and handle the complexity of the mathematical object through a network of semiotic functions that connects practices, primary entities and representations. Signs are confined to their representational function without a relation with the activities from which the object emerged, thereby leaving the students with a fragmented meaning.

Cultural-semiotic analysis

This extract shows Laura's endeavour in making sense of the mathematical object through the process of objectification described above, resorting to different semiotic means of objectification. From a semiotic point of view, in the graphic semiotic register there isn't a great difference between a circumference, a parabola and a cubic function linked through treatment transformations. The protocol also testifies a network of semiotic transformations that include treatments and conversions between different semiotic systems. Among such transformations, treatment is the main cause of difficulty when facing the meaning of the tangent. If we consider their graphs as semiotic means of objectification the reflexive activity they mediate is very different.

In the case of the circumference, the definition of the tangent allows a continuity between the use of semiotic means of objectification bound to the subjects' embodied experience as gestures and artefacts and the use of more abstract semiotic means of objectification as the graph and the specific language of Euclidean geometry. To the terms straight line, perpendicular and ray used in Laura's definition of the tangent, correspond perceptive and kinaesthetic acts, the use of artefacts as the ruler that combined also with the graph reinforce its meaning. Meaning is embodied and the concept of tangent to a circumference is strongly bound to the visual perception of the point of contact between the straight line and the graph. In this context, the concept of tangent is objectified by the student at a level of generality that Radford (2004) terms as contextual generalization, when the use of symbols are bound to the space and temporal experience of the student.

When we shift to the parabola or the cubic curve, Laura experiences a disembodiment of meaning; the reflexive activity is mediated mainly by symbolic means of objectification. The definition of a tangent to a parabola requires to introduce a linear system of the equation of the curve and the equation of the straight line or, at a higher level of generality, the calculation of the derivative. The reflexive activity is completely different from the one involved in the circumference: it doesn't make sense tracing the perpendicular to the ray. The concept has moved to a higher layer of generality. Laura experiences a cognitive rupture that obliges her to go beyond her spatial-temporal experience in order to access a more general meaning of the concept. Radford (2004) terms symbolic generalization, sense-giving activities in which the use of formal and abstract symbols require to go beyond the spatial and temporal situated personal experience. The protocol testifies Laura's endeavour to achieve higher levels of generality of the concept of tangent. She is facing difficulties in coordinating the meanings emergent from the different activities she experienced during her educational path and there is a strong resistance in moving beyond perceptual embodied meanings.

In question 1.2 she resorts to perceptive aspects to determine the tangent to the parabola. The ruler she uses to draw the straight line is the key semiotic mean of objectification that mediates her perceptive activity although to justify the unicity of the tangent she uses the derivative. There is no explanation of how she obtained the tangent to the parabola.

Notice the strength of the perceptual dimension in the objectification process that "won" on the teacher's instructional action aiming at the general mathematical concept.

The perceptive idea that the tangent is the straight line that "touches" the curve in one point resisted throughout the sequence of the 3 questions proposed to Laura. Answering to question 1.3, Laura declares that the tangents to the curve through the origin are «all the straight lines passing by (0,0) with a slope slightly smaller than the one to the straight line r». Her answer expresses the strength of her spatial and kinaesthetic experience in giving sense to the tangent in the singular point. In the interview Laura declares that she imagined the straight line "oscillating" around the singular point without touching one of the half lines of the graph. It is interesting that the student recognizes the singular point, writes the symbolic expression of the function but she doesn't think of calculating the derivative of the function in (0,0) as she did in many exercises and problems assigned by her teacher. Learning as an objectification process is not a process of construction or reconstruction of knowledge but a path that requires a deep change within the student's consciousness.

> «Learning mathematics is not simply to learn *to do* mathematics (problem solving), but rather is leraning *to be* in mathematics» (Radford, 2008, p. 226).

This idea is expressed in terms of competences by Fandiño Pinilla (D'Amore, Godino, Arrigo, Fandiño Pinilla, 2003, p. 70) who distinguishes "competence *in* mathematics" and "mathematical competence":

> «*Competence in mathematics* is centred in mathematical discipline, recognized as an established discipline, as a specific object of knowledge. […] We recognize a conceptual and affective domain as a mediator between the pupil and mathematics. Competence is seen here within the school sphere. […] We recognize *mathematical competence* when the individual sees, interprets and behaves in the world in a mathematical sense. Then analytical or synthetic attitude with which some individuals face problematic situations, is an example of this competence. Taste and valorisation of mathematics are some of the useful aspects to orient the fulfilment of mathematical competence»

This example highlights how meaning cannot be bound to the structure of the semiotic systems that reflect the structure of an ideal mathematical reality. It is necessary to focus on the mediated reflexive activity and analyze signs not only as representations but as *mediators of shared practices*. This example shows how the semiotic key element is not conversion as claimed by the structural approach and that students can encounter learning difficulties also with treatments. The key element is the underlying system of mediated reflexive activities. In a different situation that involves conversions, we could make the same kind of analysis.

Onto-semiotic analysis

We widen our perspective to analyze Laura's protocol in terms of semiotic functions. In this example an analysis based on the model many representations for one object is insufficient to understand the network of meanings that Laura has to handle to objectify the concept of tangent.

Laura is facing 3 different "linguistic games" that are behind the system of practices and configurations of objects she has to handle:

- The linguistic game of Euclidean geometry with its set of rules that allows specific activities associated with configurations of objects. In this context the concept of tangent (as a primary entity) is defined as the straight line perpendicular to the ray in a point of the circumference.
- The linguistic game of analytic geometry with its set of rules that allows specific activities associated with configurations of objects. In this context the concept of tangent to a conic (as a primary entity) is defined as the straight line whose equation in a system with the equation of the curve gives a single solution to the system.
- The linguistic game of mathematical analysis with its set of rules that allows specific activities associated with configurations of objects. In this context the concept of tangent (as a primary object) is defined as the straight line passing through the tangent point of the graph of the function whose slope is the derivative of the function in the tangent point.

To learn the concept of tangent in its broad cultural meaning, the student has to handle a network of semiotic functions that involve the pairs system of practice-configuration of objects mentioned above.

In this example, we can interpret the layers of generality of the mathematical object as the semiotic functions that connect the primary entity concept of tangent according the cognitive duality extensive-intensive. The intensive facet refers to a class of objects considered as a whole and the intensive facet refers to a particular element of the class. The student has to establish the following semiotic functions:

- A semiotic function SF1 with antecedent the concept of tangent in Euclidean geometry and consequent the concept of tangent in analytic geometry. In the cognitive duality extensive-intensive, the tangent is interpreted as an extensive object in Euclidean geometry and intensive object in analytic geometry.

- A semiotic function SF2 with antecedent the concept of tangent in analytic geometry and consequent the concept of tangent in mathematical analysis. In the cognitive duality extensive-intensive the tangent is interpreted as an extensive object in analytic geometry and as an intensive object in analysis.
- A semiotic function SF3 with antecedent the concept of tangent in Euclidean geometry and consequent the concept of tangent in mathematical analysis. In the cognitive duality intensive-extensive the tangent is interpreted as an extensive object in Euclidean geometry and as an intensive object in analysis.

Laura's difficulties in giving sense to the concept of tangent can be brought back to the absence of this network of semiotic functions. Laura is conversant with each of the above language games separately, she encounters difficulties when she has to establish semiotic functions between different pairs of systems of practices and configurations of objects. Laura "plays" the linguistic game of analytic geometry and analysis with the rules of Euclidean geometry and she inexorably falls in contradictions that the Onto-semiotic approach terms as semiotic conflicts (D'Amore, Godino, 2006).

Also within the Onto-semiotic approach the semiotic transformations as such don't play an essential role in objectifying the mathematical object. It is important recognizing the semiotic functions that are established when conversions or treatments are used in mathematical activity; from outside we recognize semiotic transformations but if we look at the process of learning from inside, the systems of practices students are involved in and the network of semiotic functions they are able to establish are the core of the sense giving activity.

We believe that, from an educational point of view, it is extremely important to understand how the students recognize the criteria that allow to relate the antecedent and the consequent in a semiotic function. The notion of objectification and semiotic means of objectification provide effective tools to understand the nature of the mathematical activity, to understand how signs mediate activity and recognize the cognitive ruptures students have to face in their learning process, for example when they have to disembody meaning.

6.3 Experiment 2: Sequences

Environment and constraints

The experiment was conducted in a primary school of Bologna. The teacher is trained in Mathematics Educations and students are acquainted with a-didactic situations and cooperative learning environments. The class was very collaborative and enthusiastic with the new activity we proposed. The classroom is characterised by high levels of communication. Students are committed to dialogue and communication and are used to express their ideas clearly and in thorough. They are willing to listen to ideas and proposals of their classmates and usually are respectful of the needs of their peers. The members of the class are able present their ideas in a clear logic and effective form using natural and symbolic language typical of their school level. The classroom is trained to work in Radford's territory of artefacts resorting to a variety of semiotic means of objectification: natural language, mathematical symbols, objects, tools…

There were no constraints in terms of the instructional design of the activity and the mathematical topic the students would have worked on. The only constraint was time. The experiment is incomplete and further investigations are needed to draw reliable conclusions.

Experimental design

The class divided in 6 working groups of 3 or 4 members that have been videotaped during the experiment. The groups were balanced in order to equally weigh more competent and weaker students. The experiment developed in three sessions organized as follows:

1) Introduction of the activity and frontal lesson on the part of the teacher.
2) Students work in small groups with the assistance of the teacher.
3) General discussion on the activity.

Students worked on sequences and the research was inspired by Radford's (2000, 2002, 2003, 2005) generalization of sequences. The difference with respect to Radford's researches is that we exposed students to several figural representations of the same sequence to test students behaviour towards meaning when facing different figural representations of the same object. Our expectation was that students wouldn't recognize the same sequence when the figural representation of the range of the sequence changed.

Below the sequences with their figural representations. Students were asked to find the number of elements in the fifth, sixth and two big number figures. They were then asked to give a general rule to construct the sequence. Below the sequences with the figural representations.

1) The sequence of odd numbers $a_n=2n+1$

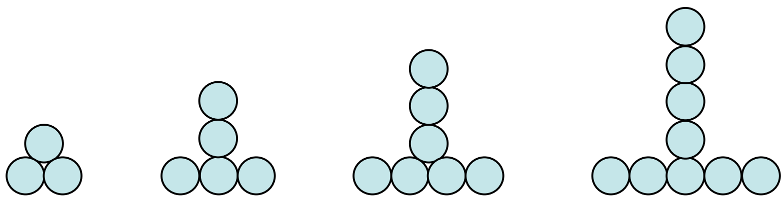


Fig. 1 Fig. 2 Fig. 3 Fig. 4

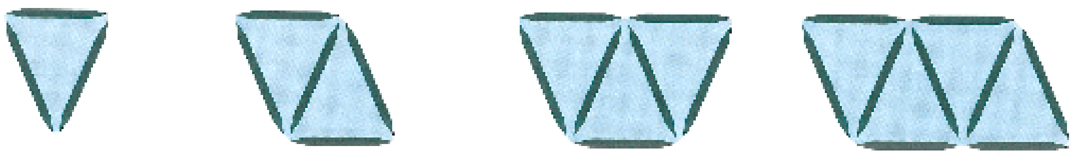


Fig. 1 Fig. 2 Fig. 3 Fig. 4

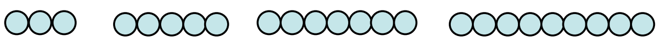


Fig. 1 Fig.2 Fig. 3 Fig. 4

2) The sequence $a_n=n^2+2n$

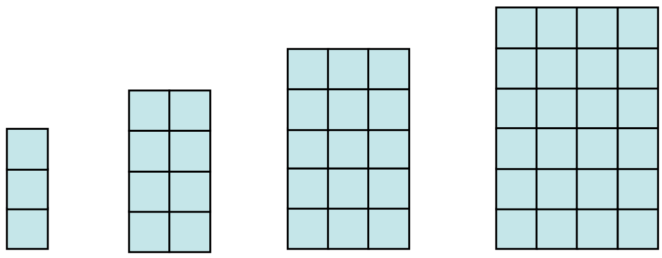


Fig. 1 Fig.2 Fig. 3 Fig. 4

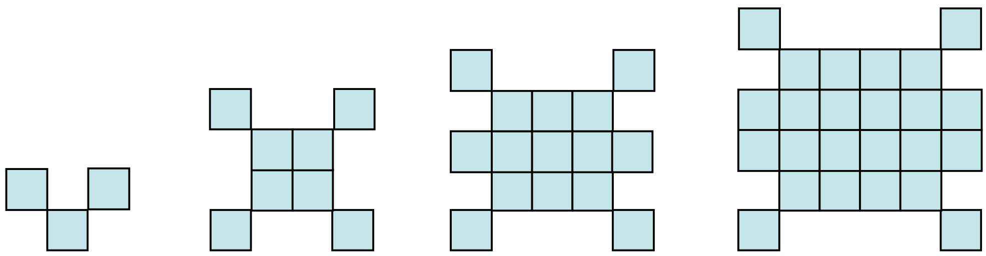


Fig. 1 Fig.2 Fig. 3 Fig. 4

Fig. 1 Fig.2 Fig. 3

Fig. 4

The sequence of even numbers $a_n=2n$

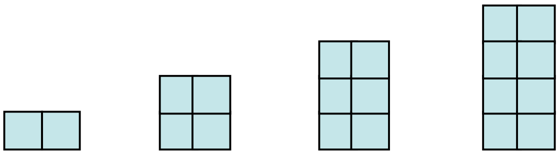


Fig. 1 Fig. 2 Fig.3 Fig. 4

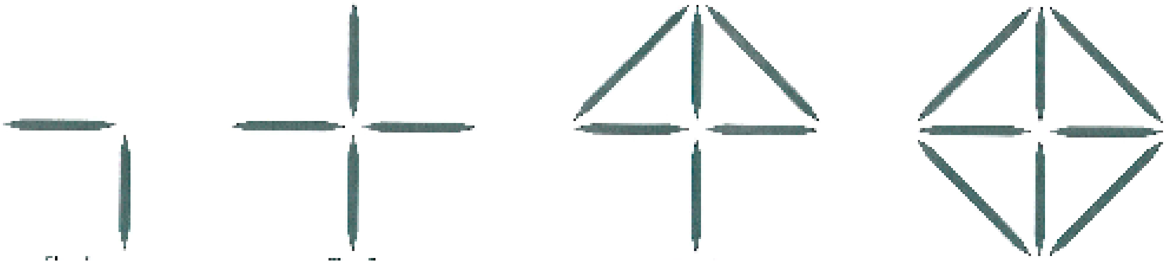


Fig. 1 Fig. 2 Fig. 3 Fig. 4

6) The sequence of $a_n = 1+2+\ldots+n$

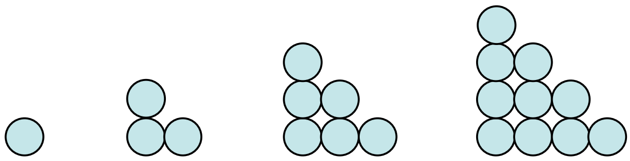


Fig. 1 Fig.2 Fig. 3 Fig. 4

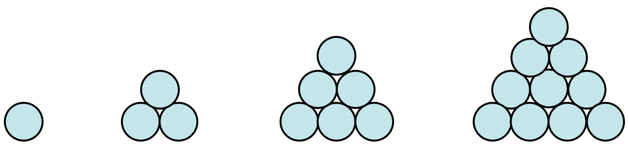


Fig. 1 Fig.2 Fig. 3 Fig. 4

Qualitative analysis

Almost all the groups were able to calculate the number of elements in the sequence and they were able to give the procedure to express the general term of the sequence.

Below part of the protocol of group 5 that is particular interesting:

Osserva le figure:

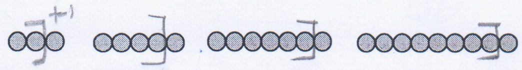
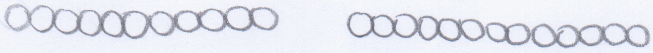


Fig. 1 Fig.2 Fig. 3 Fig. 4

Riproduci la sequenza utilizzando i "ceci"

1) Sai rappresentare la figura numero 5? E la figura numero 6?
   Disegnale e spiega come arrivi alla soluzione

5 6



Abbiamo moltiplicato il numero della figura x 2, poi, gli abbiamo aggiunto un "cecio"

2) Quanti ceci ci sono nella figura numero 191? Spiega come arrivi alla soluzione.

191 x 2 = 382 + 1 = 383

Abbiamo ~~x~~ moltiplicato il numero della figura x 2, poi abbiamo aggiunto un "cecio".

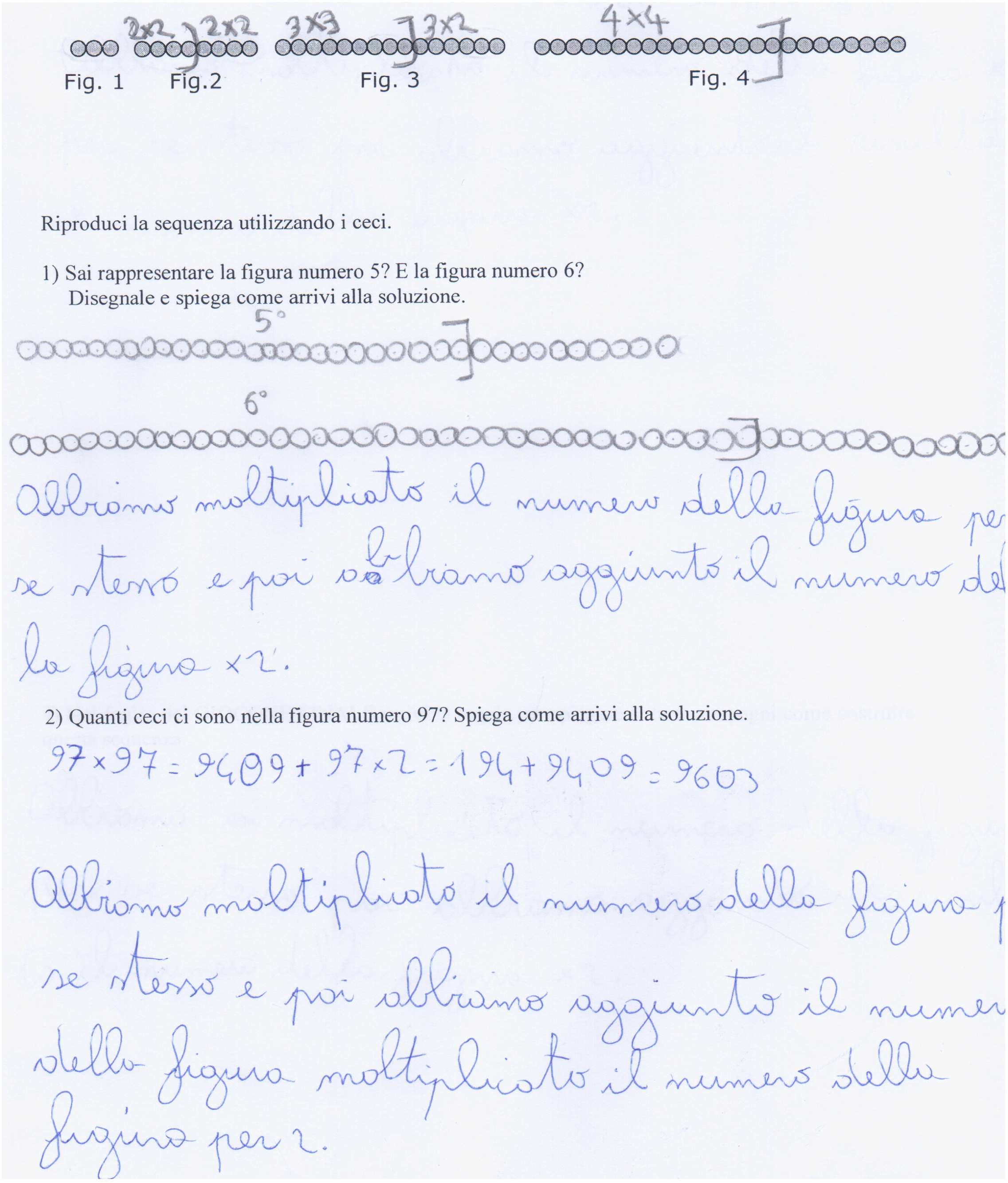


Fig. 1 2x2 2x2 Fig.2 3x3 3x2 Fig. 3 4x4 Fig. 4

Riproduci la sequenza utilizzando i ceci.

1) Sai rappresentare la figura numero 5? E la figura numero 6?
Disegnale e spiega come arrivi alla soluzione.

5°

6°

Abbiamo moltiplicato il numero della figura per se stesso e poi abbiamo aggiunto il numero della figura x2.

2) Quanti ceci ci sono nella figura numero 97? Spiega come arrivi alla soluzione.

97×97=9409+97×2=194+9409=9603

Abbiamo moltiplicato il numero della figura per se stesso e poi abbiamo aggiunto il numero della figura moltiplicato il numero della figura per 2.

In the next pages the rules for the general term of the sequence.

FOGLIO DEL GIOCONE FINALE

GRUPPO N° 5

Gioco ~~Fig.~~ n. 1°: abbiamo moltiplicato il numero della figura x 2. in modo da aggiungere sempre due quadratini alla figura precedente. La base è sempre di ~~due~~ 2 e l'altezza è sempre il numero della figura

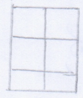


Fig. n. 3 x 2 = 6 q

Gioco n. 2°: abbiamo sommato il numero della figura + 1 per la base e il numero della figura per l'altezza.

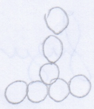


Fig. 3. + 1 alla base e numero della figura all'altezza

7 q. c.

Gioco n. 4°: abbiamo moltiplicato il numero della figura per due aggiungendo uno stuzzicadente per ottenere il numero totale degli stuzzicadenti.

