UNIVERZITA KOMENSKÉHO
FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

TEACHERS' CONVICTIONS ON MATHEMATICAL INFINITY

(dizertačná práca)

Bratislava 2004 Silvia Sbaragli

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Table of Contents

Preface

«The fear of infinity is a kind of short-sightedness that destroys the possibility of seeing actual infinity, even if infinity in its highest expression created us and sustains us and through its secondary forms of transfinite surrounds us and even dwells in our minds».

(Cantor G., *Gesammelte Abhandlungen*, 1932)

The following reflections are related to a research study on mathematical infinity carried out over several years. Such a topic represented and still represents a fascinating subject matter constituting a primal interest for and involving scholars of different branches of knowledge. In particular, as far as mathematics and didactic mathematics are concerned, the issue of infinity has been considered from different perspectives and great attention has been paid to the most delicate historical moments, the epistemological obstacles specific to this topic and the related difficulties encountered by students, attending different educational levels, to approach the issue of infinity.
The innovative and charming viewpoint characterising the present research work, within the scope of the didactics of mathematics is to focus and investigate teachers' convictions on mathematical infinity. Firstly, we analysed primary school teachers' misconceptions supported by wrong mental images conditioning their convictions and also consequently their way of teaching. Subsequently, we concentrated on secondary school teachers' convictions and came to the conclusion that there are no great variations in the false beliefs revealed by teachers teaching in different educational levels.

The present work is formed of four chapters, all of them dealing with the issue of infinity as seen from different points of view and sharing a single common thread: didactics.

The first chapter provides readers with a chronological critical- historical outline in order to allow them to focus on fractures, non-continuities, radical changes in the evolution of a mathematical concept that underline the epistemological obstacles making infinity such a difficult topic to be conceived, accepted and finally learnt.

The second chapter offers a brief outline of those elements of didactic mathematics pertaining to the treatment of this thesis. In particular, we stated our approach, that is to be considered within the scope of the present Research in Didactics of Mathematics of the French School and whose attention is focused on the phenomenon of learning considered from a foundational point of view. In this respect, we will refer to what is intended today by *fundamental didactics* (Henry, 1991; D'Amore, 1999), i.e. everything concerning the basic elements of the research in mathematical didactics deriving from the various and complex analyses of the so-called "triangle of didactics": teacher, student and knowledge. In more detail, we will provide some useful hints for the interpretation of the analyses of the following chapters.

Chapters 3 and 4 describe the core of the research work. In the third chapter, primary school teachers' misconceptions on mathematical infinity are singled out by means of qualitative methodologies: analyses of questionnaire's collected answers and the related following discussion activities. The results have shown that infinity is, in general, an unknown concept, only managed by intuition and usually banally reduced to an extension of the finite.

These reflections revealed that the major difficulties related to the understanding of the concept of mathematical infinity are not exclusively due to epistemological obstacles, but are also strengthened and magnified by didactical obstacles. Obstacles deriving from the intuitive models provided by teachers to their students. Such models represent, without teachers being aware of it, real and proper misconceptions.

The same false beliefs have to be traced back in secondary school teachers' convictions who have been asked to analyse and discuss with the researcher their students' produced TEPs (D'Amore, Maier, 2002) concerning issues related to infinity and reported in chapter 4.

Our intention was to highlight how the object of our research has been so far underestimated, especially as a subject matter for training courses addressed to teachers. It is exactly this deficiency, in our opinion, that is one of the causes of the learning problems encountered by secondary school students already possessing some previous convictions not suitable to face new cognitive situations.

Consequently, the fourth chapter is mainly based on the focus of our research, over the most recent years, that is to try to inhibit and therefore overcome those models turning into obstacles in teachers' minds and hence in turn in students' minds. The aim is to propose a learning pathway that envisages specific training courses for teachers that take into account the peculiar and intuitive aspects related to infinity, as well as the outcomes obtained by researchers in didactic mathematics. This kind of training will allow the participant teachers to properly deal with the concepts linked to infinite sets and even get their students involved in fruitful and meaningful activities in order for them to build intuitive images consistent with the theory of infinite sets.

Moreover, various present and future research studies have been introduced which we intend to carry out and that are particularly focused on teachers' and students' misconceptions on geometrical primitive entities surveyed from different points of view. This latter choice, outwardly distant from the world of infinity, actually derives from the acknowledgment that teachers' and students' misconceptions on geometrical infinity depend in most cases on those misconceptions regarding geometrical primitive entities.

The feeling pervading this dissertation is that this research work represents to the author just the beginning of a journey which has no end in sight and which is proving, year after year, to be ever more fruitful, challenging and involving.

Chapter 1. A basic critical historical approach to infinity

Before introducing the didactical aspect pertaining to this work, we would like to provide readers with a brief historical-critical excursus on the main phases that the delicate and complex concept of mathematical infinity underwent. The aim of this chapter is therefore to shed light on the origins of the epistemological obstacles related to this subject matter (see paragraph 2.5). These obstacles "justify" teachers' and students' convictions on mathematical infinity, which will be pointed out in chapters 3 and 4.

For the treatment of this chapter we refer to: Arrigo and D'Amore, 1993; Boyer, 1982; D'Amore, 1994; D'Amore and Matteuzzi, 1975, 1976; Geymonat, 1970; Lolli, 1977; Rucker, 1991; Zellini, 1993 and other names quoted throughout the text. The present research work has been primarily influenced by the singular work and personal interpretations of D'Amore (1994), which we are obliged to.

1.1 Prehistory: from 600 B.C. to 1800

«There is a concept corrupting and altering all the others. I am not talking of the Evil whose restricted realm is ethics; I am talking of infinity.» (our translation)

[Borges J.L., 1985]

1.1.1 From the Ancient Times to the Middle Ages

Thales of Miletos (624 B.C. – 548 B.C.). He identifies the *origin of all things* (*arché*) in water as according to him everything is featured by a primordial state of humidity to which all things return.

Anaximander of Miletos (610 B.C. – 547 B.C.). Pupil of Thales, he defines *arché* as something qualitatively undefined (recalling the idea of indeterminate), divine, immortal, imperishable, without any boundaries (recalling the idea of unlimited) but not consequently chaotic. He calls it *ápeiron* (infinity). According to Marchini (2001), it is reasonable to think that in Anaximander's times the concepts of infinity, unlimited and indeterminate were considered synonyms.

Anaximenes (586 B.C. - 528 B.C.). He suggests that the origin of things lies in the infinite air, since air is the substance that better represents unlimitedness and omnipresence, which are typical of the primordial principles.

Two main currents are thus created. The first considers infinity in a negative way: incomplete, imperfect, without boundaries, indeterminate and source of confusion and complications (i.e. Pythagoras' followers and Aristotle). The second holds infinity positively, as it is a concept that embraces all qualities [**Epicurus (341 B.C. -270 B.C.)**].[1]

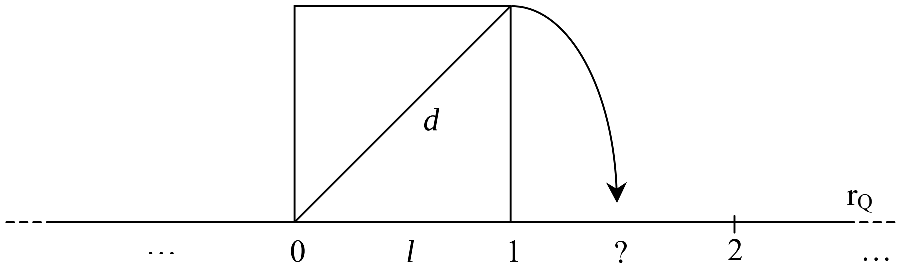
Pythagoras of Samos (580 B.C. - 504 B.C.). Mathematics is the key to the understanding of the whole universe. Everything can be described through natural numbers and their ratios which are aggregates of *monads*, which in turn are unitary corpuscles provided with size, though being so small not to be further divisible and not without dimension in any case.

Every single body is composed of monads, not randomly arranged, but on the contrary positioned according to a given geometrical-arithmetical order. Pythagoras is therefore a finitist as well as Plato.

His School is faced with the problem of incommensurability, originated from the conception of expressing everything with a natural number of monads or better said

[1] According to Epicurus, infinity is the positive principle in the becoming of bodies, whereas void represents the negative principle. This statement was drawn on also in religion and mysticism, which attached an ontological meaning to infinity.

with natural number ratios.[2] There are cases where it is not possible to express with a rational number the ratio between the lengths of two segments:[3]



The discovery of incommensurability of the square's diagonal and side (Kuyk, 1982) is to be traced back to Hippasus of Metapontum (V century B.C.) who lost his life because of his outrage to the Pythagorean School.

The dispute was between intuition and reason and represents the first case where the latter goes in the opposite direction of the former. Mathematical entities stopped being sensible and became purely intelligible, thus opening doors to infinity. This could perhaps be considered as the first step towards the conception of mathematics as belonging to the world of ideas, a conception that would dominate Greek philosophy.

Parmenides of Elea (504 B.C.).[4] In his Poem: *Perì Physeos* (*On Nature*) Parmenides introduces a dichotomy between two different ways of interpreting truth: a truth of sensible origin (*doxa*) and an opposite Truth of rational nature (*Alétheia*). The human being can use *doxa* only for the supreme goal of reaching *Alétheia*. In order to avoid paradoxes, *doxa* excludes the concept of infinity (i.e.: shooting an arrow a few steps before the end of the universe). Whereas *Alétheia* represents the spiritual height, the highest knowledge, the single immutable *being*, indivisible, eternal, immobile. Infinity

[2] In this respect, the following sentence, quoted from Plato's Theatetus, is quite remarkable: *«The ignorance of those who believe that all pairs of magnitudes are commensurable is disgraceful»* [our translation].

[3] It was Archita (430 B.C. - 360 B.C.), who first managed to demonstrate that the ratio of these two segments could not be expressed as a ratio of two natural numbers.

[4] Dates concerning Parmedides' life are quite difficult to assess. We therefore indicated 504 B.C., the date of the LIX Olympics, which according to Diogene Laertius corresponds to the most significant period of the work of Parmenides.

is thus conceived as totalising, all embracing, though *limited («The universe is limited because without limits it would be missing everything»*).

Zenon of Elea (born in 489 B.C. ca.). Parmenides' pupil inherits the knowledge of his master reinforcing the idea of immobility and immunity of the being that had raised harsh criticisms. Zenon's famous paradoxes confute the ideas of plurality and movement (i.e.: Dichotomy, Achilles and the Tortoise, The Arrow, The Stadium). As Marchini states (2001): *«To Zenon considering infinity an attribute of the being, due to inexhaustibility of infinity itself, brings about the irrationality and impossibility of the being. He is actually against this vision of infinity in act»*. All the paradoxes linked to the concept of infinity caused such confusion that Aristotle forbade the use of it, in order to avoid this *«scandal»*. It is therefore thanks to Parmenides' abstract position and to Zenon's paradoxical creations that Greek mathematicians had seriously to face the problem of infinity, though desperately trying to avoid it.[5]

Melissos of Samos (end of VI century B.C. - beginning of V century B.C.). In order to demonstrate Parmenidean theses concerning the idea of *single being*, Melissos elaborates his master's thought denying the concept that the determinate nature of being implies its finite character too. He conceives a spatially infinite being that admits nothing outside itself.

The rebellion to Parmenides started with the Pluralists and among the others **Anaxagoras of Clazomenae (500 B.C. - 428 B.C.)**. This philosopher devoted all his life to reflecting on the matter and its components creating the term *homeomeries* to indicate infinitesimal elements, not further divisible and characterised by different qualities. Interesting to the aim of present research are the following statements written in his book *On Nature*: *«In the large as well as in the small there is an equal number of parts (...) with regard to the small there is no smallest, but always an even smaller, because the existent cannot be annihilated (by division). Thus, with regard to the large there is*

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[5] From a didactical point of view, a number of research studies deal with the debate on the truth of rational nature as opposed to the truth of sensible origin: Hauchart and Rouche, 1987; Nuñez, 1994; Bernardi 1992a,b.

always a larger, and this larger is like to the small in plurality, and in itself everything considered as the sum of infinite infinitesimal parts is at the same time large and small» (in modern language, it is obvious that a shorter segment is included in a longer one, but if we think of both entities as sets of points, we will observe that in a longer segment as well as in a shorter one there is the same number of points). Mathematicians would often return to this concept during the course of history, but it will be only thanks to the German scientists of the XIX Century that the above-mentioned notion will find a rigorous systematisation. In Anaxagora's statement the ideas of infinity and infinitesimal are strictly related to one another. In some parts, it seems that the infinite subdivision is to be understood as potential, whereas at times Anaxagora seems to refer to actual infinity.

The rebellion went on with the Sophists such as: **Protagoras of Abdera (485 B.C. - 410 B.C.)** and **Gorgias of Leontini (483 B.C. - 375 B.C.).** They claimed the superiority of sensible experience towards rational truth, thus influencing the mathematical thought and the issue that is the topic of our research as well. As a matter of fact, according to the sensible experience the circumference does not touch the tangent at a point but along a segment of a certain length.

Among the Atomists we recall **Leucippus of Abdera (460 B.C.)** master of **Democritus of Abdera (460 B.C. - 360 B.C. ca.).** According to their thought the void exists and is the place in within the atoms move. Democritus, in particular, drew a distinction between two different aspects of infinite divisibility: from an abstract mathematical point of view, every entity is infinitely divisible into parts (especially segments and solids); from a physical point of view things change: there is a material limit to divisibility and the limit is a unitary indivisible material corpuscle called atom. There seem to be even more kinds of atoms with different dimensions.

Aristotle of Stagira (384 B.C. - 322 B.C.). As Plato did, and Socrates even before, Aristotle accepts the Parmenidean idea of a limited universe according to the nature of the Greek philosophy that despises disorder caused by the matter in its chaotic form. These limits surrounding the universe and arranging it at a rational level, make it acceptable to the human logic: *«…Since no sensible magnitude is infinite, it is not possible that a given magnitude could be overcome as in that case there would be something greater than the sky»*.

As to infinity, Greek philosophy and mathematics felt great embarrassment towards this subject because it was full of contradictions and paradoxes, profoundly influencing Aristotle's thought. He was the first to reveal a double nature of infinity: "in act" and "in power". "In act" means that infinity appears as a whole, given as a matter of fact, all in one go. "In power" means that infinity is referred to a situation which is finite at the moment we are talking about it, but with the certainty that the set limit could be overcome all the time (thus the limit is not definitive): *«A thing comes from another with no end and each thing is finite but of these things there are always new »*.

In short: *«[the actual infinity is] that beyond which there is nothing else; … [the potential infinity is] that beyond which there is always something else»* (*Physics*).[6]

Aristotle forbade the use of *actual* infinity to mathematicians solely allowing the use of potential infinity: *«Therefore infinity is in power and not in act»*. In Aristotle's opinion a segment is not composed of infinite parts (in act) but is divisible by infinite times (in power).

> *«In any case our debate is not intended to suppress mathematicians' research due to the fact that it excludes that the infinity by progressive growth is such that it cannot be taken in act. As a matter of fact, at present state, mathematicians themselves do not feel the need for infinity (and they do not even use it) but they only need a quantity as large as they please, though finite in any case. (…). Thus for their demonstrations' sake they will not care about the presence of infinity in real magnitudes»* [our translation] (Physics, III, ch. 7).

For a long time this prohibition was conceived as a real dogma: many scholars from the Middle Ages and the Renaissance, as well as from more recent times, were almost about

[6] There are many "Aristotelian" studies on the potential and actual use of the term infinity both in the subject form (infinity) and in the adjective form (infinite): Moreno and Waldegg, 1991; Tsamir and Tirosh, 1992.

to “master” the concept of infinity including its paradoxes, but Aristotle’s legacy was ever somewhat binding.

Aristotle also pointed out the distinction between infinity by addition and infinity by division (*Physics*) as explained in Zellini (1993): «*If you consider a length unit and you add it to itself infinite times, the result will be for sure an unlimited distance not coverable in a finite time. But if you envisage the unlimited by means of a somewhat opposite procedure, dividing by dichotomy the length unit into infinite intervals, infinity could be considered in some way exhaustible within a limited time interval*» [our translation].

Euclid (300 B.C. ca.). In his immortal and famous work *Elements* Euclid accepts Aristotle’s point of view. In other words, Euclid is well aware of the problem of infinity and strenuously tries to avoid it.

- In his definition XIV of the book I he states that figures are all finite.
- In the postulate II of the book I he does not use the term *straight line* but he talks of a geometrical entity called *eutheia grammé* (*terminated line*) which by means of a postulate can be «*continuously prolonged straight ahead*».
- The V and most known postulate still refers to *eutheia grammé* and not to *straight line*. In particular, it explicitly requires the unlimited prolongability of two terminated lines and therefore it would be as much avoided as possible by Euclid himself in his future treatment.
- In the proposition I of the book VII he applies the following procedure: «*If you take two unequal numbers and you successively subtract the minor from the major, the difference from the minor and so on, the remainder never divides the immediately preceding number until unity is obtained. The initially given numbers will be primes to each other*» [our translation]. Taking into account any two numbers the procedure always ends after a finite reiteration of operations.
- In the proposition XX of the book IX Euclid does not demonstrate that «*There exist infinite prime numbers*». On the contrary, «*Prime numbers are more than any other previously suggested total number of primes*», in accordance with the position of

Eudoxus of Cnidos (408 B.C. - 355 B.C.)[7] who deals with infinity never calling it by its name.

- One of the most famous common notions (*coinaì énnoiai*) subscribed by Euclid is: «*The whole is greater than its parts*» which is in contrast with Anaxagoras' intuition.
- The problematic nature of infinity is not always revealed by the aspect of prolongability or, as in Aristotle, by the infinity by growth. Euclid's viewpoint includes also the infinitesimal with the demonstration that *the contingency angle* is minor than any *rectilinear angle*. This denies the Eudoxus' postulate today called Eudoxus-Archimedes postulate. As a matter of fact, in the book V of Elements Euclid states: *«Two magnitudes are set into relation if each of them, multiplied by a certain appropriate number [natural], overcomes the other»*, cleverly excluding in one go mixtilinear angles from the set of rectilinear ones (thus avoiding to talk about "actual infinitesimals"), (D'Amore, 1985).

Euclid's work with regard to infinity is all based on Aristotle's philosophic choice: he completely rejects the actual infinity and accepts and uses only the potential one. By sharing this position, he is extremely rigorous and strict.

Archimedes of Syracuse (287 B.C. - 212 B.C.). He was committed with the method of exhaustion based on the division of geometrical figures (plane or solid) into infinitesimals (actual) and infinite sections. Archimedes dealt nonchalantly with very delicate matters showing to be not particularly prone to remote philosophies. He obtained significant and courageous results. At this point, it is reasonable to wonder if Archimedes knew the issue of infinity or not. Evidence of this is given in his work *The Sand-reckoner*. In this text Archimedes calculated how many grains of sand are contained in a sphere whose radius is given by the distance of the Earth from the Sun. The answer is approximately 10^{63} and Archimedes had to invent a numerical system

[7] Eudoxus of Cnidos managed to elaborate a theory on proportions, which allowed to operate on ratios without using actual infinity. We also owe him the method of exhaustion, also aiming at eliminating actual infinity. Both these methods do not abolish infinity, but they tend to prefer the potential infinity to the actual.

that goes beyond *myriads.*[8] The greatest number ever reached by Archimedes is *a myriad of myriads of unities of the myriadesimal order of the myriadesimal cycle,* i.e. $M^{(M^2)} = (10^8)^{(10^{16})}$, far larger than the "only" 10^{63} grains of sand he needed. This proves the necessity of ever-increasing numbers than the ten thousand possible in the ancient Greek language. At the same time, it has to be noted that he feared to "exaggerate", i.e. to run the risk of "involving" infinity. The need for a well-defined limit is very strongly felt.

Lucretius (10 B.C. – 55 A.C.). Known for *De Rerum Natura* (*On the Nature of the Universe*): *«Suppose for a moment that space is limited and that somebody goes up to its ultimate border and shoots an arrow…»,* this sentence embraces the idea of an unlimited universe (Book I, 968-973).

Clemens of Alexandria (150 - 215). Infinity is considered as a divine attribute. It is applied with a positive connotation to divinity and with a negative one to our unableness to understand divinity in its ineffability.

Diophantus of Alexandria (250 ca.). He introduces numerical variables using an advanced symbolism subsequently adopted and studied by his "pupil" Fermat in the XVII century. In the algebraic use of numerical variables the concept of infinity is concealed.

St. Basil the Great (330 - 379). Infinity becomes synonym of the completeness of divine perfection. From this moment onwards infinity will be always quoted in relation to divine attributes. Therefore philosophers will try, in different ways, to prove such a quality of the Supreme Being.

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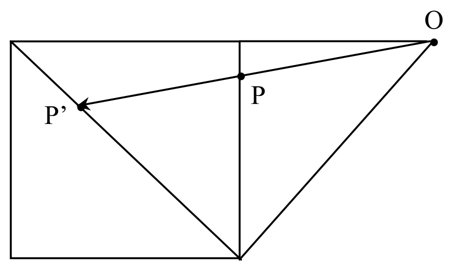
[8] Rucker reports (1991): *«The Greeks did not know the notation of exponentiation, they just used that of multiplication, moreover the maximum number they could name was a myriad, which is equal to 10,000 that is to say 10^4».*

St. Augustine of Tagaste (345 - 430). In his work *De Civitate Dei* he admits the actual infinity of natural numbers: «*God knows all numbers in an actual way. Actual infinity is in mente Dei*».[9]

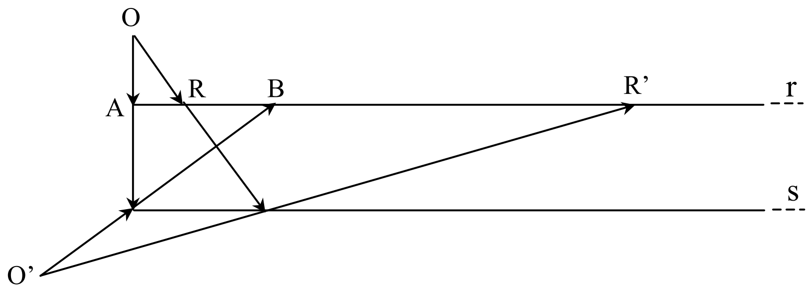
Proclus (410 - 485). Infinity is still connoted as potential when it gradually expands starting from the intelligibles, whereas Proclus seems to stand for actual infinity when he tries to convey finite and infinite principles into the One: «*Every existing thing is somewhat finite and infinite because of the first Being…* (*since*) *it is clear that the first being communicates to all things the limit as well as infinity, being itself made of these elements*» [our translation] (*Elementa teologica*).

Therefore is ever growing the importance of the distinction between philosophic and mathematical infinity.

Roger Bacon (1214 - 1292). In his work *Opus Maius* (1233) he claims that we can establish a biunivocal correspondence (as we would say today) between the points of a square side and those of the same square diagonal, although they have different lengths (the idea will be further developed by Galileo). Moreover, such a biunivocal correspondence could be established (by translation or double projection) between two half-lines (one with A origin and one with B origin) positioned on the same straight line *r*.



[9] In the second half of the 19th century, Cantor acknowledged Augustine as one of his sources of inspiration to support the set theory.



He concludes stating that mathematical infinity in act is not possible according to logic: the whole would be not greater than its parts; this principle would be anti-Euclid and therefore also anti-Aristotle, an attitude still perceived as forbidden.

St. Thomas Aquinas (1225 - 1274). In *Summa Theologiae* we find evidence of the idea of actual infinity conceived to be in *mente Dei*. In this text, Thomas admits the possibility of the existence of different levels of infinities in the infinity, but he also claims in some other passages that the only actual infinity is God. With regard to things, he talks about the infinity in power and consequently he attributes to mathematical infinity the solely potential aspect: «*… it stands out clearly that God is infinite and perfect… So even if He is God and He has an infinite power, He cannot create something un-created* (*this would be a contradiction*), *He thus cannot create any thing that is absolutely infinite*» [our translation].

William of Occam (1290 - 1350). He writes in *Questiones in quator libros sententiarum*: «*It is not incompatible that the part is equal or not minor than its whole; this is what happens every time that a part of the whole is Infinite. This is verifiable also in a discrete quantity or in any multiplicity whose part has units not minor than those contained in the whole. So in the whole Universe there are not more points than in a bean, as a bean is made of infinite parts. So the principle that the whole is greater than its parts is valid only for the things composed of finite integral parts*» [our translation].
William is accused of heresy in 1324 and is held for being questioned for 4 years in Avignon, then he escapes to take refuge first in Pisa and then in Munich. So far, it is still too dangerous to contradict Aristotle's thought.

Nicholas Oresme (1323 - 1382). He has an intuition about the coordinates to which nowadays we refer to as Cartesian. He sets the value of the following "sum" *s* implying an absolutely modern use of infinity:

$$s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots$$

$> 1 \quad > 1/2 \quad > 1/2 \quad \ldots$

$> 1 \quad \ldots$

Given any natural number M (large in any case), after a certain number of addenda, *s*>M. So: *s* is greater than any natural number though large.

Nicholas of Cusa (1400 or 1401 - 1464). He considers mathematics as an ideal of perfection and therefore he feels the need for a cosmos ordered according to "weight, number and size". He refers to infinity only from a mathematical point of view, dealing with the infinitely large and the infinitely small. Nicholas of Cusa is the last medieval neo-Platonist. Infinity is almost absent as cardinal and is considered as ordinal or as a not well-identified "vastness". In accordance with the medieval spirit, Nicholas of Cusa confuses infinity with unlimited or at times even indefinite (this confusion will last till the XIX century and even further, see teachers' statements reported in 3.7.1).

In his major work, *Docta Ignorantia* (*Learned Ignorance*), one of his most famous and beloved analogies is to be found: *«Intellect is to truth as the polygon of n sides is to the circle. When n tends to infinity, the polygon tends to the circle; the truth is therefore the limit of the intellect to infinity»* (ch. III, Book I). Moreover in this text, a paradox concerning the actual infinity, similar to those treated by Galileo and Bolzano is to be traced: *«If a line is formed by an infinite N number of one foot long segments, whereas another line is formed by an infinite M number of two feet long segments, these two lines are equally long and this length is infinite; therefore it can be concluded that "in the infinite line one foot is not less long than two feet"»* [our translation] (ch. XVI, Book I). In addition, in the *Conjectures*, an improperly carried out argumentation of Zenonian nature is to be found, it aims at demonstrating that any two lines have the same number of points (ch. IV, Book I). As already mentioned, this topic was a matter over-debated for millennia, e.g. Anaxagoras and Roger Bacon had already dealt with it and the question will be settled only at the end of the XIX century thanks to the work of Cantor. On the idea of maximum, Nicholas of Cusa claims: *«No infinite number is known and no given maximum either»* (*Conjectures*, ch. XI, Book I).