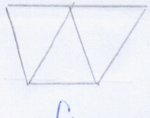
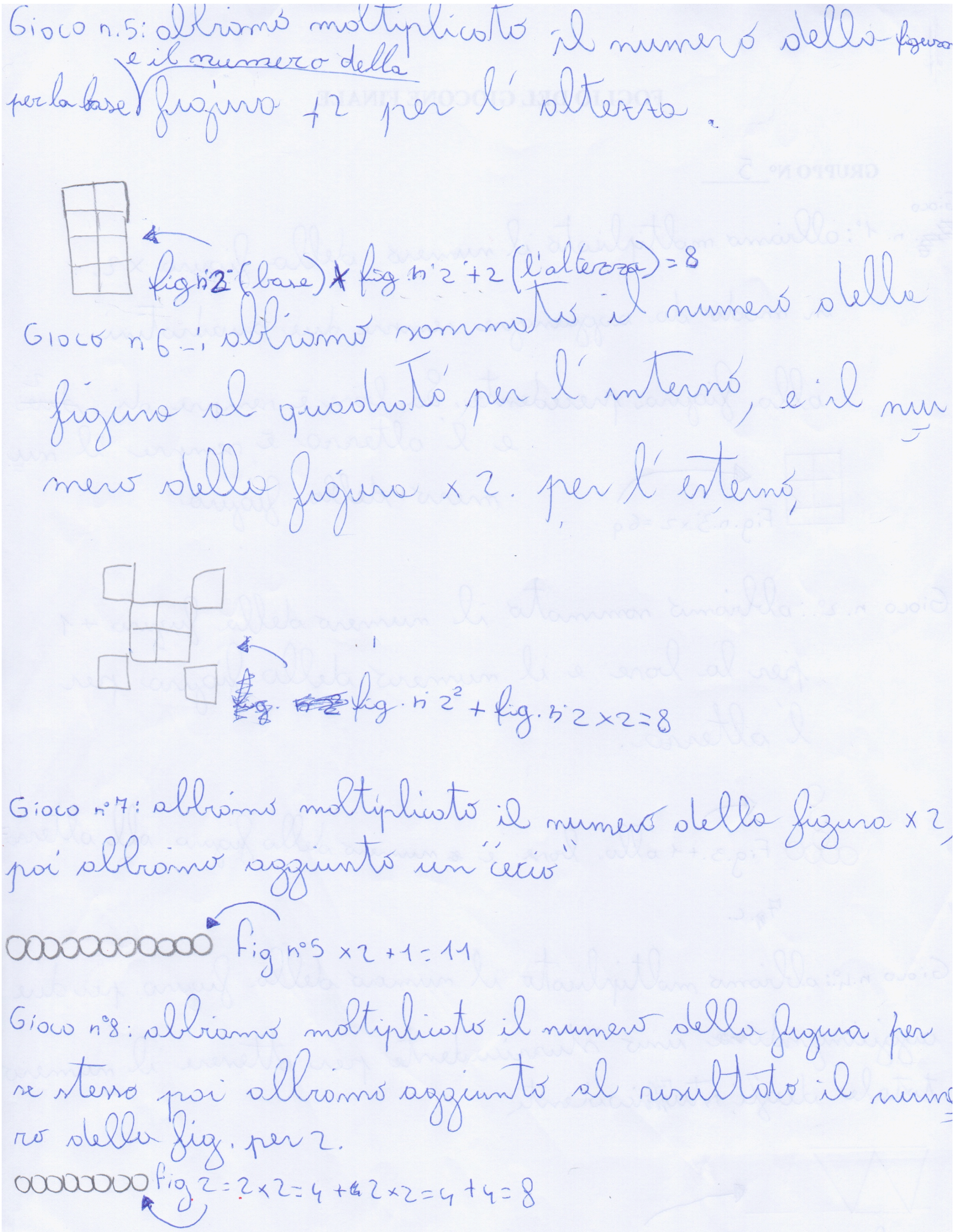


Fig. n. 3 x 2 = 6 + 1 = 7



Gioco n.5: abbiamo moltiplicato il numero della figura per la base e il numero della figura +2 per l'altezza.

fig n°2 (base) * fig n°2 +2 (l'altezza) = 8

Gioco n 6: abbiamo sommato il numero della figura al quadrato per l'interno, e il numero della figura x 2. per l'esterno,

~~fig.~~ fig. n° 2^2 + fig. n° 2 x 2 = 8

Gioco n°7: abbiamo moltiplicato il numero della figura x 2, poi abbiamo aggiunto un cerchio

Fig n°5 x 2 + 1 = 11

Gioco n°8: abbiamo moltiplicato il numero della figura per se stesso poi abbiamo aggiunto al risultato il numero della fig. per 2.

fig 2 = 2 x 2 = 4 + 2 x 2 = 4 + 4 = 8

Students were able to give the rule for the general term of the sequences and recognise the same mathematical object when they changed representation.

From a structural and functional approach, the students carry out conversions and treatments that involve the figural register, natural language, arithmetical register. Moreover, students attempt a pre-algebraic language to represent the general term of the sequence.

From a Cultural Semiotic point of view students are able to objectify the sequence as a cultural object. The interplay and synchronic use of gestures, natural language, objects and the figural representation of the sequence that is the key element of the semiotic node that allows students to objectify meaning within their embodied experience. The change of representation didn't disturb the students because, through the new figures, they were able to broaden the reflexive activity they experienced with the previous representations. In sequences 1) and 2) we proposed a representation without any structure. Our conjecture was that without an evident structure to connect the figural representation with gestures, students wouldn't recognize any rule to construct the general term of the sequence. In fact, within their socially shared reflexive activity students accessed to higher level of generality, they found by themselves the structure of the figure, thereby objectifying the general term of the sequence. The protocol shows how the students' objectification processes, that is their intentional act is culturally and socially oriented to see a structure that is not immediately perceptually evident. The reflexive mediated activity they were involved in, lead to a first algebraic representation of the sequence, although not explicitly requested.

From an ontosemiotic perspective, the students are able to establish a complicate network of semiotic functions within, the language game of natural numbers that relate the procedures, definitions, argumentations propositions, and a problem situation. This network of semiotic functions allows students to coordinate the different systems of practices and related configuration of emerging objects into a unitary and global meaning. Within the language game of natural numbers, they establish semiotic functions within the triple (SP, CO, R). The generalization process entails the relation between definitions and propositions as antecedents and procedures as consequent. Students first recognize the procedure in the operational phase that is referred through a definition and a proposition that describes and objectifies the mathematical activity. To grasp the general term of the sequences, students resort to the interplay between the semiotic function and the extensive-intensive cognitive duality. The protocol clearly show how students use the semiotic function to relate extensive and intensive facets. In the protocol we are discussing, students refer to a specific figure (figure 5, figure 2 etc) that in the language game of sequences it is used to highlight the general characteristic that allows to construct the sequence, thereby accessing the range of the sequence as a whole.

Further investigations, require to guide students spontaneous use of an algebraic language into an aware and socially and culturally recognized use in the classroom. It would be interesting to investigate students' behaviour when facing n+ (n+1), 2n+1, n+n+1 to recognize "changes" and "losses" of meaning. The same with expressions as n^2+2n, $n(n+2)$, n^2+n+n.

6.4 Answer to the research questions

Meaning in the structural and functional approach

In the structural and functional frame our research questions were stated in terms of losses meaning as follows.

1. Why do students lose the meaning of the mathematical object when changing semiotic representation and how does this phenomenon occur ?

Our conjecture that the loss of meaning stems from Duval's cognitive paradox has been confirmed in different ways in both experiments. The high students identified the mathematical object with its representation through their embodied experience. The exposure to more rich and sophisticated semiotic representations clashed with the strength of the embodied meaning they experienced in the Euclidean understanding of the tangent perceptively understood as a single point of contact between the curve and the straight line. In the case of the primary school students, their ability to coordinate the different presentations - with important structural and functional semiotic differences -of the same object, was due to the strength of their embodied activity. We didn't come to a precise answer to the "how" question in terms of a pure structural approach. What we have noticed is that pupils consider one of the representations of the object as "the object" and the others as representations of such object. A competence in coordinating semiotic systems can somehow, at same time, be a consequence of and reinforce this mechanism.

2. What is the relation between treatment and conversion when students experience a loss of meaning of the mathematical object?

Our experiments didn't give an answer to this question in terms of inferential relation between the two semiotic transformations. High school students were conversant with conversion transformations between the algebraic and Cartesian registers but treatments in the Cartesian register highlighted a missed conceptualization of the tangent outside the Euclidean context. Primary school students were skilful in performing both treatments and conversions. Our experience confirms that working with treatment is effective because it uncouples the semiotic problem and the issue of meaning; the syntax of the semiotic systems sustains the semiotic transformations and the problem of meaning emerges despite the correct coordination of the representations.

Meaning in the Cultural Semiotic approach

In the cultural semiotic approach our research questions we formulated in terms of connecting different reflexive activities and the dichotomy between the situated intentional experience of the subject and the general interpersonal meaning of the mathematical object.

1. How do students *connect* and *synthesize* several contextual meanings into a unitary meaning of the mathematical object?

Our conjecture that embodiment is the key element that allows to connect the different reflexive activities into a unitary meaning has been confirmed in both experiences. Embodiment supports a strong contiguity between mediated reflexive activities in the path towards the interpersonal and cultural object. The local meanings result on the one hand stable and on the other flexible enough to to connect with the meaning emerging from other activities.

The high school experiment clearly shows how the mathematical meaning was strongly embodied through figural representations and the use of artefacts. As they had to move to higher levels of generality, they didn't resort to other more effective means of objectification but they clung to their embodied experience.

In the primary school experience there was a good alignment between the personal embodied meaning and the cultural meaning the students had to objectify. Their generalizing process in terms of a factual and contextual generalization was sustained by their embodied experience.

The ontosemiotic approach interprets the connection of meanings emerging from different reflexive activities as a semiotic function that relates couples systems of practices-configuration of objects.

2. *How* do students objectify a general cultural and interpersonal meaning of the mathematical object, if they can only access a personal meaning obtained through their embodied space-time contextual reflexive activity mediated through semiotic means of objectification?

Our conjecture that the semiotic function is a theoretical and practical tool to overcome the dichotomy between the reflexive personal dimension and the general cultural dimension has been confirmed. Such dichotomy can be seen as an apparent conflict between the operational and the referential phases, described in chapter 3, that is healed by the semiotic function.

7

Conclusions

7.1 RESEARCH QUESTIONS AND HYPOTHESIS

The study we developed in this thesis stemmed from D'Amore and Fandiño's researches highlighting student's difficulties in dealing with the meaning of mathematical objects and their semiotic representations. Defying Duval's claim that *conversion* is the key semiotic cognitive operation that characterises both mathematical thinking and learning and is the main cause of students' failures, such researches clearly show that subjects exposed to semiotic *treatment* transformations encounter severe difficulties in dealing with the meaning of mathematical objects. An important part of the work we conducted during this doctoral research was devoted to clarify and precisely frame both the research questions and the working hypotheses. We remind the reader that the research questions were formulated within Duval's and Radford's approaches and in terms of a networking of the two theories. Facing the issue of connecting theories, we also formulated questions regarding the networking of the aforementioned theories, which, in the development of our investigation, also included the ontosemiotic approach. Below we briefly recall the research questions followed by the hypothesis.

In Duval's approach we addressed the problem as the loss of meaning understood as a loss of the common reference to the mathematical object when given object given through different semiotic representations:

3. Why do students lose the meaning of the mathematical object when changing semiotic representation and how does this phenomenon occur ?
4. What is the relation between treatment and conversion when students experience a loss of meaning of the mathematical object?

We conjectured that:

3. The loss of meaning is ascribable to the inaccessibility of the mathematical object and it is one of the possible behaviours deriving from Duval's cognitive paradox. The student is unable to disentangle the *sinn* from the *bedeutung* when he carries out a semiotic transformation thereby losing the reference to the common mathematical object or referring to a different one.
4. The phenomenon is independent from the type of transformation, but the phenomenon is more evident in the case of treatment because, from a syntactical point of view, the pupil can carry out the semiotic transformation correctly without necessarily handling appropriately the couple (sign, object).

In the cultural semiotic approach we formulated our research questions focusing on the role of activity in determining the meaning of mathematical objects:

3. How do students *connect* and *synthesize* several contextual meanings into a unitary meaning of the mathematical object?
4. How do students objectify a general cultural and interpersonal meaning of the mathematical object, if they can only access a personal meaning obtained through their embodied space-time contextual reflexive activity mediated through semiotic means of objectification?

We conjectured that:

3. When the semiotic means of objectification mediate contiguous activities there is a strong connection between the different local meanings, each of them representing a step in the path that leads to the general and cultural mathematical object; embodiment is the key element that ensures such contiguity. The transition from one situated experience into another towards

higher levels of generality can interpreted as the relation between an antecedent and a consequent of a semiotic function.

4. The semiotic function is a theoretical and practical tool to overcome the dichotomy between the reflexive personal dimension and the general cultural dimension. The disembodied meaning can be seen as a “meaning of meanings” in the sense that each meaning deriving from each couple system of practices-configuration is synthesized in a unitary meaning of object through the semiotic function

In terms of a connection of the two theories we focused our attention on the following issues:

4. What is the relation between the coordination of semiotic systems and activity in the objectification of meaning?
5. The use of semiotic systems is an outcome of the student’s learning process or it is carried out in parallel with the reflexive activity?
6. Is it possible to coordinate the diachronic use of semiotic representations and the synchronic use of semiotic means of objectification?

We conjectured that:

4. The two dimensions coexist and mathematical activity basically is a semiotic practice that allows the emergence of mathematical objects intended as primary entities.
5. The coordination of semiotic systems coexists with the reflexive activity. WE cannot disentangle mathematical cognition from the coordination of semiotic systems.
6. The two temporal dimensions are harmoniously coordinated; through the synchronic use of semiotic means of objectification, students diachronically perform semiotic transformations.

The research questions regarding the connection of the theoretical approaches we considered were:

3. What are the boundaries of each semiotic perspective we are considering and how can this inform our connecting strategies in terms of the plots of integration and differentiation?
4. What degree of integration is the most appropriate to answer our research questions? At what level is most effective networking: system of principles, methodology or research questions templates?

We conjectured that:

3. That the three theories are contiguous semiotic perspectives with frail boundaries.
4. It is possible to effectively synthesis the three perspectives at the level of the system of principles and research questions.

7.2 EXPERIMENTAL RESULTS

The aim of this section is to briefly recall the basic results of our research. We conducted two main researches; the first with students (19 years old) attending the last year of a scientific secondary school in Bologna, the second with students (10 years old) of a last year primary school in Bologna.

7.2.1 High school experiment

We start considering the results of the experiment conducted with high school students on the tangent. We remind the reader that our investigation focussed on the meaning of the tangent in different systems of practices: Euclidean Geometry, Analytic Geometry and Analysis. Our interest was in the change of meaning of the tangent as students were confronted with semiotic treatment transformations in the Cartesian register. We refer the reader to chapter 5 for the details of the test and the a-priori analysis.

The protocol was designed in order to test how the students handled the tangent in a singular point, in particular how their personal meaning was bound to embodied visual and spatial elements that act as an obstacle towards higher levels of generality. Below we synthetically recall the most important results of this first experiment.

Students' mathematical behaviour

Question 1: the circumference

As regards the tangent to the circumference we recognized three behaviours among the students. The most common was to trace the straight line perpendicular to the ray and justify the uniqueness resorting to the uniqueness of the perpendicular straight line. Another behaviour was to trace directly the ray and the perpendicular straight line without any justification to the uniqueness the tangent. A third case was to *draw* a straight line with only one *point of contact* with the curve, the uniqueness of the point of contact justified also the uniqueness of the tangent. The last rare case was to resort to algebraic or infinitesimal methods.

Question 2: graphs of real functions

As regards the tangent to the graphs of functions the most common behaviour was to *trace* the straight line that *touched* the curve in only one point, this condition also justified the uniqueness of the tangent. Some students also resorted to analytic geometry and calculus to justify the uniqueness of the tangent.

Question 3: the singular point

Only two students recognised that the tangent in the singular point doesn't exist; one of them, anyway, remarked that the tangents to the two halves lines in point (0,0) are straight lines overlapping them. The rest of the students claimed that there is one or more than one tangent, arguing in terms of strongly embodied spatial and kinaesthetic elements in the Cartesian registers; having recourse to the point of *"contact"* between the curve and the straight line and the idea that the straight line *"oscillates"* around the origin until it *"touches"* one of the halves lines of the function y=|x|; or to idea that the each tangents in the origin "*overlaps"* the halves lines of the graph of the function. Some students also resort to calculus in the algebraic register to support their claims.

Semiotics, activity and meaning

Structural and functional analysis.

This experiment shows an interesting example of what we termed as a *loss of meaning* due to *treatment* transformation in the Cartesian semiotic register. Furthermore the students' behaviour testifies a good competence in performing *conversions* between the Cartesian register and the algebraic one. Despite this cognitively sophisticated competence, the meaning of the tangent appears fragmented and inconsistent.

<u>Cultural Semiotic analysis</u>.

We testify in this experience the instrumental role of semiotic means of objectification in accomplishing reflexive activities rather than their representational one. The ruler, the compass and the figural representations strongly embody the meaning of the tangent through the students' reflexive activity, hindering the access to a more general cultural meaning of the concept and the further alignment of the personal meaning to the cultural one. The activity mediated through the algebraic symbolism is not connected to the previous reflexive activities.

<u>Ontosemiotic Analysis</u>.

The test involves three linguistic games that are behind three couples systems of practices-configurations of objects: Euclidean geometry, analytic geometry and mathematical analysis. The absence of an appropriate net of semiotic functions that through the intensive-extensive cognitive duality relates the three systems of practices leaves the students stuck to the language game of Euclidean geometry when they have to connect the systems of practices into a more general meaning. The systems of practices remain decorrelated at a personal level and the student lives a semiotic conflict between his personal dimension and the institutional one.

7.2.2 The primary school experiment

We recall here the results of the experiment conducted with a primary school class working on sequences whose domain was the number of the figure and whose range was represented by structured figures. We refer the student to chapter 5 for the for the details of the worksheets and the a-priori analysis.

Our aim was to verify if through a treatment transformation from one figure to another the students were able to find the general term of the sequence and recognize the same mathematical object. Below we synthetically recall the results of the second experiment.

Student's mathematical behaviour

Students recognized the sequences as functions that related to a natural number its corresponding figure. They distinguished the difference between the number of the figure the figure and the number of elements in the figure, thereby constructing the function with domain and codomain N. They recognised the sequence of the even and odd numbers.

Most of the groups managed to find the corresponding number of the sequences for big natural numbers. They were also able to express the general term of the sequences, in natural language and one group spontaneously attempted a fist syncopated algebraic notation. Some groups found the rule for the general term of the sequence with more than one method.

Students successfully integrated geometrical and arithmetical knowledge to sustain their first pre-algebra experience. We remark that our pupils worked on an extremely sophisticated mathematical object, usually treated at higher school levels including university.

Four of the six groups were able to recognize the same sequence when they changed the figural representation through a treatment transformation.

Semiotics, activity and meaning

<u>Structural and functional analysis</u>.

Students are able to perform conversion and treatment semiotic transformations, coordinating the natural language, figural and arithmetic semiotic registers. They were able to recognize a relation between a given natural number and the number of elements of the corresponding figure through a conversion transformation.

Treatment transformations did not cause the loss of meaning we expected in our a priori analysis. Students recognized that the different figural representations linked through treatment transformations, referred to same sequence.

<u>Cultural Semiotic analysis</u>.

Students resorted to a rich set of semiotic means of objectification: gestures, artefacts, movement, natural language, spatial perception, rhythm. The groups expressed outstanding levels of social interaction and communication. Their activity was characterized by an interplay between synchronic use of semiotic means of objectification and diachronic conversion and treatment transformation between the aforementioned semiotic registers.

This experiment confirms previous results obtained by Radford (2000, 2003, 2005) that show the role of some semiotic means of objectification in the generalization processes and the conquest of pre-algebraic thinking; gestures, rhythm, kinaesthetic activity, deictic and generative use of natural language. During their experience students became acquainted with both a factual and a contextual generalization. One of the groups attempted a transition towards a symbolic generalization, overlapping contextual and symbolic features.

The objectification process produced an harmonization between the cultural meaning (the concept of sequence) and the student's personal meaning realized at a factual and contextual level of generality.

The unexpected successful coordination of meaning emerging from different figural representations of the same sequence van be tracked back to the fact that, although the different figural representations were very different under a semiotic point of view, they mediated activities that were congruent through a strong embodied meaning.

<u>Ontosemiotic Analysis.</u>

The experience involved three couples of system of practices-configuration of objects: arithmetical, geometric and pre algebraic. The video analysis highlights the effective interplay between the primary entities that of configuration of objects; the *problem situation* contextualised the activity mediated through *semiotic* means; *arguments* justified their *procedures* and *propositions* – as schemas and descriptions of the operational invariants - introduce new general *concepts*.

The student's success in finding the number of elements in a figure corresponding to a big number and the objectification of the general rule of the schema behind the construction of the sequences is the result of a complicated net of semiotic functions. Students were able to establish semiotic functions within each system of practices and also between different systems of practices, in particular the geometric system and the arithmetical one to derive the number of elements inside the figure. A specific semiotic function allowed to relate the schema - a procedural primary entity in the geometric system of practices –with a general linguistic term in what we called a pre-algebraic system of practices. The notion of semiotic means of objectification in terms of generative and deictic use of linguistic terms was essential to identify such pre-algebraic system of practices with its configuration of objects.

The generalization process obtained through the aforementioned semiotic functions was sustained by the extensional-intensional cognitive duality. We have seen the effectiveness of the interplay between particularization and generalization, described in chapter 2, that characterizes such cognitive duality. Through the net of semiotic functions they established, students address the reasoning schema general-particular-general that allows them to grasp through the particularization process the generality and inaccessibility of the mathematical concept. For a detailed analysis of this process in we refer the reader to chapter 5.

7.3. ANSWER TO THE RESEARCH QUESTIONS

In this section we synthetically recall the answer to our research question on the basis of the theoretical and experimental investigations that we have developed in the present study. We will compare the answers with the working hypothesis. To help the reader we will recall them below followed by the answers.

7.3.1. Meaning in the structural and functional approach

In the structural and functional frame our research questions were stated in terms of losses meaning as follows.

3. Why do students lose the meaning of the mathematical object when changing semiotic representation and how does this phenomenon occur ?

Our conjecture that the loss of meaning stems from Duval's cognitive paradox has been confirmed in different ways in both experiments. The high students identified the mathematical object with its representation through their embodied experience. The exposure to more rich and sophisticated semiotic representations clashed with the strength of the embodied meaning they experienced in the Euclidean understanding of the tangent perceptively understood as a single point of contact between the curve and the straight line. In the case of the primary school students, their ability to coordinate the different presentations - with important structural and functional semiotic differences -of the same object, was due to the strength of their embodied activity. We didn't come to a precise answer to the "how" question in terms of a pure structural approach. What we have noticed is that pupils consider one of the representations of the object as "the object" and the others as representations of such object. A competence in coordinating semiotic systems can somehow, at same time, be a consequence of and reinforce this mechanism.

4. What is the relation between treatment and conversion when students experience a loss of meaning of the mathematical object?

Our experiments didn't give an answer to this question in terms of inferential relation between the two semiotic transformations. High school students were conversant with conversion transformations between the algebraic and Cartesian registers but treatments in the Cartesian register highlighted a missed conceptualization of the tangent outside the Euclidean context. Primary school students were skilful in performing both treatments and conversions. Our experience confirms that working with treatment is effective because it uncouples the semiotic problem and the issue of meaning; the syntax of the semiotic systems sustains the semiotic transformations and the problem of meaning emerges despite the correct coordination of the representations.

7.3.2. Meaning in the Cultural Semiotic approach

In the cultural semiotic approach our research questions we formulated in terms of connecting different reflexive activities and the dichotomy between the situated intentional experience of the subject and the general interpersonal meaning of the mathematical object.

3. How do students *connect* and *synthesize* several contextual meanings into a unitary meaning of the mathematical object?

Our conjecture that embodiment is the key element that allows to connect the different reflexive activities into a unitary meaning has been confirmed in both experiences. Embodiment supports a strong contiguity between mediated reflexive activities in the path towards the interpersonal and cultural object. The local meanings result on the one hand stable and on the other flexible enough to to connect with the meaning emerging from other activities.

The high school experiment clearly shows how the mathematical meaning was strongly embodied through figural representations and the use of artefacts. As they had to move to higher levels of generality, they didn't resort to other more effective means of objectification but they clung to their embodied experience.

In the primary school experience there was a good alignment between the personal embodied meaning and the cultural meaning the students had to objectify. Their generalizing process in terms of a factual and contextual generalization was sustained by their embodied experience.

The ontosemiotic approach interprets the connection of meanings emerging from different reflexive activities as a semiotic function that relates couples systems of practices-configuration of objects.

4. *How* do students objectify a general cultural and interpersonal meaning of the mathematical object, if they can only access a personal meaning obtained through their embodied space-time contextual reflexive activity mediated through semiotic means of objectification?