In conclusion, a proper and solid conscience of infinity is still to be achieved, and the history of mathematical thought has still to wait till the Renaissance when, thanks to the research of major artists in the field of perspective and Galileo's brilliant reflections, the accomplishment of such a miracle could be witnessed: Bonaventura Cavalieri and Evangelista Torricelli could finally "see" what scholars from the Middle Ages could not clearly and thoroughly "see".

1.1.2 Infinity in the Renaissance

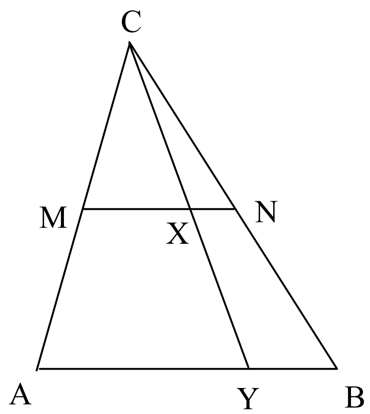
Infinity is extremely present in the Renaissance, not in the "numerical Universe" but in the world of geometry and fine arts (which happened to coincide at that time): **Piero della Francesca (1406 - 1492)** writes *De Prospectiva Pingendi*, a mathematical-pictorial work of great value; **Girolamo Cardano (1501 - 1576)** writes the treatise *De Subtilitate* (1582) on subtlety, i.e. on something that we can also call "infinitesimal magnitudes". In particular this work deals with the contingency angle.

Moreover, in the Renaissance "the method of indivisibles", already dealt with by Archimedes, is further developed thanks to: **Leonardo da Vinci (1452 - 1519)**, **Luca Valerio (1552 - 1618)**, **Galileo Galilei (1564 - 1642), Paul Guldin (1577 - 1643)**, **Bonaventura Cavalieri (1598 - 1647)**, **Evangelista Torricelli (1608 - 1647)**.

Galileo Galilei (1564 - 1642). He firstly based his work on Democritus' reflections, but he widened the scope of applicability from geometry to more extended classes of analytical problems. In his last work: *Mathematical Discourses and Demonstrations on two new Sciences* (1958) dated 1638, Galileo collected most of his major considerations on the infinity paradoxes.
Actual infinity is mentioned in several occasions. According to Galileo, lines as well as concrete objects to be found in nature are all formed by a continuum (actual infinity) of parts small as we please though measurable (hence divisible). *«Each part (if one can*

still call it a part) of infinity is infinite; since, even if a line one hundred span long is major than that of only one span of length, there are no more points in the longer than in the shorter but the points of both lines are infinite» [our translation].

Therefore his geometrical considerations envisage a concept of infinity that can collide with the VIII Euclid's common notion: «*The whole is greater than its parts*». It may suffice to draw a triangle to see that between the AB side and the MN segment that joins the midpoints of the other two sides, there is a biunivocal correspondence obtained joining the points of AB with C. This is clearly in contrast with the common intuition that being AB twice as long as MN it should be formed by a greater number of points.



> *«These are the difficulties deriving from the reasoning of our finite intellect on infinities giving to them those attributes that we assign to terminate and finite things. I consider it as inconvenient as I believe that those majority, minority and equality attributes are not suitable to infinities, about which it is not possible to say if one is major, minor or equal to the other»* [our translation] (Galileo, 1958).

In a non-geometrical field, be:

0 1 2 3 4 … the sequence of natural numbers (N)

0 1 4 9 16 … the sequence of perfect squares (Q_N)

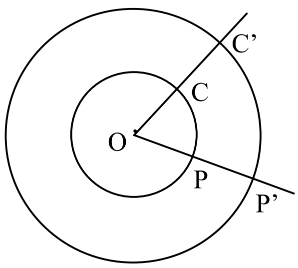
Q_N is strictly contained in N, this according to Euclid would mean that N contains more elements than Q_N, but for each natural number there is its square, that is to say a well determined element of Q_N (and vice-versa). With an obvious intuition, it can be deduced that there are as many elements in N as in Q_N (*Galileo's paradox*).[10] The present treatment appeared also in *Dialogue on the two Greatest Systems*, where the acceleration of a falling body was mentioned.

> *«I cannot come to any other decision than saying that, infinite are all numbers [natural], infinite their squares, infinite their roots, and the multiplicity of squares is not minor than that of all numbers [natural], neither the latter is major than the former, and lastly attributes such as equal, major and minor are not appropriate for infinities but only for terminate quantities»* [our translation] (Galileo, 1958).

Galileo outlined a first definition of infinite set later developed by Dedekind.

The history of infinity has come again to a delicate phase. The mechanism created by Aristotle, to protect from mathematicians one of the possible uses of infinity, has been demolished, though scholars should still go a long way before full conscience of infinity and thus the consequent ability of "dominating" it with technical means, even not extremely sophisticated ones, are reached.

Evangelista Torricelli (1608 - 1647). A pupil of Galileo's, he got in touch with the Geometry of Indivisibles of Cavalieri thanks to his master. He developed hazardous conceptions of infinity and infinitesimals (considered from the actual perspective) and he could intuitively envisage the hyperbole's improper points and consider the finite area not only strictly related to limited figures, as it was believed in his time but it is not actually. In addition, Torricelli recognised that two concentric circumferences (of different lengths) are formed of the same number of points; it is sufficient to consider the common centre as the origin of a projection.



[10] Many studies in the field of didactics concern Galileo's considerations: Duval, 1983; Tsamir and Tirosh, 1994; Waldegg, 1993.

René Descartes (1596 - 1650) and **Pierre de Fermat (1601 - 1665)**. They both deal with "infinitesimals" in order to solve the problem of the determination of the tangents to a curve.

Notable is that Descartes was able to see geometry from a totally new perspective: all geometrical entities and related properties were expressed through an algebraic language. He also dealt with the debate on infinity, but…: *«… we will never get uselessly involved in discussions on infinity. De facto, we are finite and it would therefore be absurd if we established anything on such a matter and tried to render it finite and possess it…»*. Descartes introduces a distinction between *infinity*, attribute proper to God and *indefinite* used to indicate unlimited magnitudes in quantity or in possibility.

Fermat, on the other hand, seems to make no mention of infinity.

Nevertheless the development of analytical geometry deeply influenced the issue of infinity, since it forced towards a comparison between the number infinity and the infinity of geometrical entities giving an enormous contribution to the passage from prehistory to history of the debated subject on the basis of two main reasons:

1. Mathematical Analysis is finally founded (and also infinity can find a rational systematisation);
2. Proper answers are given to the questions: How many are the points of a square and those of its side? How many are the straight lines of the plane? …

On this last point mathematicians still could not find a definite solution; Cantor and Dedekind will finally and eventually shed light on this aspect.

Blaise Pascal (1623 - 1662). He seems to stand for the actual infinity: *«The unit added to infinity does not make it any larger… Finite is annihilated by infinity and it becomes a pure nothing… We know there exists an infinity but we ignore its nature. Since we know that it is false that numbers are finite, it is therefore true that there is infinity of number… We therefore know the existence and nature of finite because we ourselves are also extended and finite in the same way. We know the existence of infinity but ignore its nature, because it has the same extension as we have, but it has no boundaries as we have. We do not know either the existence or the nature of God* *because it has neither extension nor boundaries. But it is only through faith that we know of His existence»* [our translation] (*Infinity. Nothing*).

Gottfried Wilhelm Leibniz (1646 - 1716). He suggests three kinds of infinity: *infimus*, in quantity; *medium*, as the totality of space and time; *maximum*, representing only God, as the fusion of all things into one. As in Kuyk's (1982): *«To Leibniz, each monad had an actual infinity of perceptions and each body was made of an actual infinity of monads».* Notwithstanding his confidence in dealing with infinitesimals thanks to the efforts of scholars from the Middle Ages and the Renaissance, he showed himself somewhat worried and reluctant when dealing with the above-mentioned magnitudes. Prove of this can be found in a letter addressed to Fouchet: *«Je suis tellement pour l'infini actuel, qu'au lieu d'admettre que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte partout, pour mieux marquer les perfections de son Auteur».*

We are indebted to **Isaac Newton (1642 - 1727)** for the explicit development of Mathematical Analysis which will be widely spread and developed among the others also by the great mathematician **Carl Friedrich Gauss (1777 - 1855)** who is still convinced that: *«…I protest against the use of an infinite magnitude seen as a fully accomplished whole, as this never happened in mathematics…».* Infinity is present though still not explicitly investigated.
So deeply rooted are the prehistorical convictions on infinity that they can be still traced back in present times, as we shall see in chapters 3 and 4.

Immanuel Kant (1724 - 1804). He was one of the first who "wiped out" the risk of misunderstandings deriving from the hazardous approach to the notions of infinity (actual) and infinitesimal (actual) adopted by the XVII and XVIII century mathematicians. Kant discovered antinomies in the *constitutive*[11] sense of infinity (see first and second antinomies of the Pure Reason) e.g.: when the world or anything in it contained is considered as finite, the mind can think of it as an extension; when the world or anything in it contained is considered as actually infinite, the mind cannot

[11] According to Newton and Leibniz, infinity had a *constitutive* meaning.

think of it at all. In both cases the mind is not consistent with the world: to reason, finite is too small and infinity (actual) too large (Kant, 1967). As stated by Kuyk (1982): *«Kant's solution was to consider infinity not in a constitutive but in the regulative sense. (…). By this shift of meaning, the notion of infinity goes from ontology to epistemology»*.

The dispute between actual and potential infinity continues. On the occasion of a competition promoted by the Berlin Academy [presided over by Lagrange (1736-1813)] and whose goal was to clarify the concept of infinity, the winner **S. L'Huilier (1750 - 1840)** advocated a return to classical infinity, the Aristotle's one, against the acceptance of actual infinity supported by Leibniz.

1.2 From prehistory to history of the concept of mathematical infinity

From the second half of the XIX century up to these days, the concept pertaining to actual infinity has profoundly influenced mathematical thought.

1.2.1 Bernard Bolzano (1781- 1848)

Between 1842 and 1848 Bolzano wrote *The Paradoxes of Infinity,* only posthumously published in 1851 (Bolzano, 1965). The book is a collection of 70 short paragraphs. Extracts of some of them are reported hereafter:

§13: The set of propositions and "truths in themselves" is infinite.

Wissenschaftslehre (proposition in itself): *«By W. I mean any proposition stating that a thing is or is not, without taking into account if the statement is true or false or if it has been verbally expressed or not by anyone»* [our translation]. When a W. is true is a Wahrheit an sich (truth in itself).

Be A_0 a Wahrheit an sich; be A_1 the new W. an sich: «A_0 is true»; be A_2 the new W. an sich: «A_1 is true»; …

Be $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$. $\mathcal{A}$ is "greater" than any finite set therefore is infinite. Moreover, the elements of $\mathcal{A}$ can be set in biunivocal correspondence with the elements belonging to the set N of natural numbers (by establishing a correspondence between A_i and i).

(It has to be noted that the infinite set $\mathcal{A}$ is built on the language, or better explained, on various metalinguistic levels).

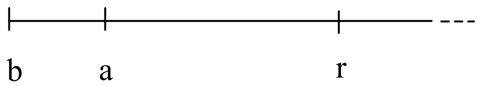
§20: A remarkable relation between the two infinite sets is the possibility to form pairs joining each object of a set with another belonging to the counter set, so that for each object of one set there is always its correspondent and no object happens to appear in two or even more pairs (biunivocal correspondence of two infinite sets).

§21: Notwithstanding their property of being of equal number, two infinite sets can be in a inequality relation as their multitudes are concerned, so that one set is a proper part of the other. (Using modern language: one set is infinite if and only if it can be put in biunivocal correspondence with one of its proper parts. This intuition was developed before Dedekind; but it was no definition and maybe there was still no fully-fledged awareness).

Bolzano was known not only for the significant results we mentioned but also for some famous errors and uncertainties. Some examples are provided here as follows:

§18: If A is a set and some elements have been subtracted from it, then A contains fewer elements than before.

§19: There are some infinite sets that are larger or smaller than other infinite sets. The half-line br is major than the half-line ar, then it can be deduced that there are infinities of different magnitudes.



§29: There is confusion between the cardinality of the {1, 2, 3, …, n, …} set and the value 1 + 2 + 3 + … + n + …

§32: Guido Grandi (1671 - 1742) raised the problem of calculating the "sum" of infinite addenda: s = a – a + a – a + a – a + … obtaining several answers:

s = (a – a) + (a – a) +(a – a) + ... = 0 + 0 + 0 + ... = 0

s = a – [(a – a) + (a – a) + (a – a) + ...] = a – [0 + 0 + 0 + …]= a – 0 = a[12]

[12] In 1703 Grandi wrote: *«If we differently position the parentheses in the expression 1 – 1 + 1 – 1 + 1 …we can obtain either 0 or 1. Therefore the principle of creation ex nihilo is perfectly plausible».*

$s = a - (a - a + a - a + a - a + \ldots) = a - s = \Rightarrow 2s = a \Rightarrow s = a/2$ (this solution proposed by Grandi[13] himself was particularly appreciated by Leibniz who defended it).

In Bolzano's time the question was still open and debated. As for this latter paradox, as Bolzano himself reported, in 1830 a writer known as M.R.S tried to provide a demonstration of the third solution publishing it on the *Annales de Mathématique de Gergonne*, 20, 12, to which Bolzano reacted in the following way: *«The series within parentheses has clearly not the same set of numbers of that originally indicated with x* (s in this case), as *the first term a is missing».*

§ 33: Precautions to be observed by the calculus of infinity in order to avoid "mistakes":

Be S_1 the sequence of numbers 1, 2, 3, …

Be S_2 the sequence of their squares $1^2, 2^2, 3^2, \ldots$

Now: since all terms in S_2 appear also in S_1 and there are terms of S_1 that do not appear in S_2, this would imply that the sum of S1 terms is major than the sum of the terms of S_2, whereas the sum of the terms of S2 is major than that of S_1, as both S_1 and S_2 can be set in biunivocal correspondence and each term of S_2 is major than (with the exception of the first term) its correspondent in S_1.

§ 40: Paradoxes on the concept of space: two segments of different lengths are formed of different number of points.

§ 48: A volume contains more points than its lateral surface and the latter more points than the curve enclosing it.

According to Cantor (1932), Bolzano's problems are due to the fact that the idea of a cardinality of a set[14] was at his time missing. There is still a long way to go, we are just at the beginning of our path.

We cannot leave **Karl Weierstrass (1815 - 1897)** out, considered by many to be the one who provided a rigorous systematisation of Mathematical Analysis. He is important for

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[13] D.J. Struik (1948) wrote: *«He* (Grandi) *obtained the value ½ on the basis of the anecdote of a father who hands down a precious stone to his two sons. Each of them has to keep it alternately for one year, so that in the end, each son will turn out to own half of the stone».*

[14] As for didactics, many recent studies aim at analysing similarities between students' "naive" remarks and some of Bolzano's statements. On this subject see for example the work by Moreno and Waldegg (1991).

our research because he conscientiously investigated the subject of infinity. Some considered the work of Analysis systematisation initiated by Cauchy (1789 – 1857) according to the modern definition of limit and continuous function (the so-called ε-δ Weierstrass' definition), as the ultimate abandonment of infinity in act in favour of the infinity in power (Marchini, 2001). Others believe that Weierstrass' work was a contribution, also from a formal perspective, to the evolution of the potential infinitesimal towards the actual infinitesimal (Arrigo and D'Amore, 1993; D'Amore, 1996; Bagni, 2001). Ideally, this evolution continued in the XX century with non-standard analysis (Robinson, 1974).[15]

1.2.2 Richard Dedekind (1831 – 1916)

In his book *Continuity and Irrational Numbers* of 1872, the fourth paragraph has a charming and meaningful title: *Creation of irrational numbers*. Creation… and as a matter of fact, thanks to his famous method of "cuts" or "sections", he creates, starting from Q, the set R adding to Q the irrational numbers.[16]

Real numbers are classes of definite sections in Q. (Q, <) is dense but not continuous (this demonstration is basically attributed to Pythagoreans); (R, <) is dense and continuous[17] (Bottazzini, 1981).

Of particular interest is the correspondence between Dedekind and Cantor that will be dealt with in paragraph 1.2.4. The necessity of defining continuum is to be traced back to that period and the above-mentioned German mathematicians provided the two most probably famous continuity axioms (Bottazzini, 1981; Kuyk, 1982).

According to Rucker (1991), in 1887 Dedekind published one of his most famous works: *Was sind und was sollen die Zahlen* (What are numbers and what should they

[15] In the sixties, Abraham Robinson (1918 - 1974) managed to form a consistent theory, based on important theorems of Mathematical Logic and some of Skolem's (1887 - 1963) ideas, to handle actual infinitesimals and infinities through non-standard analysis.

[16] Among the research studies pointing out the difficulty of the notion of irrational number, we would like to mention: Fischbein, Jehiam and Cohen, 1994, 1995.

[17] On the difficulty of the concept of density for primary school pupils see: Gimenez, 1990. Whereas on the difficulty of the notion of continuum for 16-17 year old students, see: Romero i Chesa and Azcarate Gimenez, 1994.

be), a demonstration of infinity of the *World of thoughts, Gedankenwelt* in his language. Demonstration will be shown as follows:

if *s* is a thought: "*s* is a thought" is a thought;

""*s* is a thought" is a thought" is a thought;

"""*s* is a thought" is a thought" is a thought" is a thought;

…

In a letter dated 1905 Cantor wrote on this "demonstration":

> *«A multiplicity* [set] *could be such that the assumption that all its members "are together" leads on to a contradiction, so that to conceive multiplicity as unit, a "finite thing" is impossible. I would call these multiplicities absolutely infinite or incoherent multiplicities. It is quite evident, for instance, that "the totality of thinkable things" is such a multiplicity…»* [our translation].

(The reason for excluding that the set of all thoughts is a thought is that such a set would therefore be a proper element of itself).

The infinite set definition already envisaged by Galileo is attributed to Dedekind: *"A set is infinite when it can be put in biunivocal correspondence with one of its proper parts"*.

1.2.3 Georg Cantor (1845 - 1918)

Young and brilliant mathematician, Cantor focuses his research work on those mathematical problems academic senior scholars are interested in: the problem of uniqueness of the decomposition of a real function into a trigonometric series. In 1872 (Cantor is 27 years old) he devotes his study to the infinite set of the points placed in an interval but not coinciding with the interval itself. In so doing, he analyses how the points of a straight line are positioned, the reciprocal positions between different segments; segments and straight lines, ... Everything dealt with in the actual sense with no philosophical embarrassment.

Cantor finally abandons the formal academic mathematics and starts to investigate the infinity by itself. That is the beginning of his adventure.

Some extracts from *Gesammelte Abhandlungen* (1932):

«Potential infinity has just one borrowed reality since the concept of potential infinity is to be always reconducted to that of actual infinity that logically precedes the former guaranteeing its existence.

Actual infinity manifests itself in three contexts: the first is the most accomplished form, a completely independent being transcending this world, Deo, this is what I call Absolute Infinity; the second has to do with real world, the creation; the third is when the mind grasps infinity in abstracto as a mathematical magnitude, number or type of order.

I want to clearly and firmly state the difference between the Absolute and what I call Transfinite, i.e. actual infinity in the last two forms, since it is about objects apparently limited and susceptible of growing process and thus related to finite».

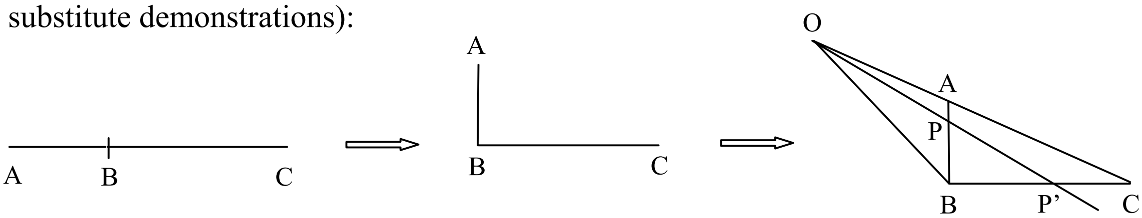
«The fear of infinity is a kind of short-sightedness that destroys the possibility of seeing actual infinity, even if infinity in its highest expression created us and sustains us and through its secondary forms of transfinite surrounds us and even dwells in our minds».

«Therefore inevitable is the need for the construction of the concept of actual infinite number obtained through the appropriate natural abstraction, as well as the concept of natural number derives from finite sets by means of an abstraction process» [our translation].

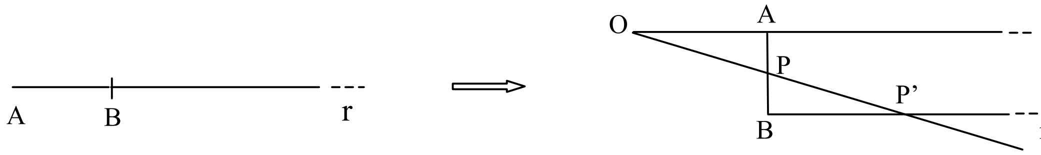
We provide a modern representation of some of Cantor's results.

- Two sets are of equal number if a biunivocal correspondence exists between them (notable is the fact that there is no distinction at all between finite and infinite sets).

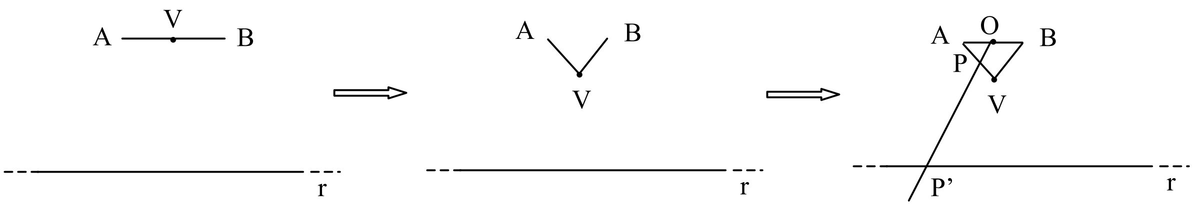
- The segments AB and AC (conceived as a set of points) are of equal number independently of their length that has no influence at all (here as follows figures substitute demonstrations):



- The set of the points of a segment has the same number of that of the points of a half-line:

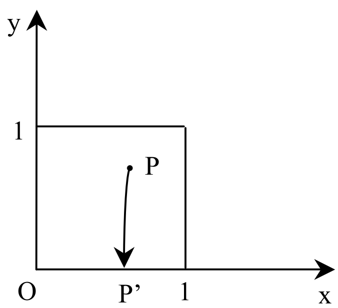


- The set of the points of a segment has the same number of that of the points of a straight line:



- The set of the points of a square and that of the points of one of its side are of equal number.

Let us consider for example the unit square in a system of Cartesian coordinates and thus having its side in abscissa coordinates:



In the duality of the possible representation for the same number, for ex.: 0.40000000… 0.399999999…, let us choose one and eliminate the other (we exclude the period 9 in this case).

Every point internal to square has coordinates such as:

P $(0.a_1a_2a_3\ldots a_n\ldots; 0.b_1b_2b_3\ldots b_n\ldots)$;

To it we make correspond a well determined point on the side (in abscissa coordinates):

P $(0.a_1b_1a_2b_2\ldots a_n b_n\ldots)$

and vice-versa.

The biunivocal correspondence between the points of a square and those of one of its side is established.[18]

(At the beginning Cantor was convinced that the cardinality **c** of the straight line was $\aleph_1$, the plane cardinality $\aleph_2$, the space cardinality $\aleph_3$ and so on. Conversely, this demonstration shows that the cardinalities pertaining to all these continuous point sets are always equal to **c**).

1.2.4 Cantor-Dedekind Correspondence

This is an extract of one of Cantor's letters to Dedekind dated 2 December 1873:

> *«As for the matters I've been occupied with lately, I realise, the following is pertinent to them:*
> *can a surface (a square including its edge for instance) be put in a univocal correspondence* [today we would call it biunivocal correspondence] *with a curve (a straight line segment with end-points included for instance) so that to each point of the surface corresponds one point of the curve and reciprocally to each point of the curve one of the surface?*
> *In this moment to answer this question seems to me very difficult and here there is so great a tendency to give a negative answer that a demonstration would seem superfluous»* [our translation].

Extract from a letter from Dedekind to Cantor dated 18 May 1874:

> *«… I talked to a friend in Berlin about the same problem and he considered the thing somewhat absurd as it goes without saying that two independent variables cannot be handled as one»* [our translation].

Extract from a letter from Cantor to Dedekind dated 20 June1877:

[18] The research work of Arrigo and D'Amore (1999) focuses on the difficulty high school students encounter in accepting this demonstration.

«I would appreciate knowing if you consider the demonstration method I used as strictly rigorous from an arithmetical point of view. It is about proving that surfaces, volumes and continuous varieties of p dimensions can be put in univocal correspondence with continuous curves, thus with only one-dimensional varieties, that surfaces, volumes, varieties of p dimensions have the same power of curves; this opinion seems to contrast with the most generally accepted, especially among founders of the new geometry according to which there are varieties once, twice, three times, … p times infinite; it is as if the infinity of points of a square surface could be obtained elevating it somehow to its square, that of a cube elevating it to the cube, the infinity of points of a line. (…). I want to talk about the hypothesis according to which a continuous multiplicity extended p times necessitates, in order to determine its elements, of p real coordinates independent of each other. This number, for the same multiplicity cannot be increased or decreased. I also came to the conclusion that this hypothesis could be correct but my point of view differed from all the others in one point. I considered this hypothesis like a theorem awaiting a proper demonstration and I expressed my point of view in the form of a question posed to some colleagues also on the occasion of Gauss' Jubilee in Göttingen».

«Can a continuous variety of p dimensions, with p >1, be put in a univocal correspondence with a continuous variety of one dimension so that to each point of one corresponds one and one only point of the other?
The majority of people whom I posed this question were quite surprised by the fact itself that I posed such a question, for they believed as obvious that, in order to determine a point in an extension of p dimensions, p independent coordinates are needed. Those who could, despite all, penetrate the question had at least to admit that the "obvious" answer "no" needed at least to be demonstrated. As I told you, I was among those who held a negative answer as probable, until very recently, when after so complex and strenuous reasoning, I came to the conclusion that the answer is affirmative and with

no restrictions. After a while, I found the demonstration that you will see hereafter» [our translation].

(Cantor showed Dedekind the above-mentioned demonstration concerning the points of the square and of its side).

The letter was sent on 20 June 1877 but Cantor was so impatient about it that he wrote to Dedekind again on 25 June 1877 urging him an answer:

«As long as you do not approve me I am bound to say: I see it but I don't believe it».

Dedekind immediately answered back on 29 June 1877:

«Once again I examined your demonstration and I found no faults. I'm convinced that your interesting theorem is correct and I congratulate you».

The route to infinity is definitely open (only the numerical infinity will be investigated in this chapter).