Our conjecture that the semiotic function is a theoretical and practical tool to overcome the dichotomy between the reflexive personal dimension and the general cultural dimension has been confirmed. Such dichotomy can be seen as an apparent conflict between the operational and the referential phases, described in chapter 3, that is healed by the semiotic function.

7.3.3. Networking questions

This research raised the problem of the interplay between the role of reflexive activity and semiotics in the learning and teaching processes.

1. What is the relation between semiotics and activity in the objectification of meaning?

The hypothesis that these two dimensions coexist has been confirmed; in fact, it is very difficult to fix a clear boundary between them. We can conclude that mathematical practice on the one hand is carried out *through* semiotic representations and on the other it is carried out *on* semiotic representations. The broadening of the notion of semiotics proposed by the Cultural Semiotic approach is effective in including this double functioning of signs in mathematics. A more exhaustive answer to this issue is obtained by recovering, through the semiotic function and the unitary-systemic duality, the notion of sign as a representation, as an instrument to carry out practices, and as the object of the mathematical practice.

2. The use of semiotic systems is an outcome of the student's learning process or intrinsically underpins the reflexive activity?

Our data confirm that also at the primary level students constantly perform treatment and conversion transformations. This confirms our conjecture that the coordination of semiotic systems underpins the reflexive activity. This is true if we confine our analysis to the cognitive functioning, thereby missing part of a more broaden and complicated picture. Our view point is that if we forget the systems of practices there is only net of semiotic systems coordinated through conversion and treatment transformation, but if we widen our view, what we termed as a semiotic transformation between two ore more representations of the same object, in fact is a semiotic function between couples systems of practices-configurations of object. In the semiotic function, a semiotic representation through the dual facet unitary-systemic plays a double role. In terms of the unitary facet, the semiotic representation plays the role of antecedent or consequent of the semiotic function and it is both an object or a representation. In terms of the systemic facet, the semiotic representation plays the role of an instrument to accomplish a particular practice that with another representation would not be possible.

3. Is it possible to coordinate the diachronic use of semiotic transformations and the synchronic use of semiotic means of objectification?

Our data confirm that the two temporal dimensions are harmoniously coordinated. In fact, it is through the synchronic use of semiotic means of objectification that students diachronically perform semiotic transformations. In terms of the semiotic function we can conclude that:

- In terms of the instrumental facet of the duality, the personal and institutional practices are carried out by synchronically using semiotics means of objectifications.

- In terms of the unitary facet, the semiotic function diachronically connects the antecedent and the consequent, one of the emerging objects of the configurations that connected; such diachronic relationship accomplishes a treatment or conversion semiotic transformation.

7.3.4 Networking theories

We recall below our issues regarding the coordination of the semiotic perspective in our study.

1. What are the boundaries of each semiotic perspective we our considering and how can this inform our connecting strategies in terms of the plots of integration and differentiation?

Our initial hypothesis that the three frameworks were contiguous, with frail boundaries was not correct. Although the three perspectives focus their attention on semiotics, its hierarchical position in the system of principles, is very different. Duval is interested in the specific cognitive functioning of mathematics that he characterises as the coordination of semiotic systems that are the only access to mathematical objects; he claims that there isn't *noesis* without *semiosis*. Radford is interested in learning as objectification-subjectification process in which individuals become a *aware* through mathematical objects of a *cultural and historical* dimension mediated by semiotic means in social and cultural modes of signification. Godino is interested in the passage from the *operational* phase to the *referential* phase in mathematical learning and generalises the notion of sign as any *antecedent* in a semiotic function. The

The nature of the three semiotic perspectives tackles the issue of learning of mathematics in different directions that in the networking semiosphere require to consider both differentiation and integration.

2. What degree of integration is the most appropriate to answer our research questions? At what level is most effective networking: system of principles, methodology or research questions templates?

Understanding/making understandable and *comparing/contrasting* were the first levels of integration we addressed to grasp the true identity of each theory and precisely identify their boundaries. Our initial hypothesis to synthesize the three perspectives was obviously too ambitious, overestimating the potentialities and possibilities of a doctoral research, and lacked the necessary theoretical knowledge about networking theories. We cannot exclude that a synthesis of these perspectives is achievable. Integrating at the level of *coordinating /combining* provided effective theoretical tools to tackle our specific issue of "changing of meaning" due to treatment semiotic transformation. It was possible to coordinate because although the boundaries are marked there is strong complementarity between the three perspectives in addressing the following dualities; personal-institutional, referential-operational, cognitive – reflexive activity etc.

We found fruitful to establish connections at the levels of the system of principles and the research questions. We didn't address the issue of integrating at the level of the methodology resorting to qualitative and ecological methods that we reckoned more appropriate to answer our research questions while focussing on the role of reflexive activity.

7.4 MEANING: CONCLUDING REMARKS

We propose the following insightful remark proposed by Anna Sierpinska that condenses the intrinsic difficulty we faced in dealing with this topic:

> «Few concepts have caused as much trouble in philosophy as the concept of meaning. There is a long history of attempts to encapsulate it into theories from which it always seemed to be able to slip away. The reason for this may lie in the unavoidable self-referential character of any theory that would pretend to speak of meaning in a general way: any definition of meaning has meaning itself, so it refers to itself as well. Rarely, therefore, was meaning considered in its full generality; different philosophers have occupied themselves with meaning of different things, and they focused their attention on different aspects of meaning» (Sierpinska, 1994, p.13).

We find ourselves in a paradoxical situation. On the one hand we cannot precisely say what meaning is, on the other meaning is "meaningful" to us, especially when we observe students' endeavour to make sense of mathematical concepts. This intrinsic paradox was the main obstacle we had to overcome all through our investigation, both theoretically and experimentally. The aim of this research was not to "encapsulate" meaning in a comprehensive theory or a comprehensive network of theories. Faithful to an anthropological and socio cultural stand our attempt was to translate meaning into something viable when our interest is in mathematics teaching-learning processes.

We have seen how the meaning of mathematical objects, according to realistic theories, is a conventional relationship between signs and a priori existing ideal entities. Pragmatic theories, instead consider the set of "uses" to establish the meaning of mathematical objects. We framed the issue of "changes of meaning" first in a realistic frame when we considered Duval's approach and shifted then to a pragmatic frame when we considered the cultural semiotic and ontosemiotic approaches. From the results of our investigation we can draw the following conclusion.

Structural approach

In terms of Duval's couple (sign, object) the problem of meaning is stated in terms of one object-many representations. Within a semoiotic system a sign is a complex structure in which we can identify a *sinn* – the way in which the object is given through the semiotic system, and a *bedeutung*- the reference to the object. Meaning is a double faced construct that includes both sinn and bedeutung.

A change of meaning (sinn) is therefore intrinsic to mathematical conceptualization and the learning of mathematics entails the ability to coordinate different meanings. A change of mathematical representation, therefore a change of meaning, is sometimes can bring a loss of the reference to the common object.

The loss of reference is a consequence of Duval's paradox that it is not overcome only by the coordination of semiotic systems through conversion and treatment transformations. Our conclusion is that a coordination of semiotic registers doesn't guarantee per se a correct conceptualization of mathematical objects, especially as far as meaning is concerned. The realistic stand that considers meaning in terms of the construct one object many representations is insufficient to account for the complexity of the mathematical object.

Cultural semiotic approach

Meaning, semiotics means of objectification, and mathematical objects are strongly entangled through the reflexive activity that encompasses both a personal dimension in the individual's intentional acts and a social-cultural dimension. The problem of meaning is viewed as an alignment of the personal embodied dimension of the subject's set of reflexive activities and the cultural dimension of the mathematical stratified in layers of generality. The focus on reflexive activity undermines the one object-many representations structure.

Then problem of changes or losses of meaning is shifted from semiotics to the activities that semiotic means of objectification mediate. The educational issue is how students coordinate and align their local activities to the general and stratified mathematical object.

Our conclusion is that the competence of coordinating semiotic systems does account for the construction of meaning if there is no alignment and coordination at the level of activities. The high school students were able to carry out the conversion that would have solved the problem of the tangent in the singular point. They didn't spontaneously resort to another semiotic register because it was meaningless to them in terms of activities. The aforementioned coordination and alignment is guaranteed as far as semiotic means of objectification mediate contiguous activities each of them representing a step in the path that leads to higher levels of generality; contiguity is sustained when activities are embodied and through social interaction in students' Zone of Proximal Development.

The cultural semiotic approach doesn't fully account for symbolic generalization when the student has to resort to semiotic means that interrupt the connection with his personal experience.

Ontosemiotic approach

Also the ontosemiotic approach undermines the one object-many representations structure through the notion of system of practices, configurations of objects and cognitive dualities. In a general, meaning is the consequent of a semiotic function. In chapter 2 we have seen how the semiotic function joins together, through the unitary-systemic duality, meaning in terms of activity and meaning in terms of reference; this is accomplished by considering, through a representation, a primary entity as antecedent and the system of practices as a consequent. We thereby have a set of local meanings given through the couples system of practices-configuration of objects with their net of semiotic functions. Linking the set of couples system of practices-configuration of objects that bring about the formation of an object through another net of semiotic functions we the meaning of "meanings" that we mentioned in chapter 2.

Our conclusion is that the semiotic function accounts for, in a specific linguist game, both the coordination of reflexive activities and the alignment of the personal meaning to the general stratified mathematical object. The semiotic function synthesizes the operational level and the referential level healing the dichotomy between personal and institutional, general and contextual, pragmatic and realistic.

We stated our research problem in Duval's framework and looked for a solution in theories whose system of principle give a prior hierarchical position to activity and not to semiotics. Our basic assumption that meaning has an anthropological and cultural origin in social activity has been confirmed during our investigation aiming at understanding the relation between meaning and semiotic transformations. Nevertheless, we cannot conclude that we can disregard Duval's approach whose analysis is extremely effective when investigating the referential phase and recognizing the rules that allow to establish the semiotic functions.

At an operational level the notion of objectification plays a central role in characterizing systems of practices, the configuration of objects and the rules that connect the antecedent with the consequent in the semiotic function. Looking at practices in terms of reflexive mediated activity provides a thorough understanding of the epistemic and institutional dimension involved in mathematical thinking and learning. Semiotic means of objectification tell us precisely the nature of the practice they accomplish thereby determining both the layers of generality and the configurations of emerging objects involved.

In conclusion, meaning can be effectively interpreted as the consequent of a semiotic function in which the antecedent and the consequent can be any kind of "object". This gives a great flexibility in analyzing the issues related to meaning and, in terms of semiotic functions, a fixed distinction between objects and representation is untenable. The analysis is shifted to the emerging configurations of objects and practices, related through the semiotic function according to one of the cognitive duality as the extensive-intensive used in our analyses.

7.5 NETWORKING THEORIES: CONCLUDING REMARKS

The present investigation rested on the assumption that it would have been possible to recognize a certain degree of connectivity between Duval's structural approach and Radford's cultural semiotic approach. Our basic naïve assumption was that Radford's approach was equivalent to Duval's approach added with socio cultural elements and gestures. A claim that would have been strongly rejected by both scholars.

As our investigation proceeded we realized the profound differences between the two approaches. This came out to be a challenge and a resource to face the problem of meaning in mathematical learning. The idea to take into account also the ontosemiotic approach was due to the theoretical effectiveness of the semiotic function as regards the understanding of meaning and the possibility of relating the personal and institutional dimensions.

We want to draw some conclusions from our investigation whose priority wasn't specifically to investigate the degree of integration of these approaches but networking was an unavoidable side effect of our research. A more structured and specific investigation on the connection of these three perspectives would certainly give more comprehensive results. Our concluding remarks will address the system of principles and the research questions of the three perspectives and their degree of integration.

System of principles and research questions

As we mentioned above our first impression was that Duval's and Radford's perspectives had contiguous system of principles. In fact, both perspectives assign semiotics with a prominent role, but the hierarchal position occupied by their system of principles and the worldview that informs the two theories is very different. As regards the Ontosemiotic approach, although the result of a sophisticated networking process, with respect to other two perspectives shares the same issues; again semiotics plays a fundamental role but both its hierarchical position and the world view are different. Looking at the relationship between semiotics and cognition in the three perspectives we can draw the following conclusions.

Based on a realistic stand, Duval's worldview is that learning can be identified with a specific cognitive functioning that characterizes mathematical thinking. Such specific cognitive functioning *is* a complex net of transformations between semiotic systems analysed from a structural and functional point of view. The basic assumption is that mathematical objects are ideal entities accessible *only* through the coordination of semiotic systems that mirror the complexity of mathematical cognition an learning.

Based on an anthropological and socio-cultural stand, Radford's worldview is that cognition is a mediated reflexive activity; when the focus is on learning such reflexive activity is regarded as an objectification process – a sense giving act, on the part of the individual's consciousness- of a cultural object. Semiotic means, bearers of cultural-historical experience, play an instrumental role to accomplish reflexive activities. The interplay between the individual consciousness and activity - considered against a social and cultural system of signification- ranks highest in the hierarchy of the system of principles.

Also Godino's approach is based on a social-cultural and anthropological stand and shares many elements with Radford's perspective. Cognition is viewed as net of semiotic functions established by an individual (personal or institutional) according to a criteria or rule within a specific language game. Semiotics loses its binary feature of a object-representation couple and is intended as a relation between an antecedent and consequent. There is no a priori distinction between an object and its representation, but according to the language game a relation between "something" called an antecedent and "something" called the consequent is established. The notion of practice and reflexive activity although very similar, at their core are different. A system of practices is essentially social and operationally defined by the solution of a field problems in a language game, whereas a reflexive activity is a culturally and historically mediated intentional act on the part of the individual's consciousness. The prominent role in the hierarchical organization of the system of principles is played by the notion of language game that sustains both the system of practices, from which it stems, and the semiotic function. Godino's concern is to objectify through the semiotic function the mathematical practices, within a linguistic game, that become referents of the institutional language.

The different hierarchical structures of the system of principles entail differences in the paradigmatic research questions of the three theories. Duval's template of questions is focused on the investigation of the representational nature of signs, their organization in structured systems of signs, their discursive functions and their transformations. Radford's template of questions is directed to the analysis of the interplay between cultural elements condensed in semiotic means, activity and the individual consciousness in students' sense making processes of general mathematical objects. Godino's template of questions addresses the nature of the language game and the primary entities that realize the systems of practices in order to identify antecedents and consequents of the net of semiotic functions that define a mathematical concept.

Degree of integration

On the basis of our analysis of the system of principles and the results of the present research, we conclude that, referring to the degrees of integration introduced in chapter 3, networking at the level coordinating the three semiotic perspectives has been effective in facing the issue of changing of meaning.

Although at the core of their system of principles, the three perspectives are very different, this resulted in a resource to face the complex and broad problem of the meaning of mathematical objects. The differences between the three perspectives allowed to tackle the problem from different directions and, limitations in one of the theories, were made up for by the others. In the terminology we introduced in chapter 3, the aforementioned differences result in a high level of complementarity that accounts for networking by *coordinating* the three perspectives. We have singled out four elements in which the three perspective complete each other:

7. Semiotics in its different acceptations: representational, instrumental, purely relational through the semiotic function. .
8. Cognitive operations specific of mathematics, identifiable with the coordination of semiotic systems and semiotic registers.
9. The role of consciousness in determining sense giving acts.
10. Social and cultural factors
11. Activity in its reflexive understanding and as a practice emerging from a language game in a field of problems.
12. The notion of mathematical object in its operational (pragmatic) and referential (realistic) sense.

Disregarding even one of the aforementioned elements would have hindered both the formulation and answer to the research questions that informed the present investigation. The networking attempt we carried out by coordinating the structural-functional, the cultural semiotic and the ontosemiotic approaches, was able to face the problem of meaning in a holistic and open way without encapsulating it in a fixed and definitive frame that would have betrayed its anthropological and socio cultural nature.

7.6 OPEN QUESTIONS

The present research opened a possible way to the solution of the changes of meaning due to semiotic transformations resorting to the notions of reflexive activity, semiotic means of objectification and semiotic function. Further and more specific research should be still carried out to address exhaustively the research questions we faced in this work and the open issues that are left unattended. Below we discuss some of the open issues we believe are more interesting.

Conversion and treatment

In Duval's perspective conversion, with respect to treatment, plays a key role in mathematical cognition and is the main cause of students' learning failures. Our research stemmed from students' difficulties in facing the meaning of mathematical objects when dealing with treatment transformations, somehow defying the claim that conversion is the main source of difficulties because it requires to overcome the cognitive paradox. Our interest on treatment derives from the fact that it uncouples the semiotic aspects from the activity that is behind such transformation. Since treatment can be easily carried out through the rules of the semiotic system, it highlights the role of activity in the objectification of meaning. It would be interesting to design a research which tests changes of meaning only with conversion transformations. It could confirm the role of reflexive activity both in establishing the connection between semiotic representations and in the objectification of meaning. Furthermore it would be interesting to determine if there is a sort of inferential relationship between conversion and treatment thereby drawing conclusions both on the role of activity and semiotics in mathematical cognition and learning.

Embodiment

Embodied experience plays an essential role in the objectification of meaning and we have seen how the local meanings are easily coordinated when there is a spatial and kinaesthetic link between them. Embodiment accounts for factual and contextual generalization but symbolic generalization requires a disembodiment of meaning. Can we claim that symbolic generalization *necessarily* disembodies meaning? Our doubt is if, at higher levels of generality, embodiment assumes other characteristics that, at the moment, we are not able to identify. We can address the extreme case of pure mathematics, taking into account fields like algebraic structures, Hilbert spaces, etc. If we open a functional analysis or algebra university textbook there are only symbolic representations and English linguist terms that seem to have little to do with embodied experience; are we sure we can exclude kinaesthetic and sensimotor features or embodiment is still present in other forms?

Semiotic function

The semiotic function is an extremely effective rule to generalize the notion of meaning synthesizing the operational and referential phases. We have detailed the complicated net of semiotic functions that connect a variety of entities: primary entities, system of practices, representations, configurations of objects. The semiotic function accounts for both the coordination of system of practices and the alignment of the personal meaning with the cultural meaning. A semiotic function is established within a language rule by a subject according to certain criteria or codes. The recognition of a code is usually carried out a posteriori as a further institutional or personal practice. But how do students recognize the code and the criteria to establish the semiotic function when they are facing new concept or a new learning situation? Duaval's and Radford's approaches may provide elements to answer this question considering intentional acts, the role of semiotic means of objectification and the structural and functional characteristics of semiotic systems and their transformations through conversion and treatment. For a better understanding of the problem it is necessary to carry out specific investigations on the role of the aforementioned elements in establishing a semiotic function.

Activity and semiotics

We pivoted our analysis about the meaning of mathematical objects on the role of activity. We highlighted the strong connection between activity sings and activity. We have seen how activity is mediated through sign and how activity is also performed on sign. The onotsemiotic approach through the notion of primary entities identifies six specific forms of activity; language, procedure, problem-situations, argumentations, propositions and definitions. If we look at the six primary elements they always entail the use of signs. Is there a precise boundary in mathematics between activity and semiotics or can we say that a mathematical activity is intrinsically a semiotic activity?

7.7. EDUCATIONAL PERSPECTIVES

The present worked developed at a high theoretical level involving more theoretical perspectives and addressing a very general issue although through a specific research problem. A possible objection could be that the result of our work has a low impact on educational design and the classroom concrete reality. This kind of investigation certainly has less immediate educational spin off compared to researches that address specific didactical issues concentrating on a particular mathematical topic, but there are significant didactic insights that can be drawn anyway.

Semiotics and meaning are transversal to mathematical cognition and learning. Students continuously have to face a complex net of semiotic transformations in their mathematical practice. Duval's claim that there isn't noesis without semiosis is certainly true and dismissing a specific semiotic instruction hinders mathematical learning. This research has shown the underlying role of semiotic in mathematical teaching and learning.

The present work has also described the prominent role of reflexive activity in welding meaning with semiotics. Understanding the role of activity is a powerful educational resource for didactical engineering. Semiotic competence is not sufficient to reach a stable learning if is not meaningful for the students through the objectification process. In particular, awareness of the phenomenon of the changing of meaning due to semiotic transformations and of possible routes to overcome the problem is important when teaching mathematics. We propose some mathematical topics that can be particularly sensible to this phenomenon: transition from arithmetic to algebra, arithmetical operations, properties of geometrical figures in relation to their position in space, analytic geometry.

We briefly describe the situation of algebra. Students are constantly involved in very complicated and heavy semiotic transformations of algebraic symbolism that is usually senseless to them, failing to achieve the fundamental goal of algebra, generalisation and relational thinking. Algebra is an empty manipulation of signs that remains unrelated to to any significant context. Physics teachers usually clash with the problem of changes of meaning when students cannot solve simple cinematic problems because in the equations as unknown there is an “s” or a “t” instead of the “x” , thereby they cannot recognise an equation.

The issue of meaning is at the core of mathematical teaching and learning. In line with the cultural semiotic perspective we have considered learning as a meaning making process of a cultural object on the part of the student. Many of the important themes in mathematics education can be tracked back to the issue meaning: didactical contract, misconceptions, cognitive conflicts, problem solving etc. Our claim is that a teaching experience is successful if brings students to objectify mathematical concepts. From this point of view learning has not to do only with an efficient cognitive functioning in terms of problem solving, modelling, reconstructing knowledge etc. There is much more than that. Learning mathematics entails an ethical dimension, in which the student, through mathematical forms of thinking and rationality, becomes part of a social and cultural reality. The objectification process entails a subjectification process (Radford, 2008) in which the student notices himself both as an intentionally acting individual and as a communitarian self responsible for the social and cultural growth of the community he belongs to. We conclude with two quotations of Luis Radford that explain this extensive understanding of learning as an ethics of being and learning:

> «The classroom is a symbolic space in which the student elaborates a communal and active relation with his/her historical-cultural reality. It is here that the aforementioned encounter between the subject and the object of knowledge occurs. The objectification that allows for this encounter is not an individual process but a social one. The sociability of the process, nevertheless, cannot be understood as a simple business “negotiation” during which the stake holder invests some capital (e.g. some meaning) in the hopes of ending with more of it. Here, sociability means the process of the formation of consciousness which Leont’ev (1978), paraphrasing

Vygotsky , characterised as *co-sapientia,* that is to say, as knowing in common or knowing-with others» (Radford, 2008, p. 227).

«Instead of a self-regulated Enlightened individual common to many contemporary theories in education, the theory of knowledge objectification suggests the idea of communitarian self, one busy with learning how to live in the community that is the classroom, learning how to interact with others, to opening oneself up to understanding other voices and other consciousnesses, in brief, *being-with-others*. (Radford, 2008, 229).

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APPENDIX

Objectification and Semiotic Function

Bruno D'Amore
NRD, Department of mathematics, University of Bologna, Italy
MESCUD, Distrital University, "F. José de Caldas", Bogotà, Colombia
Email: damore@dm.unibo.it Phone: (++39)0512094446
Fax: (++39)0512094490

George Santi
NRD, Department of Mathematics, University of Bologna, Italy
Department of Mathematics, University of Palermo, Italy
Email: grpsanti@gmail.com Phone: (++39)0512094446
Fax: (++39)0512094490

Abstract: The objective of this article is to study student's difficulties when they have to ascribe the same meaning to different representations of the same mathematical object. We address two theoretical tools that are at the core of Radford's cultural semiotic and Godino's onto-semiotic approaches: objectification and the semiotic function. We drop the realistic idea that meaning is the relation between an object and its possible representation, instead, addressing a pragmatic viewpoint, we consider meaning as a complex network of mathematical activities mediated by signs. We show how, through objectification processes and the semiotic function, mathematical practices, signs, objects and meaning are strictly interwoven. The analysis of a teaching experiment involving high school students working on the geometric interpretation of the first derivative, shows how students' difficulties in ascribing sense to different representation of a common mathematical object can be traced back to the kind of objectification process and semiotic functions they are able to establish.