1.2.5 Cardinality

Let us consider the N set of natural numbers being **n** its cardinality or numerousity or power that we will call "of the numerable"; **n** is an infinite cardinality as it is larger than any given finite cardinality.

Be N_s the set (Galileo's) of perfect squares, N_e the set of even numbers, N_o of odd numbers, N_{Pr} of primes, … Each one of these sets can be put in biunivocal correspondence with N and therefore has the cardinality of the numerable **n**.

If A is a subset of N, infinite, then the cardinality of A is **n**.

In fact, as supposed, $A = \{a_1, a_2, \ldots, a_m, \ldots\}$ where a_i are elements of N. Let us consider the biunivocal correspondence $a_1 \leftrightarrow 0$, $a_2 \leftrightarrow 1$, …, $a_m \leftrightarrow m - 1$, …

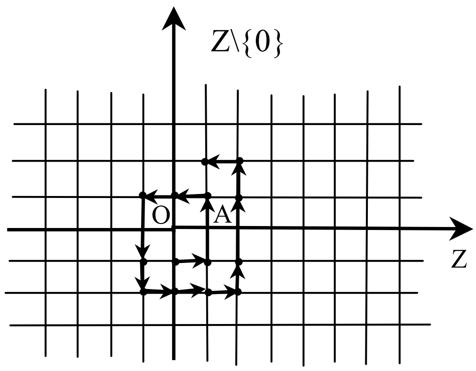
Thus: **n** is the *smallest* infinite cardinal.

Let us consider the set Z of whole numbers; the biunivocal correspondence with N is created:

$0 \leftrightarrow 0$, $1 \leftrightarrow {}^{+}1$, $2 \leftrightarrow {}^{-}1$, $3 \leftrightarrow {}^{+}2$, $4 \leftrightarrow {}^{-}2$, …

Thus the cardinality of Z is still **n**.

In search of those values of infinity to which St.Thomas Aquinas referred to (see paragraph 1.1.1), let us have a try with Q, the set of rational numbers. Every rational number $\frac{^{+}a}{^{-}b}$ can be represented as a point P of ($^{+}$a; $^{-}$b) coordinates placed on a Cartesian coordinate system.
Then, all rational numbers can be positioned as shown in the figure below starting from origin in O.



Only some of these points represent rational numbers [the point A (1; 0) does not represent any rational, for instance]. The points representing rationals have been deeply marked in the figure. The outcoming rational sequence has as the first element (0 ; $^{-}$1), the second (1 ; $^{-}$1), the third (1 ; 1); …that represent 0, $^{-}$1, 1 ... So there is a biunivocal correspondence of Q with N since we "numbered" Q.
[As for the problem of "doubles" as in the cases of ($^{-}$2 ; 3) and (2 ; $^{-}$3) that are different points representing the same rational; you may either count them twice or just skip them the second time you encounter them].
Thus the cardinality of Q is still **n**.

In addition, Cantor proved that the set of algebraic numbers (equation solutions) has against all expectations **n** cardinality.

It is precisely at the moment when **n** seems to be impossible to overcome that the key demonstration is achieved.
The set of real numbers included between 0 and 1 (end-points excluded) has a cardinality superior to **n**.

The demonstration is performed per absurdum. Assume per absurdum:

$0.a_{11}\ a_{12} \ldots a_{1n} \ldots$

$0.a_{21}\ a_{22} \ldots a_{2n} \ldots$

…

$0.a_{n1}\ a_{n2} \ldots a_{nn} \ldots$

…

all real numbers included between 0 and 1 (that is to say: let us suppose that they are a denumerable quantity). Let us consider the notation:

$0.b_1\ b_2 \ldots b_n \ldots$

such that $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, …, $b_n \neq a_{nn}$, …;
then it is obvious that this notation:

- *is not* included in the preceding list of *all* real numbers between 0 and 1;
- is a real number included between 0 and1;

we found therefore a contradiction due to the assumption that real numbers between 0 and 1 would have **n** cardinality.
(Once again considerations on the double writing of rational numbers should be taken into account).

Thus: real numbers included between 0 and 1 are infinite although they do not form a denumerable infinity.
n is the smallest infinity and real numbers included between 0 and 1 constitute a *larger* infinity.
The exact date of Cantor's discovery is 7 December 1873. The date is known because on the following day he wrote a letter to the friend Dedekind to communicate his demonstration.
Observing that such a cardinality, that of reals between 0 and 1, is the same for all reals is banal. We would call it *cardinality of the continuum* and indicate it with **c**.
With a little abuse of symbolic language, we would write:

$$\mathbf{n} < \mathbf{c}$$

But **c** is also the cardinality of the points of a straight line, of those of a plane, of those of any continuous variety of *m* dimensions.

> *«It can be with no doubt affirmed that the theory of transfinite numbers works out or collapses together with irrational numbers; they share the same essence because these are anyway all examples or variants of actual infinity»* (Cantor, 1932). [our translation]

Therefore Cantor was at that moment aware that there were at least two infinite numbers: **n** and **c**. His aim was to find a set S of **s** cardinality such that $\mathbf{n} < \mathbf{s} < \mathbf{c}$.
He spent a long time working on that, but then a peculiar analogy raised his attention.

1.2.6 The Continuum Hypothesis

Let us consider the finite set I and its so-called power-set: P(I). From now on we will indicate the cardinality of a set with: | I |.
It can be demonstrated that:
$| P(I) | = 2^{| I |}$

Let us extend this concept to infinite sets.
According to Dedekind's method of cuts (or sections) used to introduce real numbers, R is nothing else but a class of classes of cuts in Q;
and therefore | R | = | P(Q) |
But then:

$$\mathbf{c} = 2^{\mathbf{n}}$$

this writing introduces an interesting demonstration:

$$\mathbf{c} \cdot \mathbf{c} = 2^{\mathbf{n}} \cdot 2^{\mathbf{n}} = 2^{\mathbf{n} + \mathbf{n}} = 2^{\mathbf{n}} = \mathbf{c}$$

(therefore the plane that is the set of all ordered pairs of real numbers and whose cardinality is **c** · **c** has **c** cardinality. This has been previously proved through the biunivocal correspondence between the points of a square and those of its side).

Once the meaning of the order of transfinite numbers has become clear, to continue with this procedure is an easier task. Let us consider the set F of functions from R in R.
We call **f** the cardinality of F:

$$\mathbf{f} = 2^{\mathbf{c}}$$

as well as the sequence of natural numbers 0, 1, 2, 3, … goes on adding 1 all the time, also the sequence **n**, **c**, **f**, **g**, … of transfinite numbers works in the following way:

$$\mathbf{n}, \qquad \mathbf{c} = 2^{\mathbf{n}}, \qquad \mathbf{f} = 2^{\mathbf{c}}, \qquad \mathbf{g} = 2^{\mathbf{f}}, \qquad \dots$$

in a never ending process. If there were an end to it, we would in fact find out a paradox: an entity of the maximum possible cardinality, G for instance, which admits an increase going through its power -set P(G).

However, the aim was to find a set S such that: $\mathbf{n} < | S | < \mathbf{c}$:
Let us put it into more general terms:
we can try to find a set S_1 such that $\mathbf{c} < | S_1 | < \mathbf{f}$; and then another S_2 such that $\mathbf{f} < | S_2 | < \mathbf{g}$; and so on.
In 1883 Cantor wrote that he wished to be soon able to demonstrate that the continuum cardinality is the same of the second numerical class, that is to say that such a set S does not exist. His research produced no results: he could not prove it; nor could he prove the opposite (to demonstrate such S).
Then he developed a conjecture:
Cantor's hypothesis or *continuum hypothesis:*

c strictly follows **n** that is to say that there is no cardinal **s** such that $\mathbf{n} < \mathbf{s} < \mathbf{c}$.[19]

Now, if it is supposed that **c** strictly follows **n**, then why not generalise it?
Cantor's hypothesis or *generalised continuum hypothesis*:

c strictly follows **n**, **f** strictly follows **c**, **g** strictly follows **f**, and so on.

Therefore these are elements of a new sequence that can be re-named as follows:

$$\mathbf{n} = \aleph_0, \qquad \mathbf{c} = \aleph_1, \qquad \mathbf{f} = \aleph_2, \qquad \mathbf{g} = \aleph_3, \qquad \dots$$

Thus $\aleph_{n+1} = 2^{\aleph_n}$

[Between 1938 and 1940 **Kurt Gödel** would demonstrate that, assuming the continuum hypothesis (CH) in the *set theory* (we will call it ZF by the name of the creators:

[19] As Rucker reports (1991): *«Cantor firmly believed that* $\mathbf{c} = \aleph_1$ *was valid. Gödel, at a certain stage of his studies, believed that* **c** *was* $\aleph_2$ *and D.A. Martin wrote an article from which we could deduce that* **c** *is* $\aleph_3$*».*

Zermelo (who developed the axioms in 1908) and Frankel (who further investigated the above-mentioned axioms in 1922 and then transcribed them in the language of the Calculus of Predicates), no contradictions are introduced (in other words, CH is compatible with ZF). Therefore: CH is either independent from ZF axioms or it can be demonstrated on their basis. To say it differently, this means that Cantor was not mistaken, i.e. from ZF is not possible to deduce that **c** is different from $\aleph_1$. At the same time to prove that Cantor was right is also not possible. In 1963 **Paul J. Cohen** showed (by means of a method called "forcing") that no contradictions to ZF are introduced if we assume the negation of CH. Thus, CH negation is compatible with ZF. Therefore it cannot be demonstrated if Cantor was right or wrong. In conclusion, CH has to be dealt with as a new axiom: if we add ZF it to we have the "*Cantorian* set theory"; whereas if we add its negation the "*non-Cantorian* set theory" (Gödel, 1940; Cohen, 1973)].

1.2.7 Giuseppe Peano (1858 - 1932)

Let us make a little digression with Peano. He also committed himself with questions related to infinity. His famous systematisation of natural numbers needed at some point an Axiom of Induction.[20] It can be even said that this is a basic and fundamental feature of the concept of natural numbers itself (Borga et al., 1985). Today the induction principle is a fundamental support for arithmetic and logical demonstrations and it recalls potential infinity.[21]

1.2.8 Cantor and the ordinals

Let us go back to Cantor (this part is an extract from Rucker, 1991; D'Amore, 1994).

Let us define ordinals by repeated steps:

0 is an ordinal

Principle 1:[22]: every ordinal number a has an immediate successor $a + 1$

---

[20] Be P a property that can apply to natural numbers. Let us assume that 0 possesses this property P, let us assume that for every natural number x, if x has the property P, then x + 1 will also have the property P, under these conditions we can state that every natural number has the property P.

[21] As far as didactics is concerned, Fischbein and Engel (1989) and Morschovitz Hadar (1991) worked on high school students' difficulty to accept the induction principle.

[22] The basic concept behind this Principle is that: *no ordinal number is minor than itself.*

Principle 2: given an increasing sequence of ordinals a_n, the minimum ordinal is defined [indicated as $\lim(a_n)$] that follows all the ordinals in the given sequence.

Starting from 0 and repeatedly applying the Principle 1, we obtain the ordinals 0, 1, 2, 3, …

Now if we want to overcome the infinite sequence of finite ordinals we need to use the Principle 2 to get lim(n) that we indicate with ω:

$$0 \quad 1 \quad 2 \quad \dots \quad n \quad n+1 \quad \dots \quad \omega$$

This is in its turn a new sequence of ordinals and consequently applying progressively the Principle 1 many more times, you obtain:

$$0 \quad 1 \quad 2 \quad \dots \quad n \quad n+1 \quad \dots \quad \omega \quad \omega+1 \quad \omega+2 \quad \omega+3 \quad \dots$$

This is the new increasing sequence of ordinals ($\omega + n$) and therefore applying to it the Principle 2 lim(ω+n) is created:

$$0 \quad 1 \quad 2 \quad \dots \quad \omega \quad \omega+1 \quad \omega+2 \quad \omega+3 \quad \dots \quad \omega+\omega$$

it can be also written down in this way: ω+ω or ω · 2.

Adding and multiplying ordinals could be written down in this way:

$a+b$ ·= counting starting from a + 1 for b times

$a \cdot b$ ·= juxtaposing b copies of a

When we deal with finite ordinals, these operations will coincide with the usual sum or product and are commutative, but when it comes to their extension to transfinite ordinals the commutative property is not maintained.

Some examples:

1 + ω = 1 0 1 2 … = (counting again from the beginning) 0 1 2 3… = ω

ω+1 = 0 1 2 … 1 = ω+1

thus: 1 + ω = ω ≠ ω+1

2 · ω = 2 2 2 … = (counting) 0 1 2 … = ω

ω · 2 = (double juxtaposition of ω) =

= 0 1 2 … 0 1 2 … = 0 1 2 … ω ω + 1 ω + 2 … = ω + ω

and thus: 2 · ω = ω ≠ ω + ω = ω · 2.

We came to $\omega \cdot 2$; applying many times the Principle 1 we get:

0   1   2   … ω $\omega+1$ $\omega+2$ … $\omega\cdot 2$ $\omega\cdot 2+1$ $\omega\cdot 2+2$ …

and again the Principle 2, obtaining $\lim(\omega \cdot 2 + n) = \omega \cdot 2 + \omega$ that will be also called $\omega \cdot 3$.

Operating in the same way we get to $\omega \cdot n$, for every finite n and consequently we could use the Principle 2 to obtain $\lim(\omega \cdot n)$ i.e. ω copies of ω, that is to say $\omega\cdot\omega$ that we will also call ω^2. Continuing you easily get to a ω^3 and progressively to:

$$\omega^{\omega}$$

ω^2 can be conceived as the first ordinal a for which: $\omega + a = a$.

De facto: ω^2 is like $\omega + \omega + \omega + \omega + \omega + \ldots$ and so it will make no difference if we put before another ω as addendum.

Analogously, the first ordinal a for which: $\omega \cdot a = a$ is ω^{ω}.

As a matter of fact, ω^{ω} can be thought as $\omega \cdot \omega \cdot \omega \cdot \omega\ldots$, obtained by the juxtaposition of ω for ω times; therefore it will make no difference if we put before another ω as factor: $\omega \cdot \omega^{\omega} = \omega^1 \cdot \omega^{\omega} = \omega^{1+\omega} = \omega^{\omega}$, since that $1 + \omega = \omega$.

Let us consider the first ordinal a for which the equality: $\omega^a = a$ is valid

This number is:

$\omega^{\omega^{\omega^{\cdots}}}$ (in it the raising to power is performed ω times).

Nothing will change if we put at the base of this notation another ω: the exponents will be $1 + \omega$ that is always ω.

To this number it has been given the name ε_0 and for nearly 200 years it has been indicating real numbers small "as we please".

We introduce a new operation:

${}^a b \cdot = b^{b^{\cdot^{\cdot^{b}}}}$ (i.e. b elevated to itself, for a times counting the base).

Some examples will be provided only to give an idea of how numbers grow by means of this new operation.

${}^{3}3 = 3^{(3^3)} = 3^{27}$ nearly eight thousand billions.

According to the new operation, ω^{ω} is nothing else than ${}^{2}\omega$.

And therefore ${}^{3}\omega$ $\omega^{(\omega^{\omega})}$ is a number which is very difficult to imagine.

Let us go back to ε_0 that according to the new notation is ${}^{\omega}\omega$.

ε_0 is not the last ordinal, here we have an even larger one:

$$\omega^{\omega^{\omega^{\omega^{\cdots\omega}}}}$$

Every time that you come to larger ordinals, you need to stop for a while before envisaging the way of producing even greater ones, and this is only the beginning[23] (for a further investigation of this topic see Rucker, 1991).

1.2.9 Ordinals as cardinals

Let us go back to our subject matter using alephs.

ω is exactly $\aleph_0$, the first infinite cardinal.

But $\omega+1$, $\omega+2$, …, $2 \cdot \omega$ are also all $\aleph_0$.

$n \cdot \omega$, …, ω^{ω} are also $\aleph_0$

$\omega^{\omega}+1$, …, ${}^{\omega}\omega$ are also $\aleph_0$

Thus, ω is the smallest ordinal equal to $\aleph_0$; so far there was no growth however. Also ε_0 is nothing else than a "banal" $\aleph_0$. Every ordered set with cardinality ω, …, ${}^{\omega}\omega$ can be always put in biunivocal correspondence with N.

However, since it is possible to find ever-increasing ordinals, it is also possible to find ever-increasing cardinals:

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \ldots,\ \aleph_{\omega},\ \aleph_{\omega+1},\ \ldots,\aleph_{\aleph_1},\ \ldots,\ \aleph_{\aleph_{\omega}},\ \ldots$$

---

[23] In Bachmann (1967) we are provided with probably the most exhaustive description of notation systems for denumerable ordinals. Whereas in Cantor (1955) we can find the clearest description of transfinite ordinals, he had ever produced.

We can also find a number θ such that $\theta = \aleph_\theta$

This θ is of this kind:

$$\theta = \aleph_{\aleph_{\aleph_{\aleph_{\aleph_{\ddots}}}}}$$

θ ends a cycle.

So far, as after $\aleph_\theta$ comes $\aleph_{\theta+1}$, …, $\aleph_{\theta+\omega}$, …

You never come to an end in the discovery of transfinite numbers.

Let us prove it:

- A set S is finite or denumerable if and only if $|S| \leq \aleph_0$
- The Principles 1 and 2 induce a stronger Principle i.e. the no. 3: for each set of *A* ordinals there exists the minimum ordinal that is major than every element of A and that we will call *supA*.

Let us consider the collection O*n* of all ordinals. If O*n* were a set, then according to the Principle 3 there would be an ordinal supO*n* (we call it Ω, the Absolute Infinity that is positioned at the end of the sequence of ordinals). But this is impossible because if Ω were an ordinal, then Ω would be an element of the collection O*n* of all ordinals and then it will be Ω < sup O*n* = Ω, a fact that contradicts a fundamental property of the ordinals according to which no ordinal can be minor than itself.[24]

The Principle 3 states that no set of ordinals can reach Ω.

We conclude with a passage from a Cantor's letter to Dedekind dated 28 August 1899:

> *«It may be legitimate to wonder if well-ordered sets or sequences corresponding to cardinal numbers* $\aleph_0$, $\aleph_1$, …, $\aleph_\omega$, … $\aleph_{\aleph_1}$, … *are real sets in the sense of being "consistent multiplicities". Is it not possible that these multiplicities are "inconsistent" and that the contradiction deriving from the assumption that these multiplicities exist in the form of unified sets has not been acknowledged yet? My answer is that the same question could be posed with regard to finite sets, and if you properly focus on it, it stands*

[24] Cesare Burali and Forti disclosed this situation in 1897, but Cantor had noticed it even before.

> *out clearly that not even for finite multiplicities a demonstration of consistency is possible. In other words: the consistency of finite multiplicities is a simple and improvable truth that we can call "axiom of arithmetic" (in the old meaning of the word). Analogously, the consistency of those multiplicities that have aleph cardinality constitutes "the axiom of arithmetic extended to transfinite"*» [our translation] (Meschkowski, 1967).

It seemed that here Cantor referred to simple and direct perception of the reality of cardinal numbers in the Realm of Thoughts. Moreover, in 1899 he proved his intellectual courage stating that: «*A number such as* $\aleph_2$ *is much easier to be perceived than a casual natural number of ten million digits*» (see Cantor, 1932). This Cantor's daring affirmation will prove quite difficult to be shared after the results regarding teachers' convictions will be illustrated in chapters 3 and 4.

Chapter 2. International research context

The research work on mathematical infinity, which will be described in chapters 3 and 4, is to be considered within today's scenario of mathematical didactics. This seems to be aiming at focusing on the phenomenon of learning, the latter seen from the point of view of *fundamental didactics* (Henry, 1991; D'Amore, 1999). This notion includes all the basic elements related to the research in mathematical didactics, deriving from the numerous and complex analyses of so-called "triangle of didactics" (see paragraph 2.4). A brief outline of the major topics concerning this research field will be provided in the next chapters and will recall mainly the writings of D'Amore (1999, 2002, 2003) where the author builds up a personal trajectory in the field of didactics of mathematics. We fully subscribe to the ideas developed by D'Amore.

2.1 The didactical contract

The first attempt "to define" the didactical contract is the following: *«During a teaching class prepared and held by a teacher, the student is generally given the task to solve a problem (a mathematical one), but access to the assigned task is made possible through interpretation of the questions posed, the pieces of information provided and through the fixed steps imposed by the teacher's method. The (specific) teacher's behaviours expected by the student and the student's behaviours expected by the teacher constitute together the didactical contract»* (Brousseau, 1980a; our translation). The latter idea has been shared by various scholars from all over the world and has become part of the language spoken by the whole international community since the late Eighties (Brousseau, 1980b, 1986; Brousseau and Pères, 1981; Chevallard, 1988; Sarrazy, 1995; Schubauer-Leoni, 1996). The original idea of didactical contract has been often reinterpreted and modified by various authors over the years, even with very different modalities and approaches as stated by Sarrazy (1995). Going back to the original principle, the "expectations" to which Brousseau refers to, are in most cases not due to explicit agreements imposed by school or teachers and negotiated with the students but they are strictly related to the way school, mathematics and the repetition of modalities are conceived (D'Amore, 1999, 2002, 2003). Over the last decades, the analysis of phenomena related to such students' behaviours has yielded significant results favouring the interpretation and explanation of various behaviours that were still considered inexplicable or due to lack of interest, ignorance or students' immaturity until recent times (Baruk, 1985; Spagnolo, 1998; Polo, 1999; D'Amore, 1999). The above-mentioned research study revealed that children and young people have specific expectations, general schemes and behaviours having no relation with mathematics though depending on much more complex and interesting motivations emerging from the didactical contract set up in the classroom (D'Amore, 1993b; D'Amore and Martini, 1997; D'Amore and Sandri, 1998). In order to modify these behaviours, students should be able *to break the didactical contract* (Brousseau, 1988; Chevallard, 1988), being personally responsible for their choices. As a matter of fact, through the breaking of the didactical contract students create a new situation that contrasts their expectations, habits and all the clauses that have been set so far in didactical situations. To achieve this goal students should be determined enough to try themselves out and be in the front line, going against the given contract clauses. This phenomenon can happen only if the teacher favours such a breaking.

2.2 Images and models

With respect to *"image"* and *"model",* we will use the following terminology and treatment adopted by D'Amore (1999, 2002, 2003):

The *mental image* is the figural or propositional result produced by an (external or internal) impulse; cultural influence, personal style[25] and feature condition it. In short, it

[25] By cognitive style we intend all those personal features that an individual, more or less consciously, has got and implements, when involved in a learning process; these characteristics seem not to depend just on "natural" proclivities, but also on mood and temporary situations, disposition, interest, motivation, … For example, one can gradually get to know how to learn acoustically or visually and get familiar with learning by manipulating images or symbols, … (De La Garanderie, 1980; Gardner, 1993; Sternberg, 1996).

is the typical product of the individual but it still presents common and constant connotations shared with other individuals. The mental image undergoes different levels of conscious elaboration (this skill is also related with the individual).

All elaborated mental images (more or less consciously) connected to the same concept form the (internal) *mental model* of the concept itself. As a matter of fact, students build for themselves the image of a concept. They believe it to be stable and definitive but at a certain point in their cognitive history they receive information on the concept that is not included in the image they have constructed. Therefore students have to adjust the "old" image to a new wider image that contains both the previous one and new pieces of information. This fact is caused by a *cognitive conflict* triggered by the teacher (see paragraph 2.3). The process can take place many times during the student's "educational history". Most concepts in mathematics are formed only through the constant transit, over years, from an image to the other, the latter being more powerful than the former. One can visualise these subsequent conceptual constructions as a sequence of images, which get "closer and closer" to the concept.

During the sequence of images you reach a certain point when the image you have come to after several passages "resists" different stimuli, and turns out to be "strong" enough to contain the new argumentation and pieces of information gradually encountered. These are related to the concept, which is represented by the image itself. Such a stable and no longer changing image can be addressed as *model* of the concept. Therefore, "to construct a model out of a concept," means to successively revise several (weak and unstable) images to come to an ultimate strong and stable image.

It can be verified that:

- *the model is created at the right time*, i.e. it is just the correct model aimed at for that specific concept of mathematical knowledge. The didactical action has worked out: the student has built a correct model of the concept;

or:

- *the model is created too early*, i.e. the image is still weak and needs to be widened. In this case reaching the concept turns out to be difficult because the stability of the model is an obstacle to future learning.

The name *intuitive model* is given to those models that fully respond to intuitive stimuli and are immediately and strongly accepted. That is to say that there is direct correspondence between the suggested situation and the mathematical concept used. When a teacher suggests a strong and convincing image that becomes persisting and is continuously confirmed by numerous examples and experiences of a concept, the image develops into an *intuitive model* (Fischbein, 1985, 1992). Still this model could not correspond to the model of concept expected. There is also the category of *parasite models*, created through repetition, but not at all desired (Fischbein, 1985). Examples of this kind can be found in D'Amore (1999).
From a didactical point of view, it is advisable that the misconception-image does not become a model (see paragraph 2.3), for, due to its own nature, it is awaiting a definitive collocation. In this case, assimilating the new situation to adjust the former model (strong and stable) to the new one proves quite a difficult task. It is advisable to let students keep unstable images until proper and meaningful models, which are suitable to the expected level of mathematical competence, are created. Thus, it is important that the teacher avoids providing explicitly unreliable and wrong information as well as autonomous building of information helping create parasite models in students' minds. In order to succeed in reaching this difficult goal, the teacher should be confident and skilled not only in the field of mathematics but also in didactics of mathematics.

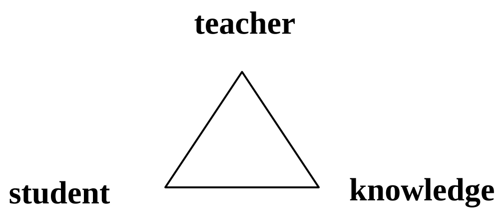
2.3 Conflicts and misconceptions

Another subject dealt with in didactics of mathematics and pertaining to this work is that of *cognitive conflicts* (Spagnolo, 1998; D'Amore, 1999, 2003). Over time the student constructs a concept and then builds an image of that concept; during the school years this image can become stronger and be validated through tests, repeated experiences, figures and exercises, especially those assessed and marked as correct by the teacher. It can also happen that such an image turns out to be inadequate sooner or later when compared to another one relating to the same concept. This latter image can be suggested by the teacher or anybody else, it is unexpected and in contrast with the previous one. A *conflict* is born between the previous image, which the student thought to be definitive, and the new one; this generally happens when the new image expands the scope of applicability of the concept or provides a comprehensive version of it. Therefore, the *cognitive conflict* is an "internal" conflict between two concepts, two images or a concept and an image.

Misconceptions are at the basis of conflicts. These are temporary, incorrect conceptions, which await a more elaborated and critical cognitive collocation (D'Amore, 1999). A misconception is a wrong concept and therefore it normally represents an event that must be avoided; this situation should not be considered completely negative: a temporary misconception, which is undergoing the process of finding its cognitive collocation, can be necessary to reach the construction of a concept. In some cases images become real misconceptions, i.e. wrong interpretations of the information received. To call them *mistakes* is somehow to make things too easy and banal, as even very young children have naive but deep mathematical conceptions (Aglì and D'Amore, 1995) that are obtained empirically or through social exchanges. As to mathematical knowledge, it is reasonable to think the whole school career of an individual as a continuous transit from misconceptions to correct conceptions. The passage from a first elementary conception (naive, spontaneous, primitive, etc.) to a more elaborated and correct one is a delicate and necessary phase.
Examples of conflicts and misconceptions can be found in D'Amore (1999).

2.4 The triangle: teacher, student, knowledge

Over the last twenty years, research in the field of didactics of mathematics has deeply and profoundly investigated the aspect of what is hidden behind the "triangle" whose "vertices" are: student, teacher and knowledge (Chevallard and Joshua, 1982; Chevallard, 1985; D'Amore, 1999; D'Amore and Fandiño, 2002).



According to *fundamental didactics*, this is a *systemic model* used to locate and analyse the many relationships established among the three "subjects" that represent the "vertices" of the triangle. The complex nature of the systemic model is due to the necessity of simultaneously considering all the mutual relationships among the "vertices" including all the implications of different natures.

"Vertices"

In this paragraph we shall refer to D'Amore and Fandiño's synthesis (2002) where every "vertex" of the triangle acts as a pole:

- *knowledge,* be it academic, official or university, represents the ontogenetic or epistemological pole. It is around this vertex that the *epistemological obstacles* theory (see paragraph 2.5.) is situated. Those obstacles are the ones related to the concept's intrinsic nature, to its evolution and to formal complexity of its structures.

- *student* represents the genetic or psychological pole. This vertex is about personal, cultural or cognitive projects filtered by the *scholarisation*[26] relationship that makes learning subject's personal experiences not free from constraints. It is around this pole that *ontogenetic obstacles* theory is situated (see paragraph 2.5).

- the *teacher* is the functional or pedagogical pole. This vertex is about those cognitive and cultural projects which are highly influenced by all pedagogical expectations (not always explicit), beliefs linked to knowledge, professional convictions and "implicit philosophies" (Speranza, 1992).[27] It is around this pole that the *didactical obstacles*

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[26] Referring back to D'Amore's idea (1999): *«By "knowledge scholarisation" I refer to the mostly unaware act in the social and school life of a student (occurring almost always during primary school), through which s/he delegates to school (as an institution) and to the school teacher (as a representative of this institution) the task of selecting relevant knowledge issues (relevant from a social point of view, for acknowledged status and approved by the noosphere), thus giving up direct responsibility to choose learning contents according to personal criteria (taste, interest, motivation,…)»* [our translation].

[27] We are referring to the "philosophies" that Speranza describes as "implicit", in other words to those philosophies that exist and are influential, although they are not implemented in didactical praxis.

theory (see paragraph 2.5) is situated, as the teacher is responsible for didactical projects' choices.

"Sides"

D'Amore and Fandiño (2002) provide an explanation of the "sides" that highlights the relationships between pairs of poles:

• *teacher-student* could be summarised with the verb *"animate"* (a term linked to *motivation*, interest, *volition,*[28] ...). This verb recalls the following concepts:

- *devolution* is the action of the teacher on the student. The teacher tries to involve students in the didactical project proposed. Therefore this is the process or the responsibilisation activity that the teacher uses in order to get students personally involved in a cognitive activity that consequently becomes one of the cognitive activities of students themselves;

- *involvement* is the students' action exerted on themselves: students accept devolution, i.e. they become personally responsible for the construction of their own knowledge;

"To animate" can be therefore interpreted as a thrust towards personal involvement favouring devolution.
Midway between devolution and involvement, *adidactical situations*[29] (Brousseau, 1986) are to be found. These are situations favouring the "passage" from devolution to

[28] It is important to draw a distinction between *motivation* and *volition* as in Pellerey (1993). The former refers to: *«The formation of intentions, that is to say the elaboration of reasons inveigling someone into doing something»,* whereas the latter refers to *«The concrete will to achieve the aim expressed in the intentions»*. Being motivated to do something, like learning for example, does not necessarily mean being ready to do that or able to persevere when facing up to the first difficulties or failures. This distinction was introduced in the didactical context by the work of D'Amore and Fandiño (2002).

[29] In an environment which has been organised for the purpose of learning a special subject, we can talk about an *adidactical situation,* when the didactical intention is no longer explicit. The teacher suggests an activity without declaring the purpose of it; the student is well-aware that all activities in the classroom are meant to build up new knowledge, but in this case s/he does not know exactly what s/he is going to

involvement. When students are faced with a *didactical situation*[30] structured according to specific "rules of the game", the knowledge acquisition is not guaranteed unless a confrontation of students with an *adidactical* situation is not foreseen. It is as if the teacher-student relationship were interrupted in favour of the student-situation relationship: students produce their knowledge as a personal response to *milieu*[31] requirements rather than to teacher's expectations. The milieu is not "constructed" by the teacher. It preexists to the didactical situation and in general terms is referred to the collection of objects (mental and concrete) known to system-subjects independently of the fact that those objects are at that moment part of the knowledge acquisition process in act.

Elements characterising this side are the following:
-. didactical contract (see paragraph 2.1);
- didactical obstacles (see paragraph 2.5);
- pedagogical relationships;
- valuation (Fandiño Pinilla, 2002);
- scholarisation;
- devolution or lack of it;
…

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learn. If s/he decides to participate, accepting to get involved, then s/he frees her/himself from "contract" constraints (see paragraph 2.1) and participates in an adidactical activity. In this case, the teacher is just a spectator, that is to say, s/he is not explicitly involved in the knowledge management. The teacher dissimulates her/his didactical purpose and her/his will to teach, in order to make the student accept the cognitive situation as her/his own responsibility.

[30] We talk about *didactical situation* when we analyse an explicit education context, for example when a teacher playing in the role of a teacher openly informs her/his students about the knowledge content that is at issue in that moment.

[31] In the Theory of Didactical Situations, Brousseau (1989) introduces the notion of *milieu*, in order to stress the systemic nature of his approach: *«For the researcher a modelling of the environment and of its pertinent responses as far as a specific learning process is concerned, is just one of the components of a [didactical] situation. (…). It plays a fundamental role in the learning, as it is the cause of adaptation (for the student), and in the teaching, as reference and epistemolocical object »*. (our translation)

• *student-knowledge* is characterised by the verb *"to learn"*. The prevailing activity is *involvement*. It favours the access to "personal knowledge" which will be *institutionalised* (see teacher-knowledge side) by the teacher through the implementation of knowledge construction. On this side are positioned the images students possess of school, culture, ...; the specific personal relationship with mathematics and overall with knowledge institutionalisation (mainly depending on age), previous experiences; family and society, …

Elements characterising this side are:
- various learning theories;
- role and nature of conceptions;
- epistemological obstacles theory;
…

• *teacher-knowledge.* The main verb is *"to teach"* and the featuring activities are: *knowledge institutionalisation* (Chevallard, 1992) and *didactical transposition* (Chevallard, 1985, 1994; Cornu and Vergnioux, 1992).

Knowledge institutionalisation [32] is a process complementary to devolution and involvement that takes place when the teacher recognises that the student's personal acquisition of knowledge is legitimate and usable in the school context, once devolution and student's involvement have been verified.

The more general activity characterising this side is the *didactical transposition* (Chevallard, 1985) that is intended as the adaptation activity, transformation of knowledge into a teaching object according to place, audience and the didactical goals expected. The latter aspect will be fundamental for the treatment of this thesis (see ch. 4). The teacher should therefore operate a transposition from *knowledge* (originating

[32] According to Brousseau (1994): *«Knowledge institutionalisation is the social act through which teacher and student recognise devolution»*.

from research) to *taught knowledge* (knowledge taking place in the classroom)[33]. As a matter of fact, the passage is much more complex because it goes from *knowledge* (that of the discipline experts that structure and organise such knowledge) to *knowledge to be taught* (that decided by institutions) to *taught knowledge* (chosen by teachers as specific object of their didactical intervention).

The passage from *knowledge* to *knowledge to be taught* is filtered by teachers' epistemological choices which depend on their convictions, on their "implicit philosophies", on their idea of didactical transposition, on the influence of the *noosphere*[34], ...

Therefore elements characterising this side are teachers' beliefs about knowledge, pupils, learning, educational goals, school, …

In this analysis the function of the "triangle" is not explicative or descriptive of educational experience, but mainly methodological: each "vertex" of the system is the observer that looks at the relation between the other two. Though, none of the elements involved can be completely separated from the others. Furthermore, its implicit effort is to fill this scheme with as many elements (or variables) concerning the educational experience as possible. This experience has to be understood as problematic.

In this systemic model act at least three categories of entities:

- *elements* (which are "vertices" or "poles");
- *relationships* among elements (which are the "sides");
- *processes* which are the modalities for the system to function (e.g.: devolution, didactical transposition, didactical engineering, ...).

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[33] The teacher is never an isolated individual, when extracting a knowledge item from her/his social or university context to adapt it to the always unique context of her/his classroom. In fact it is the collective community, the institution that provides an objective definition of school knowledge in its specificity, its methods and rationality. The didactical transposition produces a certain number of effects: simplification, de-dogmatisation, creation of fake objects or production of totally new ones.

[34] The *noosphere* is a sort of intermediate zone between the school system (and the teacher's choices) and the wider social system (outside the school). In this zone, relationships as well as their conflicts between these two systems operate. The noosphere could be described as *«The external sphere containing all the people who think about the teaching contents and methods»* (Godino, 1993).

Over the whole triangle gravitates the noosphere with its burden of expectations, pressures and choices.

2.5 Obstacles

Building models, especially models concerning mathematical infinity, is not an easy task, as we shall see in the later chapters. This depends on the fact that every concept, even if it seems an easy one at a first glance, is wrapped in fluctuating and complex surroundings of associated representations, creating multiple levels of formulations and integration of the concept (Gordon and De Vecchi, 1987). Therefore the first step is to "clean up" the concept from this halo that seems to conceal its intimate meaning. And this is what we tried to do with teachers when dealing with mathematical infinity (paragraph 4.1).

Moreover, the *obstacles* to learning that should be taken into account, as firstly described by Guy Brousseau (1983, 1986), are of primal interest for this research (Ferreri and Spagnolo, 1994; Spagnolo, 1998).

«Obstacle is an idea that, at the moment of formation of a concept, has been able to cope with the previous problem (even if this has a cognitive nature), but has failed to cope with a new problem. Given the success obtained at this stage (in fact, because of this), there is a tendency to keep the idea already acquired and tested and save it, despite its failure. This ends up by being a barrier to following learning processes». (D'Amore, 1999; our translation).

Brousseau makes a distinction among three types of *obstacles*:

- *obstacles of ontogenetic nature*;
- *obstacles of didactical nature*;
- *obstacles of epistemological nature.*

• *Ontogenetic obstacles* are linked to pupils and their maturity. During the learning process every individual develops skills and competences suitable to their mental age (which is different from the chronological age). As for the acquisition of some concepts, these skills and competences can not be sufficient and create obstacles of ontogenetic nature. For example the student can have neurophysiological limitations, which may even depend only on their chronological age (Spagnolo, 1998).

• *Didactical obstacles* depend on the teacher's strategical choices. Every teacher chooses a project, a curriculum, a method, personally interpreting the didactical transposition, according to personal, scientific and didactical beliefs. The teacher believes in the choice made, considers it to be effective and thus proposes it to the class; but what has proved effective for some students, may not be effective for all the others. For some others the choice of that particular project may turn out to be a didactical obstacle. This kind of obstacles would be the core of our research (see chs. 3 and 4).

• *Epistemological obstacles* depend on the nature of the subject itself. For instance, when in the evolution history of a mathematical concept a non-continuity, a fracture, or some radical changes of the concept are singled out, then that concept presumably bears internal obstacles of epistemological nature, as far as understanding, acceptance and finally learning by the mathematicians' community are concerned (Spagnolo and Margolinas, 1993; Spagnolo, 1998; D'Amore, 1999). Mathematical infinity provides an emblematic example (see ch. 1). This last point is manifested, for instance, in typical and recurrent mistakes made by different students in different classes over the years (see ch. 4). Discontinuity is revealed not only in the concept of mathematical infinity but also in teachers' convictions (see chs. 3 and 4) or in the beliefs of anybody else that has dealt with this subject (Spagnolo, 1995).

Chapter 3. Primary school teachers' convictions [35] on mathematical infinity[36]

The following reflections refer to a research study on mathematical infinity, which has been carried out over many years. As we have seen in chapter 1, this subject is still fascinating and provides the humankind with an opportunity for deep reflection.

One wonders why the specific and difficult subject of this research is addressed to primary schools. It is in primary school that pupils get in touch with infinite sets like the sequence of natural numbers 0, 1, 2, 3, … which is maybe the first and more spontaneous example of these kinds of sets.

From the early years of primary school onwards, teachers explain that this sequence does not finish, i.e. it has no "end". There will always be a "greater" number than the one taken into consideration: one has just to add a unit. This process can go on forever to "infinity". Teachers affirm that if we take into consideration any natural number n we will always be able to find the next natural number $n + 1$; this process gives birth, step by step, to the sequence of natural numbers and represents the basis of one of the fundamental schemes of the mathematical reasoning: the principle of mathematical induction constituting a "delicate" axiom of Peano's axiomatic system (see paragraph 1.2.7) (Borga, Freguglia and Palladino, 1985).

Primary school children often talk of infinity with reference to numbers. For instance, during an experiment in the primary school of Mirano (Venice), Marco (an 8 year old pupil) wrote the following letter addressed to his classmates attending the first year, after the teacher had asked pupils to describe what most arose their curiosity:

Dear children of the first year, do you know what counting to infinity means? It means that if you count for 1000 years without a break, there will always be a

[35] We have chosen to talk about *convictions* instead of *conceptions* because we think that the interpretation of the first term that is generally provided is more consonant with our research. By *conviction (belief)* we mean: «*an opinion, a set of judgements/expectations, what is thought about something*» (D'Amore e Fandiño, 2004) whereas the interpretaion of conception that we make our own, and also more and more widespread and shared is the following: «*the set of convictions of somebody (A) on something (T) gives the conception (K) of A relatively to T; if A belongs to a social group (S) and shares with the others members of S that set of convictions relatively to T, then K is the conception of S relatively to T*» (D'Amore e Fandiño, 2004).

[36] This chapter has been published in Sbaragli (2003a).

greater number than the number you have just counted up! There will always be a further number and it will go on like this forever. Close your eyes and count. When you grow as old as your grandfathers you will be still counting. And you will be old men with beards; you will be so old that your parents will not recognise you anymore!

This word, infinity, is therefore fascinating even for primary school children. Already at this age children can perceive the mystery that goes along with this term.

Pupils, even in the early school years, often talk about this still unknown word, they feel its "power" and charm and this term will be present until secondary school or even at university. Still it often remains a concept that is not understood in a mathematical sense.

3.1 The mathematical infinity and the different nature of "obstacles"

At the root of the following considerations about infinity, there are studies in this field surveyed by many researchers in didactics of mathematics. They have analysed the problem of teaching and learning this subject, pointing out the mental processes of the students, their convictions and intuitions that are the results of widely spread misconceptions about different aspects of mathematical infinity [among many other examples, there are the classic works of Tall (1980), of Waldegg (1993) and the more recent ones of Fischbein, Jehiam and Cohen (1994, 1995), of Tsamir and Tirosh (1994, 1997), of D'Amore (1996, 1997), of Arrigo and D'Amore (1999, 2002), of Tsamir (2000)]. These researches involve different approaches to the theme of infinity and share the common aspect of "looking through the eyes of the students" in order to examine the reasons that render infinity such a complex subject to be learnt.

It is necessary to refer to the important field of didactics of mathematics concerning the study of the so-called *obstacles* hindering the construction of knowledge: ontogenetic, didactical and epistemological obstacles (Brousseau, 1983; Perrin Glorian, 1994; D'Amore, 1999), (see paragraph 2.5).

As to the treatment of mathematical infinity in primary schools, there are for sure *ontogenetic obstacles* bound to conceptual and critical immaturity. This is mostly due to

the age of the pupils (Spagnolo, 1998). This is not a good reason though to underestimate the first intuitions, the first images, the first models that take form in the mind of children since primary school as a consequence also of the spur of teachers.
Furthermore, international literature, starting from the historical development of this controversial subject (see ch. 1), managed to point out *epistemological obstacles* hindering the learning of mathematical infinity. This makes it possible to understand some of the difficulties encountered by students [see for instance Schneider (1991)].
In this work we aim to establish if it is possible to encounter *didactical obstacles*, perhaps even more influential than the epistemological, due to didactical choices of teachers which condition and strengthen pupils' first misconceptions (see paragraph 2.3). The presence of didactical obstacles in the learning process of mathematical infinity has already been noticed by Arrigo and D'Amore (1999 and, most of all, 2002).

In order to explain the aim of this work, we will make some considerations on epistemological obstacles. As to the historical development of a concept, we can assume that there has been a gradual shift in history from an intuitive "initial" phase to a final phase of the concept itself (maybe it would be better to call it "actual" or "advanced"), mature and structural (at the time of reference). It is of course only a scheme, since there are many other fundamental transformations, which allow us to reach the "actual" phase of the concept (Sfard, 1991) between the two phases considered as the starting and arriving point (when speaking about it).
What has happened in the history of mathematics can be also said for didactics. As a matter of fact, the first historical naive intuitions on infinity usually recur in the first considerations and convictions expressed by students in classroom.
From a didactical point of view, a similar situation is observed: during a first phase students approach intuitively a mathematical concept without possessing a complete and developed understanding of it.[37] Only successively, learning turns out to be fully-fledged and more mature (Sfard, 1991).

[37] This is also due to the necessity in mathematics of a coordination of semiotic registers, to be acquired only in the long term, which is a condition for the mastery of comprehension, being it the essential condition for a real differentiation between mathematical objects and their representations. (Duval, 1995).

Two "parallel" patterns can be envisaged: the first is related to the historical development of knowledge; the second concerns a pattern that is similar to what happens in didactics (Sfard, 1991; Bagni, 2001).

In didactics the transit from the "initial" phase to the "advanced" phase of knowledge can provoke doubts and reactions in students' minds. These can be also found in the corresponding transit of formation of knowledge.

It is important to underline that the "naive intuitive" phase seems to be in opposition to the "advanced" phase. Both in the history of mathematics and in the processes of learning and teaching, in primary school as well as in secondary school and in some cases even in further years of study, as models are still present in further education (Arrigo and D'Amore 1999, 2002).

These considerations could be useful when dealing with *didactical transposition* (see 2.4) that should begin with a first intuitive knowledge on the part of students, and successively address the students' initial convictions towards the "advanced" phase of the concept itself.

3.2 First research questions and related hypotheses

The first research questions and the corresponding formulation of hypotheses emerge from the above debated considerations concerning didactical transposition:

- Do primary school teachers know and are they aware of the "advanced" phase of the concept of mathematical infinity?
- In the didactical transposition, do teachers base teaching on real results reached in the "advanced" phase of the development of knowledge? Or do they strengthen the students' "naive intuitive" phase instead?
- And however, have teachers ever accessed the knowledge on infinity?

The hypothesis proposed here is that primary school teachers do not know the "advanced" phase of mathematical infinity concept. This is the reason why they are stuck to the "naive intuitive" phase of knowledge. In so doing, they strengthen the students' initial intuitive convictions without helping them in the transit to the "advanced" phase of the concept. Therefore, the didactical transposition, instead of moving from the students' "naive intuitive" phase to the concept "advanced" phase (meaning the "advanced" phase of knowledge), reinforces their naive convictions and keeps them to the intuitive phase. This attitude is considered in our opinion the source of didactical obstacles hindering the comprehension of the infinity concept.

The present research was firstly addressed to students attending the last year of primary school. Our intention was to retrieve the first images, the first intuitions and possibly the first difficulties encountered when students have to cope with the subject of mathematical infinity. These experiences, on which the following chapter is based, showed that already starting from the last years of primary school, the intuitive ideas possessed by students on this topic turn out to be most of the time false convictions. These beliefs are usually explained away with sentences like: *«The teacher told me that…», «In class we saw that…»* i.e. with attitudes similar to the famous case of Gaël[38] (Brousseau and Pères, 1981) that definitely confirmed the idea of didactical contract in the field of didactics of mathematics (see paragraph 2.1). Moreover, teachers were sometimes curious about the things we wanted to show children. They got informed about the object of our research and they frankly and honestly exposed their wrong beliefs on the matter. On the basis of such considerations, the core of our research shifted from students' convictions to teachers' convictions and consequently to possible didactical obstacles verifiable when introducing the concept of mathematical infinity.

3.3 Description of theoretical framework

Among the many publications in the international context, D'Amore (1996, 1997) has given an outstanding contribution, providing an accurate outlook on different research "categories" and a vast bibliography with more than 300 titles.

[38] The case of Gaël was significant for the study of the causes of the elective failure in mathematics; researchers described Gaël's case as follows: instead of consciously expressing his knowledge he always just uttered it referring himself to terms that involved the teacher. The child experienced every didactical situation through the eyes of his teacher, till the researchers, thanks to adidactical situations, managed to get more personal interventions from him and, all in all, more effective from a cognitive point of view.

More precisely, the theoretical framework on which this work is based is mainly constituted by the fundamental considerations and results of Arrigo and D'Amore (1999, 2002) offering a crucial reference to this research study.
In particular, in the first work two phenomena have been described, based on the generalisation to infinite cases of what has been learnt on the biunivocal correspondence of finite cases and to which we will refer in the present work (Shama and Movshovitz Hadar, 1994; Arrigo and D'Amore, 2002):

- the first phenomenon is called by Arrigo and D'Amore *"flattening"* and has been already dealt with in other publications [Waldegg (1993), Tsamir and Tirosh (1994), Fischbein, Jehiam and Cohen (1994, 1995)]. This is about considering all infinite sets as having the same cardinality, that is to say that a biunivocal correspondence could be established between all infinite sets. In more detail, literature on this subject has showed that once the students have accepted that two sets such as N and Z for instance, have the same cardinality (thanks to the help of the researcher or teacher showing them the biunivocal correspondence between the two given sets), it is much more common that students tend to consider as true the generalisation that all infinite sets must have the same cardinality, which is not the case. The latter misconception is not only due to epistemological obstacles, of which we found evidence in the history of mathematics, but also to didactical obstacles as pointed out by Arrigo and D'Amore (1999 and in particular, 2002).

- the second phenomenon is that of *"dependence",* as named by the two authors, according to which there are more points in a long segment than in a shorter one (Tall, 1980). This phenomenon can be observed not only in geometrical milieu, but it is also valid when referring to *dependence* of the cardinality on the "size" of numerical sets. For example, since the set of even numbers represents a sub-set of the natural numbers set, the former seems to be by implication formed of a smaller number of elements.

The above-mentioned attitudes have been surveyed and analysed in detail by Arrigo and D'Amore (2002). The two authors also pointed out that most difficulties encountered in the understanding of infinity are strictly related to students' intuitive models of geometrical entities (Fischbein, 1985) (see 2.2), in particular the point and the segment (see ch. 4). In our research work, we also based ourselves on the considerations reported in Fischbein (1993). He revealed, by means of some examples (some of these concerning the point), the complex nature of the relationships between figural and conceptual aspects, pertaining to the organisation of *figural concepts* and the fragility of such an organisation in the students' minds, has been underlined. On this latter aspect and from a didactical standpoint, Fischbein believes that teachers should systematically point out to their students the various contradictory situations in order to stress the predominance of definition on the figure. That is to say, students should be made aware of conflicts and of their origins, so that they can start being confident with the necessity for mathematical reasoning to depend on formal constraints. In addition, Fischbein (1993) claims that the integration of conceptual and figural properties into unitary mental structures, with the predominance of conceptual constraints on figural ones, is far from being a spontaneous process and in fact this could constitute a major continuous and systematic concern of teachers. To achieve this Arrigo and D'Amore (2002) suggest intervening in primary school teachers' preparation in this specific field. This latter aspect represents a crucial point in the present work, which is based on primary school teachers' beliefs on mathematical infinity; these convictions influence students' intuitive models resulting in situations of cognitive disadvantage. In order to modify and re-adjust these convictions, a new way of learning, only attainable thanks to suitable training courses for teachers enhancing a closer examination of the above – mentioned topics, is therefore required (see 4.1).

Another subject pertaining to this research is the classic philosophical debate about infinity in the actual and potential sense inspiring many authors: Moreno and Waldegg (1991), Tsamir and Tirosh (1992), Shama and Movshovitz Hadar (1994), Bagni (1998, 2001), Tsamir (2000). These authors point out that from both a historical point of view and that of the learning of infinity, the evolution of the actual conception of infinity is extremely slow and frequently contradictory and this is only possible thanks to a cognitive process involving cognitive maturation and systematisation of learning (for a historical excursus see in ch 1 the reflections of Aristotle, Euclid, Augustine of Tagaste, Thomas Aquinas, Galileo, Torricelli, Descartes, Gauss, Cantor). More specifically in

Tsamir (2000), difficulties encountered by teachers in training when faced with actual, rather than potential infinity have been highlighted. This is to be traced back to the previous considerations on the necessity of introducing these contents in the training of primary school teachers too.

3.4 Description of problems

This section provides a description of the problems inspiring the present research.

P.1 Are primary school teachers aware of the concept of mathematical infinity and of its epistemological and cognitive meaning?

P.2 Do teachers provide their students with some intuitive models on the topic since the first years of primary school? If they do, are they aware that these are misconceptions that will be awaiting a further systematisation, or do they believe these to be correct models that should accompany their students during their whole future educational career?

P.3 Could teachers' convictions be the cause of didactical obstacles responsible for the strengthening of the epistemological obstacles already pointed out in the research at international level?

3.5 Research Hypotheses

Here as follows we report the hypotheses related to problems described in 3.4:

H.1 We believe that mathematical infinity is a rather unfamiliar subject for most primary school teachers, both from an epistemological and from a cognitive point of view. We therefore thought that teachers would not be able to handle infinity and to conceive it as a mathematical object. Consequently, we assumed that teachers would stick to naive convictions as for example: infinity is nothing but indefinite, or infinity is synonymous with unlimited, or else infinity is a very large finite number [convictions that were present over the centuries throughout the history of this topic, see chapter 1, in particular they are to be traced back in the statements of Nicholas of Cusa (1400 or 1401-1464)].

H.2 We believe that primary schoolteachers normally provide pupils with intuitive models of mathematical infinity, starting from the early years of primary school.
Moreover, if teachers' naive convictions, assumed in H.1, were verified, they would condition (in our opinion) the models provided to pupils. We assumed that teachers provided intuitive models that they considered correct, but in fact they were based on misconceptions. In order to verify this hypothesis, we judged that it would be interesting to analyse accurately the teachers' statements and their way of expressing ideas.