Keywords: mathematical objects, semiotics, meaning, activity, objectification, semiotic function.

1 Introduction

In this paper we face the issue of changes of meaning due to treatment semiotic transformations introduced by D'Amore (2006). This research highlights unexpected behaviours on the part of students that defy Duval's claim that conversion is the most difficult cognitive function which alone ensures a correct conceptualization of mathematical objects. At all school levels, we testify students' and trainee teachers' difficulty in handling the meaning of mathematical objects when dealing with different representations of the same object. In a treatment transformation, students ascribe different objects to different representations of the same object. For example, primary school students recognize that 1/2 is the probability of an even throw on a six sided die but 4/8, obtained after a treatment transformation, doesn't represent the same probability; university students claim that (n-1) + n + (n+1) is the sum of three consecutive numbers but 3n, obtained after algebraic treatment they performed by themselves, is the triple of a number and in no way can be interpreted as the sum of three consecutive numbers.
In this article, analysing an experimentation with high school students working on the geometric interpretation of the derivative, we will show how the approach to meaning based on the idea that there are many representations for the same object is inadequate to frame students' learning behaviour. We will move from the realistic ontological stand that considers mathematical objects as ideal *a priori* entities and we will go beyond the epistemology that conceives meaning within the structure of semiotic systems, assuming that the meaning of a semiotic representation is the object it refers to.
A more comprehensive notion of mathematical knowledge and signs, that takes into account the role of mathematical activity, is necessary to tackle the issue of meaning in learning environments; we will show the effectiveness of the notion of Objectification and of Semiotic Function, theoretical tools that are the core of the Cultural Semiotic and Onto-semiotic approaches respectively.

> «[The onto-semiotic approach] assumes a certain socio-epistemic relativity […] for mathematical knowledge, since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the institutions and the social context of which it forms a part» (Font, Godino, Contreras, 2008, p. 160).

> «The point is that processes of knowledge production are embedded in systems of activity that include other physical and sensual means of objectification than writing (like tools and speech) and that give a corporeal and tangible form to knowledge as well. Within this perspective and from a psychological view point, the objectification of mathematical objects appears linked to the individuals' mediated and reflexive efforts aimed at the attainment of the goal of their activity. To arrive at it, usually the individuals have recourse to a broad set of means» (Radford, 2003, p. 41).

2 Theoretical background

In this section we present Objectification and Semiotic Function as theoretical tools that allow a thorough analysis of the relationship between mathematical objects, semiotic representations and meaning.

The Semiotic Cultural (Radford, 2003, 2004, 2005) and Onto-semiotic (Godino,2002; D'Amore, Godino, 2006) approaches go beyond a structural and functional approach to semiotics that considers the meaning of signs as the relation between elementary signs in a semiotic system. Both the Semiotic Cultural and the Ontosemiotic approaches still recognize the same fundamental role to signs in mathematical thinking and learning but in a more comprehensive way. They claim is that the use of formal systems of signs is an emergent phenomenon arising from culturally and socially framed systems of practices. To understand the meaning of signs, we cannot reduce them to what they represent but we must understand the kind of activity they accomplish. We will show that students' difficulties root not only in the complicated semiotic structures they have to handle but mainly in the systems of practices associated with semiotic representations.

> «We take signs here not as mere accessories of the mind but as concrete components of 'mentation'. [...]instead of seeing signs as the reflecting mirrors of internal cognitive processes, we consider them as tools or prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage. As a result, there is a theoretical shift from what signs represent to what they enable us to do» (Radford, 2000, p. 240-241).

> «To Ernest's question if "semiotics potentially offers the base to a unified theory in mathematics education (and mathematics)" we answer affirmatively, under the condition to adopt (and elaborate) an appropriate semiotics and to complement it with other theoretical tools, in particular an ontology that takes into account the variety of objects that are involved in mathematical activity» (Godino, 2002, p. 262)

2.1 Duval's structural and functional semiotic approach

Before we start analyzing the role of activity to understand the meaning of signs and mathematical objects, we step back to recall the role of semiotics in mathematical thinking and learning. The aforementioned approaches stem from and broaden Duval's (1993) previous cutting-edge studies that introduced semiotics in mathematics education to single out the specific cognitive functioning in mathematics, thereby broadening our horizon when looking at learning and teaching processes. When we face issues regarding mathematics conceptualization and teaching-learning processes it is necessary to take into account that:

> «the special epistemological situation of mathematics compared to other fields of knowledge leads to endow semiotic representations a fundamental role. In the first place they are the only way to access mathematical objects which raises the cognitive issue of the passage form one representation of the object to another of the same object» (Duval, 2006, p. 586).

The special epistemological situation that distinguishes the cognitive functioning in mathematics from other fields of knowledge is the intrinsic inaccessibility of mathematical objects. Mathematical *cognitive processes* are therefore intrinsically semiotic processes that involve a complicated network of signs. In mathematics, signs

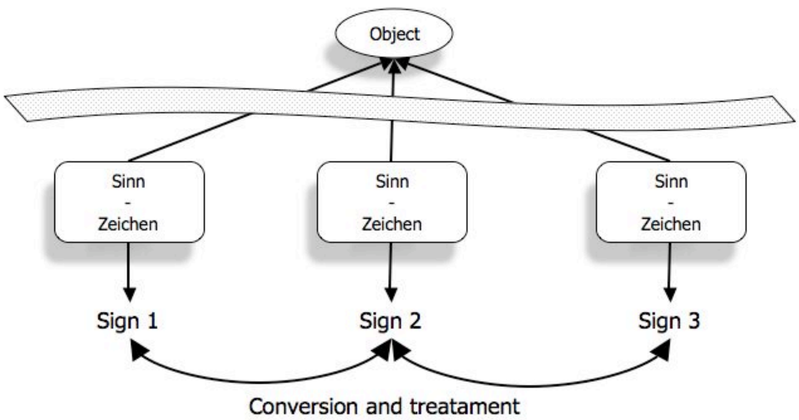
cannot be considered surrogates of mathematical objects or expressions used to handle and communicate internal mental images or models that have been previously formed: they are constitutive of mathematical thinking and learning.
In Duval's perspective there are no signs outside a system of signs, in their broader sense termed as semiotic systems (Duval, 2006, p. 608), whose *structure* allows production and transformations of signs along with discursive or meta-discursive *functions* (Duval, 1995, pp. 88-98).
The main feature that characterizes the specific *cognitive functioning* and therefore the development of knowledge in mathematics is that semiotic systems allow transformations and comparisons between signs. The *structural and functional* approach identifies mathematical thinking and learning with the coordination of semiotic representations pursued by the following *cognitive processes* (Duval, 1993, 1995, 2006; D'Amore, 2001):

- Choice of the distinctive features which involves a comparison between representations related to the socially shared mathematical activity.
- Treatment consisting in transforming a representation into another in the *same* semiotic system.
- Conversion which consists in transforming a representation into another in an *other* semiotic system.

Transformations and comparisons between representations calls out an important both cognitive and didactical issues: when we juxtapose different representations, especially if belonging to different semiotic systems, how can we *connect* one representation to the other and *recognize the common reference* to the same mathematical object if we have no access to the object but with representations? How can we endow *meaning* to semiotic representations if we have no access to the mathematical object but with semiotic representations?
As regards the issue of meaning Duval's perspective overcomes the problem of meaning considering the relationship between representations within the structure and the syntax of the semiotic system they belong to. Again, the semiotic system allows to overcome the problem of connecting different representations when dealing with treatment because the rules of the semiotic system link the different representations of the same object.
There is no way out to the problem of connecting and comparing appropriately representations of mathematical objects when dealing with conversion; there are no rules that connect representations of different semiotic systems, conversion is not a symmetric transformation, and we must deal with non-congruence phenomena (Duval, 1993, 1995, 2006). Duval endows conversion with a central role in respect to other cognitive processes, in particular compared with treatment and considers it a *cognitive threshold* that characterizes conceptualization in mathematics and the main cause of learning failures.

The structural and functional approach provides an extremely refined and powerful tool to understand the cognitive processes that underlie mathematical thinking and learning. The unexpected behaviour of students facing semiotic treatments that we presented in the introduction, testify that when dealing with the sense of mathematical objects we cannot bound meaning to the structure of semiotic systems and single out conversion as the most important cause of learning difficulties in mathematics; in

specific learning situations also treatment can seriously puzzle students when they try to give sense to semiotic representations. Conceiving meaning in terms of the pair one object-many representation, resorting to the reference to the object and the structure of the semiotic system, gives a partial picture of the problem. It is necessary to understand where the specific cognitive processes highlighted by Duval root and scrutinize the notion of meaning and mathematical objects. To accomplish this we turn to the notions of objectification and semiotic function developed by the Semiotic Cultural and Onto-semiotic approaches .
We propose the following schema to frame the issue of meaning of mathematical objects in Duval's perspective.



2.2 Cultural-semiotic approach

Mathematical objects
Before analyzing the objectification of knowledge it is necessary to characterize thinking and the nature of mathematical objects according to the cultural semiotic approach.

> «This theory suggests that thinking is a type of a social practice (Wartofsky, 1979), praxis cogitans. To be more precise thinking is considered to be a mediated reflection in accordance with the form or mode of activity of individuals» (Radford, 2008, p.218).

Within a pragmatic view and continuing Vygotsky's (1986) path, signs are constituents of thinking because they *mediate* the social activity and they bind the individual and historical and cultural dimensions. Such mediators are termed as *artefacts* in a general sense and as semiotic means of objectification when the cognitive activity is addressed to learning: objects , instruments, gestures, words etc.

Thinking is not an isolated activity in which the individual assimilates knowledge, but it is a *reflection* on the part of the subject, accomplished in a socially shared *activity*, of a cultural and historical reality; the term reflection refers to the manner in which the individual intentional acts are directed towards reality, according to cultural and social criteria.

Within this mediated reflexive dimension,

> «mathematical objects are fixed patterns of reflexive activity incrusted in the ever changing world of social practice mediated by artefacts» (Radford, 2008, p. 222).

Objects are strongly embedded in a pragmatic view in which both the individual and social activity play a prominent role, and lose any character of a-priori identities. This is a key point when discussing the relation between meaning and semiotic representations of mathematical object. We cannot confine the issue of meaning to the relation between signs in a semiotic system and the coordination of different semiotic representations, referring to a common somehow *a priori* object, through treatment and conversion. Each representation is imbued with personal and social practices that oblige to broaden meaning beyond the symbolic structure.

In Radford's approach mathematical objects, concepts, signs and meaning are entangled through reflexive activity triggered to solve set of problems that are culturally and socially significant. We underline that when dealing with mathematics, activity is necessarily a linguistic activity that requires the coordination of a wide set of mediators that includes along with symbolic semiotic systems, objects artefacts, gestures etc.

Learning as an objectification process

Learning is considered a mediated reflexive activity but addressed to the mathematical objects that bare a cultural and historical dimension. The cognitive and epistemological situation is very different when we consider learning in respect to the historical and cultural construction of the mathematical objects. In the historical development of mathematics, mathematicians' reflexive activity aims at creating new object, while learners' reflexive activity addresses an object that already exists, not in a realistic sense, but as a culturally and socially recognized entity.

«Students' acquisition of a mathematical concept is a process of becoming aware of something that is already there, in the culture, but that the students still find difficult to notice. The awareness of the object is not a passive process. The students have to actively engage in mathematical activities not to "construct" the object (for the object is already there, in the culture) but to make sense of it. This process of meaning-making is an active process based on understandings and interpretations where individual biographies and conceptual cultural categories encounter each other – a process that, resorting to the etymology of the word, I call *objectification*. To learn, then, is to objectify something» (Radford, 2005, p. 117).

Learning is an *intentional act* in which the subject encounters and puts in "front" of his consciousness the mathematical object through a mediated activity that gives sense to the learned object.

In this perspective, signs cannot be reduced to a purely representational function but they *culturally* mediate the reflexive activity that brings to the objectification of the mathematical objects. The way learners intend the mathematical object through their intentional acts is not a neutral subject-object relationship, but it is intrinsically “tainted” by culture, history and social structures through the semiotic mediators that direct our intention:

> «Sense-giving acts and all that makes them possible are essentially cultural. […]What appears in front of us in our intentional experience is consequently ubiquitously framed by the culearltural history of the means that we use to apprehend it. Sense-giving acts and all that makes them possible are essentially cultural. […] In giving meaning to something, we have recourse to language, to gestures, signs or concrete objects through which we make our intentions apparent […]. Language, signs, and objects are bearers of an embodied intelligence (Pea, 1993) and carry in themselves, in a compressed way, cultural-historical experiences of cognitive activity. […] I termed the whole arsenal of signs and objects that we use to make our intentions apparent semiotic means of objectification» (Radford, 2006, p. 52).

When we focus our attention to learning, the objectification process obliges to broaden our notion of meaning of a mathematical object. Indeed, in the objectification process meaning entails a relationship between a cultural dimension and a personal dimension, between a cultural meaning and a personal meaning. On the one hand the student is the protagonist of learning through his sense-giving intentional acts, on the other hand such intentional acts through social activity are directed to an interpersonal and general cultural object.

> «I want to suggest that it is advantageous to think of meaning as a double-sided construct, as two sides of the same coin. On one side, meaning is a subjective construct: it is the subjective content as intended by the individual’s intentions. On the other side and at the same time, meaning is also a cultural construct in that, prior to the subjective experience, the intended object of the individual’s intention (l’object visé) has been endowed with cultural values and theoretical content that are reflected and refracted in the semiotic means to attend to it» Radford (2006, p. 53).

The sense giving activity students are involved in can be seen as a convergence of the cultural meaning with the personal meaning. At an ontogenetic level the personal activity mediated by the semiotic means of objectification traces out the phylogenetic activity culturally condensed in the mathematical object.

> «I believe that mathematical learning of an object O on the part of an individual I within a society S is nothing else but the adhesion of I to the practices that other members of S develop around the object O. How do we express such adhesion? Accepting the practices that are mainly linguistic» (D’Amore, in: Bagni, D’Amore, Radford, 2006, p. 22)

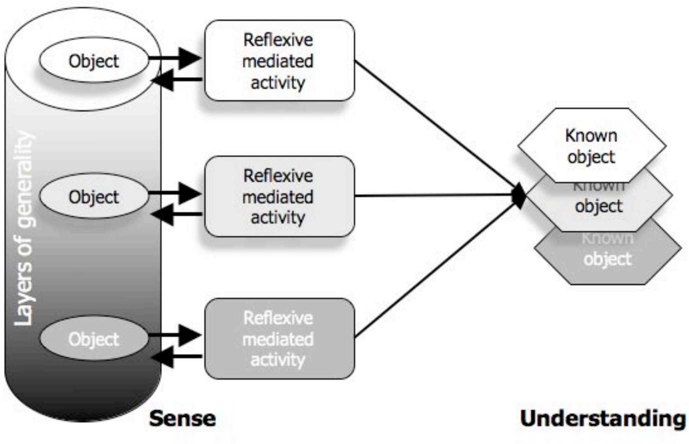
The objectification process entails mainly two difficulties on the part of the student:

1. The mathematical object is an entity stratified in *layers of generality*. Each layer of generality is associated with a particular reflexive activity determined by the characteristics of the semiotic means of objectification that mediate it. The diversity of the student's reflexive activities splits his intentional acts towards objects that he considers disconnected but, at an interpersonal level, are recognized as belonging to the same cultural entity. The objectification process therefore doesn't require a coordination of semiotic representations as such but of the different activities mediated by those representations.

2. Meaning has a strongly *embodied* nature (Radford, 2003; Lakoff, Nuñez) but at higher levels of generality the student has to employ formal and abstract symbols that brake the relationship with his *spatial and temporal experience*. Students have to experience a *disembodiment of meaning* that hinders the objectification of the interpersonal and general aspects of the mathematical object.

As we already mentioned above, when considering learning processes in the cultural semiotic perspective, thinking, mathematical objects, signs and meaning are indissolubly entangled through the reflexive activity. Analyzing students' sense-giving acts therefore requires to shift our focus from the duality object-representation to the reflexive activity that entangles objects, signs and meaning.

We propose the following schema to frame the issue of meaning of mathematical objects in the Semiotic Cultural approach



2.3 Ontosemiotic approach

Operational and referential phases

We have seen that Duval's approach recognizes an a-priori inaccessible mathematical object to which semiotic representations refer and that in Radford's approach mathematical objects lose such ideal existence, as they are bound to individuals' culturally framed reflexive activity.

The onto-semiotic approach (Godino, 2002; D'Amore, Godino, 2006; Font, Godino, 2007) also develops within a pragmatic theory of mathematical objects and generalizes the notion of representation, through the notion of semiotic function which relates an antecedent (signifier) with a consequent (signified): the role of representation is not played only by language but any object emerging from mathematical practices can be antecedent of a semiotic function; the ontosemiotic approach thus endows mathematics with its essentially relational character.

Such generalization of representations stems from a distinction, in the development of mathematical activity, between an *operational phase and a referential phase* (Ullman, 1962). The recognition of an operational and referential phases allows to overcome the alleged opposition between realistic and pragmatic points of view. Mathematical objects acquire meaning within a system of practices and it doesn't make sense considering them with an independent existence, nevertheless at a cultural and historical level it is possible to refer to the object and consider learning as an objectification process.

> «The meaning of mathematical objects starts in a pragmatic sense, relative to a specific context; but, amongst the types of use relative to that meaning, there are some that allow to orient the learning-teaching processes of mathematics. These types of uses are objectified through language and end up being referents of the institutional vocabulary» (D'Amore, Godino, 2006, p. 27).

With a different terminology, the same development of mathematical objects is recognized by the cultural semiotic approach:

«I mentioned previously, in addition to its social dimension, meaning also has a cultural-historical dimension which pulls the interaction up in a certain direction – more precisely, in the direction of the cultural conceptual object. […] Cultural conceptual objects are like lighthouses that orient navigators' sailing boats. They impress classroom interaction with a specific teleology» (Radfrod, 2006, p. 58).

The structural and functional approach is extremely effective in describing cognitive processes at a referential level. The coordination of semiotic systems, that characterizes mathematical thinking is the outcome of the operational phase; to understand its nature and functioning and learning difficulties we must resort also to the system of practices from which representations and their use originate.

Systems of practices and mathematical objects

In the Ontosemiotic approach the role of practice and systems of practices plays a central role in understanding the development of mathematics and its learning. Practices can be carried out at a personal or institutional level, giving rise to cognitive and epistemic dimensions respectively.

> «All kinds of performances or expressions (e.g., verbal and graphic), carried out by someone in order to solve mathematics problems, communicate the solution obtained to others, validate it or generalise it to other contexts and problems, are considered to be mathematical practice (Godino and Batanero, 1998). These practices might be idiosyncratic (e.g., the students'answers in Figure 1) or be shared within an institution (e.g., the teacher's practices implemented in the mathematics class). An institution is constituted by the people involved in the same class of problem-situations, whose solution implies the carrying out of certain shared social practices and the common use of particular instruments and tools» (Font, Godino, D'Amore 2007).

In the Ontosemiotic perspective the mathematical object is an emerging entity from the systems of practices that loses its a priori and ideal character but it is not understood only as a conceptual object, it is seen also with other attributes. According to the characteristics of the systems of practices, emerging mathematical objects are conceived as the following *primary entities*: situations, procedures, definitions, properties, arguments and languages. The primary entities are interrelated through the mathematical activity, forming a network of objects called *cognitive configurations*, if related to personal activity, or *epistemic configurations*, if related to institutional practices (D'Amore, Godino, 2006).

«For a more precise description of mathematics activity it is necessary to introduce six types of primary entities: situations, procedures, languages, concepts, properties and arguments. In each case, these objects will be related among themselves forming configurations, defined as the network of emerging and intervening objects of the systems of practices and the relations established between them» (Font, Godino, D'Amore, 2007)

Semiotic function

Wittgenstein's notion of "language game" (Wittgenstein, 1953) plays a central role in the Onto-semiotic approach with a normative role in the shared practices. Primary entities according to the language game they belong to can be seen as the following *cognitive dualities*: personal-institutional, unitary-systemic, expression-content, ostensive-non-ostensive and extensive-intensive (D'Amore, Godino, 2006).

We focus our attention on the expression-content duality to introduce the Semiotic Function. Hjelmslev's (1943) introduced the notion of *function of signs*, called by Eco (1979) *semiotic function*, the dependence between a text and its components and the components themselves. The Onto-semiotic approach generalizes such dependence also to the primary entities:

> «In the onto-semiotic approach a semiotic function is conceived, interpreting this idea, as the correspondences (relations of dependence or function between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by subject (person or institution) according to a certain criteria or corresponding code. These codes can be rules (habits, agreements) that inform the subjects about the terms that should be put in correspondence in the fixed circumstances. In this way, semiotic functions and the associated mathematics ontology take into account the essentially relational nature of mathematics and

generalize the notion of representation: the role of representation is not totally undertaken by language (oral, written, graphical, gestures, …)» (Font, Godino, D'Amore, 2007, pp. 3-4)

In the Onto-semiotic approach

«Meaning is the content of any semiotic function, that is to say, the content of the correspondences (relations of dependence) between an antecedent (expression, signifier) and a consequent (content, signifier, or meaning), established by a subject (person or institution, according to a distinct criteria or a corresponding code» (Font, Godino, Contreras, 2008, p.161).

This approach goes beyond the idea that meaning stems from a referential relation between an independent object and one of its possible representations. According to the Onto-semiotic approach we must think of meaning in terms of an object O_1 (antecedent), an object O_2 (consequent) and the rule that allows to establish the semiotic function between O_1 and O_2 considered as emerging primary entities.

Meaning is a relation established through the semiotic function between two pairs constituted by a system of practices and a configuration of objects. Semiotic transformations can be seen as the emerging aspect of a semiotic function that relates a representation R (antecedent) in a pair, system of practice – configuration of objects, with a representation S in another pair, system of practices- configuration of objects. If we generalize, meaning can be conceived as a relationship between a pair $P_1(SP_1,CO_1)$ and a pair $P_2(SP_2,CO_2)$ established by a semiotic function. The pairs are formed by systems of practices and configurations of objects and they allow both a macro analysis if we consider relations between the whole configuration and a micro analysis if we consider relations between primary entities of such configuration.