H.3 We assumed that, if the two above-mentioned hypotheses had been proved true, beside epistemological obstacles that the study of mathematical history and the criticism of its fundamentals have highlighted, we would have been able to trace obstacles of didactical nature too. One of the obstacles we thought we would encounter is bound to a naive idea of infinity as a synonym for unlimited, a conviction which is in contrast with the concept of the infinity of points in a segment, a segment being limited though constituted of infinite points. One more obstacle we thought we would find is bound to the idea of infinity, considered as a large natural number [see ch. 1: Anaximander of Miletus (610 B.C. – 547 B.C.) and Nicholas of Cusa (1400 or 1401-1464)], it follows that the same procedures applied to finite sets are automatically transferred to infinite sets, which are seen as very large finite sets. Another didactical obstacle, often highlighted by Arrigo and D'Amore (1999, 2002), that we were confident we would come across, is the "model of the necklace" as the two authors call it. Students often point it out as a suitable model to visualise the points on a straight line, and they indicate their primary school teachers as the source of this model that withstands all subsequent attacks. (Arrigo and D'Amore, 1999; 2002) Our hypothesis was therefore that we would encounter didactical obstacles, deriving from typical models, usually introduced by primary school teachers.

If the above-mentioned hypotheses had been proved true, we would have gone ahead in our investigation and would have considered the possibility and the necessity of revising the didactical contents of primary school teachers' training courses. This is not meant to force teachers to change the contents of their didactical activity, but to prevent them from building intuitive models that could bring about situations of cognitive disadvantage for their students.

3.6 Research Methodology

3.6.1 Teachers participating in the research and methodology

As a consequence of the shift of present research focus on primary school teachers' convictions, our idea was to develop a questionnaire to use as a starting point for reflections and opinion exchange among teachers on matters related to mathematical infinity. The aim was to let their convictions, misconceptions and intuitive models with regard to this concept emerge.

For the outline of the questionnaire, several informal interviews with teachers were held to verify the text readability and understandability. The questions were all about those concepts that are usually dealt with in primary school and that create in students' minds, even without teachers' awareness, the first images to be transformed in intuitive models of geometrical entities or more in general of infinity.

The questionnaire and the following opinion exchange were administered to 16 Italian teachers of primary schools, different from those already involved in the initial phase [4 from Venice, 8 from Forlì (Emilia Romagna), 4 from Bologna].

The research was been carried out according to the following modalities: six meetings were organised. Two teachers attended each of the first four sessions, whereas four teachers at a time attended the two meetings left (16 teachers in total). Each meeting started with the questionnaire proposal: teachers were asked to read through the questionnaire and then to fill it in individually. After everybody had handed it in, open discussion among pairs or group of four people would start. During the debate teachers could express their convictions, doubts and perplexities in the presence of the researcher who intervened in the conversation only on certain occasions, in order to stimulate the discussion on some relevant aspects, but firmly trying not to modify teachers' ingenuous convictions. Discussion groups were organised as to allow confrontation between teachers that could already get on well together and were used to discuss and exchange opinions.

However, it was clearly stated right from the beginning that their names would not appear in the research work.

The teachers have judged the questionnaire easily "comprehensible". As a matter of fact, after a first reading of the questions, teachers unanimously affirmed that it was clear and of accessible interpretation, even though when it came to answer the very first question, 13 teachers out of 16, manifested great embarrassment: *«I don't know what to write, I never reasoned on this topic»*. Only after some self-assuring statements such as: *«I will write down what comes up to mind, even if it won't be well expressed»*, they started answering the first question.

Teachers had one hour for the questionnaire, so that they could read it through, reflect, think it over again and organise their answers with no pressure and taking their time. None of the teachers involved used all the time available.

As to the second phase, based on discussion and confrontation, there were no time restrictions; we adopted the technique of active open debate in groups of different size, using the tape recorder and leaving the researcher the task of highlighting contradictions and deeply rooted intuitive models.

This last discussion phase was the most fruitful and significant. As a matter of fact, since the very first interviews it was clear that a written text is not a suitable means to make real intuitive models emerge. A single answer, synthesised in most cases, is not enough to interpret teachers' real convictions. Such a complex and delicate topic needs a further and deeper investigation into teachers' individual and single convictions. To this aim, opinion exchange has proved to be a very useful means of revising and reworking the questionnaire's answers, to understand their intimate meaning, to verify their stability and to point out possible contradictions.

The decision of implementing confrontation between teachers, rather than between a single teacher and the researcher, is based on the necessity of collecting teachers' real convictions, otherwise difficult to be identified. When teachers are asked to express or defend their own opinions, in front of other colleagues with whom they feel confident and are used to arguing and sharing more or less the same knowledge, the expected outcome is that they would feel freer to manifest their ideas.

The applied strategy also served the aim of reducing some teachers' reactions such as: "trust in the researcher" or "trust in what mathematicians affirm" [often reported in literature; for example: Perret Clermont, Schubauer Leoni and Trognon (1992)], emerging not only when research is addressed to students but also when teachers are involved.

The complete documentation of these exchanges will not be provided in this thesis, only the most significant and recurrent sentences will be reported. Questionnaires and complete recordings will be at disposal of whoever is interested in further researching this topic.

3.6.2 Questionnaire content

The questionnaire contained 15 A4 sheets, one sheet for each question (with space for teachers to write their answers).

Here as follows the 15 questions will be transcribed together with some explanations on the methodology used for the compilation of the questionnaire:

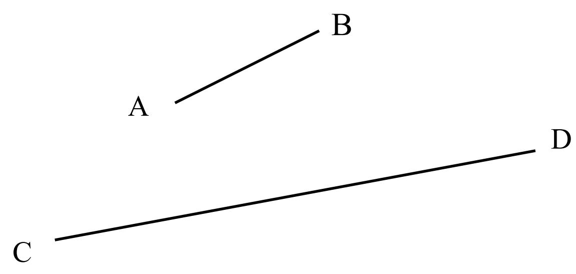
1) What do you think mathematical infinity means?

2) Has it ever happened to you to talk of infinity during the five years of primary school teaching? When? In what sense? How? Using what kind of support?

3) Does the term "infinity" in mathematics exist both as an adjective and as a noun?[39]

4) Are there more points in the AB segment or in the CD segment? (Write down on the sheet of paper everything that comes to mind).

[39] Translator's note: the term *infinito* is used in Italian both as a noun and as an adjective, thus covering both meanings of the English words *infinity* and *infinite.*



5) How many even numbers are there: 0, 2, 4, 6, 8, …?

6) How many odd numbers are there: 1, 3, 5, 7…?

7) How many natural numbers are there: 0, 1, 2, 3…?

8) How many multiples of 15 are there?

9) Are there more odd or even numbers?

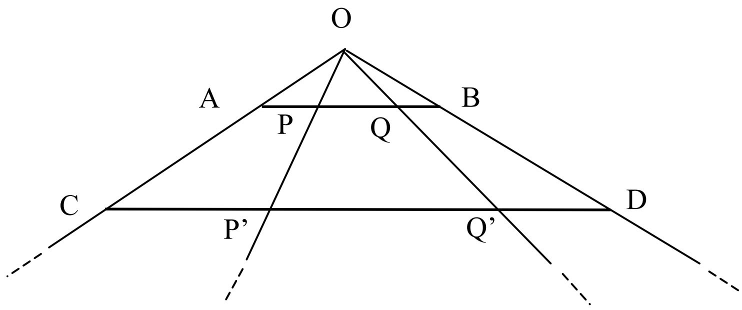
10) Are there more even or natural numbers?

11) Are there more odd or natural numbers?

12) Are there more multiples of 15 or natural numbers?

13) Has it ever happened during your primary school teaching to compare the quantities of these numerical sets (even with odd, even with natural, odd with natural)? How? On which occasion?

After teachers handed in the first 13 answers, they were showed Georg Cantor's demonstration (1845-1918) (see paragraph 1.2.3) related to question no. 4 that proves that there is the same number of points in two segments of different length. In order to illustrate it, teachers were showed the biunivocal correspondence on a sheet of paper containing two segments AB and CD (differently positioned on the plane from those concerning question number no. 4; they were shifted by hypsometry so that they appeared parallel and "centred" with respect to one another). At the beginning, with the help of a ruler, the point O of intersection between the straight lines AC and BD was drawn; successively from O the points of the segment AB were projected on the segment CD and vice-versa. In so doing, the biunivocal correspondence between the sets of points belonging respectively to segments AB and CD was demonstrated. It was therefore easy to observe that there is the same number of points in segments of different length.



Successively teachers received a sheet of paper with the following question:

14) Try to be as honest as possible answering the following question: were you convinced by the demonstration that there are as many points in AB as in CD?

After answer no. 14 had been handed in, teachers were shown the demonstration of question no. 10, that proves that the set of even numbers (E) is formed by the same number of elements than that of natural numbers (N), showing the related biunivocal correspondence (Tall, 2001a).
Let us illustrate the biunivocal correspondence showed to teachers:

N	0	1	2	3	4	5	…	n	…
	↕	↕	↕	↕	↕	↕	↕	↕	↕
E	0	2	4	6	8	10	…	2n	…

This idea developed from the consideration made by Galileo Galilei (1564-1642) (even if Galileo talked about square numbers and not even numbers) (see paragraph 1.1.2): to each natural number corresponds a determined square and vice-versa. To each square number corresponds a determined natural number (its «arithmetic root»); thus there are as many natural numbers as square numbers.

As in the case of the previous question, teachers received a sheet of paper with the following question:

15) Try to be as honest as possible answering the following question: were you convinced by the demonstration that there are as many numbers in the set of even as in that of natural numbers?

Only after all sheets had been handed in, debate and opinion exchange between groups of two or four people started.

In consideration of the nature of this research and most of all the involved subjects' competences on this specific topic, the choice was not to establish a determined order of the questions to pose taking into account the preliminarity of concepts as the discrete that should precede the continuum. As a matter of fact, only question no. 4 seems to pertain explicitly and specifically to the field of "continuum".

3.7 Description of test results, opinion exchange and verification of hypotheses outlined in 3.5

From the answers given to the questionnaire, some rather generical affirmations emerged which have undergone further investigation thanks to the opinion exchange between teachers. Some of the answers given to each question have been selected and are provided below. Integrations to such answers obtained through verbal exchange during discussion are provided as well. The aim is to offer the widest and most representative view as possible of the respondents' convictions. Researcher's interventions and comments have been indicated in bold. It ought to be remembered that the researcher intervened only to stimulate conversation and to go more deeply into teachers' convictions.

3.7.1 Description of test results and related opinion exchange

1) As for the answers to the first question of the questionnaire, they can be all classified one way or another into convictions reported here as follows. It ought to be noted that none of the 16 interviewed teachers was aware of the "advanced" conception of mathematical infinity. It is important to remember that the operated classification should not be considered as definitive, since as we shall see later, some of the teachers' affirmations, that were at the beginning dealt with as belonging to a specific category, have been also successively inserted in other ones as a direct consequence of the outcome of successive conversations.

• ***Infinity as indefinite.*** 7 teachers tend to consider infinity as indefinite, that is to say they do not know how much it is, what it is exactly, what it represents.

R.: «To me it means without boundaries, with no limits like the space»

R.: «In the sense of indefinite?»

R.: «Yes without borders»

C.: «Something that you cannot say»

R.: «In what sense?»

C.: «You don't know how much it is»

A.: «Something that cannot be written down»

• ***Infinity as a finite large number.*** 3 teachers affirm that infinity is nothing but a very large finite number.

A.: «To me it's a large number, so large that you cannot say its exact value»

B.: «After a while, when you are tired of counting you say infinity meaning an ever-increasing number».

• ***Infinity as unlimited.*** 5 teachers confuse infinity with unlimited, they think the term infinity can be attributed exclusively to the straight line, half-line and plane, i.e. everything unlimited. Therefore it is not possible to talk of infinity with regard to the points of a segment that is a limited entity. Curious enough is that if the researcher intervenes asking: *«How many points are there in a segment?»* teachers show to know the answer to the question that is: *«Infinite»*, but without understanding the real meaning of this statement. As a matter of fact, further investigating it, 3 of the 5 respondents affirm that in the case of the number of points of a segment, infinity is considered a large finite number whose exact value is unknown, whereas the remaining 2 see infinity as indefinite: you do not know exactly how much it is. So, all of the 5 answers are pertinent also to the other categories indicated for point no.1. To these teachers the following relationship seemed to be valid: in the cases of lines, planes and space, infinity and unlimited seem to be synonyms; in the cases of the quantity of numbers or points, they refer to infinity as a very large finite number or indefinite number.

A.: «With no limits»

M.: «Something that I cannot quantitatively measure»

(Also in this case, teacher M. associates the term infinity with unlimited without considering that a segment, for instance, though being limited and measurable in the sense understood by M., contains infinite points).

N.: «Something unlimited»

R.: «So you will never use the word infinity referring to a segment?»

N.: «No, because it has a beginning and an end»

R.: «How many points do you think there are in a segment?»

N.: «Oh you're right, infinite. But it's just to say a large number, not as large as in the straight line. Even if you make very little points, you cannot fit in it more than that».

[Teacher N. already reveals the conviction emerged also in answers to question no. 4, that is to say, that there are more points in a straight line than in a segment stressing the idea that to greater length corresponds a greater number of points. Points are therefore conceived not as abstract entities but as objects that should have a certain dimension in order to be represented (see ch. 4). These misconceptions are derived from teachers' models of fundamental geometrical entities such as point, straight line and segment].

G.: «Unlimited»

R.: «How many points do you think there are in a segment?»

G.: «You say that in a segment there are infinite points because it is not known how many there are exactly».

• ***Infinity as procedure.*** Only a single teacher talks of infinity in the first question referring to a never-ending process:

B.: «I know infinity, it means to keep going on as with numbers… for ever».

This conviction recalls the idea of potential infinity that we shall see illustrated in 3.7.3. Analysing the collected answers in more detail, it can be observed that also the answer given by B. belongs to the category of infinity as a large finite number. In B.'s answer (and also in answers of other teachers, which we will see later on) the conviction is to be traced back to potential infinity considered as an ever-lasting process.

2) Answers concerning the second question unanimously reveal, that the concept of infinity, under different forms, is dealt with by teachers from the early years of primary school, thus creating images of what is intended by this term. All of the 16 teachers affirmed that they mention and talk about infinity in primary school.

A.: «Talking about the number, I show them the line with numbers and I say that they never end. And talking about infinity I show the difference among segment, half-line and straight- line».

Once again the conviction of infinity as unlimited emerges. As a matter of fact, it was the same teacher affirming that infinity means lacking at least one limit. Therefore she explicitly says: *«You can talk of infinity only in the cases of the half-line and the straight line, but not in that of the segment»*, the segment being limited.

G.: «I talk about it when we do numbers. I always say they are infinite»

A.: «In the third year I usually say that the straight line is infinite trying to evoke mental images rendering the idea of infinity like the laser beam»

M.: « I also use it for the parts I can make out of a quantity, I can keep on dividing always the same quantity».

These are only some examples of the teachers' statements showing that they deal with the term infinity since the very early years of primary school, even without a complete awareness and correctness of its mathematical meaning.

3) The aim of the third question was to discover if awareness that infinity represents a mathematical object existed among teachers (Moreno and Waldegg, 1991).

For 13 teachers "infinity" in mathematics is only an adjective, the remaining 3 believe it is also a noun, but out of the latter 3, 2 of them conceive infinity as indefinite whereas the other one believes that you can use this word also as a noun but only meaning a very large finite number whose value is unknown.

N.: «As an adjective»

M.: «In mathematics it exists only as an adjective, in the Italian language also as a noun»

A.: «In mathematics it is used as an adjective: infinite numbers, infinite space. As a noun in the Italian language: Leopardi's "Infinity"[40]*; "I see infinity"; "I lose myself in infinity"»*

B.: «Also as a noun to mean a large number»

4) The fourth question was about the teachers' supposed conviction that two segments of different length should correspond to a different number of points [this idea already emerged from the answer to the first question provided by the teacher N. and illustrated in point 1) of this paragraph].

All of the 16 interviewed teachers affirmed that in two segments of different length there is a different number of points and more specifically, to a greater length corresponds a greater number of points (Fischbein, 2001). It stands out clearly that visually one segment appears to be included in the other and therefore in this case the figural model is predominant. This model negatively influences the answer and in fact the Euclidean notion: *«The whole is greater than its parts»* (see 1.1.1) cannot be applied to infinity.

Here as follows, we have reported some of the answers pertaining to the above-mentioned conviction:

N.: «This makes me think that the different length of two segments should have some influence on the number of points»

B.: «In the segment CD; of course, it's longer»

G.: «In AB there should be a lot, in CD many more»

A.: «I'm not sure. Given that the segment can be considered as a series of points in line, I think that CD has more points than AB, even if I have learnt that the point is a

[40] Translator's note: Giacomo Leopardi's *Infinity* is one of the most famous Italian poems of all times.

geometrical entity, which, being abstract, is not possible to quantify it because it's not measurable. I would say CD, anyway»

[The teacher A. showed inconsistency between what she affirms she had studied in order to take an Analysis exam at university and what she believes to be the most plausible answer according to common sense. Once again, the intuitive model persists and predominates. In this situation it is quite evident that there is no correspondence between the formal and the intuitive meaning (Fischbein, 1985, 1992; D'Amore, 1999)].

The above discussed intuitive conception represents a widespread misconception, it has already been mentioned in 3.3 and is called *dependence* of transfinite cardinals on factors related to magnitudes (the set of greater size has more elements). The teacher in question is therefore convinced that a greater length implies also a greater cardinality of the set of points. Accurate surveys have largely proved that mature students (those attending the last year of higher secondary school and the first years of university) do not succeed in mastering the concept of continuity because of the persisting intuitive model of a segment seen as a "necklace of beads" (Tall, 1980; Gimenez, 1990; Romero i Chesa and Azcárate Giménez, 1994; Arrigo and D'Amore, 1999, 2002).

This misconception will return also in the answers to questions no. 10-11-12, where *dependence* is to be intended as *dependence* of the cardinality on the "size" of numerical sets.

This conviction, as we have already observed in chapter 1 (that in a *longer* segment there are more points than in a *shorter* one) and notwithstanding several occasional episodes, has been definitely eradicated only in the XIX century therefore rather recently. Once again the history of mathematics has witnessed the presence of an epistemological obstacle highlighted in several research studies (Tall, 1980; Arrigo and D'Amore, 1999). The latter obstacle represents a misconception belonging to the common sense outside the mathematical world. Therefore, this phenomenon is to be traced back even within teachers' convictions, teachers who have not been given the opportunity to reflect on the "advanced" conception of this topic.

As a matter of fact, the epistemological obstacle, considered according to Brousseau (1983) (see 2.5) in its classic meaning, is a stable item of knowledge that has worked out correctly in previous contexts, but that is a source of problems and mistakes when

trying to adjust it to new situations (dis-knowledge or *parasite model*). Furthermore, as stated by Arrigo and D'Amore (1999): *«… in order to overcome this kind of obstacle a new item of learning is needed»*, in many cases this learning did not take place during the educational career and neither is favoured in further years of study.
Nevertheless, it seems quite difficult to figure out that teachers who have never reflected on such topics could possess an image of the topology of the set of the points of the straight line (and therefore at least their density) that enables them to understand the specific case of two segments of different length, for instance. To avoid that the above-mentioned convictions turn into incorrect models producing didactical obstacles (that in turn magnify the already highlighted epistemological obstacle), it is important to help the subject in question to detach her/himself from the model of the segment seen as a "necklace". In this way s/he comes to more appropriate images for the comprehension of the concept of non-dimensional points (see ch. 4). In order to do this, the subject should enlarge previous knowledge and build new items of knowledge but the only way to achieve this goal for her/him is to study theorems concerning the already mentioned topics.

5) – 6) – 7) – 8) As to the four following questions, 15 teachers answered in this way: *«Infinite»*, with the exception of one of them who, after some hesitation, wrote: *«Quite a few!»*, being afraid of saying something wrong. In mathematics it is a common and most widespread attitude to answer with "learnt by heart sentences" without proper awareness, or understanding of its real meaning according to the "advanced" conception of a concept (see 4.1). All of them remember that these sets are infinite, but they actually ignore the sense of such an affirmation. Almost everybody has memories of having studied that the point has zero dimension, but they do not know what this means, since the intuitive model of the point as the mark left by the pencil is still predominant in many cases.

9) This question and the following four envisaged the task of comparing some infinite set cardinalities that are often dealt with in primary schools. The collected answers have been classified in the following three categories:

• ***There are as many even numbers as odd numbers.*** 12 teachers out of 16 have this opinion.
C.: «It's the same number to me»

• ***It is impossible to make a comparison of the cardinalities of infinite sets.*** To 3 teachers a comparison of the cardinalities of infinite sets is not conceivable. As a matter of fact, in the logic of those who conceive infinity as indefinite or as something finite, very large but with an undetermined value, it is rather difficult if not impossible to make a comparison between cardinalities of infinite sets.
R.: «You can not answer that, it is not possible to compare infinities»

• ***The unsure.*** One teacher answered back with a question:
A.: «I would say they are of equal number, the even and the odd numbers; but I have a major doubt: If they are infinite how can I quantify them?»
(From this answer emerges the idea of infinity seen as indefinite).

10) – 11) – 12) The answers provided to these three questions belong to the following four categories. All of the 16 interviewed teachers are consistent always replying in the same way to all three questions:

• ***There are more natural numbers.*** 10 teachers answered that there are more natural numbers, supporting the common Euclidean notion: *«The whole is greater than its parts».*
C.: «The natural numbers»

• ***You cannot compare infinite sets.*** The same 3 who in reply to question no. 9 could not conceive a comparison between cardinalities of infinite sets, remained firm in this opinion; this results in the idea that you can refer to cardinality only when dealing with finite:
R.: «You cannot answer that, you can't make a comparison».

• ***The unsure.*** The same teacher who answered to question no. 9 with another question, replied in the same way which shows consistency:

A.: «I would say the natural numbers, but how can I quantify them? To say infinity means nothing».

• ***They are all infinite sets.*** 2 teachers affirmed that all the sets in question are infinite and therefore they have all the same cardinality.

B.: «They are both infinite. If two sets are infinite, they're just infinite and that's it».

From the interview of these two teachers emerges the misconception of the *flattening* of transfinite cardinals illustrated in paragraph 3.3, resulting in the belief of considering all infinite sets of equal power. In other words, these teachers came spontaneously to the conclusion that being all the above-mentioned sets infinite, the attribute "major", in compliance with a passage of Galileo's, cannot be used when dealing with infinities (see 1.1.2). The direct consequence is that all of the sets of this type are nothing else than - banally - infinity.

R.: «According to you, do all infinite sets have the same cardinality?»

B.: «What do you mean? The same number? Yes, if they are infinite!"

13) This question has been posed in order to point out if some of the teachers interviewed had ever proposed the topic of the comparison of the cardinalities of infinite sets during the didactical activity in class. All of the 16 teachers answered that they had never proposed specific activities on that topic even if 3 of them admitted, during the open discussion, that they might ingenuously have said to their students that there are more natural numbers than even numbers. Such an affirmation is definitely a didactical obstacle to students' future learning.

14) In order to test to what extent teachers are convinced of their affirmations regarding the idea of point and segment (on which in particular question no. 4 is based) they were provided with the construction described in 3.6.2. This shows that there is the same number of points in two segments of different length and only afterwards question no. 14 had been distributed.

Answers have been classified according to the following categories:

• ***Not convinced by the demonstration.*** 5 out of 16 respondents were not convinced by the demonstration:

R.: «Were you convinced by this demonstration?»

M.: «Well, not really; to me a point is a point, even if I make it smaller, it's still a point. Look! (Drawing it on the sheet). *Then, if I make them all of the same size, how can they be of the same number?»*

R.: «According to you, between two points is there always another one?»

M.: «No, no if draw two points one next to the other, very close, so close, practically stuck to one another there won't be any in between them»

B.: «Ummh! But in the segment AB you go over the same point when lines get thicker. I'm not convinced».

As a matter of fact, to grasp the exact meaning of this construction has proved quite a difficult task for those teachers to whom the point is not conceived as an abstract entity with no dimension, but rather as the mark left by the pencil and therefore with its own dimension. More in general terms, teachers rejecting the above discussed construction are those who imagine the segment as the "model of the necklace of beads".

• ***Convinced by demonstration.*** 9 were convinced by the demonstration. The teachers A. and C., in particular, considered it crystal clear and extremely effective:

A.: «That's nice! You convinced me»

C.: «You convinced me, it's exactly like that»

G.: «Yes, I'm convinced».

Although these 9 teachers were promptly and immediately confident with the demonstration correctness some doubts and perplexities were provoked by questions such as: *«Are you really sure about it?»*. Our intention was to observe if teachers were inclined to change their mind showing by that not a profound and stable conviction. As a matter of fact, 3 admitted not being thoroughly convinced, returning to the initial affirmation that there are more points in CD. [On this aspect consult: Arrigo and D'Amore (1999, 2002)].

R.: «Are you really sure about it?»

G.: «No, no! I'm still convinced that there are more of them in CD, you can see it»

R.: «I'm not so sure».

• ***Trust in mathematicians.*** One teacher showed a sort of "trust in mathematicians", though not being totally convinced by the demonstration:

A.: «If you mathematicians say that, we trust you. Me for sure, I won't get into these problems!»

• ***The unsure.*** One teacher seemed to be in need of some kind of explanation, but after a little discussion claimed to be convinced:

M.: «It's because you took that point over there, if you had taken another one it wouldn't have worked out… look!»

(The teacher drew another point different from the projection point identified by the researcher and then drew lines intersecting the longer segment and not the shorter one. These considerations mirror the difficulty in understanding what it is and how a mathematical demonstration works).

R.: «Yes, but if you want the projection point to be exactly the point you drew you have to perform a translation of the two segments and project right from that point (the translation on M.'s drawing was performed)***, however, the translation will not alter the number of points of the two segments»***

M.: «Ok, you convinced me».

15) The biunivocal correspondence was subsequently demonstrated to the 16 teachers. The biunivocal correspondence proved that the cardinality of even numbers is the same as that of natural numbers and then the question 15 was asked.

Teachers reacted in two different ways:

• ***The dubious.*** 6 expressed themselves as not being particularly convinced:

M.: «Well, it's kind of a "strain"»

N.: «It's strange, in the set of even numbers all the odd numbers are missing to obtain the natural ones».

• ***Those affirming to be convinced.*** 10 claimed to be convinced, but 2 in particular showed some trust in the researcher as the one who possesses Knowledge.

Furthermore, during interviews, it emerged that all the teachers who accepted the idea that some infinite sets are of equal power (as in the case of the even and natural numbers) are now convinced that this is bound to infinity and as a consequence they generalise that all infinite sets are also of equal number. This *flattening* misconception is seen as an "improvement" in comparison to the *dependence* misconception of the cardinality on the set "size". This change in attitude seems a slow and gradual approach towards "the correct and advanced model of infinity". The appearance of the *flattening* misconception of transfinite cardinals was not unexpected since primary school teachers ignore the set of real numbers, and therefore they opt for a generalisation of the notions related to the sets known to them.

Prove of that is given by the following conversation:

A.: «Therefore all infinite sets are equal»

R.: «What do you mean? Do whole numbers have the same cardinality as natural numbers?»

A.: «Uhm, yes»

R.: «And the rationals? The fractions»

A.: «I think so»

R.: «And real numbers? The roots»

A.: «Yes all, all of them, they are either all equal, that is to say infinite or none of them is so».

The aim of the proposed demonstrations was to show teachers that the primitive Euclidean property: «*The whole is greater than each of its parts*» cannot be applied to infinite sets: neither in the ambit of geometry [look at the proofs of: Roger Bacon (1214-1292), Galileo Galilei (1564-1642), Evangelista Torriccelli (1608-1647) and Georg Cantor (1845-1918)] nor to infinite numerical sets where one is a proper subset of the other.

Teachers' intuitive affirmations (misconceptions) seemed to be inconsistent, as the two contradictory misconceptions of *flattening* and *dependence* coexist in their mind. It has been observed a generalised difficulty of the teachers to realise when two affirmations are contradictory and we believe it to be the result of their lack of knowledge and of mastery of the concept of mathematical infinity.

In addition, it has also to be noted that the discussions among teachers brought no change of opinion when it had to do with the infinity issue. Some of the participants changed their mind only as a consequence of the two demonstrations showed by the researcher, whereas they showed somehow reluctant when stimuli to reflections came from the other colleagues.

3.7.2 The idea of point

Many of the teachers' affirmations, especially those related to the question no. 4 (reported in paragraph 3.6.2) based on the misconception that to a different segment length corresponds a different number of points, revealed how some of the convictions in question are related to the idea of point seen as a geometrical entity provided with a certain dimension, though small. This belief originates from the most commonly adopted representation of point conditioning the building of this mathematical object related image. As a matter of fact, this misconception seemed to be shared also by those not explicitly expressing it though stating with regard to question 4 that a longer segment has more points than a shorter one. In so doing they revealed "naive" interpretation of the idea of segment and point possessed.

Hereafter some of the affirmations related to question no. 4 are reported:

B.: «In the segment CD, of course it's longer»

R.: «How many more?»

B.: «It depends on how big you make them»

M.: «It depends on how you draw them: distant or very close to one another; but if you make them as close as possible and all of the same size then there are more in CD»

G.: «In CD, it's longer»

R.: «But can you really see the points as graphically represented here?»

G.: «Yes, it's the kind of geometry we do that makes us see the points».

Hereafter we report once again the statement mentioned in 3.7.1 concerning the question no. 1:

N.: «Oh you're right, infinite. But it's just to say a large number, not as large as in the straight line. Even if you do very little points, you cannot fit in more than that».

These affirmations are strongly influenced by the so-called "necklace model" to which we mainly referred to as source of obstacles in the understanding of the concept of mathematical infinity and of the straight line topology. De facto, a *parasite model* has been built in the students' minds (Fischbein, 1985) (see 2.2) as a consequence of the acceptance of the intuitive model seen as a thread of little beads. Most striking is that the "necklace model" represents not only a didactical device ingenuously invented by teachers in order to provide their students with just an idea of a segment, though being aware that the image in question is an imprecise, rough and quite distant representation of the real mathematical concept related to segment. On the contrary, this unfortunately represents the real model teachers possess of a segment and point. In addition, as emerged from discussion most of the teachers' deficiencies are particularly linked to the concepts of the straight line density and continuity.

3.7.3 Potential and actual infinity

The opinion exchange revealed how some of the teachers' convictions are definitely referable to the potential view of infinity. As a matter of fact, also in those cases when they adopted definitions ascribable to actual infinity such as: *«The straight line is formed of infinite points»*, they successively turned out to be inconsistent when declaring also that the term straight line is used only to indicate an ever longer segment, returning once again to the potential vision (in compliance with the Euclidean thought, see 1.1.1).

As pertaining to the potential use of the infinity concept the two following examples are reported:

R.: «We use to say that natural numbers are infinite, but we know that this doesn't mean a thing as they can't be quantified! It's like saying a very large number that you cannot even say; that you can go on forever I mean. To say straight line is like saying nothing, it doesn't really exist, it's another way of saying an ever longer line».

A further aspect originates from R.'s affirmation: the term infinity is mentioned but does not represent a quantity. Stating that natural numbers are infinite (a very commonly used expression pertaining, at a first glance, to infinity in its actual sense) is just another way to say a large finite number. Moreover, this affirmation seems to support the conviction that everything concerning the unlimited and infinity is perceived as non-existing since it is not to be traced in the sensible world. On the other hand, concepts such as segment, square, rectangle, for which it is possible to locate some approximated "models" surrounding us, are perceived as existing. The presence of such a conception implies that the real sense of mathematics and its related concepts too is mislaid. As a matter of fact, if you do not perceive mathematical entities as abstract but you remain stuck with the attitude of envisaging them as things existing in the sensible world, then to think of concepts such as mathematical infinity or the straight line topology happens to be cause of major disadvantage. The resulting problem is that some teachers think that most branches of mathematics are related to the concrete and sensible world and that there are some other concepts such as infinity or the straight line which are detached from the world of things and hence according to opinion not suitable to be dealt with in primary school. A teacher expressed this idea with the following words: *«If a thing doesn't exist as in the case of the straight line there is therefore no meaning to teach it?»*. The same considerations are also applicable to the following affirmation:

N.: «I say that numbers are infinite, but I know it's only imagination, you'll never get to have them all, you use infinity to mean an ever-increasing number. No way you can reach infinity».

The main consequence of such conceptions in the teaching activity is the risk of providing the students with images completely extraneous to mathematics and possibly turning into an obstacle to future learning both in analysis courses of higher school and even before in the lower secondary when concepts such as the density of Q, irrational numbers such as π, the ratio between the square side and its diagonal and many more, are introduced.

Interviewing teachers revealed the prevailing use of the potential infinity and with respect to this the outcomes of the discussions that have proved extremely interesting were aimed by the researcher at the comprehension of the double nature of infinity: actual and potential, in the same way as it appeared to Aristotle (see 1.1.1).

• 10 teachers are still stuck to potential infinity as shown in the following passages:

M.: «To me there exists only the potential infinity, the other doesn't exist, it's pure fantasy, tell me, where is it?»

S.: «When talking of the straight line»

M.: «But where is the straight line? There is none. So actual infinity does not exist»

S.: «What do you think of the straight line?»

M.: «I think these kinds of things shouldn't be taught, at least not in primary school, poor children what can they do! Yes, of course you can also say that the straight line is formed of infinite points, but how are they supposed to understand that? (I don't believe it myself!), at their age they have to see things. They have to touch things with their own hands»

N.: «I think very large things though still finite exist, all the rest does not exist».

• 6 teachers seemed to grasp the idea of actual infinity. In particular, three teachers showed a very enthusiastic reaction to their discovery of the distinction between the two conceptions of infinity: potential and actual.

A.: «I never though about this distinction, but now I got it, I can imagine it»

B.: «I never even thought about it, nobody gave them the possibility of reflecting on this topic, but to be honest I always thought that it was meant only in the sense of a continuous and constant process. But now I've understood the difference».

The latter statement shows the embarrassment felt by the teachers who were not given any possibility of reflecting on such fundamental topics they should be able to master in order to prevent the creation of students' misconceptions.

The crucial point is that "no sensible magnitude is infinite" and therefore the comprehension of such topics seems to go against intuition and everyday experience (Gilbert and Rouche, 2001). With reference to this, various research works [Moreno and Waldegg (1991), Tsamir and Tirosh (1992), Shama and Movshovitz Hadar (1994), D'Amore (1996, 1997), Bagni (1998, 2001)] pointed out that when acquiring the concept of *actual infinity* epistemological obstacles, deriving from an initial intuition, have to be encountered (and the history of mathematics itself confirms that). As a matter of fact, as the first chapter of the present work is meant to demonstrate, during the 2200 years from Aristotle till present time, the treatise of the concept of infinity underwent a very slow and not homogeneous evolution process.

Up until the XVIII century, infinity was considered only in its potential sense, and potential is still the approach of those who are led by intuition and lack an appropriate reflection on this topic. Yet the conception of actual infinity is fundamental for the study of Analysis, even if teachers tend to convey to their students only the potential use, as if it were the only way to conceive this concept. But problems come later, when students attending higher secondary school have to face the actual aspect of infinity, which may at that point turn out to be extremely difficult to accept. This as a result of the learning - in the previous years - of an intuitive model of infinity so deeply rooted and only representing its potential aspect. This model is only based on students' and their teachers' intuitions, but very distant from the world of mathematics.

Tsamir (2000) states: *«Cantor's set theory and the concept of actual infinity are considered as opposite to intuition and can raise perplexities. Therefore they are not easy to be acquired and some special didactical sensitivity is necessary to teach them»*. Unfortunately, when the above-mentioned concepts have not been properly investigated in higher secondary school, the corresponding image, mainly based on initial intuition, remains linked to potential infinity.

In other words, if primary school teachers (and not only them) have never been taught the topic in question, they are obviously bound to refer when teaching these concepts to their intuitions. The history of mathematics has vastly proved these intuitions to be opposite to theory. Consequently, Tsamir's didactical sensitivity would hardly be developed, which causes didactical obstacles strictly related to the inevitable epistemological obstacles.

Didactical ones worsen epistemological obstacles. Intuition plays a dominant role, but it is also confirmed by the knowledge taught at school. This could be the case of a self-sustaining chain: teachers base their teaching actions on their intuitions. These were in turn strengthened by their teachers who in turn had previously based their teaching on their intuition and so on. Therefore there is an urgent need for breaking this chain. This goal can be achieved by highlighting teachers' deficiencies and introducing an infinity-targeted didactical activity addressed to both teachers with experience and those without, so that high school students' cognitive disadvantages and obstacles - pointed out by several research works - can be avoided.

3.7.4 The need for "concreteness"

During interviews with teachers a commonly shared opinion emerged. According to this view, primary school children need concrete models in order to understand mathematical concepts. That justifies the didactical choice of using a necklace of beads as a model of segment; or the mark left by the pencil or the grain of sand as a model of mathematical point. Unfortunately, not everything can undergo the process of modelling without consequences. It is not rare to verify that in the didactical transposition the teaching choices, based on major reference to everyday world, negatively condition students' future learning. Tests carried out with primary school children and with teachers willing to change their way of teaching, showed that children enjoy and find it easy to enter a dimension so far from the sensible world. It has also been observed working in this way, that teachers themselves find it easier to deal with mathematical concepts not making any more reference to the concrete world, as mathematical entities are abstract by nature. Such results raise the question whether children or rather teachers are those who feel this need for concreteness. And in fact, it stands out clearly that teachers find it difficult not to make reference to real things, whereas children at times are delighted in leaving the sensible world and feel completely at ease with that.

Two significant statements made by teachers during the interviews give evidence of this. These examples mirror two opposite points of view. The first one was already quoted and analysed from a different perspective in paragraph 3.7.3:

M.: «I think these kinds of things shouldn't be taught, at least not in primary school, poor children what can they do! Yes, of course you can also say that the straight line is formed of infinite points, but how are they supposed to understand that? (I don't believe it myself!), at their age they have to see things. They have to touch things with their own hands»

A.: «You have to imagine these concepts rather than find them, I believe; the only way of doing that in Primary School is to make them use their imagination, which is so rich: "A straight line is a line that goes as far as the most remote infinite space", and they start imagining it … I tell them that you can't measure or weigh a point. It *exists, but you can't see it, it's like magic. So it works, because they enter a world which is not any more that of concreteness. They need to enter the world of imagination, in order to make it».*

(The latter teacher took an Analysis exam at university).

3.8 Answers to questions formulated in 3.4

We are finally able to provide answers to the research questions formulated in 3.4.

P.1 The answer is with no doubt negative. There is a total absence of knowledge of what is intended by mathematical infinity, both in the epistemological and cognitive meaning. This deficiency surely derives from the problematic aspect of the subject matter thoroughly featured by epistemological obstacles and the lack of a targeted formation on this topic. To primary school teachers, infinity is an unknown concept, solely managed by intuition and for this reason considered as a banal extension of finite. That causes the creation of intuitive models that turn out to be thorough misconceptions. Teachers accept namely the Euclidean notion: *«the whole is greater than its parts»* for the finite and tend to consider it also valid for infinity, which is a *dependence* misconception. Expressions such as "to be a proper subset" and "to have less elements" should not be confused when dealing with infinite sets. Nevertheless primary school teachers, during their educational training, have only found evidence of what happens when dealing with finite and accepted it as an absolute intuitive model and consequently transferred to their pupils. In other words, if an A set is a proper subset of a B set, then the cardinality of B is automatically greater than the cardinality of A. In building such a misconception, teachers' intuitive model of the segment seen as a necklace of beads also plays a role, thus leading to the phenomenon of *dependence* on magnitudes. The *flattening* misconception also participates in this mechanism, but in the didactical repercussion it brings about less affecting consequences than *dependence* to primary school pupils. Also the straight line seen as an unlimited figure and the prolonged counting of natural numbers seem to make teachers consider infinity only in power and not in act, which results in major didactical obstacles (Tsamir and Tirosh, 1992; Shama and Movshovitz Hadar, 1994; Bagni, 1998, 2001; Tsamir, 2000).

P.2 The answer is affirmative. Students' intuitive images concerning infinity are continuously strengthened by teachers' stimuli, who tend to transmit to their students their own intuitive models, which are – without their being aware of that – thorough misconceptions (see P.1). Such convictions persist in students' minds and become so strong that they create an obstacle difficult to be overcome when facing the concept of actual infinity in higher secondary school. Intuitive models such as the segment seen as a lace for instance make the conception and understanding of the idea of density impossible. The latter is already introduced in lower secondary schools or even before, in primary schools. For example, when the so-called fractional numbers are positioned on the "rational straight line" r_Q, the necklace model resists and the density is limited to its potential aspect. To many students, density seems to be sufficient to fill the straight line and therefore the difference between r_Q and r results incomprehensible, even when the set R and the definition of continuity are introduced a couple of years later: the intuitive necklace model still dominates.

P.3 The present research has clearly shown that, besides epistemological obstacles already pointed out in the international literature, there are serious didactical obstacles deriving from teachers' wrong intuitive models, that are in their turn transferred to students. In order to avoid such obstacles a better training is needed, so that a purely and exclusively intuitive approach to infinity can be averted. It is therefore necessary to reconsider the teaching contents for teachers in training (working for any educational level). In so doing, it could be avoided that students in higher secondary school would have to face the study of analysis with an improper background of misconceptions. The treatment of problems concerning actual infinity requires the development of different intuitive models, if not even opposite to those regarding finite. We believe that a suitable education on the subject of infinite sets should start at primary school, in order for students to start handling the basic differences existing between finite and infinite field, both in a geometrical and a numerical context.

3.9 Chapter conclusions

Many research works on the topic of mathematical infinity have revealed that the obstacles impeding the comprehension of this subject are mainly of epistemological nature.

The present research has identified primary school teachers' beliefs on infinity, which are supported by erroneous mental images, firstly influencing their convictions and subsequently their teaching activity, too. Many of the teachers involved in this research study, after some explanations were provided to them, have firmly admitted - and in that they showed a great professionalism - that their teaching was rich in wrong models. Such models were confirmed year after year, but they might have been - according to teachers themselves - the source of future didactical obstacles. We want to thank these teachers for their honesty and professionalism.

We believe that the difficulties encountered in the understanding of the concept of mathematical infinity are not exclusively due to epistemological obstacles, but didactical obstacles resulting from the teachers' intuitive ideas magnify them also. It is also very likely that surveyed deficiencies on this specific topic are not a problem exclusively affecting primary schools, but are instead rather widespread at every school level, among all those teachers who have never been given the opportunity to properly reflect on mathematical infinity.

So far it seems as if such a topic has been very much underestimated, above all as a subject for teachers' training. This deficiency is the main cause for the problems encountered by high secondary school students already possessing previous and strong convictions, which are unsuitable to face new cognitive situations. Models provoking obstacles in the teachers' as well as students' minds are necessarily to be inhibited and overcome. As we have seen many a time in this chapter, primary school teachers targeted training courses are required. These courses should take into account the several intuitive aspects and peculiarities of infinity as well as the outcomes collected by the researchers of didactic mathematics. They should be mainly based on open and free discussion; the historical aspects of the subject should be outlined too. They should start from initial intuitive ideas in order to transform them into new and fully-fledged convictions.

All of the teachers involved in this research have clearly voiced this necessity. In this respect, we report two teachers' opinions:

M.: «Yes, it's the kind of geometry that we do that gets us to see the points. We need someone to help us reflect on such things and on the importance of transposing them in a correct way. In the mathematics we learned, they did not make us think about these things. We need some basic theory»

A.: «Our problem is that we try to simplify things, without some previous theory. We are sure we've got it, but in fact we don't have it. We are concerned with transferring it in a tangible way, without deeply investigating how it works».

Such a specific training will enable primary school teachers to properly master concepts regarding the infinite sets, getting their students involved in meaningful experiences and activities implementing the building of intuitive images which are pertinent to infinite sets theory.

Chapter 4. Present and future research

4.1 The first training course on this topic

Conclusions to the preceding chapter underlined the fundamental importance of specific training courses addressed to teachers on mathematical infinity in order to achieve an “advanced” awareness of the topic. To reach this goal, in the last two years a tailored training “trajectory” has been implemented. The course has been addressed to 37 primary school teachers and 8 lower secondary school teachers of Milan. This has turned out to be an occasion for us to reflect on new aspects pertaining to the debated subject. The selection of participants, teachers, was the result of the attendance, within the framework of a series of conferences held in 2001 in Castel San Pietro Terme (Bologna): “Meetings with mathematics no. 15”, to a seminar for primary and lower secondary school teachers called: *“Infinities and infinitesimals in primary and lower secondary school”*. At the end of the seminar a large group of teachers, curious about the topic dealt with during the meeting and whose convictions perfectly mirrored those misconceptions described in chapter 3, openly showed the need for a better understanding of the discussed subject matter which they were never given the opportunity to reflect upon. They have been chosen because they have turned out to be highly motivated and have a serious interest in that, up to that moment, unknown subject. From the researchers point of view, this situation represented such an ideal fertile ground to start up not only a training “trajectory” but also a real action research that day-by-day is proving fruitful and rich in stimuli. Before starting in 2001 the above-mentioned training course, teachers were asked to fill in the same questionnaire described and commented in paragraph 3.6.2 adopting the same methodology as in 3.6, in order to assess if these teachers’ convictions were to be considered similar to those already collected and classified in chapter 3. With no hesitation, we can therefore assert, that the latter results go in the same direction as the former ones. Furthermore, we observed no relevant differences between primary school teachers’ convictions and lower secondary ones. This latter aspect showed how deeply rooted and difficult naive intuitions are to be eradicated, because of the complex nature of concepts characterised

by epistemological obstacles. Only a proper and targeted training activity could help modify such convictions. Out of 8 people teaching in the lower secondary, 2 have a degree in mathematics and 6 in scientific subjects. None has ever attended a course on this topic at university or even successively. As a consequence, there was actually no difference whatsoever related to the different kinds of degrees of the participants. But the most striking aspect was that, teachers with a mathematics degree and people who had only attended a high school specific for teachers (until recent times in Italy such a high school diploma was a sufficient prerequisite to teach in primary schools) shared the same awareness, or better to say lack of it of with respect to these specific topics. It is really surprising to note how the epistemological complex nature of this subject really undoes all other knowledge items, creating a levelling of convictions.
The collected outcomes are made available by the author but the reader can also refer to the reflections reported in chapter 3.

The only real dissimilarity was mainly based on the different linguistic expressions, which emerged during discussions between groups of 4 secondary school teachers. They basically used "definitions" derived from the adopted textbooks and consequently passed on to their students. Definitions that turned out to be most of the times improper, badly expressed and managed.
We provide an example taken from the answer given by a lower secondary school teacher to question no. 4 of the questionnaire described in 3.6.2 and asking if there are more points in a longer segment rather than in a shorter one:

S.: «Well… provided that ***a point is a dimensionless fundamental geometrical entity****, I would say that it is not possible to establish if there are more points in AB rather than in CD. By the way, it is also true that* ***a straight line is formed of infinite points****, but we are talking about segments. Well, the fact of being of different length must mean something so I'd say there are a greater number of points in CD. Yes, yes, there should be more in CD».*

The bold type has been used to underline those expressions commonly considered by teachers themselves as "definitions" and not only in the lower secondary but also in the higher secondary schools as we shall see in paragraph 4.3.

This answer provides a good example that the management of such factual "knowings" at times improper, misunderstood in their real meaning and not internalised does not result in differences from those collected with primary school teachers. *«Knowledge is not in books, it is the understanding of books. If you consider scientific results, it has to be admitted that normally the one who is able to enunciate them without being aware of them, does not know them (…). Knowledge is neither a substance or an object, it is an activity of the human intellect performed by subjects that try to substantiate what they do and say (by means of demonstration and reasoning)»* (Cornu and Vergnioux, 1992) [our translation]. The secondary school teacher F. though not sharing the same view as the above-mentioned colleague's of primary school and contained in 3.7.2: *« Even if you do very little points, you cannot fit in more than that»*, she ends up saying all the same that there are more points in a longer segment than in a shorter, attributing to the point a nature that cannot be a-dimensional if it depends on size, as she affirmed at the beginning repeating a notion learnt by heart.

It clearly stands out that teachers are not aware, especially when dealing with delicate topics such as those concerning geometrical primitive entities, that in most cases they think they know some concepts but they actually do not. These considerations have inspired a new research work, still in progress at the moment, based on geometrical primitive entities and strictly linked to the concept of mathematical infinity treated in paragraph 4.3.

Cognitive deficiencies deeply influence the *didactical transposition* (see paragraph 2.4) whose choices may be resulting in misconceptions or even wrong models. These conceptions and models are at the basis of didactical experiences badly managed by teachers and presented year after year in the same way. As observed in paragraph 3.8, to many teachers and consequently also to many students, density seems to be sufficient to fill the straight line and therefore the difference between r_Q and r results incomprehensible even when the set R and the definition of continuity are introduced. The distinction between density and continuity is however not favoured by the a-critical use of the entity straight line that starts with the introduction of N since primary schools causing several didactical problems (Gagatsis and Panaoura, 2000) and continues in the following educational levels (Arrigo and D'Amore, 2002).

Another problem, common to all educational levels, is represented by the “natural” model of the order of Z, that due to its prompt understanding and extremely conceptual and above all graphic simplicity, at the end turns out to be univocal and impossible to overcome even when the biunivocal correspondence between the set Z and the set N, requiring a different order of the elements of Z in comparison with the “natural” order, is introduced (Arrigo and D’Amore, 2002) (see 1.2.5).

A further hint is provided by a lower secondary school teacher’s answer to question no. 7 of the mentioned questionnaire: *How many natural numbers are there: 0, 1, 2, 3…?*:

F.: «Natural numbers are infinite because ***a set is infinite when is formed of infinite elements*** *and 0, 1, 2, 3, … are infinite».*

The same teacher confronted with question no. 10: *Are there more even numbers or natural numbers?,* affirms:

F.: «There are more natural numbers than the even, it’s logic they are the double».

It is important to restate that most of the times “definitions” provided by textbooks are improper. A good example is contained in a lower secondary schoolbook in the section of arithmetic with the title*:* ***finite sets, infinite sets***:

“The sets we referred to are formed of a well determined ***finite*** *number of elements”,* (this suggests that an infinite set like the natural numbers one is not formed of a specific number of elements as it is actually: a denumerable infinity. These considerations inevitably imply that infinity is associated with the indeterminate).

And furthermore:

“In mathematics there exist sets of an ***infinite*** *number of elements”* (in the text the term infinite was underlined). Therefore, infinite sets are introduced as sets of an infinite number of elements. The latter statement is to be found in many lower secondary school textbooks and is perceived by teachers as a “definition” (F.: «*It’s in the textbook*») whereas other books used in higher secondary schools refer to the same affirmation classifying it as among the “primitive ideas”.

The previously cited paragraph on finite and infinite sets continues and ends in the following way: *“A kind of infinite set is that of whole numbers for instance: no matter which you consider to be a finite set of whole numbers, it is always possible to find another whole number different from those already taken into account”* (this idea embraces an exclusively potential vision of infinity recalling the Euclidean approach of the “Elements”). The potential vision can be found in many books as well as in one in particular adopted in Italian lower secondary schools. In this book, just to give an example, the chapter on numbers begins in this way: *“The ultimate number would never be reached even if 1 keeps being added on and on and on…”* and follows with: *“the set of natural numbers is **infinite**”* (idea that could result in those misconceptions described in 3.7.1 such as: infinity as indefinite or infinity as a large finite number). As a matter of fact, the exclusively potential treatment of infinity will be passed on to teachers who in turn will pass it on to their students making them all think that infinity cannot be conceived as an object by itself, something definite and possible to grasp, “reach” and dominate.

Furthermore, some textbooks start with finite sets to successively define the infinite ones, others use the opposite method successively defining a *“finite set as a set non-infinite”*. Our objection is not addressed to the definition by negation, since in this case we subscribe to Bolzano’s thought (1781-1848), reported in the bibliographical reference of 1985 and based on the consideration that if the so-called “positive concepts” exist, there should be no impediment for the existence of “negative” ones and for these latter concepts a definition in the negative is possible. As a matter of fact, the definition of an infinite set has in general a positive character, whereas the negative is attributed to finite sets, although philosophical texts usually attribute to the term “finite” the “positive” concept and to “infinity” (meaning non-finite) the “negative”. [For a better understanding of the difficulties of defining the concept of “finite” see Marchini (1992)].

The crucial problem is to choose what definition of infinite set to adopt and to avoid the vicious circles that are triggered as a consequence of an initial definition for the concept not properly representing the concept itself. It has already been stated that false definitions magnify both teachers’ and students’ misconceptions.

To clarify the goals of a textbook may be of some help towards a better understanding of the issue at stake. As a matter of fact, a textbook is nothing but the result of a didactical transposition chosen by the authors and is therefore not to be interpreted by teachers as a book of mathematics where one can learn concepts. Knowledge should be already possessed and mastered by teachers when adopting textbooks. All the pieces of knowledge should only be refreshed and reinterpreted in the specific case of the didactical transposition decided by the text author and therefore and only successively personally adjusted to the specific case of class-context. With regards to mathematical infinity, very often teachers themselves do not seem to be confident at all with this knowledge and so they only attribute to the didactical transposition contained in the text the function of mathematical contents transmitters. Evidence of this is given by the frequent attempts to justify their answers concerning mathematical concepts with affirmations such as: *«It's written in the book we use»*. But when a concept or domain of knowledge is inserted into a textbook, it undergoes a massive transformation, i.e. its nature is changed in order to respond to another statute, another logic, and another rationality, influenced by school pedagogy requirements imposing a different form.