When facing the issue of meaning, considering mathematical knowledge in terms of one object–many representations is insufficient to grasp the whole of its complexity. As we mentioned above, mathematical objects, representations and meaning are entangled through activity. Such a net distinction between the mathematical object and its possible representations is effective when devoted to the cognitive operations in mathematics, but when investigating the teaching and learning processes as a whole such distinction is untenable for the following reasons:

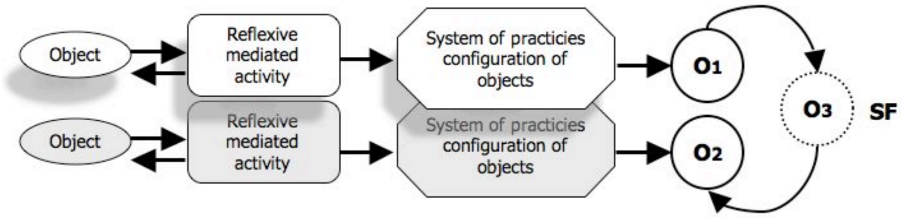
- It is very difficult to identify "the" mathematical object. Mathematical objects are stratified in layers of generality and organized in epistemic and cognitive configurations of primary entities.
- As an emergent from a system of practices, it is difficult to recognize a clear boundary between the object and the representations that mediate the practice. Of course, on the one hand we mustn't confuse the object with it's representation, but on the other if we try to separate the object from its representation we exclude the practices it emerged from. The unitary-systemic cognitive duality allows to take into account both the need to refer to the object and the social activity the objects comes from: a representation has a representational value, as something that stands for something else in a unitary sense; a representation has an instrumental value, as it sustains specific practices in a systemic sense (Font, Godino, D'Amore, 2007).
- In a semiotic function a representation can play both the role of expression and content. In a semiotic function relating two pairs systems of practices-configurations

of objects, a representation can play the role of expression in one of the configurations and of content in the other.

In the Onto-semiotic approach, meaning is a complex and holistic construct that, in a linguistic game, binds systems of practices, configurations of objects, cognitive dualities and the semiotic function. Meaning has a local value when we consider a particular system of practices obtained by a specific representation and it has a global value when we relate through the semiotic function the possible systems of practices involved in the emergence of a mathematical object.

> «We can decide that the meaning of a mathematical concept is the pair "epistemic configuration/practices it entails", where the definition of the concept (explicit or implicit) is one of the components of the epistemic configuration. When the concept has another equivalent definition the concept can be built into another pair "epistemic configuration/practices it entails", different from the pair considered before. In this case, each pair can be considered as a different "sense" of the concept, while the meaning of the concept is the set of all the pairs "epistemic configuration/practices it entails"» (Godino, 2002, p. 5-6, appendix).

The analysis in terms of one object-many representations is extremely effective at a referential level. The coordination of semiotic systems is at the core of mathematical thinking but we cannot disregard the operative phase in terms of systems of practices and configurations of objects in which such coordination is rooted. To understand how and why semiotic transformations occur, we resort to the semiotic function that relates pairs "systems of practices-configurations of objects".
We believe that the notion of objectification plays a central role in characterizing systems of practices, the configuration of objects and the criteria that connects the antecedent with the consequent in the semiotic function. Looking at practices in terms of reflexive mediated activity, provides a thorough understanding of the epistemic and institutional dimension involved in mathematical thinking and learning. Semiotic means of objectification tell us precisely the nature of the practice they accomplish thereby determining both the layers of generality and the configurations of emerging objects involved. The introduction of the semiotic function provides a more refined theoretical tool to analyse the issue of meaning in mathematical thinking and learning, that connects both practices and the relative emerging objects.
We propose the following schema integrating the notion of objectification and semiotic function to frame the issue of meaning in mathematics.

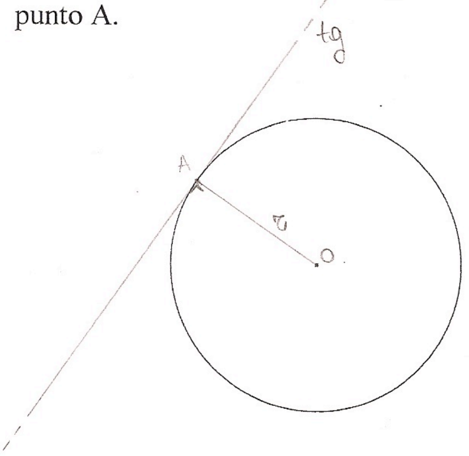


3 Analysis of a classroom episode

We propose the analysis of a protocol of a student taken from an experimentation carried out in a scientific secondary class in their final year (18-19 years old students), dealing with the concept of derivative of a real function. Without any influence in the design of the lesson, students were first videotaped during classes, than a test was given to them and finally they were interviewed.
After introducing the derivative as the limit of the incremental ratio, the teacher provided students with the geometrical interpretation: the slope of the tangent to the graph of the function. When dealing with the geometrical interpretation of the derivative in singular points, it came out that students didn't have a correct and comprehensive conceptualization of the tangent to the graph of a function.
We will analyse the answers of Laura to the questions of the test. The questions were designed in order to investigate if the meaning of the tangent to a curve changed as the graph changed after a semiotic treatment.
Below Laura's answer to question 1. The scan of the protocol is in Italian, the reader finds the questions and the student's answers translated in English.

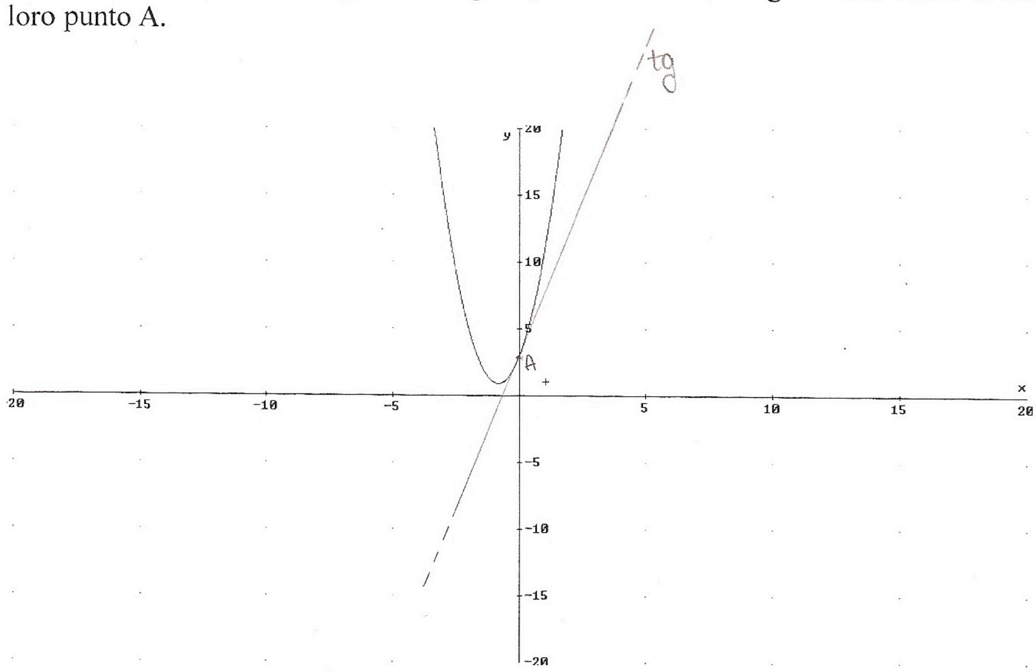
Domanda 1

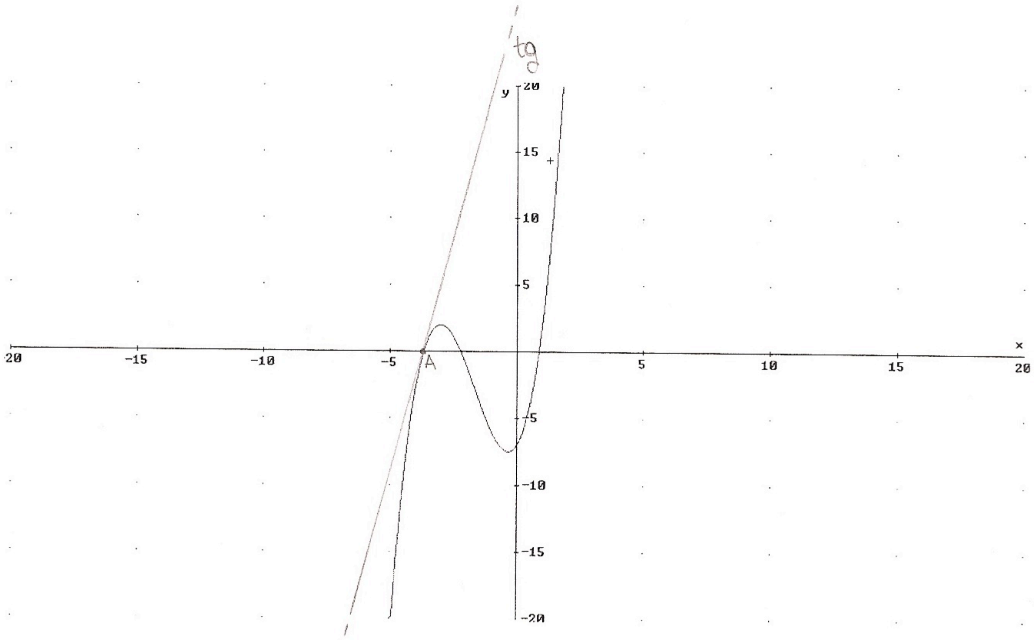
1.1 Data la circonferenza disegnata di seguito, traccia la retta tangente ad un suo punto A.

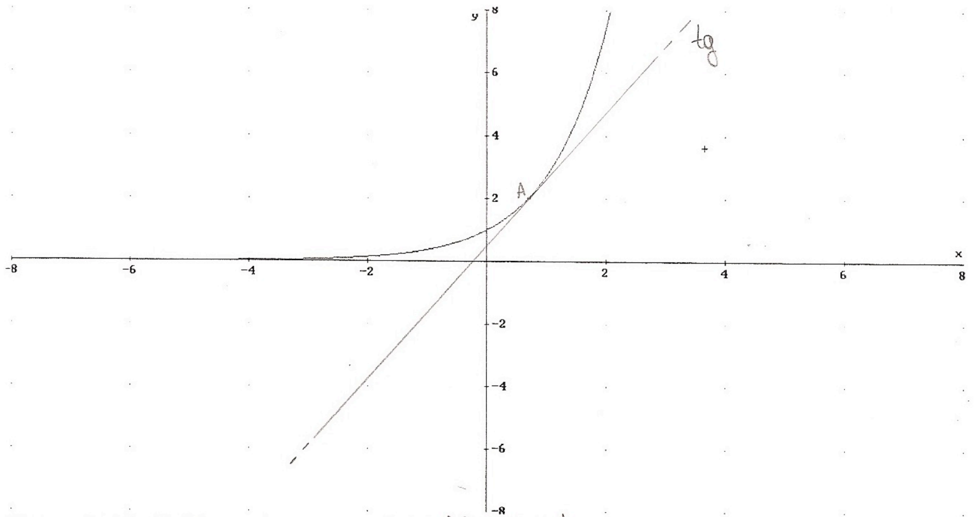


Come hai individuato la tangente? geometricamente, la tangente è la retta ⊥ al raggio nel punto A.
La retta tangente è unica nel punto A? tecnicamente sì,
Motiva le tue risposte. anche se il disegno è impreciso.

1.2 Considera le curve disegnate di seguito, traccia la retta tangente alle curve in un loro punto A.



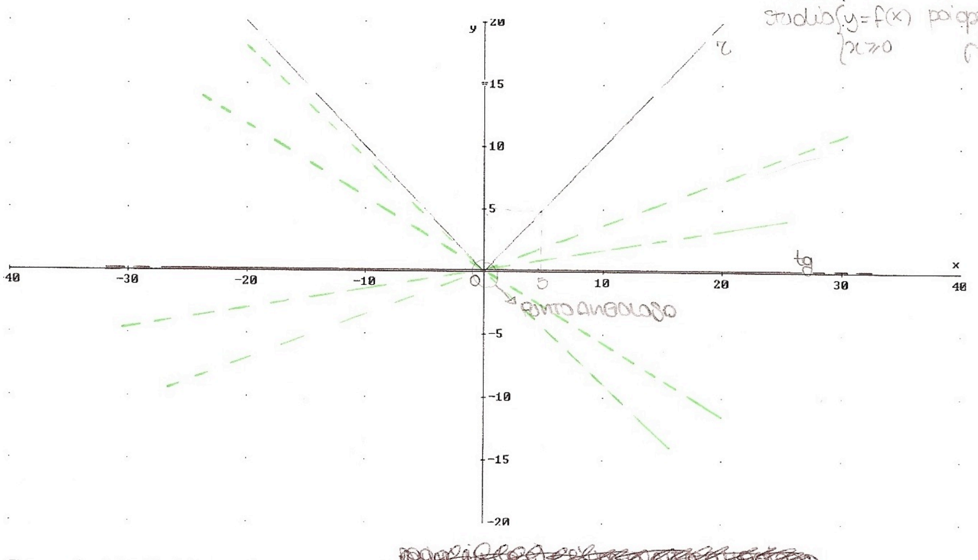




Come hai individuato la tangente? graficamente
La tangente è unica nel punto A. Sì, avendo l'equazione delle curve e calcolando la $f^I(x)$ dovrei ottenere l'equazione di una retta (che è la retta tg).
Motiva le tue risposte.

1.3 Considera la curva disegnata di seguito, traccia la retta tangente alla curva nel punto di coordinate (0,0).

r: y=|x|
è una f(x) del tipo y=f|x|.
studio {y=f(x), x≥0 poi specchio rispetto a y.



Come hai individuato la tangente ~~[illegible]~~
La tangente è unica nell'origine? Non lo so, graficamente mi sembra che ci possano essere più tg nel punto O ~~[illegible]~~ (tutte le rette passanti per (0;0) fino a quella con inclinazione appena inferiore alla retta r). Tuttavia credo che la tg debba essere unica, e che sia quella che coincide con l'asse x.
Motiva le tue risposte.

Question 1
1.1 Trace the straight line tangent to the circumference through a point A.
How did you determine the tangent?
L: *From a geometric point of view the tangent is the straight line perpendicular to the ray in point A.*
The tangent is unique in point A?
L: *Theoretically yes, even if the figure is imprecise.*

1.2 Trace the straight line tangent to the following curves in a point A.
How did you determine the tangent?
L: *Graphically*.
The tangent is unique in point A?
L: *Yes, with the equation of the curve and calculating the derivative in A, I should obtain the equation of a straight line, the tangent.*

1.3 Trace the straight line tangent to the curve through the point (0,0).
How did you determine the tangent?
The tangent is unique in point A?
L: *I don't know, it looks as if there could be more than a tangent in point O (all the straight lines passing through (0,0) with a slope slightly smaller than the one to the straight line r*). *Nevertheless I believe that the tangent must be unique, the one passing through the x axis.*

Laura's protocol highlights the difficulty in handling the meaning of the tangent, a difficulty shared with the other students of the classroom whose behaviour was contradictory when facing the singular point. We will analyse this protocol first from the Cultural Semiotic approach and after from the Onto-semiotic one.

Cultural-semiotic analysis
This extract shows Laura's endeavour in making sense of the mathematical object through the process of objectification described above, resorting to different semiotic means of objectification. From a semiotic point of view, in the graphic semiotic register there isn't a great difference between a circumference, a parabola and a cubic function linked through treatment transformations. The protocol also testifies a network of semiotic transformations that include treatments and conversions between different semiotic systems. Among such transformations, treatment is the main cause of difficulty when facing the meaning of the tangent. If we consider their graphs as semiotic means of objectification the reflexive activity they mediate is very different.
In the case of the circumference, the definition of the tangent allows a continuity between the use of semiotic means of objectification bound to the subjects' embodied experience as gestures and artefacts and the use of more abstract semiotic means of objectification as the graph and the specific language of Euclidean geometry. To the terms straight line, perpendicular and ray used in Laura's definition of the tangent, correspond perceptive and kinaesthetic acts, the use of artefacts as the ruler that combined also with the graph reinforce its meaning. Meaning is embodied and the concept of tangent to a circumference is strongly bound to the visual perception of the point of contact between the straight line and the graph. In this context, the concept of tangent is objectified by the student at a level of generality that Radford (2004) terms as contextual generalization, when the use of symbols are bound to the space and temporal experience of the student.
When we shift to the parabola or the cubic curve, Laura experiences a disembodiment of meaning; the reflexive activity is mediated mainly by symbolic means of objectification. The definition of a tangent to a parabola requires to introduce a linear system of the equation of the curve and the equation of the straight line or, at a higher level of generality, the calculation of the derivative. The reflexive activity is completely different from the one involved in the circumference: it doesn't make sense tracing the

perpendicular to the ray. The concept has moved to a higher layer of generality. Laura experiences a cognitive rupture that obliges her to go beyond her spatial-temporal experience in order to access a more general meaning of the concept. Radford (2004) terms symbolic generalization, sense-giving activities in which the use of formal and abstract symbols require to go beyond the spatial and temporal situated personal experience. The protocol testifies Laura's endeavour to achieve higher levels of generality of the concept of tangent. She is facing difficulties in coordinating the meanings emergent from the different activities she experienced during her educational path and there is a strong resistance in moving beyond perceptual embodied meanings.
In question 1.2 she resorts to perceptive aspects to determine the tangent to the parabola. The ruler she uses to draw the straight line is the key semiotic mean of objectification that mediates her perceptive activity although to justify the unicity of the tangent she uses the derivative. There is no explanation of how she obtained the tangent to the parabola.
Notice the strength of the perceptual dimension in the objectification process that "won" on the teacher's instructional action aiming at the general mathematical concept.
The perceptive idea that the tangent is the straight line that "touches" the curve in one point resisted throughout the sequence of the 3 questions proposed to Laura. Answering to question 1.3, Laura declares that the tangents to the curve through the origin are «all the straight lines passing by (0,0) with a slope slightly smaller than the one to the straight line r». Her answer expresses the strength of her spatial and kinaesthetic experience in giving sense to the tangent in the singular point. In the interview Laura declares that she imagined the straight line "oscillating" around the singular point without touching one of the half lines of the graph. It is interesting that the student recognizes the singular point, writes the symbolic expression of the function but she doesn't think of calculating the derivative of the function in (0,0) as she did in many exercises and problems assigned by her teacher. Learning as an objectification process is not a process of construction or reconstruction of knowledge but a path that requires a deep change within the student's consciousness.

> «Learning mathematics is not simply to learn *to do* mathematics (problem solving), but rather is leraning *to be* in mathematics» (Radford, 2008, p. 226).

This idea is expressed in terms of competences by Fandiño Pinilla (D'Amore, Godino, Arrigo, Fandiño Pinilla, 2003, p. 70) who distinguishes "competence *in* mathematics" and "mathematical competence":

> «*Competence in mathematics* is centred in mathematical discipline, recognized as an established discipline, as a specific object of knowledge. […] We recognize a conceptual and affective domain as a mediator between the pupil and mathematics. Competence is seen here within the school sphere. […] We recognize *mathematical competence* when the individual sees, interprets and behaves in the world in a mathematical sense. Then analytical or synthetic attitude with which some individuals face problematic situations, is an example of this competence. Taste and valorisation of mathematics are some of the useful aspects to orient the fulfilment of mathematical competence»

This example highlights how meaning cannot be bound to the structure of the semiotic systems that reflect the structure of an ideal mathematical reality. It is necessary to focus on the mediated reflexive activity and analyze signs not only as representations but as *mediators of shared practices*. This example shows how the semiotic key element is not conversion as claimed by the structural approach and that students can encounter learning difficulties also with treatments. The key element is the underlying system of mediated reflexive activities. In a different situation that involves conversions, we could make the same kind of analysis.

Onto-semiotic analysis

We widen our perspective to analyze Laura's protocol in terms of semiotic functions. In this example an analysis based on the model many representations for one object is insufficient to understand the network of meanings that Laura has to handle to objectify the concept of tangent.

Laura is facing 3 different "linguistic games" that are behind the system of practices and configurations of objects she has to handle:

- The linguistic game of Euclidean geometry with its set of rules that allows specific activities associated with configurations of objects. In this context the concept of tangent (as a primary entity) is defined as the straight line perpendicular to the ray in a point of the circumference.
- The linguistic game of analytic geometry with its set of rules that allows specific activities associated with configurations of objects. In this context the concept of tangent to a conic (as a primary entity) is defined as the straight line whose equation in a system with the equation of the curve gives a single solution to the system.
- The linguistic game of mathematical analysis with its set of rules that allows specific activities associated with configurations of objects. In this context the concept of tangent (as a primary object) is defined as the straight line passing through the tangent point of the graph of the function whose slope is the derivative of the function in the tangent point.

To learn the concept of tangent in its broad cultural meaning, the student has to handle a network of semiotic functions that involve the pairs system of practice-configuration of objects mentioned above.

In this example, we can interpret the layers of generality of the mathematical object as the semiotic functions that connect the primary entity concept of tangent according the cognitive duality extensive-intensive. The intensive facet refers to a class of objects considered as a whole and the intensive facet refers to a particular element of the class.

The student has to establish the following semiotic functions:

- A semiotic function SF1 with antecedent the concept of tangent in Euclidean geometry and consequent the concept of tangent in analytic geometry. In the cognitive duality extensive-intensive, the tangent is interpreted as an extensive object in Euclidean geometry and intensive object in analytic geometry.
- A semiotic function SF2 with antecedent the concept of tangent in analytic geometry and consequent the concept of tangent in mathematical analysis. In the cognitive duality extensive-intensive the tangent is interpreted as an extensive object in analytic geometry and as an intensive object in analysis.
- A semiotic function SF3 with antecedent the concept of tangent in Euclidean geometry and consequent the concept of tangent in mathematical analysis. In the cognitive duality intensive-extensive the tangent is interpreted as an extensive object in Euclidean geometry and as an intensive object in analysis.

Laura's difficulties in giving sense to the concept of tangent can be brought back to the absence of this network of semiotic functions. Laura is conversant with each of the above language games separately, she encounters difficulties when she has to establish semiotic functions between different pairs of systems of practices and configurations of objects. Laura "plays" the linguistic game of analytic geometry and analysis with the rules of Euclidean geometry and she inexorably falls in contradictions that the Onto-semiotic approach terms as semiotic conflicts (D'Amore, Godino, 2006).
Also within the Onto-semiotic approach the semiotic transformations as such don't play an essential role in objectifying the mathematical object. It is important recognizing the semiotic functions that are established when conversions or treatments are used in mathematical activity; from outside we recognize semiotic transformations but if we look at the process of learning from inside, the systems of practices students are involved in and the network of semiotic functions they are able to establish are the core of the sense giving activity.
We believe that, from an educational point of view, it is extremely important to understand how the students recognize the criteria that allow to relate the antecedent and the consequent in a semiotic function. The notion of objectification and semiotic means of objectification provide effective tools to understand the nature of the mathematical activity, to understand how signs mediate activity and recognize the cognitive ruptures students have to face in their learning process, for example when they have to disembody meaning.

4 Conclusions

In this paper we discussed the relation between mathematical objects, representations and meaning. We have shown how the idea that the meaning of a semiotic representation is its object of reference and reducing mathematical thinking and learning to a coordination of many representations with a common denotation is insufficient to account for the complexity of mathematics as an individual and cultural endeavour. The structural and functional approach to semiotics is an effective tool to understand mathematical thinking at a referential level: when analyzing students' behaviour we cannot skip the complicated network of semiotic systems they have to handle. Mathematical activity and therefore also mathematical learning is intrinsically a semiotic activity; without resorting to the transformation of signs within semiotic systems, mathematics wouldn't have developed into the refined form of rationality we know today. To understand how signs are used, we must consider, at an operational level, the coordination of semiotic registers as an emergent effect underlain by social and cultural elements termed as mediated reflexive activity in the Cultural-semiotic approach and systems of discursive and operative practices in the Ontosemiotic approach.
Mathematical objects, representations and meaning are entangled through activity that breaks the dual structure, object-representation and the many representations for a single object model.
We have singled out two theoretical tools that we believe provide insights to understand the issue of meaning in mathematics, the notion of objectification and semiotic function. The notion of objectification is an effective lens when we analyse the nature of the mathematical activity mediated by semiotic tools that allows to interpret meaning as a

relationship between a cultural dimension in which the mathematical object lives and the personal one.
The semiotic function coordinates the pairs systems of practices-emerging configurations of objects. Meaning is the consequent of a semiotic function in which the antecedent and the consequent can be any kind of "object". This gives a great flexibility in analyzing the issues related to meaning and, in terms of semiotic functions, a fixed distinction between objects and representation is untenable. The analysis is shifted to the systems of practices and their emerging configurations of objects, connected by the semiotic function according to one of the cognitive duality as the extensive-intensive used in our analysis.