Returning to the "definition" of infinite set expressed by the concept that a set is infinite if it is formed of infinite elements, it clearly stands out that the latter cannot be considered a reliable definition, thus impeding the understanding that two infinite sets, as the natural and even numbers for instance are formed of the same number of elements. On the contrary, a definition that in the beginning may be somehow twisted and complex, but is in fact appropriate for defining infinite sets, is the definition called Galileo-Dedekind's (see paragraph 1.2.2): *"A set is infinite when it can be put in biunivocal correspondence with one of its proper parts"*. This implies, in fact, that the sets of natural numbers and even numbers may be formed of the same number of elements provided that the correct biunivocal correspondence is established between them (see 3.6.2).

The treatment of these subjects in textbooks is clearly problematic. Authors tend to diffuse rather delicate subject matters without the necessary critical caution that people who produce material destined for didactical use should have. Unfortunately readers, both students and teachers, do not display a sufficiently critical approach towards what is published. As we observed, they tend to accept everything they find in any textbook as trustworthy. Our future intention is also to analyse textbooks, in particular those generally chosen by teachers, to test them on the topic in question and to evaluate inaccuracy and defects, to find out, with suitable methodologies, the extent of teachers' reliance on this didactical instrument.

In 2001, after the questionnaire had been proposed, a training course was created on this topic. Initially 45 teachers, that have the same misconceptions, began to attend the course, recognising that, as was well expressed by the words of a lower secondary school teacher: *«As far as this is concerned, we are all in the same boat!»*. The course was organised in different meetings and is still running today involving a more limited number of participants. The course was initially conceived as based on the history of mathematics concerning this topic (see chapter 1).

This was due to the awareness that some convictions, influenced by strong epistemological obstacles, had to be eradicated. We were well aware of that, as we had encountered two of the features highlighted by the research body of Bordeaux and referred to by D'Amore (1999) as useful to spot epistemological obstacles:

• in the historical analysis of an idea, a fracture, a sharp gap, non-continuity in the historical-critical evolution of the idea must be traced; (the history of infinity is a good example of that).

• a mistake must recur over and over again, always in similar terms; (the same mistakes, coinciding also with the historical fractures, were traceable in the convictions of the involved teachers).

On the basis of this awareness, we thought it fundamental to build a strict connection between the history of mathematics and the didactical aspects during the course. We tried to join the two subjects through discussion and confrontation starting from teachers' primary intuitive ideas to develop them into new, more advanced convictions. This strategy proved fundamental to enable teachers to make a critical reading of their ideas, as they recognised them in some statements of the mathematicians of the past. This confrontation facilitated the eradication of the misconceptions that had emerged from the initial questionnaire. Moreover, the description of these historical fractures and discontinuity, highlighted some erroneous situations the mathematicians found themselves in and let some of the teachers understand the meaning of mistake in mathematics. (D'Amore and Speranza, 1989, 1992, 1995). The study of history turned out to be a sort of essential keystone for teachers' critical self-analysis. In particular, three primary school teachers, over recent past years, have spontaneously decided to keep track of their ongoing progression, writing down step by step the evolution of their convictions. It is our intention to publish the outcomes of the training course soon, seen through the eyes of one or more primary school teachers (in paragraph 4.4.3 you will read some extracts of self-evaluation made by two of the teachers who attended the course).

The effectiveness of the course can be seen in some of the following sentences by primary school teachers: *«I've learned more during these lessons, than I've ever learned in a whole life of mathematics teaching and refresher courses. I have the feeling I understand now for the first time what mathematics is, and think! I've been teaching mathematics for 27 years. This discovery has really upset my life»*, *«I've understood what actual infinity is, and I've accepted it easily, since I had the courage to conceive it as a self-standing object, as a whole. Now I feel stronger»*; *«I've realised I'm much more attentive to my pupils and much more open to discussion. Above all, I try to work on their intuitions, as you did with us. They are happy with that»*. Lower secondary school teachers said: *«You illuminated me! At last I've understood what I've kept on repeating and teaching without having seriously thought about it. I'm so thankful to you»*; *«I can't teach as I used to any more. I'm not satisfied with the way I taught before… I can't go back any more»*; *«What surprised me most, was to find out that I didn't even know what I taught. Do you know what in particular? I lost sleep over the discovery that* $3.\overline{9}$ *really equals 4 and there is absolutely nothing missing. It doesn't approximate it; it really equals it. I've always introduced the "rule of the recurring numbers". But I've never applied it to specific cases. To tell the truth I skipped them on purpose, in order to avoid confusion, I had therefore never noticed that myself»*. Out of 8 lower secondary school teachers, just one, with a degree in mathematics knew that $3.\overline{9}=4$, although at the beginning she honestly admitted that in her opinion this represented an exaggeration she accepted as a fact, whereas all the others initially argued something like: *«Over there* (indicating a point at "infinity"), *there must be something missing!»*. Later on, when little by little they managed to accept the view of actual infinity, they succeeded also in conceiving and accepting this new discovery. Checking the results of the evolution in teachers' conceptions has been a slow, suffered but constant process. Signs of their progress have emerged and still emerge from the training course discussions. This experience has turned out to be really rich and significant from a scientific point of view and it has led to: the discovery that in training courses targeting schoolteachers there is a tendency on our part to take some fundamental pieces of knowledge for granted, that in fact are badly interpreted by our interlocutors; the discovery that teachers are sometimes totally unfamiliar with some subjects, and that this can cause fractions and incoherence in teaching and the basis for the creation of didactical obstacles; the discovery of the vital importance in this matter of addressing research to teachers convictions first, to focus subsequently on those of the pupils; the discovery that from a didactical point of view, there is a whole new world around infinity opening up that is still to discover.

We are still in touch with all 45 teachers, but we are cooperating in particular with a working-group of 5 primary school teachers with whom we have chosen to operate on a deeper level, from different points of view.

With these teachers we at last moved from teachers to students, that is to say, we went back to where we had started with our observations, back in 1996. As we showed in chapter 3, that year we investigated the convictions of primary school children,[41] who could not help reporting on this subject the knowledge they had learned from their teachers. [In this respect, on the connection between students' and teachers' convictions see the famous example of El Bouazzaoni (1988), dealing with the notion of continuity of one function]. Working with children we realised how big the potentialities of dealing with primary school children are, not only when referring to the "concrete" world, but also when having the courage of letting pupils explore the world of the "extra physical", as that of infinity for example, growing away from the physical world.

We will not go further into this matter in the present work, as we are focusing on teachers' convictions rather than on students'. However, we will just make some brief reference to the outcome of our previous research.

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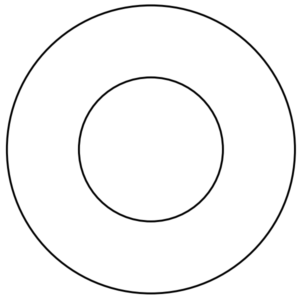
[41] As for specific research works on the subject of infinity for primary school children, please refer to following bibliography: Bartolini Bussi (1987, 1989), specific for primary and even nursery schools; Gimenez (1990) focussing on the difficulty of the density concept for primary school children; Tall (2001b) dealing with the evolution of the concept of infinity, from nursery school onwards, reporting the case study of a child called Nic.

4.2 Brief description of the research carried out with primary school children in 1996

This inquiry has been carried out with two classes of 10-year-old children in Forlì (Emilia Romagna). A total number of 38 children were asked to come out of the classroom in pairs and to work with the researcher. This the explicit agreement arranged with the children from the very beginning: all they would say outside the classroom would neither be evaluated nor told to their teachers. This in order to avoid that the stipulation of the didactical contract, depending on a classroom situation, could influence the experimental contract (Schubauer Leoni, 1988, 1989; Schubauer Leoni and Ntamakiliro, 1994) that students were establishing with the researcher. Our choice to work with pairs instead of with individuals was meant both to encourage children to undergo our inquiries and to trigger discussions, that could enable us to examine in depth the real convictions of interviewed children. The methodology we adopted consisted of letting two children enter a classroom and having them sit down at a table next to each other, in front of the researcher, who had a tape recorder without the children knowing. None of the classes that underwent our research had previously been proposed by their teacher specific activities on infinity.

The researcher started by handing out a sheet of paper where following two segments of different length were drawn and by asking: «*What do you think they represent?*»

After we got the answer and we clarified they were two segments, we carried on with the second question: *«Do you believe there are more points in this segment or in this other segment?* (pointing at the two segments)»; (this question is practically the same as no. 4 of the questionnaire in paragraph 3.6.2). After the children had given their answer and had discussed on it, they were presented Cantor's demonstration, showing that there is the same number of points in two segments of different length (see paragraph 3.6.2). Afterwards, the researcher handed out a second sheet where two concentric circumferences of different length were drawn and asked: *«Are there more points in this circumference or in this other one?* (pointing at the two circumferences)».



From time to time, during the discussions the researcher could make some more questions or remarks, with the aim of stimulating confrontation, but being careful not to influence the opinion of the interviewed children.

The children were subsequently asked: *«What is mathematical infinity in your opinion?»* and they were let free to discuss until the confrontation would stop.

Let us briefly report some of the results we gathered. In bold type there are the researcher's interventions during the discussion, meant to stimulate conversation and to inquire in depth into the children's convictions.

• Many children when asked to answer the first question: «*What do you think they represent?»* (showing them the two segments of different length) did not reply: *«Two segments»,* but often generically said: *«Lines»* or *«Straight lines»,* others noticed things based on knowings that were not included in our area of interest as:

S.: «They are bases, we revised them on Friday»

R.: «Bases of what?»

S.: «Of a rhombus, or better said, of a rhomboid»

R.: «Can you draw it?»

S. The child drew a trapezium.

Another child answered as follows:

F.: «They are parallel lines. Parallel lines aren't like concurrent lines that meet only in one point. They are infinite (in the sense that they do not meet)»; (here we can trace the use of "mathematese").[42]

[42] This word was minted by D'Amore (1993a) and refers to a sort of "mathematical dialect" used in classroom. A special language that the student considers correct, right, and appropriate to use in maths classes to fulfil "contractual" duties. Oppressed by the "burden" of this new language, the student often gives up the sense of the question or of her/his discourse.

• After having explained to the interviewees that they should concentrate their attention on the two segments, they were asked: «*Do you believe there are more points in this segment or in the other one?*».

Most of those interviewed answered something like: in the longer segment. Just one child claimed that there were more points in the shorter one G.: «*You just need to stretch it out and make it longer than the other one*». Whereas 16 children replied: «*Equal*»; 2 of them affirmed it without supporting their opinion with any reason and without saying that there are infinite points in both segments. Whereas as many as 14 children, all belonging to the same class, claimed that there is the same number of points in two segments of different length and more precisely, two points in both of them that mark the two end points of the segments. That highlights how teachers' didactical attention and consequently children's didactical attention often focuses on small details, conventions and non-significant formalisms, when dealing with the description of a concept. [On teachers' view on mathematics please refer to: D'Amore, 1987; Speranza, 1992; Furinghetti, 2002].

Let us now look at the extract of a conversation between two children, one of whom had expressed the above-mentioned interpretation.

R.: «Do you believe there are more points in this segment or in this other segment?»

M.: «In this one (pointing at the longer one)»

I.: «No, they are equal. There are two points»

R.: «What do you mean?»

I.: «In both of them there are the two end points that delimitate the segment. The teacher told us that»

R.: «So, how many points are there in a segment in your opinion?»

I.: «The same, they are always two».

Then there was the significant case of a child that, though he answered correctly to the first question: «*They are two segments. The teacher told us to write down that a segment is a set of infinite geometrical points*», he then claimed that there are more points in the longer segment, thus showing that he had failed to grasp the meaning of his previous statement. This reveals that one must be very careful when proposing definitions and above all when considering as satisfying the answers of a student, just because they coincide with the given or expected definition: repeating a definition does not necessarily mean understanding its meaning.

• Many misconceptions on geometric primitive entities emerge from children's conversations; these are false beliefs that negatively affect the subsequent learning process and that we are going to analyse in more detail in the next paragraph. We believe that this outcome underlines the importance of not leaving concepts to the sphere of mere intuition, and shows the importance to work on these pieces of knowledge, thinking of specific and structured activities, as those we suggested in paragraphs: 4.4.3 and 4.5.3.

• After having shown Cantor's biunivocal correspondence between the two segments of different length (see paragraph 3.6.2), most of the children immediately intuitively understood that both segments were formed of the same number of points, whereas in other cases, the discovery process was somehow slower, but it nonetheless occurred:

G.: «Here you are a triangle. Do you want to know the perimeter?» (G. had recognised a triangle in the figure after the researcher had drawn two semi-straight lines originating from the point of projection O and intersecting the two segments; the child was trying to use the knowledge acquired in class, and undervalued the question that was actually asked).

R.: «No, not the perimeter»

G.: «So the points are two in both of them»

R.: «Look!» (the researcher showed two more corresponding points of the biunivocal correspondence)

G.: «Then there are 3»

R.: «But there are also these two!» (the researcher showed two more points)

G.: «Then there are 4»

R.: «But there are also these ones» (the researcher showed two more points)

G.: «Ah, now I know: there are infinite»

R.: «Are there more points in this segment or in this one?»

G.: «The same»

R.: «Are you sure?»

Both of them: «Yes, yes»

Thanks to the demonstration some of the children understood that the two segments were formed of the same number of points, but they could not say the exact number, so they answered *F.: «They are the same. There are many points, but I don't know exactly how many»*.

Except for 4 children, all pupils said they were persuaded by the truth of the new discovery, that is to say that two segments of different length are formed of the same number of points. Some pupils affirmed that in both segments there are infinite points, although when asked to explain what infinity is, they did not seem to be familiar with the advanced idea of the concept. It is interesting to notice that the explanation of a biunivocal correspondence did not surprise children, whereas it did surprise teachers, when some years later the same construction was proposed to them.
Only 4 children were perplexed and affirmed they were not convinced by the demonstration, this was mainly due to their strong misconceptions on the mathematical point:

A.: «I think it always depends. If you make smaller points here and larger points there, you never know how many they are» (from this reply we came to the conclusion that is important to focus on children's idea of mathematical point. Read more on this subject in paragraph 4.4).

• When the researcher showed the two concentric circumferences, almost every child immediately concluded that the number of points forming them was the same. Most of the children succeeded in building the biunivocal correspondence autonomously, starting from the central point, thus transferring a piece of knowledge they had learned before.

R.: «Now look at this (the researcher showed the sheet with two concentric circumferences of different length)»

M.: «This is a circular crown, we haven't done it» [manifesting a clause of the didactical contract (see paragraph 2.1) like: *«Only questions on subjects handled in class are allowed»*]

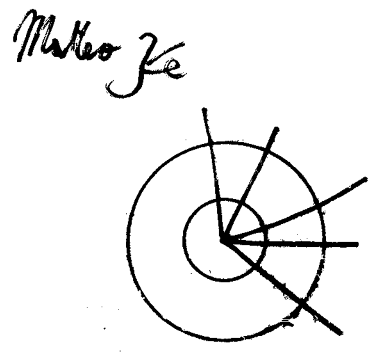
R.: « Are there more points in this circumference or in this other one?»

M.: «There is the same number of points, as before, it doesn't matter if one is small and the other is large»

S.: «I agree»

R.: «Why do you think it is so?»

M.: «It's like before, it goes like: you… you… you» (the child drew the biunivocal correspondence between the two concentric circumferences, starting from the centre)



R.: «What are you trying to show me?»

M.: «This little point corresponds to this little point, this… to this. Therefore they are perfectly the same. If one understands that, one understands this too»

S.: «You just need to understand one and you have understood them all»

R.: «Which one do you like most?»

M.: «The wheel, because I invented it» (this reply highlights how personally acquired pieces of knowledge are much more motivating and meaningful to a learner than any other proposed directly by the teacher).

In some cases the search for a demonstration has proved more difficult:

R.: « Are there more points in this circumference or in this other one?»

F.: «For me in this larger one»

M.: «No, in both of them (with a finger the child points at the biunivocal correspondence, but then covers it with the hand). *I just want to see one thing. I'll try»*

F.: «Yes, but if you make the little points smaller here and larger there»

M.: «It's different here, because it's a circle, it's closed»

F.: «You just need to make them more tightened here»

M.: «I just wanted to see how I had done the thing before. A thing like that (showing the biunivocal correspondence between the two segments). *If we do the same thing now.*

I wanted to see if we can do that, but here there must be something, because even if you try… I think they are the same and that's it»

R.: «You can draw, if you want»

M.: «This time it's different, because the circle is closed. I wanted to see how I had done the thing before like that, if we do the same thing now, I wanted to see if we can do that. But here there must be something, because even if you try. In that one there was a smaller one and a larger one while here there is a smaller one and a larger one, maybe it's more difficult to have the same number of points, in my opinion» (M. was considering an external point to both circumferences, not being able to find the biunivocal correspondence, he changed his mind and withdrew what he was saying at the beginning).

R.: «Why did you start from this point? Can't you think of another point from where it is more convenient to start?»

L.: «And if we draw a straight line?»

M.: «Ah, wait. We just need to do it in the middle, don't we?»

L.: «What are you doing, making a cross in the middle?»

M. built the biunivocal correspondence, discovering that there is the same number of points in two circumferences of different length.

Only 4 children, the same who claimed that there are more points in the longer segment, kept on arguing that the longer circumference is formed of a larger number of points. The reason for their choice derived from an erroneous idea of the mathematical point.

• 8 children spontaneously transferred the knowledge they had learnt in other contexts:

M.: «It's also like that and from the point you have always to go through both of them, so they have the same number of points (the child draws two concentric squares of different perimeters and builds the biunivocal correspondence)»

R.: «So in your opinion, there are as many points in a little square as in a big square»

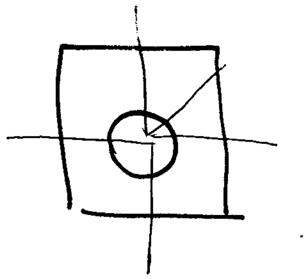
M.: «Yes, didn't you know that? No, because it seems like you don't know it»

R.: «It's that I didn't expect it, I couldn't believe you could have such an intuition»

M.: «I got so many intuitions, that's why I invented the wheel before» (meaning the demonstration related to the two concentric circumferences)

R.: «Now I ask you: are there more points in a " little circle" or in a "big square"?»

M. draws a "little circle" and a "big square" one inside the other, then he builds the biunivocal correspondence and answers:



M.: «These are the same too, because they have always to pass through here, here and here. It's easy though!»

• Only two children, particularly involved and open to discussion, have been further asked:

R.: «According to you, are there more points in this segment or in a straight line?» (on a sheet of paper a segment and a straight line parallel to one another have been drawn)

D.: «Now it's different! I don't know. Let's try»

F.: «You'd better use the ruler. But it's impossible to do it as before because there are some empty spaces here» (the child indicates that the segment is limited at both extremities)

D.: «Help, how can we do it?»

F.: «To me they are the same»

D.: «It's the same thing, you just have to join the points»

F.: «But the straight line never ends, there are more in the straight line»

R.: «How many points are there in a straight line?»

D.: «So many»

F.: «Infinite»

R.: «And in the segment?»

D.: «A lot but I think in the straight line there are always more, because the straight line continues to infinity whereas the segment stops» (here it is particularly evident the misconception of infinity as unlimited, see paragraph 3.7.1)

R.: «What if I told you that there are in both infinite?»

F.: «I wouldn't believe it. Here there are not infinite (indicating the segment)*, they will end sooner or later»*

R.: «But you told me before that in the segment there are infinite points»

F.: «Just to say so many»

R.: «So many? How many?»

F.: «Hey, do I have to count them?»

D.: «You think you can count them, but it's not because the point can be also extremely small, like that ·» (everything can be traced back to the misconception of point).

The researcher shows Cantor's biunivocal correspondence (see paragraph 1.2.3).

D.: «Then there are infinite, here and here»

F.: «So each straight line has the same points of the segment, because both of them have the same number of points»

D.: «Therefore for each line there is the same number of points because they are both infinite. So you cannot count them»

F.: «It's enough to say that the number of points is always the same even without looking at the length, one can be like that and the other like that (indicating different distances) *and you say infinite».*

It ought to be observed that the word infinity is very frequently used, but from the investigation of children's convictions with respect to such topics emerged the same misconceptions showed by primary school teachers in paragraph 3.7.1.

Some of the answers collected are provided here as follows:

G.: «Something that never ends, the teacher told me. It's like a track with a beginning and an end but that you can go along it as many times as you want»

S.: «They are the lines, the straight lines, the curves, the polygonal lines»

F.: «So many as the points we talked about before»

M.: «To me they are the normal numbers 1, 2, 3, 4… that never end. Our teacher always tells us so»

I.: «It's a sphere getting bigger and bigger» (the potential infinity idea)

S.: «The darkness and the points of before»

A.: «Something that has a beginning, but the teacher says that it can't have an end»

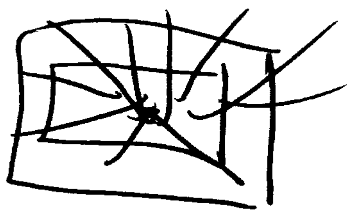
R.: «For example?»

A.: «Numbers, they don't have an end»

R.: «Is there not a last number?»

A.: «Infinity, it's the longest».

A significant case. During one of the first experiences carried out in order to test children's reactions before starting present research, we met Marco, a ten-year-old boy attending a different class from the others involved in the research. After he had been showed the biunivocal correspondence between two segments of different lengths, Marco, spontaneously, without having even seen the two concentric circumferences of different length, made the following drawing and affirmed by that:
«So, also this works»



Marco is of course an isolated case that therefore cannot be considered a prototype of what normally happens, but he deserves a mention as he inspired us researchers with trust and enthusiasm to embark on this project. We never came across any more "Marcos", especially since we shifted the focus of our attention from children's to teachers' convictions, in the latter case resulting much more difficult to try to break the erroneous models previously formed. Children showed to be extremely open-minded, flexible, willing to cooperate and to learn. Unfortunately all these attitudes were often negatively influenced by the kind of teaching received. It was not rarely observed that children, after having uttered sentences revealing misconceptions, also added: *«My teacher told me that»* and from the successive interviews conducted with primary school teachers we had evidence of this. That was the main reason why we have devoted our research to teachers whose convictions turned out to be more rigid and stereotyped.

4.3 Primitive entities of geometry

One of the research aspect we are currently investigating concerns students' and teachers' convictions not only regarding infinity but also geometrical primitive entities. This need originates from the frequent observation, when dealing with infinity, of the presence of misconceptions related to the point, the straight line, …

To achieve this goal, our decision was to use the methodology of TEPs: *«By TEPs we mean literally: students' autonomous text productions»*[43] (D'Amore and Maier, 2002). TEPs are therefore about written texts autonomously produced by students and regarding mathematical topics. TEPs are not to be confused with non-autonomous written productions (class tests, notes, procedure descriptions, …) since these productions are bound to certain constraints more or less explicitly given like such as direct or indirect assessments. In short, TEPs have to be considered as those productions that induce students to express themselves in a comprehensible way and use a personal language, accepting in this way to set themselves free from linguistic constraints and to employ spontaneous expressions instead.

In the article written by D'Amore and Maier (2002) some of the TEPs effects are listed and the following are the most interesting:

• TEPs production stimulates students to analyse and reflect on mathematical concepts, relationships, operations and procedures, researches and problem solving processes, which they get in contact with. In this way, every student can reach a better awareness as well as a deeper mathematical understanding of these concepts;

• TEPs give students the opportunity to constantly monitor their comprehension about mathematical topics by means of a fundamental and reasoned feedback with their teacher and classmates (self-evaluation);

• TEPs allow teachers to evaluate students' real personal knowledge and their understanding of mathematical ideas in a more detailed and accurate way than it would be possible with the analysis of common written texts, solely performed as not-commented problem solving activities.

TEPs production should provide the student's profound vision, detailed and explicit of her/his way of thinking and understanding mathematics, it is therefore necessary that the

[43] The German original term comes from Selter (1994).

student addresses her/his TEPs to whom needs all the pieces of information related to the subject of the writing. The addresser should be obviously someone but not her/his teacher.

The TEPs collected within the scope of this research are students' production starting from the nursery school (3 years old) and up to the higher secondary (19 years old). The idea was to start from nursery school in order to investigate if children of 3-5 years of age already possessed some primitive naive ideas related to these concepts. Additionally, our aim was to monitor the evolution of these ideas over the course of time, to this end nearly 350 TEPs have been produced and distributed to students of different educational levels. On the basis of the survey of these TEPs, our intention was to pursue the research object, more interesting in our opinion, of investigating teachers' real convictions concerning these mathematical objects, as direct consequence of the interpretation of the written performances of their students. In other words, after we handed to teachers the TEPs written by their students, transcribed on PC so that teachers could not be able to recognise students' handwriting, teachers were asked to read them, provide an interpretation of them and analyse them in detail on their own. Starting from the analysis carried out by teachers, the researchers' investigation could begin with the aim to assess teachers' convictions on the mathematical objects we proposed. The research had been performed by means of interviews, since we were afraid of exclusively taking into consideration the written answers that might be subjected to those factors already pointed out in international literature and namely: time pressure in finishing the assigned task, superficial answers, fear of being judged, etc. The joint use of the TEP and the interview, instead, especially if performed with all the necessary calmness and with no time pressure, has the advantage of making the subject feel at ease and therefore favour the investigation of the real, deep and hidden competences of the subject in question.

At the same time, our aim was to evaluate, in general, how teachers analyse and interpret a TEP, what their point of view is and finally their skills and ability in interpreting them.

Before handing out the TEPs, students were explained that no evaluation on the part of teachers had been envisaged for that work, and only after that clarification pupils, starting from those attending primary schools, were asked to provide written answers to the following questions:

• *Imagine you have to explain to one of your classmates what mathematical infinity is. What would you say?*

• *How would you explain it to a classmate of… years old?* (primary school pupils were asked to consider children two years older than them, whereas secondary school students had to think of younger students)

• *Imagine you have to explain to one of your classmates what a point in mathematics is. What would you say?*

• *Now explain it to a child of ... years old*

• *Explain to one of your classmates what a straight line in mathematics is*

• *Now explain it to a child of ... years old*

• *Finally, imagine you have to explain to one of your classmates what a line in mathematics is. What would you say?*

• *Now explain it to a child of ... years old*

As for nursery school the decision was to pose the following explicit questions: What is mathematical infinity for you? What is a point in mathematics for you? What is a straight line in mathematics for you? What is a line in mathematics for you? In addition, for each question, children were asked to draw if they felt like it.