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Le rôle de l'épistémologie de l'enseignant dans les pratiques d'enseignement

Bruno D'Amore, Martha Isabel Fandiño Pinilla, Ines Marazzani, George Santi, Silvia Sbaragli
N.R.D. – Université de Bologna

Résumé. Plusieurs épistémologies entrent en jeu dans l'action didactique; une formation épistémologique devrait donc jouer un rôle fondamental dans la formation des enseignants. Des concepts comme ceux de milieu, d'obstacles et de contrat didactique (dans toute leur épaisseur à la fois épistémologiques et didactiques) se prêtent à cette formation.
Cette question est ici discutée et exemplifiée par l'analyse de cas qui témoigneront des effets d'une formation épistémologique lacunaire sur l'action didactique des professeurs.

Mots-clés:
Epistémologie de l'enseignant; milieu; obstacles épistémologiques; obstacles didactiques; contrat didactique.

1. Vers une didactique définie comme épistémologie de l'enseignant

Le terme "épistémologie" et ses différentes acceptions ont été introduits en didactique des mathématiques à la fin des années 1960 et ont donné lieu à une multiplicité de "définitions" et d'interprétations dans le monde, dans les contextes les plus divers. Pour une analyse critique et comparée du terme et de ses occurrences, nous renvoyons à Brousseau (2006a, b).
Une analyse fine de l'épistémologie de l'enseignant permet d'opèrer un rapprochement entre la notion d'épistémologie et les notions de conviction, conception, savoir et connaissance.
En effet par *conception épistémologique,* nous désignons un ensemble de convictions, de connaissances et de savoirs scientifiques qui cherchent à définir les connaissances d'un individu ou d'un groupe d'individus et leur fonctionnement ainsi que les moyens d'évaluer leur validité, de les acquérir et donc de les enseigner et de les apprendre. L'épistémologie cherche à identifier et à unifier différentes conceptions épistémologiques relatives à certaines sciences, à des mouvements de pensée, à des groupes d'individus, à des institutions ou des cultures.
Pour les termes suivants, nous reprendrons les définitions de D'Amore, Fandiño Pinilla (2004):

- *Conviction* (ou croyance): opinion, ensemble de jugements et d'attentes, ce que l'on pense à propos de quelque chose;
- l'ensemble des convictions d'un sujet (A) sur quelque chose (T) donne la *conception* (K) de A à propos de T; si A appartient à un groupe social (S) et s'il partage avec les autres membres de S les mêmes convictions à propos de T, alors K est la conception que S a de T. Néanmoins, au lieu de la "conception que A a de T", on a tendance à parler de l'"image que A a de T".

Par *savoir*, nous entendons un ensemble de connaissances et de comportements qui peuvent être reproduits et qui ont été acquis à travers l'étude ou l'expérience.
On distingue les *savoirs* des *connaissances*:

- on entend par *savoirs*, les données, les concepts, les procédures, les méthodes qui existent en dehors de tout sujet connaissant et qui sont généralement codifiés dans des ouvrages de référence, des manuels, des encyclopédies ou des dictionnaires;
- les *connaissances* sont indissociables d'un sujet connaissant; en d'autres termes, il n'existe pas de connaissances a-personnelles; un individu qui intériorise un savoir *consciemment* transforme ce savoir en connaissance.

Brousseau introduit la notion d'*épistémologie scolaire* pour désigner l'ensemble des convictions -explicites ou implicites- qui circulent au sein de l'école, sur les méthodes, les objets et la finalité des connaissances, des enseignements et des apprentissages. L'*épistémologie scolaire* influe sur l'activité didactique et les programmes dans la mesure où elle influence profondément le choix des savoirs à enseigner, la méthodologie à adopter, les modèles d'apprentissage sur la base desquels l'enseignement doit être organisé.
Celle-ci doit être distinguée de l'*épistémologie de la société* qui se manifeste à travers certaines obligations comme par exemple l'obligation de résultat, la règle des conditions préliminaires suffisantes, la règle d'optimisation et le passage d'une étape à une autre; ces notions sont approfondies dans (Brousseau, 2008).
Les conceptions épistémologiques poussent les enseignants, souvent inconsciemment, à mettre en place des pratiques d'enseignement inadaptées qui renvoient l'apprenant en difficulté à un apprentissage personnel laborieux et qui finira pas l'éloigner des aapprentissage. Les conceptions épistémologiques des enseignants se manifestent à travers une série de comportements et de croyances comme par exemple:

- l'enseignant doit avoir enseigné tout ce qui, selon lui, doit être su;
- l'apprenant doit se souvenir de tout ce que l'enseignant a dit;
- et par conséquent tout devrait être appris par coeur;
- ou bien, l'apprenant devrait être à même d'inventer ou de deviner la réponse exacte le moment venu;
- ou bien, on suppose au contraire que ce qui a été compris est su et donc qu'il n'y a rien à étudier;
- ou que chercher une solution consiste à attendre la réponse miracle…

Ces comportements didactiques non reconnus comme tels , dans la plupart des cas, tendent à éloigner l'activité didactique de sa finalité spécifiquement mathématique et engendrent des stratégies d'évitement que Brousseau a systématisées en termes d'"effets": les enseignants cherchent et acceptent des réponses qui sont formellement correctes mais qui sont obtenues via des moyens rhétoriques dépourvus de toute valeur cognitive et didactique, comme par exemple suggérer la réponse à l'élève (effet Topaze), accepter une mauvaise raison ou une paraphrase (effet Jourdain), abuser des analogies ou de l'ostension, fragmenter le savoir à l'infini…
Nous remarquerons que l'évaluation est également fortement influencée par les conceptions épistémologiques des enseignants. A titre d'exemple, l'idée selon laquelle l'évaluation est influencée par l'épistémologie de la société a entraîné la diffusion de tests formels standardisés, plus simples à réaliser, à compléter et à analyser de manière superficielle. Cette approche de l'évaluation a conduit aux effets signalés par Brousseau dans la pratique scolaire. En voici quelques uns:
i) *La sous évaluation des apprenants*. En effet, par définition, les connaissances ne peuvent pas être évaluées en dehors des situations et notamment par des tests standards.

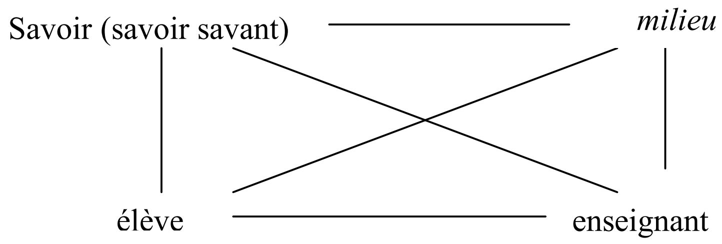
Aujourd'hui, l'évaluation interprète comme un échec le moindre écart par rapport à la norme d'apprentissage, d'où une multiplication dramatique des "cas" d'échec.
ii) *L'allongement illimité du temps d'enseignement.* Devant chaque "échec", l'enseignant se sent contraint de reprendre l'apprentissage dans sa totalité jusqu'à atteindre la forme de "savoir" de la connaissance. À cela s'ajoutent d'autres causes d'allongement du temps d'enseignement: la définition de l'enseignement et la fragmentation du savoir.
iii) *La définition de l'enseignement.* En réalité, cet allongement des temps d'apprentissage individuel augmente car l'enseignant *doit* créer, officiellement ou dans les faits, des groupes de niveau. Le processus débouche sur un enseignement de type individuel. Le temps que l'enseignant peut consacrer à chaque apprenant est alors insignifiant s'il ne s'agit pas d'un précepteur, à savoir d'un enseignant de cours particuliers à domicile. (Et les précepteurs ne peuvent bénéficier des processus réels de construction des mathématiques).
iv) *La fragmentation du savoir.* Tout "échec" porte à une décomposition en savoirs "plus élémentaires". Les liens entre les savoirs décomposés sont alors de plus en plus difficiles à établir. L'allongement du temps d'enseignement entraîne des conséquences catastrophiques en situation d'enseignement/apprentissage.
v) La concentration sur les savoirs (de bas niveau taxonomique) et donc sur les processus d'apprentissage à bas rendement (comportementalisme) accroît encore davantage le temps d'enseignement.
vi) *Des conséquences sociales* démontrées par les demandes réitérées d'alléger les programmes ou le nombre des objectifs de la part des enseignants.
Après cet aperçu général sur le rôle des conceptions épistémologiques, nous nous attacherons à décrire certains éléments de la didactique des mathématiques fortement liés à l'épistémologie des enseignants et susceptibles de la modifier positivement. Nous analyserons les concepts de milieu et de situation didactique, d'obstacle épistémologique par rapport à l'épistémologie spontanée des enseignants, et celui de contrat didactique afin de montrer combien les systèmes de convictions influent de manière décisive sur les processus d'enseignement/apprentissage.

2. Le milieu

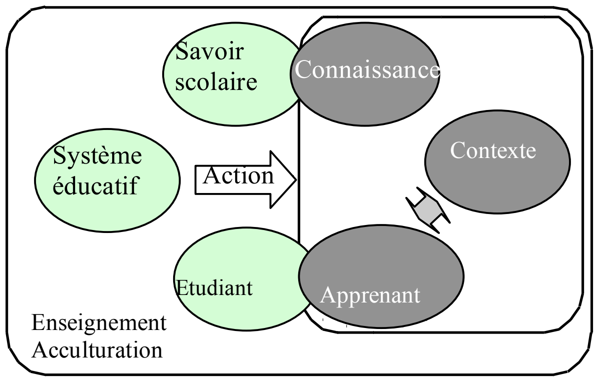
La théorie des situations nous apprend que l'enseignant doit savoir susciter chez l'apprenant des comportements que ce dernier, pour afficher sa connaissance, devrait acquérir de manière autonome. Ce qui semble paradoxal. Ou mieux: il s'agit bel et bien d'un paradoxe. La solution de la théorie des situations est d'impliquer un troisième élément, le *milieu*, et faire en sorte que la réponse de l'apprenant se rapporte exclusivement aux besoins du *milieu*.
L'art de l'enseignant consiste alors d'établir une relation entre l'apprenant et le *milieu* qui:

- d'une part, laisse une incertitude raisonnable que les connaissances du sujet doivent permettre de réduire;
- et qui, d'autre part, permette que cette réduction puisse effectivement se réaliser avec un degré d'incertitude limité, du point de vue de l'enseignant.

Le concept de milieu permet d'élaborer le schéma ci-dessous que nous appellerons le "quadrilatère" didactique:



Ce schéma présente des lacunes dans la mesure où il ne permet pas de distinguer les “savoirs” scolaires à enseigner ou déjà enseignés, des “connaissances” de l’apprenant qui ne coïncident pas et qui fonctionnent selon des modalités différentes. En outre, les sujets apprenants ne présentent pas les mêmes caractéristiques. Ainsi l’“hexagone didactique” proposé par G. Brousseau nous semble donc plus fonctionnel.



3. Les obstacles épistémologiques

Les études sur l’apprentissage des nombres naturels menées par Brousseau, dès le début des années 1960 jusqu’à la fin des années 1980, ont permis de démontrer que l’apprentissage se construit à travers des sauts de complexité “informationnelle” et que ce phénomène pouvait être généralisé en mathématiques.
Contrairement à ce qu’avançait Gaston Bachelard (1938) à propos de l’absence en mathématiques d’obstacles de type épistémologique, les recherches menées dans ce domaine ont justement permis d’aboutir à ce concept au sein de la recherche scientifique. La compréhension des nombres naturels exige, par exemple, de concevoir les nombres et leurs opérations selon une approche bien précise: un nombre naturel comme 4 a un successeur, son produit par un autre nombre naturel sera plus grand etc. Ces mêmes propriétés sont parfois erronées si 4 est un nombre rationnel n’a pas de successeur. Néanmoins, l’apprenant ne se rend pas compte de ce passage et continue à “forcer” les propriétés de N en les appliquant à Q; certains soutiennent, pour Q, que 2,33 est le successeur de 2,32 (encouragés par ailleurs par certains manuels). Prenons enfin l’exemple de $0{,}7 \times 0{,}8 = 0{,}56$ où 0,56 est plus petit que chacun des deux facteurs, résultat déconcertant qui remet en question les connaissances acquises précédemment.

L'apprenant ne remarque quasiment pas cette transformation du savoir. L'enseignant nomme multiplication ou division les nouvelles opérations que les apprenants devraient reconnaître et assimiler aux précédentes. La connaissance des nombres naturels est indispensable pour acquérir celle des nombres rationnels mais parallèlement elle représente un obstacle à l'apprentissage. Ce phénomène est à l'origine de malentendus et de difficultés à la fois importantes et invisibles dans la mesure où l'obstacle se cache, certes, à l'intérieur d'un savoir qui fonctionne mais il s'agit d'un savoir "local" qui ne peut être généralisé à l'objet mathématique à acquérir.
Voilà le sens même de la notion d'*obstacle épistémologique*. Ce concept participe à la formation de la conception épistémologique de l'enseignant et joue un rôle clé dans la transformation du savoir en connaissance. Il est donc essentiel de garantir aux futurs enseignants de mathématiques une préparation adéquate, à la fois historique et épistémologique. Il ne faut toutefois pas oublier qu'elle se greffe sur une épistémologie que l'on peut appeler *épistémologie spontanée des enseignants* (Speranza, 1997; Brousseau, 2006a).
Au moment de prendre leurs décisions en salle de classe, les enseignants ont recours de manière explicite ou implicite à tout type de connaissances, méthodes, convictions sur la manière de trouver, d'apprendre ou d'organiser un savoir. Ce bagage épistémologique est construit essentiellement de manière empirique afin de répondre aux besoins didactiques. Il s'agit parfois du seul moyen dont les enseignants disposent pour proposer les procédés didactiques qu'ils ont précédemment retenus et pour les faire accepter par les apprenants et leur environnement. L'ensemble des convictions des enseignants, des apprenants ou de leurs parents sur ce qu'il convient de faire pour enseigner, apprendre et comprendre les savoirs en jeu constitue une *épistémologie* pratique que l'on ne peut en aucun cas ignorer et rejeter. L'épistémologie philosophique ou scientifique peut difficilement prétendre endosser ce rôle.
L'épistémologie spontanée puise ses racines dans des pratiques ancestrales: la tendance à communiquer des expériences d'une génération à l'autre est un trait spécifique à l'humanité. L'opposer aux connaissances scientifiques serait absurde: il faut la respecter, la comprendre et l'utiliser de manière expérimentale comme tout phénomène naturel.
L'introduction de l'épistémologie et des théories scientifiques relatives à la formation des enseignants présente un nouvel avantage (D'Amore, 2004).
On assiste à l'application des deux formes d'épistémologie lorsque l'enseignant a recours à l'analogie pour aider l'élève en difficulté. Après des activités de soutien opportunes, l'enseignant propose une situation analogue où l'apprenant, convenablement "formé", est à même de résoudre le problème avec succès. On assiste à une fraude épistémologique car l'élève répond correctement sans qu'il y ait un véritable apprentissage à la fois solide et conscient qui puisse répondre aux attentes de l'enseignant. Nous retrouvons l'effet Jourdain, mentionné plus haut.
L'activité de l'apprenant doit répondre à deux contraintes incompatibles:

- une contrainte déterminée par les conditions adidactiques qui impliquent une réponse originale et l'organisation de connaissances spécifiques;
- une contrainte déterminée par les conditions didactiques qui ont pour but de générer la réponse attendue indépendamment des modalités de production.

Cet exemple montre que si l'épistémologie et les sciences cognitives peuvent étudier ou rendre compte des réponses des apprenants sous la première contrainte, elles ne peuvent prétendre aider les enseignants en ignorant la deuxième. Les contraintes didactiques finiront par opprimer les contraintes cognitives. Elles transforment la nature même des connaissances et leur fonctionnement. L'enseignement devient ainsi une simulation de la genèse des connaissances. Cette thématique illustre la complexité de l'épistémologie de l'enseignant qui ne peut se réduire à une dimension purement cognitive ou épistémologique mais qui remet en cause la complexité des processus d'enseignement/apprentissage que l'enseignant doit savoir gérer.

4. Le contrat didactique

Le contrat didactique, par sa force et ses implications, montre comment un système d'attentes, de convictions et d'interprétations sur les mathématiques, influencées également par l'épistémologie de l'enseignant, ont des répercussions lourdes, inattendues et surprenantes dans l'apprentissage des mathématiques.
Une expérience a été menée en classe de CE2 (élèves entre 8 et 9 ans) et de 5ème (12 - 13 ans) afin d'étudier les comportements des apprenants face à un problème où certaines données ont été omises (D'Amore, Sandri, 1998).

> Exemple:
> «Giovanna et Paola vont faire les courses. Giovanna dépense 10.000 lires et Paola dépense 20.000 lires. Après les achats, qui a le plus d'argent dans son porte-monnaie, Giovanna ou Paola?».

Ci-après un prototype du genre de réponses fournies par les élèves de CE2. Nous analyserons le protocole de réponse de Stefania, que nous reproduisons ci-dessous:

> Stefania:
>
> C'est Giovanna qui a le plus d'argent dans son porte-monnaie:
> 30-10=20
> 10×10=100

Dans la mesure où il s'agit d'un "contrat", nous avons identifié au fil du temps des *constantes de comportement* que l'on appellera "clauses".
Dans le cas présent, deux clauses jouent un rôle fondamental:

- *la clause des attentes*: l'enseignante attend une réponse, je dois donc fournir cette réponse, peu importe le sens du texte;
- *la clause de la constance*: l'enseignante a *toujours* donné des problèmes à résoudre sous la forme d'un texte rédigé avec des nombres, et pour fournir un résultat j'ai *toujours* effectué des opérations à partir de ces nombres; nous avons *toujours* travaillé comme ça, je dois forcément faire la même chose ici *encore*.

La réponse «Giovanna» (réponse fournie par 58,4% des apprenants de CE2 entre 8 et 9 ans) est justifiée dans la mesure où l'apprenant estime que si l'enseignant donne un problème, celui-ci *doit pouvoir être résolu*; ainsi, quand bien même l'apprenant se rend compte qu'il manque la somme de départ, il finit par l'inventer de manière implicite, plus ou moins de la sorte: «Je *dois* pouvoir résoudre ce problème; par conséquent, Giovanna et Paola avait sans doute la même somme de départ». Dans ce cas-là, *la réponse est correcte*: dans la mesure où Giovanna dépense moins, il lui reste forcément plus d'argent. Ce qui justifie la partie rédigée dans la réponse de Stefania. Un autre mécanisme se met alors en place qui est lié à une autre clause (du type: représentations des mathématiques, attentes supposées de la part de l'enseignant): «Ça ne peut pas

suffire, en mathématiques il faut faire des calculs, la prof attend sûrement des calculs». Dès lors, le contrôle critique s'écroule et... tous les calculs sont bons.
Dans D'Amore, Sandri (1998) (et dans d'autres études ultérieures), nous avons étudié dans les détails cette clause du contrat didactique que nous avons appelée "exigence de la justification formelle" ou ejf. Cette clause est également très présente au collège (apprenants entre 11et 14 ans): [58,4% des élèves en CE2 (8-9 ans) ont répondu «Giovanna» contre 24,4% en 5ème (12-13 ans); mais seuls 63,5% des élèves de 5ème affirment que le problème est impossible à résoudre; donc 36,7% des élèves fournissent une réponse: plus d' 1/3 en moyenne].
Nous reproduisons ci-dessous la réponse fournie par une élève de 5ème pour le même problème:

Silvia:

D'après moi, c'est Giovanna qui a le plus d'argent dans son porte-monnaie car:
Giovanna dépense 10.000 tandis que Paola dépense 20.000.

10.000	20.00
Giovanna	Paola

20.000-10.000=10.000 (Giovanna)
10.000+10.000=20.000 (Paola)

Dans le protocole de Silvia nous retrouvons les mêmes clauses du contrat didactique mises en oeuvre dans le protocole de Stefania mais son analyse est plus complexe. En premier lieu, on remarque un effort d'organisation logique et formelle. Tout d'abord, Silvia a fait le même raisonnement que Stefania: elle a répondu spontanément «Giovanna» sans faire de calculs; puis, en raison de la clause ejf, elle estime qu'elle *doit* fournir des calculs. Elle se rend probablement compte, sans doute de manière confuse, que ses opérations sont détachées de la logique du problème; elle effectue ces opérations uniquement parce qu'elle estime *devoir* le faire. Mais, aussi absurde que cela puisse paraître, l'élève finit par juger ses calculs plausibles. En effet, dans la mesure où elle est parvenue à tirer un résultat à partir de calculs –insensés- qui contrastent avec le résultat donné de manière intuitive, elle préfère remettre en question sa propre intuition et accepter la réponse obtenue de manière formelle. Après ses calculs, elle conclut que c'est Paola qui a le plus d'argent dans son porte-monnaie et non plus Giovanna, comme elle l'avait préalablement supposé; elle finit donc par barrer Giovanna et par ajouter Paola:

D'après moi, c'est ~~Giovanna~~ Paula qui a le plus d'argent dans son porte-monnaie car:
Giovanna dépense 10.000 tandis que Paola dépense 20.000.

10.000	20.00
Giovanna	Paola

20.000-10.000=10.000 (Giovanna)
10.000+10.000=20.000 (Paola)

C'est le contrat didactique, dicté ici par une représentation formelle (inefficace et nuisible) des mathématiques, qui l'a remporté sur la raison...

5. Convictions erronées des enseignants: quelques exemples

Au cours des dernières années, de nombreuses recherches se sont intéressées à l'analyse des convictions et des changements de convictions des enseignants sur différentes notions mathématiques. Ces études ont montré combien les convictions des enseignants influencent les pratiques de classe. On remarque en effet un lien de cause entre les convictions et les *misconceptions* dans la mesure où les *misconceptions* des apprenants découlent souvent directement des *misconceptions* des enseignants et de leurs convictions, selon la séquence suivante: conviction de l'enseignant → *misconception* de l'enseignant → *misconception* de l'apprenant → conviction de l'apprenant.
Analysons quelques exemples.

5.1. *Infini*
Dans Sbaragli (2006), on propose la synthèse d'une recherche menée sur plusieurs années sur les convictions et leurs changements chez des enseignants de l'école primaire en matière d'infini mathématique. Cette recherche a permis de montrer que cette notion est inconnue des enseignants de ce niveau scolaire aux plans mathématique et épistémologique comme au le plan cognitif. On retrouve ainsi, parmi les conceptions des enseignants, de nombreuses misconceptions qui touchent plusieurs domaines des mathématiques. En outre, le changement éventuel de convictions qui peut se produire chez certains enseignants face à un élémentaire traitement mathématique de l'infini mathématique montre bien combien les connaissances sur l'infini mathématique s'appuient uniquement sur des convictions spontanées et intuitives basées sur le bon sens. Il en découle, de la part des enseignants, un fort sentiment de gêne par rapport à ce type de savoir, une gêne qui a des répercussions négatives sur la transposition didactique.
Voyons ci-après un exemple de misconception. À la question: *Y a-t-il plus de points dans le segment AB ou dans le segment CD?* (les segments ont été tracés sur une feuille de sorte que CD soit plus long que AB), les 16 enseignants interrogés ont fourni les réponses suivantes:

> B.: *Dans le segment CD, forcément, il est plus long.*
> Chercheur: *Combien en plus*?
> B.: *Tout dépend de la longueur*.
> M.: *Tout dépend aussi de la largeur et s'ils sont juxtaposés. Mais si les deux segments sont rapprochés au maximum et s'ils sont de la même longueur, alors il y a plus de points dans le segment CD.*

Ces affirmations laissent apparaître le soi-disant "modèle du collier" selon lequel un segment est un fil formé de minuscules perles-points au contact les unes des autres; ce modèle a déjà été mis en évidence dans de nombreuses recherches (Arrigo, D'Amore, 1999, 2002).
Ces recherches ont permis de démontrer que des étudiants plus mûrs (en lycée) ne parviennent pas à s'approprier le concept de continuité en raison de ce modèle intuitif persistant. Grâce aux informations fournies par les enseignants, nous avons pu démontrer que ce modèle ne représente pas seulement un stratagème didactique auquel les enseignants ont recours pour donner aux apprenants une première idée de segment avant une démonstration plus correcte - conscients toutefois qu'il s'agit d'une représentation approximative très éloignée du concept mathématique de segment - mais

qu'il s'agit du modèle que les enseignants même ont du segment et qu'ils donnent donc comme modèle définitif à leurs propres apprenants. En outre, il ressort des conversations plusieurs lacunes au niveau des compétences des enseignants qui sont liées notamment aux concepts de densité et de continuité de l'ensemble ordonné des points de la ligne droite. Les lacunes n'apparaissent pas seulement à l'école élémentaire mais on les retrouve à tous les niveaux scolaires et chez tous les enseignants qui n'ont pas été amenés à réfléchir sur cette notion sous un angle épistémologique.
Lors d'une recherche menée ultérieurement dans ce domaine, (Sbaragli, 2007), l'auteur a recueilli des signes d'embarras de la part d'enseignants dans la construction conceptuelle de cette notion. Par exemple, certains enseignants déclarent expliciter à leurs propres élèves des affirmations que l'on retrouve dans la misconception de *dépendance* du cardinal des ensembles numériques:

> A.: *Je dis à mes élèves que tous les nombres: 0, 1, 2, 3, ... sont le double des pairs parce qu'il manque tous les impairs. Et puis, je leur dis que si nous ajoutons les négatifs, nous avons encore une quantité infinie de nombres par rapport à 0, 1, 2, ….*

Ce phénomène de dépendance s'explique par une approche qui reconnaît, dans tous les cas, la validité de la notion commune n° 8 d'Euclide: *Le tout est plus grand que la partie*, pour le fini comme pour l'infini.
Ces exemples montrent combien les intuitions des enseignants sont éloignées du "savoir institutionnel" souhaité en mathématiques. Les misconceptions sont transmises aux apprenants durant les pratiques de classe, ce qui entraînera des conséquences négatives en termes d'apprentissage aux niveaux scolaires plus avancés.

5.2. *Périmètre et Aire*
Dans Fandiño Pinilla, D'Amore (2007) et dans la recherche qui l'a précédé et rendu possible, (D'Amore, Fandiño Pinilla, 2005), on a relevé les erreurs des apprenants lorsqu'il s'agit d'évaluer les rapports entre l'aire et le périmètre des figures planes: ils ont tendance en effet à déduire de manière irrationnelle des majorations et des minorations entre des entités en rapport.
Par exemple, la littérature a amplement démontré que de nombreux apprenants de tout âge sont convaincus qu'il existe un lien étroit de dépendance entre les deux concepts sur le plan relationnel, du type:

Si A et B sont deux figures planes, alors:
- si (périmètre de A > périmètre de B) alors (aire de A > aire B)
- idem avec <
- idem avec = (ainsi: deux figures isopérimétriques ont forcément la même aire);
- et vice-versa, en inversant l'ordre périmètre – aire en aire – périmètre.

Une recherche menée sur ces enseignants a permis de démontrer que ce préconcept était également présent dans les convictions des professeurs. Nous observons ainsi que d'une part, les convictions des enseignants influencent nettement celles des apprenants; de l'autre, on retrouve une certaine disponibilité à modifier ses propres convictions, voire même au niveau du contenu.
On retrouve ces rapports forcés entre le périmètre et l'aire des figures planes dans l'histoire la plus reculée, dans le mythe et dans la légende à tel point que l'on peut

affirmer que le périmètre, l'aire et leurs relations réciproques représentent des obstacles épistémologiques. Si l'on examine les convictions des enseignants (de tout niveau scolaire) à ce sujet, on comprend très bien pourquoi ces objets mathématiques sont souvent traités de manière telle à représenter des obstacles didactiques.

5.3. *Fractions*

Dans Fandiño Pinilla (2005) et dans les travaux de recherche précédents et ultérieurs (cf. à titre d'exemple Campolucci, Fandiño Pinilla, Maori, Sbaragli, 2006), on a répertorié et classé d'un point de vue purement mathématique -sans aucune proposition efficace sur le plan didactique- une infinité d' "erreurs" que l'on retrouve de manière diffuse et qui, depuis des décennies, font l'objet d'études.

Les recherches préliminaires, et sans doute encore davantage les suivantes, ont démontré encore une fois que l'erreur est motivée et causée par les convictions des enseignants.

En effet, dans Campolucci, Fandiño Pinilla, Maori, Sbaragli (2006), on propose à ce sujet le compte-rendu d'une expérience d'apprentissage et de recherche mise en acte par un groupe de 36 enseignants (école maternelle, école primaire et collège). La notion de fractions jugée comme particulièrement complexe sur le plan conceptuel de la part des apprenants mais qui ne présente aucune difficulté d'un point de vue mathématique, a été abordée en premier lieu à l'occasion de cours de formation et lors de travaux collectifs en suivant l'ouvrage de Fandiño Pinilla (2005). L'approche consciente et adulte du point de vue mathématique, épistémologique et didactique a poussé les membres du groupe à exprimer leurs premières convictions mathématiques, épistémologiques et didactiques puis à prendre conscience des changements même notables qu'ils ont pu remarquer. Ils ont ainsi été amenés, toujours en termes de recherche-action, à revoir leurs propres positions en ce qui concerne la transposition didactique des fractions. La méthodologie retenue pour ce compte-rendu est celle de la réflexion personnelle (que certains appellent "autobiographie").

Par exemple, seuls quelques enseignants avaient d'ores et déjà réfléchi sur la définition de fraction, à savoir une unité divisée en parties "égales", et notamment sur le fait qu'ils utilisaient un terme plutôt générique qui doit être interprété selon les contextes. Le premier ouvrage répertorie 12 contextes très différents.

Par exemple, si on divise une figure plane en parties "égales", on entend "qui ont la même aire"; si on divise un nombre de personnes en parties "égales", on fait référence uniquement à un nombre; si on divise un nombre en parties "égales", alors il faut effectuer une opération de division (et le doute s'installe: parle-t-on de N ou de Q vu que l'opération de division n'est pas interne à N?); etc.

Et pourtant les enseignants affirmaient initialement:

> S.: *La fraction est une opération qui permet de diviser un entier en parties égales.*
>
> A.: *Pour moi, la fraction est quelque chose que l'on divise en parties égales, une division en quelque sorte. Mais la fraction divise 1 chose (un gâteau, un cercle, un bonbon, un objet), tandis que la division divise plusieurs choses (des nombres, des objets).*
>
> C.: *Le jour où je me suis présentée avec une tarte, j'avais 17 élèves en classe, j'ai donc décidé, pour un meilleur partage, de rajouter une part pour moi!*

Suite à ce parcours qui a débouché sur un changement notable au niveau des convictions des enseignants, voilà les affirmations qui ont été relevées:

A.: *D'après nous, l'image du gâteau partagé en plusieurs parties "égales" était efficace, elle permettait aux élèves de comprendre le rapport entre l'entier et ses parties. Cette image était aussitôt ancrée dans la tête de nos élèves et on pensait pouvoir passer immédiatement à la définition qui cristallisait le concept de fraction. Je me rends compte aujourd'hui que cette définition n'est pas assez précise et qu'elle ne tient pas compte des différents sens de fraction et des différents contextes d'utilisation. De plus, cette image est si simple qu'elle se fixe immédiatement: je pensais que c'était un avantage, mais je comprends maintenant qu'elle est source de difficultés.*

D.: *C'est vrai, ce maudit gâteau que j'apportais à l'école car je pensais que ça marchait très bien, il s'est inscrit de manière indélébile dans leur mémoire. J'étais persuadée qu'il suffisait d'enseigner les fractions comme je les avais moi-même apprises, mais je me trompais…Oh que je me suis trompée!*

Nous pourrions émettre les mêmes considérations que dans les exemples précédents. Pour les fractions, il semble n'y avoir aucun signe d'obstacles épistémologiques puisque la prise en charge des fractions par la communauté mathématique s'est vérifiée dans les temps les plus reculés (fin de l'Egypte ¯ 2000 et sans doute plus tôt). Néanmoins, une étude attentive et critique démontre le contraire. L'idée de fraction a constitué un moment de rupture remarquable et de crise dans l'évolution de l'histoire des mathématiques (Fandiño Pinilla, 2005). De plus, comme nous l'avons noté, les fractions représentent un obstacle didactique considérable.

7. Conclusion

La formation des enseignants est un sujet qui revête de plus en plus d'importance, non seulement pour la recherche en Didactique des Mathématiques mais également pour ses implications pédagogiques et sociales qui influent sur la société. En l'absence de résultats satisfaisants dans ce domaine de recherche, on pourra difficilement dépasser les difficultés cognitives et le sentiment d'aversion affective que la plupart des étudiants éprouvent pour les mathématiques. Le développement de la didactique des mathématiques comme épistémologie des mathématiques semble représenter un cadre théorique à même d'accueillir et de gérer la complexité de ce courant de recherche. L'épistémologie de l'enseignant que nous avons définie comme un système de convictions qui influe lourdement sur les processus d'enseignement/apprentissage des mathématiques interagit avec toutes les variables du système didactique.

Dans le présent travail, nous avons montré le rapport entre les conceptions épistémologiques de l'enseignant et certains éléments caractéristiques de la didactique des mathématiques et nous avons souligné que l'absence d'une culture épistémologiques adéquate risque d'éloigner l'enseignant des objectifs de la didactique. L'enseignement des mathématiques se réduit alors à un ensemble de techniques détachées les unes des autres qui permettent tout au plus d'atteindre de maigres résultats peu significatifs.

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Misconceptions and Semiotics: a comparison

George Santi	Silvia Sbaragli
N.R.D., Bologna, Italia *Università di Palermo, Italia*	*N.R.D., Bologna, Italia* *Università di Bologna e Bolzano, Italia* *Alta Scuola Pedagogica, Locarno, Svizzera*

Summary. *Following D'Amore's constructive interpretation for the term misconception, we propose a semiotic approach to misconceptions, within the theoretical frameworks proposed by Raymond Duval and Luis Radford.*

1. Introduction

In this article we deal with one of the most common terms for decades in Mathematics Education research, the word "*misconception*", interpreted according to a constructive perspective proposed by D'Amore (1999: p. 124): «A misconception is a wrong concept and therefore it is an event to avoid; but it must not be seen as a totally and certainly negative situation: we cannot exclude that to reach the construction of a concept, it is *necessary* to go through a temporary misconception that is being arranged». According to this choice, misconceptions are considered as steps the students must go through, that must be controlled under a didactic point of view and that are not an obstacle for students' future learning if they are bound to *weak and unstable* images of the concept; they represent, instead, an obstacle to learning if they are rooted in *strong and stable models*. For further investigation into this interpretation, look in D'Amore, Sbaragli (2005).

To understand what a misconception is, we believe it is necessary to make clear what is a concept and a conceptualization. Taking the special epistemological and ontological nature of mathematical objects as a starting point, we will show that mathematics requires a specific cognitive functioning that coincides with a complex semiotic activity immersed in systems of historical and cultural signification. This paper highlights that handling the semiotic activity is bristling with difficulties that hinder correct conceptual acquisition.

We will follow a constructive approach to misconceptions, analyzing them within the semiotic-cognitive and semiotic-cultural frameworks, upheld by Raymond Duval and Luis Radford respectively.

2. Theoretical framework

2.1 D'Amore's constructive approach to misconceptions

The problem of misconceptions developed within cognitive psychology studies, aiming at understanding the formation of concepts. In what follows, we refer to D'Amore (1999), but for the sake of brevity, we will not quote him.

This kind of approach focusses on the cognitive activity of the individual who is exposed to adequate stimuli and solicitations, and adapts his cognitive structures through assimilation and accommodation processes. The cognitive structures we mentioned above are characterised by two important functions that the human mind is able to perform: images and models formation.

The main characteristics of images and models are:

- Subjectivness, i.e. a strong relationship with individual experiences and characteristics.
- Absence of a proper sensorial productive input.
- Relation to a thought, therefore it does not exist per se, as a unique entity.
- Sensory and bound to senses.

An image is *weak* and transitory and accounts for the mathematical activity the pupil is exposed to in the learning process; it undergoes changes to adapt to more complex and rich mathematical situations set by didactical engineering as a path to reach a concept C.

A model has a *dynamical* character and it is seen as a limit image of successive adaptations to richer and richer mathematical situations. We recognise the limit image when a particular image doesn't need further modifications as it encounters new and more difficult situations. We can conclude that a model is a strong and *stable* image of the concept C the teacher wants the pupil to learn. A model among the images is the definitive one which contains the maximum of information and it is stable when facing many further solicitations. When an image is formed there are two possibilities:

- The model M is the correct representation for the concept C.
- The model M is formed when the image is incomplete and it had to be further broadened. At this point it is more difficult to reach the concept C, because of the strength of M towards changes.

The adaptive process the student has to handle in his path towards the construction of a concept gives rise to a cognitive and emotional *conflict,* since he has to move to a new cognitive tool when the one he was using was working well; we usually call such conflict an error and the student requires specific support on the part of the teacher.

An image that worked well, has become inappropriate in a new situation and needs to be broadened for further use of the concept, is called *a misconception.* In the constructive perspective we have chosen, a misconception is not seen as a negative phenomenon, as long as it is bound to weak images. As we have already said, misconceptions are necessary stages the pupil has to go through in his learning process, and they must be controlled under a didactic point of view to ensure they are bound to modifiable images, and not to stable models that would hinder the student's conceptual acquisition.

We propose a classical primary school example of this path that leads the pupil towards the conceptualisation, starting from an image and ending with a model, passing through a cognitive conflict.

A grade 1 primary school student has always seen the drawing of a rectangle "lying" on its horizontal base with its height vertical and shorter. He constructed this image of the concept "rectangle" that has always been confirmed by experience. Most textbooks propose this prototypical image:

At a certain point the teacher proposes a different image of the rectangle that has the base smaller than its height.

The pupil's spontaneous denomination in order to adapt the concept already assumed is extremely meaningful: he defines this new shape as "standing rectangle", opposed to the former "lying rectangle", which expresses the more inclusive character of this image.

This denomination testifies the positive outcome of a cognitive conflict between a misconception (an improper fixed image of the concept "rectangle") and the new image wisely proposed by the teacher. The student already had an image bound to his embodied sensorial activity and the teacher's new proposal, obliging the student to move to a higher level of generality of this mathematical object.

An example of a misconception bound to a model that hinders the pupil's cognitive development is of a grade 11 high school pupil dealing with second degree equations.

We propose the solution in an assessment of the following equation:

$2x^2+3x+5 = 0$

The student behaves as follows:

$2x^2 = -3x-5$

$x = \pm\sqrt{(-3x-5)/2}$

At this point, he is unable to go further, even with the teacher's help. We highlight that the solution of second degree equations had already been explained to the class.

In this example, we can see how the procedure for the solution of first degree equations condensed into a strong model that didn't change even after the teacher's further explanations and mathematical activities.

This example shows that a misconception is not a lack of knowledge or a wrong concept, but knowledge that doesn't work in a broader situation.

In this purely psychological perspective, the construction of concepts in mathematics is independent of the semiotic activity. Signs are used only for appropriation and communication of the concept, *after* it has been obtained by other means. In mathematics, both when dealing with the production of new knowledge and with teaching-learning processes, this position is untenable, due to the ontological and epistemological nature of its objects.

In fact, we witness a reverse phenomenon: «Of course, we can always have the "feeling" that we perform treatments at the level of mental representations without explicitly mobilising semiotic representations. This introspective illusion is related to the lack of knowledge of a fundamental cultural and genetic fact: the development of mental representations is bound to the acquisition and interiorisation of semiotic systems and representations, starting with natural language» (Duval, 1995, p. 29).

2.2 Duval's semiotic-cognitive approach

Every mathematical concept refers to "non objects" that do not belong to our concrete experience; in mathematics ostensive referrals are impossible, therefore every mathematical concept intrinsically requires to work with semiotic representations, since we cannot display "objects" that are directly accessible.

The lack of *ostensive* referrals led Duval to assign the use of representations, organized in semiotic systems, a constitutive role in mathematical thinking; from this point of view he claims that there *isn't noetics without semiotics*. «The special epistemological situation of mathematics compared to other fields of knowledge leads to bestow upon semiotic representations a fundamental role. First of all they are the only way to access mathematical objects» (Duval, 2006).

The peculiar nature of mathematical objects allows outlining a specific cognitive functioning that characterises the evolution and the learning of mathematics. The cognitive processes that underlay mathematical practice are strictly bound to a complex semiotic activity that involves the coordination of at least two semiotic systems. We can say that conceptualisation itself, in Mathematics, can be identified with this complex coordination of several semiotic systems.

Semiotic systems are recognizable by:

- Organizing rules to combine or to assemble significant elements, for example letters, words, figural units.
- Elements that have a meaning only when opposed to or in relation with other elements (for example decimal numeration system) and by their use according to the organizing rules to designate objects (Duval, 2006).

Duval (1995a) identifies conceptualisation with the following cognitive-semiotic activities, specific for Mathematics:

- *formation* of the semiotic representation of the object, respecting the constraints of the semiotic system;
- *treatment* i.e. transformation of a representation into another representation in the same semiotic system;
- *conversion* i.e. the transformation of a representation into another representation in a new semiotic system.

The very combination of these three "actions" on a concept can be considered as the "construction of knowledge in mathematics"; but the coordination of these three actions is not spontaneous nor easily managed; this represents the cause for many difficulties in the learning of mathematics.

Duval bestows upon conversion a central role in the conceptual acquisition of mathematical objects:

«(…) registers coordination is the condition for the mastering of understanding since it is the condition for a real differentiation between mathematical objects and their representation. It is a threshold that changes the attitude towards an activity or a domain when it is overcome. (…) Now, in this coordination there is nothing spontaneous» (Duval, 1995b).

The coordination of semiotic systems, through the three cognitive activities mentioned above, broaden our cognitive possibilities because they allow transformations and operations on the mathematical object. When the object is accessible, distinguishing the representative from its representation and recognizing the common reference of several representations bound by semiotic transformations is guaranteed by the comparison between each single representation with the object. In Mathematics the situation is more complicated, because there is no object to carry out the distinction mentioned above and to guarantee the common reference of different representations to the object. The lack of ostensive referrals makes the semiotic activity problematic in terms of production, transformation and interpretation of signs.

From an educational point of view, this is a fundamental issue that leads the student to confuse the mathematical object with its representations and requires a conceptual acquisition of the object itself to govern the semiotic activity that in turn allows the development of mathematical knowledge. This self-referential situation is known as *Duval's cognitive paradox:* «(...) on one hand the learning of mathematical objects cannot be but a conceptual learning, on the other an activity on the objects is possible only through semiotic representations. This paradox can be for learning a true vicious circle. How could learners not confuse mathematical objects if they cannot have relationships but with semiotic representations? The impossibility of a direct access to mathematical objects, which can only take place through a semiotic representation leads to an *unavoidable* confusion. And, on the other hand, how can learners master mathematical procedures, necessarily bound to semiotic representations, if they do not already possess a conceptual learning of the represented objects?» (Duval, 1993, p. 38).

In the example that follows, given by Duval (2006) at the beginning of high school, we can see how the semiotic activity, in this case conversion, is crucial for the solution of the problem. Students encounter difficulties finding the solution because they are stuck on the fractional representation of rational numbers or, worse, they consider fractions and decimal representation different numbers. The mathematical procedure is grounded on the cognitive semiotic activity. The mathematics involved is very simple but the semiotic task is not trivial.

$1+1/2+1/4+1/5+\ldots = 2$

The following conversion solves the problem brilliantly, shifting from the fraction representation of rational numbers to the decimal one.

$1+0.5+0.25+0.20+\ldots = 2$

2.3 Radfrod's semiotic-cultural approach

Within the semiotic path we follow to understand mathematical thinking, we make a step forward and move on to Radford's semiotic- cultural framework.

Radford's theory of knowledge objectification, considers thinking a *mediated reflection* that takes place in accordance with the mode or form of *individuals' activity* (Radford, 2005):

- The *reflexive* nature refers to the relationship between the individual consciousness and a culturally constructed reality.

- The *mediated* nature refers to the means that orient thinking and allows consciousness to become aware of and understand the cultural reality; Radford calls such means *Semiotic Means of Objectification* (Radford, 2002). The word semiotic is used in a broader sense to include the whole of the individuals embodied experience that develops in terms of bodily actions, use of artifacts and symbolic activity: artifacts, gestures, deictic and generative use of natural language, kinaesthetic activity, feelings, sensations and Duval's semiotic systems. Semiotic Means of Objectification mustn't be considered as practical and neutral technical tools, but they incarnate historically constituted knowledge. They bare the culture in which they have been developed and used. The semiotic means determine the way we interpret and understand reality that is given through our senses. The mediated nature of thinking is constitutive of our cognitive capabilities and makes thinking culturally dependent.
- *Activity* refers to the fact that mediated reflection is not considered here a solitary purely mental process, but it involves shared practices that the cultural and social environment considers relevant.

Before analyzing the learning process, we need to deal with the notion of mathematical object in Radford's objectification theory. Going beyond realist and empiristic ontologies, the theory of knowledge objectification considers mathematical objects culturally and historically generated by the mathematical activity of individuals. In agreement with the mediated reflexive nature of thinking and from the viewpoint of an anthropological epistemology Radford claims that «(…) Mathematical objects are fixed patterns of activity embedded in the always changing realm of reflective and mediated social practice» (Radford, 2004; p.21).

Learning is an objectification process that allows the pupil to become aware of the mathematical object that is culturally already there, but it is not evident to the student. Ontogenetically speaking, the student carries out a reflection on reality, not to construct and generate the object as it happens phylogentically, but to make sense of it. Learning is therefore an *objectification* process that transforms *conceptual and cultural* objects into objects of our *consciousness*. In this meaning-making process, the semiotic means of objectification within socially shared practices allow the student's individual space-time experience to encounter the general disembodied cultural object.

The access to the object and its conceptualization is only possible within a semiotic process and it is forged out of the multifaceted dialectical interplay of various semiotic means, with their range of possibilities and limitations. This multifaceted interplay synchronically involves, within reflexive activity, bodily actions, artefacts, language and symbols. At different levels of generality these three elements are always present. For example, at the first stage of generalization in algebra students have mainly recourse to gestures and deictic use of natural language, whereas in dealing with calculus the use of formal symbolism will be predominant, nevertheless without disregarding the kinaesthetic activity or the use of artefacts.

In the objectification process the student lives a conflict between his reflexive activity situated in his personal space-time embodied experience and the disembodied meaning of the general and ideal cultural object. The teaching-learning process has to face the dichotomy between the phylogenesis of the mathematical object and the ontogenesis of the learning process. The cognitive processes phylogenetically and ontogenetically involve the same reflexive activity, but with a significant difference: in the first case the mathematical object emerges as a fixed pattern; in the second case the object has its independent existence and the didactic engineering has to devise specific practices to allow the student becoming aware of such object.

To heal the conflict between embodied and disembodied meaning, the student has to handle more complex and advanced forms of representation «that require a kind of *rupture* with the ostensive gestures and contextually based key linguistic terms that underpin presymbolic generalizations» (Radford, 2003: p. 37).

The following example proposed by Radford (2005) shows the difficulty students encounter when they have to use algebraic symbolism that cannot directly incorporate their bodily experience. Students were asked to find the number of toothpicks for the n-th figure of the following sequence.

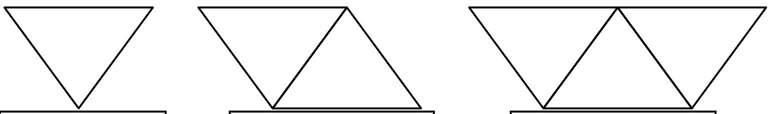


Fig. 1 Fig. 2 Fig. 3

After resorting to gestures, deictic use of natural language, students manage to write the algebraic expression n+(n+1) [n is the number of the figure in the sequence], but they are not ready to carry out the trivial algebraic transformation that leads to 2n+1. The parentheses have a strong power in relating the algebraic representation to their visual and spatial designation of the figure, disregarding them implies a disembodiment of meaning that it is not easily accepted. Even though 2n+1 is synthactically equivalent to n+(n+1), the former expression requires a rupture with spatial based semiotic means of objectification and a leap to higher levels of generality.

3. Misconceptions: a semiotic interpretation

The semiotic approach we have outlined in the previous sections provides powerful tools to understand the nature of misconceptions. From what we said, the path that from weak images leads to strong models can be seen as the interiorisation of a complex semiotic activity; the student has acquired a correct *model* of the concept when he masters the coordination of a *set of representations*, relative to that concept, that is *stable* and effective in facing diverse mathematical situations. The student acquires control of an adequate set of representations, through an adaptation process that enlarges the representations of the set and coordinates them in terms of semiotic activity. From a semiotic point of view an i*mage*, is a *temporary* set of representations that needs to be developed, both in terms of representations and of their coordination, as the student faces new and more exhaustive solicitations.

A misconception is a set of representations that worked well in previous situations but it is inappropriate in a new one. If a misconception is relative to a weak image the student is able to enlarge the set of representations and he is also ready to carry out more complex semiotic operations. In this case a misconception is a necessary and useful step the student must go through. If, instead, a misconception is related to a strong model the student will refuse to incorporate new representations and commit himself to more elaborated semiotic transformations. At this point, the pupil's cognitive functioning is stuck and he is unable to solve problems, deal with non standard mathematical situations and broaden his conceptual horizon. His reasoning is bridled in repetitive cognitive paths related to the same representations and transformations. In this case, a misconception is a negative event that must be avoided.

The representations we mentioned above are Radford's Semiotic Means of Objectification, including also Duval's semiotic systems. We can broaden D'Amore's (2003, p.55-56) *constructivist* view point of mathematical knowledge based on Duval's semiotic operations (formation, treatment and conversion) on semiotic systems, to include also bodily activity and artefacts and deal with more general Semiotic Means of Objectification. The positive outcome of the construction of a mathematical concept is therefore the dialectical interplay of Semiotic Means of Objectification that includes also treatments and conversions on semiotic systems. Such positive outcome is not a plain solitary process but it is culturally embedded in shared activity and it must overcome three synchronically entangled turning points that give rise to misconceptions; for sake of clarity we will discuss them separately but to show how entangled they are we will propose always the same example to explicit them.

- The first turning point we discuss is the *cognitive paradox*. The first and only possible approach to the mathematical object the student has is with a particular semiotic means of objectification. It can be an artefact, a drawing or a linguistic expression. He necessarily identifies the object with the first representation he encounters and connecting it with others is not spontaneous and requires a specific didactic action to go through this misconception. The student spontaneously sticks to the first representation that worked well in the situation devised by the teacher, but he is in trouble when a new situation requires to connect the first representation to a new one, because he believes that such representation *is* the mathematical object.

We can take the prototypical example of the rectangle we analyzed in section 2.1. In primary school the first access to the rectangle usually is a drawing with the base longer than the height. The student thinks that the object rectangle is *that* drawing with *those* specific perceptual characteristics. He is in trouble when the teacher proposes the new representation; he calls it "standing rectangle". If the teacher hadn't exposed the student to a new solicitation that first misconception would have condensed into a model, hindering the pupil's further cognitive development.

- The second turning point is the *coordination* of a variety of representations. In terms of Semiotic Means of Objectification the student has to handle a very complicated situation. First of all, the semiotic means can be very different from each other in terms both of their characteristics and the way they are employed. For instance a gesture is very different from an algebraic expression. The first one is used spontaneously, while the second is submitted to strict syntactic rules. The first one is related to the kinaesthetic activity, whereas the second one is a semiotic system that does not incorporate the students' kinaesthetic experience in a direct manner. An algebraic expression requires treatment and conversion transformations, while these operations are impossible with gestures. The student has to handle a semiotic complexity that leads to misconceptions mainly related to the coordination of semiotic means. The interplay of heterogynous Semiotic Means of Objectification is not spontaneous and it requires a specific didactic action.

Let us turn back to the example of the rectangle. In his cognitive history the student will have to coordinate more and more representations of this object. We have seen that he started with a very simple drawing, perceptively effective. The teacher proposes a treatment that leads the student to consider a new representation that is in conflict with the previous one. This is not enough to construct a model of the rectangle. As the mathematical problems become more complicated he will need to resort, through conversions, to other semiotic systems like natural language, the cartesain system or the algebraic one. We can ask him if a square is a rectangle, at this point he needs to combine his perceptual experience bound to the figural semiotic system with the definition given in natural language. Many students cannot accept that a square is a rectangle. In high school we could ask him to calculate the area of a rectangle obtained by the intersection of four straight lines given as first degree equations. Although the problem is simple from a mathematical point of view, it puzzles the student because of a complex semiotic activity that involves conversions between cartesian and algebraic representations. In this case, conversion is a heavy task to accomplish because of non congruence phenomena (Duval, 2005a, pp. 55-59). The student has to face a misconception that will cause a compartmentalization of semiotic systems, hindering his semiotic degree of freedom.

The coordination of many representations is a source of misconception, also because, as recent researches in the field conducted by (D'Amore, 2006) show, semiotic transformations change the sense of mathematical objects. For the student each representation has its own meaning related to the nature of the semiotic means of objectification and to the shared practices on the object carried out through such representation. The misconception of the rectangle is a good example of this phenomenon. Students bestow different senses upon each representation, at such a point that the child calls them "lying" and "standing" rectangles, as if they were different objects. It turns out that keeping the same denotation of different representations is a cognitive objective difficult to acquire because it demands to handle many representations without accessing what is represented.

- The last turning point we want to discuss regards the *disembodiment of meaning*. We have seen that there is a dichotomy between the space-time situated embodied experience of the pupil and the disembodied general mathematical object. The student lives a conflict between the embodied and situated nature of his personal learning experience and the disembodied general nature of the mathematical object. The mathematical cognitive activity of the child cannot start but in an embodied manner resorting mainly to Semiotic Means of Objectification related to bodily actions and the use of artefacts. But, when the mathematical activity requires a higher level of generality, the student must also engage in abstract symbols; the toothpicks shows how difficult it is for the student to give up his space situated experience, and how the algebraic language is meaningful to him as long as it describes his contextual activity. The conflict between situated experience and the generality and abstraction of the mathematical object is a source of misconceptions. At present, it is not completely clear how the disembodiment of meaning takes place. We know that the disembodiment of meaning requires the coordination of Duval's semiotic systems, in terms of treatment and conversion, and what we usually do is to expose students to an abstract symbolic activity, aware that we must handle the rise of misconceptions. Turning back to the example of the rectangle, the "lying" rectangle and the "standing" one are symptoms of the embodied meaning bound to the student's perceptual and sensorial experience. The treatment between the two figurative representations implies a disembodiment of meaning that must continue as natural language and other semiotic systems will be introduced so that the pupil can grasp the general and abstract sense of the rectangle.

We have presented a thorough analysis of misconceptions from a semiotic perspective. Anyway, it is possible to single out from what we have said a pivot upon which the issue of conceptualization and misconception turns, i.e. the lack of ostensive referrals of mathematical objects. The inaccessibility of mathematical objects both imposes the use semiotic representations and makes the semiotic activity intrinsically problematic.

4. A first classification of misconceptions

From what we have said above, on the one hand it seems that misconceptions are somehow a necessary element of the learning of mathematics and on the other the role of the teacher is crucial to overcome them by supporting the student's ability to handle the semiotic activity. We have, hence, divided misconceptions into two big categories: *"unavoidable"* and *"avoidable"* (Sbaragli, 2005); the first *does not depend directly on the teacher's didactic transposition*, whereas the second *depends exactly on the didactic choices and didactic engineering devised by the teacher.* Avoidable misconceptions derive directly from teachers' choices and improper habits proposed to pupils by didactic praxis. Unavoidable misconceptions derive only *indirectly from teachers choices* and are bound to the need of beginning from a starting knowledge that, in general is not exhaustive of the whole mathematical concept we want to present.

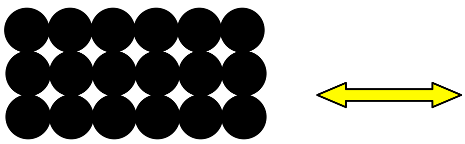
We will analyze avoidable and unavoidable misconceptions referring to the three turning points mentioned above.

4.1 "Unavoidableness"

"Unavoidable" misconceptions, that do not derive from didactical transposition and didactic engineering, depend mainly on the intrinsic unapproachableness of mathematical objects. Duval's (1993) paradox is a source of misconceptions that gives rise to an *unavoidable* confusion between semiotic representations and the object itself, especially when the concept is proposed for the first time. Another source of unavoidable misconceptions derives from the conflict between embodied and disembodied meaning of the mathematical concept. When the student learns a new mathematical concept he cannot begin to approach it with Semiotic means of Objectification related to his practical sensory-motor intelligence. These Semiotic Means of Objectification can lead the student to consider relevant "parasitical information", in contrast with the generality of the concept, bound to the specific representation and the perceptive and motor factors involved in his mathematical activity. The student *unavoidably* misses the generality of the mathematical object and grounds his learning only on his sensual experience.

The following example highlights an unavoidable misconception.

We know from literature (D'Amore and Sbaragli, 2005) that a typical misconception, rooted in the learning of natural numbers is that the product of two numbers is always greater than its factors. When students pass to the multiplication the product of two numbers is always greater than its factors. They are stuck to the misconception that "multiplication always increases". This is true in N and it is reinforced by the embodied meaning enhanced by the array model of multiplication perceptually very strong, effective in the first stages of students' learning of arithmetic and in strong agreement with the idea of multiplication as a repeated sum. We can see that there is strong congruence between the figural representation and the symbolic one that makes conversion very natural.



$6\times3=6+6+6$

When we pass to Q and consider 6×0.2 what does it mean to sum 6, 0.2 times, and what is an array with 0.2 rows and 6 columns?

We can see how the strong identification of the mathematical object with its representation hinders the development of the concept, and it is also clear that this identification is an unavoidable passage.

This example clearly shows, on the one hand, the rupture that leads from embodied to disembodied general meaning, the student has to go through when he faces rational numbers and how difficult it is to give up the perceptual and sensory evocative power of the array. On the other hand, it is also evident that we cannot avoid the embodied meaning skipping directly to a general and formal definition of multiplication.

The array is an effective Semiotic Means of Objectification when the student *begins* to learn multiplication in N, but if there is no *specific didactic action* that fosters the generalizing process towards the mathematical concept, it condensates into a strong model, difficult to uproot. The array image of multiplication is a typical example of a parasitical model. This last remark opens the way for the discussion of avoidable misconceptions.

4.2 "Avoidableness"

Avoidable misconceptions derive directly from *didactic transposition and didactic engineering*, since they are a direct consequence of the teachers' choices.

We have seen that the cognitive paradox and disembodiment of meaning give rise to unavoidable misconceptions. Nevertheless the teacher has an important degree of freedom to intervene in the students' ability to handle the semiotic activity. Even if misconceptions are unavoidable they must be related to images without becoming stable models. This is possible if the student is supported in handling the complex semiotic activity, within socially shared practices, that fosters the *cognitive rupture*, allowing the pupil to incorporate his kinaesthetic experience in more complex and abstract semiotic means. The student thus goes beyond the embodied meaning of the object and endows it with its cultural interpersonal value. In this perspective, Duval (1995) offers important didactic indications to manage the rupture described above, when he highlights the importance of exposing the student, in a critical and aware manner, to many representations in different semiotic registers, overcoming also the cognitive paradox. Nevertheless didactic praxis is "undermined" by improper habits that expose pupils to univocal and inadequate semiotic representations, transforming avoidable misconceptions in strong models or giving rise to new ones.

An emblematic example of an inadequate semiotic choice that brings to improper and misleading information relative to the proposed concept, regards the habit of indicating the angle with a "little arc" between the two half-lines that determine it. Indeed, the limitedness of the "little arc" is in contrast with the boundlessness of the angle as a mathematical abstract "object". This implies that in a research involving students of the Faculty of Education, most of the persons interviewed claimed that the angle corresponds to the length of the little arc or to the limited part of the plane that it identifies, falling into an embarrassing contradiction; two half lines starting from a common point determine infinite angles! (Sbaragli, 2005).

An inadequate didactical transposition or didactic engineering can in fact strengthen the confusion, lived by the student, between the symbolic representations and the mathematical object. The result is that «the student is unaware that he is learning signs that stand for concepts and that he should instead learn concepts; if the teacher has never thought over this issue, he will believe that the student is learning concepts, while in fact he is only "learning" to use signs» (D'Amore, 2003; p. 43).

It thus emerges how often the choice of the representation, is not an aware didactical choice but it derives from teachers' wrong models. And yet, in order to avoid creating strong misunderstandings it is first required that the teacher knows the "institutional" meaning of the mathematical object that she wants her students to learn, secondly she must direct the didactical methods in a critical and aware manner.

From a didactical point of view, it is therefore absolutely necessary to overcome "unavoidable" misconceptions and prevent the "avoidable" ones, with particular attention to the Semiotic Means of Objectification, providing a great variety of representations appropriately organized and integrated into a social system of meaning production, in which students experience shared mathematical practices.

From what we have said, learning turns out to be a constructive semiotic process that entangles representations and concepts in a complex network, with the rise of misconceptions. Therefore the task of the teacher is to be extremely sensible towards misconceptions that can come out during the teaching-learning process. The teacher must be aware that what the student thinks as a correct concept, it can be a misconception rooted in an improper semiotic activity.

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Semiotic representations, "avoidable" and "unavoidable" misconceptions

George Santi	Silvia Sbaragli
N.R.D., Bologna, Italia	N.R.D., Bologna, Italia Università di Bologna e Bolzano, Italia Alta Scuola Pedagogica, Locarno, Svizzera

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Summary. *Following D'Amore's constructive interpretation for the term misconception, we present a distinction between "unavoidable" and "avoidable" misconceptions from a semiotic point of view, within the theoretical frameworks proposed by Raymond Duval and Luis Radford.*

1. Introduction

In this article we deal with one of the most used terms for decades in Mathematics Education research, the word "*misconception*", interpreted according to a constructive perspective proposed by D'Amore (1999): «A misconception is a wrong concept and therefore it is an event to avoid; but it must not be seen as a totally and certainly negative situation: we cannot exclude that to reach the construction of a concept, it is *necessary* to go through a temporary misconception that is being arranged». According to this choice, misconceptions are considered as steps the students must go through, that must be controlled under a didactic point of view and that are not an obstacle for students' future learning if they are bound to *weak and unstable* images of the concept; they represent an obstacle to learning if they are rooted in *strong and stable models* of a concept. For further investigation into this interpretation look D'Amore, Sbaragli (2005).

This semantic proposal is analogous with Brousseau's use of the term *obstacle,* starting from 1976 (Brousseau, 1976-1983), to which he gave a constructive role in Mathematics Education, interpreting it as knowledge that was successful in previous situations, but it does not "hold" in new situations.

Within this interpretation misconceptions have been divided into two big categories: *"avoidable"* and *"unavoidable"* (Sbaragli, 2005a); the first *do not depend directly on the teacher's didactic transposition*, whereas the second *depend exactly on the didactic choices performed by the teacher.*

We will analyze these categories within the theoretical frameworks upholded by Raymond Duval and Luis Radford.

2. Reference theoretical frameworks

According to Duval's formulation the use of signs, organized in semiotic registers, is constitutive of mathematical thinking since mathematical objects do not allow *ostensive* referrals; from this point of view he claims that there *isn't noetics without semiotics.* «The special epistemological situation of mathematics compared to other fields of knowledge leads to bestow upon semiotic representations a fundamental role. First of all they are the only way to access mathematical objects» (Duval, 2006).

This lack of ostensive referrals to concrete mathematical objects obliges also to face *Duval's cognitive paradox:* «(...) How learners could not confuse mathematical objects if they cannot have relationships but with semiotic representations? The impossibility of a direct access to mathematical objects, which can only take place through a semiotic representation leads to an *unavoidable* confusion» (Duval, 1993).
In particular, conceptual appropriation in mathematics requires to manage the following semiotic functions: the choice of the *distinguishing features* of the concept we represent, *treatment* i.e. transformation in the same register and *conversion* i.e. change of representation into another register. The very combination of these three "actions" on a concept represents the "construction of knowledge in mathematics"; but the coordination of these three actions is not spontaneous nor easily managed; this represents the cause for many difficulties in the learning of mathematics.
To better understand the learning processes it is suitable to integrate Duval's theoretical frame with the one proposed by Radford who enlarges the notion of sign incorporating in the learning processes also the sensory and kinaesthetic activities of the body. Radford (2005) considers learning an *objectification* process that transforms *conceptual and cultural* objects into objects of our *consciousness*. This objectification process is possible only by turning to culturally constructed forms of mediation that Radford (2002) calls *semiotic means of objectification*; i.e. gestures, artifacts, semiotic registers, in general signs used to make an intention visible and to carry out an action.
Like Duval, also Radford (2005) underlines the importance of the *coordination* between representation systems, when he claims that conceptualization is forged out of the dialectical interplay of various semiotic systems, with their range of *possibilities and limitations*, mobilized by students and teachers in their *culturally mediated social practices*. In the continuation of the article we will read "avoidable" and "unavoidable" misconceptions according to these theoretical frameworks.

3. "Unavoidableness"

"Unavoidable" misconceptions, that do not derive from didactical transposition, can depend on the representations teachers are obliged to provide in order to explain a concept because of the intrinsic unapproachableness of mathematical objects. These representations, according to Duval's paradox, can be confused with the object itself especially when a concept is proposed for the fist time. These representations can lead the student to consider valid "parasitical information" bound to the specific representation, in contrast with the generality of the concept. This "parasitical information" for example can stem from sensory, perceptive and motor factors of the specific representation since Radford (2003) claims that cognition is embodied in the subject's spatial and temporal experience and therefore requires to mobilize semiotic means bound to the practical sensory-motor intelligence.

The "embodied" character of cognition and the use of semiotics makes these "*misconceptions unavoidable*" and interpretable as steps the student must go through in the construction of concepts.

As we will show in the following example, these particular misconceptions can also be put down to the *necessary gradualness of knowledge*. In fourth primary school one day the teacher shows how the request that highlights the "specific difference" between the "close genus" rectangles and the "subgenus" squares regards only the length of the sides (that must all be congruent). After drawing a square on the blackboard, the teacher claims that it is a particular rectangle. The possible misconception created in the mind of the student that the prototype image of a rectangle is a figure that must have consecutive sides with different lengths, may create at this stage a cognitive conflict with the new image proposed by the teacher. This example highlights that it is unthinkable to propose initially all the necessary considerations to characterize a concept from the mathematical point of view, not only for the necessary gradualness of knowledge, but also because in order to propose mathematical objects, they must be anchored to semiotic representations that often hide the totality and complexity of the concept.

These examples of "avoidable" misconceptions seem to be bound to *ontogenetic* (that originate in the student) and *epistemological* (that depend on intrinsic facts to mathematics) obstacles (Brousseau, 1986); the last are considered by Luis Radford related to the social "practices" (D'Amore, Radford, Bagni, 2006).

4. "Avoidableness"

In the appropriation of a mathematical concept the pupil performs a desubjectification process, that leads him beyond the body spatial temporal dimension of his personal experience. The teacher has the delicate task of fostering a *cognitive rupture* to allow the pupil to incorporate his kinaesthetic experience in more complex and abstract semiotic means. The student thus goes beyond the embodied meaning of the object and endows it with its cultural interpersonal value (Radford, 2003). In this perspective, Duval (2006) offers important didactic indications to manage the rupture described above, when he highlights the importance of exposing the student, in a critical and aware manner, to many representations in different semiotic registers. Nevertheless didactic praxis is "undermined" by improper habits that expose pupils to univocal and inadequate semiotic representations. These habits cause misconceptions considered "avoidable", since they are ascribable to the *didactic transposition.*

An emblematic example of inadequate choice of the distinguishing features that brings to improper and misleading information relative to the proposed concept, regards the habit of indicating the angle with a "little arc" between the two half-lines that determine it. Indeed, the limitedness of the "little arc" is in contrast with the boundlessness of the angle as a mathematical "object". This implies that in a research involving students of the Faculty of Education, most of the persons interviewed claimed that the angle corresponds to the length of the little arc or to the limited part of the plane that it identifies.

An inadequate didactical transposition can in fact strengthen the confusion, lived by the student, between the symbolic representations and the mathematical object. The result is that «the student is unaware that he is learning signs that stand for concepts and that he should instead learn concepts; if the teacher has never thought over this issue, he will believe that the student is learning concepts, while in fact he is only "learning" to use signs» (D'Amore, 2003).

This misunderstanding derives also from the univocity of the representations that teachers usually provide students with, as is the case of geometry's primitive entities. Researches aiming at detecting incorrect models built on image-misconceptions relative to these mathematical concepts show that as regards the mathematical point, some pupils and teachers ascribe to this mathematical entity a "roundish" shape (bidimensional o tridimensional) that derives from the univocal and conventional representations they have always encountered (Sbaragli, 2005b). Moreover, some students and teachers are led to associate with the wrong idea bound to the unique shape of mathematical points also a certain variable dimension.

From these results it emerges how often the choice of the representation, is not an aware didactical choice but it derives from teachers' wrong models. And yet, to not create strong misunderstandings it is first required that the teacher knows the "institutional" meaning of the mathematical object that she wants her students to learn, secondly she must direct the didactical methods in a critical and aware manner. "Avoidable" misconceptions seem to be bound to the classical *didactic obstacles* (Brousseau, 1986) that originate in the didactic and methodological choices of the teacher.

From a didactical point of view, it is therefore absolutely necessary to overcome "unavoidable" misconceptions and prevent the "avoidable" ones, with particular attention to the semiotic means of objectification, providing a great variety of representations appropriately organized and integrated into a social system of meaning production, in which students experience shared mathematical practices.

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