Here as follows the outcomes of this research will be provided followed by only some general remarks, as the above-mentioned results have been not yet analysed in detail. It is however important to underline that the original texts handed in by children contain some grammatical mistakes that might not appear in the translation. These works are made available by the author to whom is interested in consulting them.

♦ The results collected in nursery school showed that 4-5 year old children already possess some first intuitive ideas related to these concepts, these are convictions that can serve as the basis for future misconceptions. For instance, the majority of children tend to associate the mathematical point with the graphic sign of a pen and answer with sentences such as:

«They're little spots» (Loris, 4 years old)

«They're some small and big dots» (Andrea, 5 years old)

Here are some of the answers concerning mathematical infinity:

«It's infinite line. That never ends. Universe is infinite. Numbers go to infinity» (Federico, 5 years old)

«It's when one never stops doing maths» (Riccardo, 5 years old)

Answers concerning the straight line:

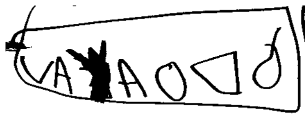
«It's a line that is straight» (Marco, 5 years old)

«When I'm hungry and I ask my grandpas and they don't give me anything, I have to wait till the cooking is ready»[44] (Riccardo, 5 years old)

Answers concerning the line:

«It's a line dividing the numbers» (Anna, 4 years old)

«The mathematical line is a meter» (Riccardo, 5 years old)



♦ It has been noted that starting from the lower secondary school, the texts produced by the students did not really assume the form of real TEPs, even if the motivation was to explain some concepts to one of their classmates. As a matter of fact, students tend to answer in a direct and concise way, adopting most of the time some supposed definitions, even when they have to address their explanations to pupils younger than them. Only in some specific cases, students decided to use the drawing for younger pupils, as a privileged form for making themselves understood and in so doing revealing severe misconceptions.

This phenomenon may be depending on the kind of topics dealt with, so specific and targeted or on the motivation chosen. In order to discover this, our future aim is to try to change the students' motivational aspect using a different strategy that has often proved itself extremely involving: "Pretend to be a teacher, a mother, a child of … years old

[44] Translator's note: in Italian the term straight line is "retta". This word is also used in an idiomatic expression and namely "dare retta" which means to obey to someone.

…" (D'Amore and Sandri, 1996; D'Amore and Giovannoni, 1997) to verify if the students' approach and consequently also their related written productions change.

♦ It has been revealed strong misconceptions belonging both to students and teachers concerning these mathematical objects and deriving from the visual images and the use of these terms in other contexts different from the mathematical one (for a better treatment of this aspect see paragraph 4.4). Researchers were considerably surprised by a particular aspect and namely the fact that formulating the questions as belonging to the mathematical field rather than the more specific one of geometry, has turned out to be misleading for students as well as for some teachers. Let us try to shed light on this latter aspect. In Italy, in primary school there is the most widespread attitude of creating at least two different subjects of study within the field of mathematics: geometry and the so-called "mathematics", meaning arithmetic. When activities are introduced in primary school one of the children's most common attitudes is to ask: *«Are we going to do mathematics or geometry?»*, *«Do we have to take the exercise book for mathematics or for geometry?»*. There exist, consequently for both children and teachers two separate worlds and according to the world chosen, there are different behaviours and attitudes to adopt: you are ready to do calculation if the field is that of "mathematics" or you expect to make a drawing if the subject is that of geometry.
Therefore to the question:
Imagine you have to explain to one of your classmates what a point in mathematics is. What would you say?
Children provided answers such as:
«To me the point in mathematics is an important thing. But it can mean three things to me:
a) The point in a large number like 143.965.270.890 in such a number the points are useful to be able to read the number;[45]
b) Someone, instead of × uses the point for example 144 · 5 = 620 in this multiplication the point is used as abbreviation;
c) Somebody else uses instead the point as a comma, for example194,6 or 194.6

[45] Translator's note: in the Italian language the comma is used to separate decimals instead of the decimal point, whereas the dot is used to separate large numbers with more than three figures.

To me the most useful of all is the first case» (10 years old)

The majority of children attending the last years of primary school make no reference to the point in its geometrical sense, as they believe it as not a part of mathematics, but they rather look for the use of point in the field of arithmetic. In the nursery school instead, as well as in the first years of primary school, no distinction has yet taken place between "mathematics" and geometry as a consequence of the teaching received and the choice is mainly for the geometrical field.

Teacher's comment: *«This* (referring to the child who wrote the above quoted TEP) *has correctly identified the point in mathematics. If he were asked in geometry then it would have been another matter, but as for mathematics he is right: the point is this one».*

Here as follows there is an attempt made by a child to join the two fields: geometrical and "mathematical":

«The point in mathematics is a very, very little spot that can become a very high number» (10 years old).

Among the few children of the last years of primary school that opt for the geometrical field, the largely discussed and pointed out misconceptions are to be traced:

«I would say that the point is a small element, round, the beginning and the end of a straight line» (10 years old).

Teacher' comment: *«If he is referring to the point in geometry, then what he says is ok, he explained it in detail, but the question was about the point in mathematics».* The teacher demonstrates misconceptions related to the point and distorted ideas of mathematics.

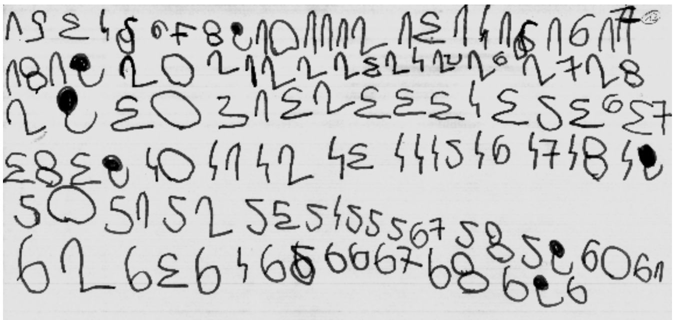
So being the straight line included in the arithmetic field, it becomes: *«The line of numbers»* and the line becomes: *«A symbol used in operations or in fractions».*

We believe these considerations are crucial, from a didactical point of view, to highlight the importance of the context that will be dealt with later in 4.4.

♦ Student produced TEPs do not show any evolution in the course of years concerning what is intended by mathematical infinity: the misconceptions underlined are always the same as the teachers' ones on which paragraph 3.7.1 is about: accepted, shared and confirmed by teachers themselves. Here some of most significant examples are provided:

«They are the angels that live for ever» (6 years old)

«I thought about numbers» (6 years old)



«Difficult works like doing 60 sheets of exercises in one day» (7 years old)

«To me mathematical infinity is an infinity of numbers and problems to solve. I'm not very good at it and so to me it never ends» (8 years old)

«To me there is no infinity in mathematics, because numbers in mathematics do not start and they never end» (9 years old)

«In my opinion mathematical infinity is like space, it never ends, numbers cannot end, combinations of numbers cannot end. But I think that the characteristics of mathematical infinity are not only numbers, they can be also shapes, and we know some of the many geometrical shapes. Infinity in mathematics is difficult to explain because mathematics is everywhere, even only to calculate the depth of a picture you need mathematics, to see how large a classroom is, you need to calculate the perimeter or the area.

There is one thing I've always asked myself: who's got evidence that mathematics is infinite? I know well that it is infinite but is there any evidence?» (10 years old)
«*It's a thing that goes on forever, it gets so far*» (11 years old)

«*I'm sorry but nobody has ever taught me what infinity is, I think it's something whose well defined quantity is not known*» (12 years old)

«*Infinity is something which has no end, e.g. numbers, after the last number you think there is always another one and you can get to count with no end (that is to say to infinity)*» (13 years old)

«*I would say it's nothing but it's everything at the same time. That is why is not possible to imagine it*» (first year of gymnasium)

«*Infinity in mathematics in an undetermined set, like that of natural numbers or of the points of a letter of the alphabet*» (second year of gymnasium)

«*It's a set whose elements are uncountable*» (third year of gymnasium)

«*Infinity, vast concept pertaining to the mathematical field and constituting a conceptual limit*» (fourth year of gymnasium)

«*Think about the greatest number you can ever conceive. Imagine to surpass it and to make it grow as much as you can: that number tends to infinity*» (last year of gymnasium)

♦ The TEPs obtained at the higher secondary school concerning primitive entities are mainly based on the use of supposed "definitions" proposed or accepted by the teacher, that have however in most cases not a proper and real meaning in the mathematical sense, or even if they have, are not thoroughly internalised and accepted by students.
Here are some examples:

«The point is a geometrical entity belonging to a set defined as space. It is indicated with capital letters» (second year of scientific high school).

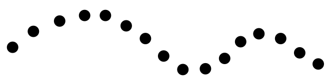
There is no actual explanation of the specific characteristics of mathematical point; in the first part of the above quoted affirmation the straight line, the plane, … may be included; nevertheless the teacher commented in the following way: *«I believe this is an acceptable definition of point, to me it's clear that the student understood what it is»*.

Another example:

«The line is an infinite set of points» (second year of scientific high school).

Teacher's comment: *«This is not good, I would not accept it because it doesn't say how the points are located»*, therefore the teacher assumes this statement as incomplete.

On this reply: *«I would say that it is an infinite set of points not necessarily in line»* (second year of scientific high school) the teacher commented in this way: *«This is ok, it's the one written in the book and that I asked them to write in the exercise book. I accept this one, because it makes clear that the points can be not in line»*. And yet this way of conceiving the line, could make one even think of a plane or points arranged in the following way:



A further example: *«The straight line is the set of points joined to one another so as to stay aligned»* (second year of scientific high school)

Teacher's comment: *«This is ok for me, I would accept it because it is clear that she understood what is meant by straight line, even if she uses some improper terms»*. The largely mentioned "model of the necklace" is clearly revealed by this way of conceiving the straight line.

Let us put an end here to such considerations that are still largely up for discussion. The aim of present work was to simply underline how TEPs are a useful device for researchers in order to obtain more detailed information concerning students' as well as teachers' knowledge and comprehension of mathematical concepts. We intend to publish the results of this research as soon as possible.

4.4. The discovery of the relevance of context: the point in different contexts

4.4.1. Where the idea of point in different contexts originates from

In consequence of the training course involving the teachers of Milan and the research in progress on the primitive entities of geometry, significant points for reflection have been emerged leading the analysis towards several directions: among these the investigation of the point used in different contexts. Starting with the *dependence* misconception (see 3.3), which was initially manifested by the teachers involved and according to which there are more points in a longer segment than in a shorter one, it has been highlighted how influential the figural model is, negatively conditioning in this case answers such as: to a greater length correspond a major number of points. This phenomenon is connected to the idea of straight line seen as "the necklace model" (Arrigo and D'Amore, 1999; 2002), being the latter proposed by the teachers as the suitable model to mentally represent the points on the straight line.

It is exactly in this model that the different convictions pertaining both to students and teachers can be traced back. Such convictions are related to the idea of point as having a certain dimension, though very small; such beliefs derive from the commonly used representation of point (for a better treatment of this specific topic see 4.5) that influences the image for this mathematical object.

The TEPs, collected during the research work concerning the primitive entities of geometry and reported in the preceding paragraph, have shown that the ideas of point manifested by students of whatever class-age are usually linked to the graphic mark left by the pencil or to their personal idea of point that can be traced in different contexts. These contexts are at times very far from the world of mathematics and students tend to directly transfer them in the mathematical field:

«A point in mathematics is a point with some numbers inside» (6 years old)

«I think that the mathematical point is a point that makes a mathematical sentence end and also makes numbers finish» (8 years old)

«It's not yet exactly known what a point is but to me it's just a point on a sheet that can have different dimensions» (9 years old)

«The point in mathematics is a little sign like that · or the question to solve.
The point in mathematics is also the one that is put on certain numbers for ex. 1˙000.
The point in calculators is considered a comma.
The point is also that of the equation for ex. 100 ×…= 200» (10 years old)

«One point in mathematics is important to get a good mark and be happy» (11 years old)

«The point in geometry is the reference point of a figure» (12 years old)

«It's a round point that forms the lines» (13 years old)

«A point is a part of an undetermined plane because it can have different dimensions, it constitutes the beginning, the end or both of a segment, a straight line, etc» (13 years old)

«A point is a small sign and is a fundamental geometrical entity» (first year of scientific high school)

«A point is the smallest element taken into consideration» (second year of scientific high school)

«A point is a geometrical entity, the smallest conceivable one tending to 0. Between two points there is always a third one» (third year of scientific high school)

«A minimum point · » (fourth year of scientific high school)

« • ⟵ *this is a point»* (last year of scientific high school)

«A geometrical entity infinitely small, that located on a Cartesian plane has 2 coordinates (x, y)» (last year of scientific high school)

As we have underlined in 4.3, these ideas are in some cases accepted and even shared by teachers of different educational levels (as to the idea of point showed by primary school teachers see 3.7.2).

In order to avoid that these convictions become the basis of incorrect models possessed by both teachers and students, it is therefore necessary to help subjects take the distance from the model of the segment as a "necklace" and from distorted visions concerning geometrical primitive entities, creating more suitable images allowing them, for instance, to conceive points without thickness. To this end, subjects should be supported in overcoming their previous knowing in order to build a new knowing. The questions arisen by this consideration were the following: When should this knowing be introduced and how? Which is the right direction to follow when introducing it? Where are the learning difficulties for these "delicate" mathematical objects mostly hidden?

4.4.2 Reference theoretical framework

Both teachers' and students' affirmations made us focus on the importance of context, following a situational and socio-cultural approach of social constructivism. According to this view knowledge, in particular the mathematical one, should:

- be the product of the student's active construction (Brousseau, 1986);
- have the characteristics of referring to a specific social and cultural context, though remaining in constant relation to other contexts;
- be the result of special models of cooperation and social negotiation (Brousseau, 1986);
- be used and further readjusted to other social and cultural contexts (Jonnasen, 1994).

On the basis of the above-mentioned considerations, we embraced an “anthropological” vision thoroughly oriented on the learning subject (D’Amore, 2001a; D’Amore and Fandiño Pinilla, 2001; D’Amore, 2003), rather than on the discipline, favouring “the relation and use of knowledge”, rather than the “knowledge”. This kind of approach is a philosophical choice of pragmatic nature (D’Amore, 2003). It is de facto the “use” that conditions the meaning and therefore the value of a given content and in this specific case, we would deal with the points used in different contexts. However this idea could be enlarged, in general, also to all geometry primitive entities and not only to them. In this perspective, we perceived the necessity and importance for the teaching activity to focus on the different contexts and forms of “uses” of an item of knowledge that determine the meaning of objects.

As a matter of fact, within the pragmatic theory we opted for as possible reference for the analysis, linguistic expressions, single terms, concepts and the different strategies to solve a problem, etc. assume different meanings according to the context in which they are used. And that is why they should be properly decoded, interpreted, selected and managed by the student. As stated by D’Amore (2003), according to this theory no scientific observation is possible since the only kind of possible analysis is “personal” or subjective and in any case circumstantial and not to be generalised. The only way is to examine the different “uses”: the set of the “uses” determines in fact the objects meaning. This however should not mean, according to our view, which the teacher has to address the learning activity towards a mere act of intuition or a student’s mere personal interpretation. Especially when dealing with mathematical concepts which entail the risk that the student’s intuitive image turns into a parasite model (D’Amore, 1999), as it has been largely proved in this research work. As stated by D’Amore (2003): *«One of the main difficulties is that in the idea of “concept” participate several factors and causes; to express it briefly and therefore also incompletely, it seems not correct to affirm for instance that “the concept of straight line” (assuming that it exists)* (the example could be obviously also generalised to the point) *is that inhabiting the scholars’ minds who have dedicated to this topic their life made of study and reflection; it seems rather more correct to affirm instead that there is a predominant so-called “anthropological” component that stresses the importance of the relations between* $R_I(X,O)$ *[institutional relation with that object of knowledge] and* $R(X,O)$ *[personal*

relation with that object of knowledge] (in this case D'Amore explicitly refers to those symbols and terms dealt with by Chevallard, 1992) (…) *Therefore, according to the direction I chose, in the "building" of a "concept" would participate the institutional part (the Knowledge) as well as the personal part (belonging to anyone who accesses the Knowledge and thus not only to the scholar but also the student)»* [our translation].
But what does traditionally happen for the geometrical primitive entities? In particular, in the cases of the point and the straight line? The feeling is that in the case of these mathematical objects, the subject in question is simply left to the "personal aspect" and its comprehension is simply due to an act of intuition.[46]
Unfortunately, this approach bears the risk of severely reinforcing in the students' minds some parasite models such as the so-called "necklace model" which turns out to be binding for future mathematical learning, with a predominance of the figural aspect on the conceptual one (Fischbein, 1993) and being source of intuitive distorted ideas that will constantly and continuously clash throughout the student's education pathway and even get in conflict with the other branches of knowledge. We believe, from a didactical point of view, that it is important to follow a pragmatic approach with a constant mediation activity on the part of the teacher, in order for mathematical objects and their related meanings to overcome the "personal" phase and become "institutional" (Chevallard, 1992; D'Amore 2001a, 2003; Godino and Batanero, 1994). In order to obtain this result, teachers should be aware of the "institutional" aspect of knowledge, but this phenomenon as we have observed in the preceding paragraph does no take place in the case of the geometrical primitive entities. Once again it has to be noted that the choice of leaving primitive entities to the mere "personal" aspect is not a conscious didactical choice, aimed at sidestepping very delicate questions related to the attempt to "define" such objects. This choice actually derives from the passive acceptance of well-

[46] In Borga et al. (1985) it is highlighted how Pasch, already in 1882, clearly called for the opportunity to avoid any recourse to the meaning of geometrical concepts and to refer only to mutual relations among them, explicitly formulated in axioms. Peano's contributions on the fundaments of elementary geometry are bound, not only ideally, to the works of Pasch. This leads to the creation of a hypothetic-deductive system, where primitive concepts, generally without any meaning, are considered as implicitly defined by axioms. Bertrand Russel had exactly this in mind, when he expressed following paradoxical sentence: *«Mathematics is the science, where one does not know what s/he is talking about and does not know if what is said is true»* (Enriques, 1971).

established misconceptions, which have turned into wrong models held by teachers themselves. G.: *«For thirty years I've been telling my children that the point is what you draw with a pencil, I cannot change it now. And after all, I'm convinced that this is the real meaning of a point. Why, isn't it like this anymore?»* (primary school teacher). Rather curious is the question posed by the teacher G.: «*Why, isn't it like this anymore?*» that highlights not only the false convictions related to the idea of point, but also the personal beliefs concerning the idea of mathematics [on teachers' personal view and their "implicit philosophies" see Speranza (1992), on the ideological beliefs see Porlán et al., 1996)]. It seems as if the idea possessed by the teacher exactly coincided with the *didactical transposition* of mathematical knowledge that is usually proposed by the *noosphere* (see 2.4). For the teacher G. there is therefore no distinction between a mathematical concept and consequently its transposition deriving from a particular didactical choice: these two aspects are one single thing to her/him.

The direction we wish teachers would adopt, for themselves as well as for their pupils, follows a "pragmatic" approach according to which the notion of the object meaning (knowledge in general, mathematical knowledge in the specific case) is not more interesting than that of relation, "relation to the object". The latter should however be consistent with the basics of the reference discipline. For more than 2000 years, mathematicians have been trying to introduce the linguistic device of simply using words such as "point", "line", "straight line", "surface", "plane", "space" without providing an explicit definition, basing themselves on the hypothesis according to which more or less all the people that use them (children included) have an idea of their meaning: as a matter of fact, they will learn what they are just by using them. But is it sensible to consider this strategy a winning one, after having analysed children's and most of all teachers' affirmations? To avoid severe misunderstandings such as those revealed in these chapters, teachers should be firstly aware of the "institutional" meaning for a particular mathematical object that they intend to implicitly define, secondly they should convey the use of such objects into a critical and confident way so as to remain consistent with respect to the related discipline.

Also Fischbein's considerations embrace this view (1993): *«High school students should be made aware of the conflict and of its origin, so as to keep the emphasis in their minds on the necessity to base themselves, when dealing with mathematical* *reasoning, mostly on formal constraints. All this leads to the conclusion that the process of building in students' minds figural concepts should not be considered as a spontaneous effect of plain and common geometry courses. The integration of the conceptual and figural properties into some unitary mental structures, with the predominance of the conceptual constraints on the figural ones, is not a natural process. This should be a major systematic and continuous concern of the teacher»* [our translation]. The considerations of Fischbein referred to high school students, should be in our opinion transferred also to all the other educational levels, or better still, we do believe it is essential that teachers pay this didactical attention already since primary school.

In a fourth grade of a primary school of Rescaldina (Milan) after having asked children: *«How big is a mathematical point?»*, we obtained the following answers: *«A point could be big, it depends on the felt-tip because it has different sizes»; «To me the point could be a very, very big thing or microscopic because it is like a circle of different sizes»; «It depends on how you make it»; «To me the point is big according to what you compare to it. If you compare it to an atom, it's very big. If you compare it to a wardrobe, very little»*.

Moreover, to the question: *«How many points are there in a plane?»* it has been answered in the following way: *«It depends, if the little points are very close to one another there could be 100, even more»; «It depends on how many of them we want to make, we can make them very close and they become quite a lot. If we want to make them distant there are a few»; «It depends on the plane to me, the bigger it is the more there can be»; «In my opinion they can be infinite, in this plane they can be infinite, because a little point always finds some room»; «No plane is made of points, this sheet has been printed as a whole, it is not made by little points»; «According to me there can be more little points if the plane is large, it depends on how we draw them. There can be a big one and many little ones»*.

What has emerged from all these reflections is the awareness of the necessity of not taking for granted the ideas pertaining to both teachers and students about infinity and the other primitive entities of geometry. Another fundamental aspect that has emerged from this research concerns the fact that it is essential, from a didactical point of view,

that these ideas are conveyed towards the "institutional" facet, showing the featured properties and the relationships that connect them.
In addition, we do consider as crucial that the teaching activity starting from primary school should focuses on several aspects such as: the importance played by the different contexts which is bound to the different "uses" of the knowledge on the part of students. It has therefore been identified as necessary for teachers who participated in the training course we organised, to put together with the researchers some activities to be practiced with their students. Some interesting proposals have been developed all regarding this topic and addressed to both primary and lower secondary school pupils. The choice was to start with these educational levels since as clearly revealed by the outcomes collected, misconceptions concerning the several geometrical entities are already to be spotted in primary school. It is about naive ideas most of the time linked to different contexts but which are nevertheless nonchalantly transferred to the mathematical world especially for the linguistic analogy.
Our educational and didactical goals regarding the structuring of activities with teachers are not limited to the students' acquisition of knowings, skills and competences but are targeted to develop one's own individual "use" of knowledge. The activities are meant not only to teach something but also to teach students how to manage one's knowledge and consequently to allow them to be able to make the right choices when confronted with a complex amount of information or a problematic event. All this in accordance with Gardner's words (1993): *«One of the basic targets when educating to understand or teaching to understand is: to train the child's skill to transfer and apply the acquired item of knowledge to various situations and contexts»* [our translation].
With regard to the different contexts, already pointed out by students' TEPs, and in particular as to the item of point, it suffices to look up in any Italian dictionary as for instance "Il Grande Dizionario della Lingua Italiana" (The Great Dictionary of the Italian Language) published by UTET, to find nearly 40 different meanings for the word "point". Additionally, if you look up for idioms and common expressions, at least 200 different contexts for the use of this term are to be found. Among all these, there is obviously also the definition of mathematical point but from a didactical point of view the latter is usually left to intuition, dealt with only successively and almost neglected in

some respects when compared to the other uses. The main effect is an exclusive sedimentation of all the other meanings for the term in question.
As a matter of fact in primary school, the difference between the mathematical point and the point used in other contexts (e.g. the figural one) is seldom highlighted. As a main consequence, when the point is finally dealt with in a more sophisticated way during high school, it is too late for students: all other meanings prevail and as a result the idea of a new meaning is unacceptable contradicting those already designated up to that moment (we recall the distinction between image and model reported in 2.2).

4.4.3 A provocation

In an article due to be published: *"The discovery of the relevance of context: the point in different contexts"* (Sbaragli, 2003b), we started off the treatment in question with a provocation we held as particularly significant. The reader is firstly asked to observe the following two figures and subsequently to answer the following questions borrowed from the fundamental work on figural concepts of Fischbein (1993):

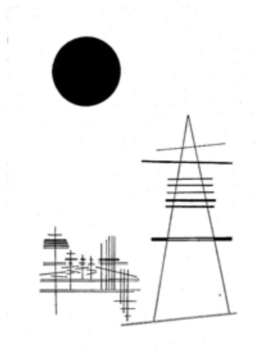
3a 3b

«In 3a there are four intersecting lines (point 1). In 3b, there are two lines (point 2). Compare the two points 1 and 2. Are these two points different? Is it one of them larger? If so, which one? Is it one of them heavier? If so, which one? Have the two points got the same shape?».

The experiment goes on with a series of reflections and provocations such as:
- by answering the preceding questions, what context would the reader have been thinking of?;

- how would Seurat, the "pointillist" painter have reacted to these stimuli? To Seurat, would the point have been conceived as an abstract concept or would its dimension have assumed a great importance? Can we affirm that Seurat did not succeed in “conceptualising”[47] as intended by Fischbein? Let us reflect on the effect that the illustrated below Seurat’s painting would have had if it were “represented” with a conception of the point only as a position but dimensionless;

- what would Kandinsky (1989) have thought of the questions formulated by Fischbein, since he called one of his paintings: “*The subtle lines stand up to the heaviness of the point*?”;



- what would the Australian Aboriginals think of the point since they use it as the basis to represent every single image?;

- in addition it is reported the answer given by a land surveyor of “the old guard” with more than 40 years of experience to the questions posed by Fischbein: *«It’s obvious that a point is… a point, but in drawing it changes according to which kind of pen-nib you*

[47] We are well-aware that we are using the rather complex and delicate term of *conceptualisation* in a rather simplicistic way: *«Entering this adventure leads one to become at least aware of something that is to say that the question: What is it or how does conceptualisation occur? remains basically a mystery»* (D’Amore, 2003). We owe our choice to the use that Fischbein (1993) made of the term in question in the example of a point individuated by the intersection of different segments, which we have transferred in our considerations.

use and so a point can get larger or smaller. If you use different pen-nibs or if you go over it with an ever-increasing number of lines, the point becomes visually bigger». And it is legitimate to ask oneself if the surveyor has still to conceptualise or if the conceptualisation depends on the context.

The above-mentioned provocations offer a possibility to reflect on the context importance which seems to be even more evident thanks to the accurate analysis of Fischbein's (1993) above-mentioned article, where the experimental situation of the two points is introduced: one identified by the intersection of four segments and the other by two segments. This research was addressed to subjects whose ages are included between 6 and 11 years old and who had been asked the above-mentioned questions with intentional ambiguity. Fischbein himself affirms that these questions could have been considered either from a geometrical point of view or from a material (graphic) one. The research aim was to reveal the evolution linked to the age in the subject's interpretation and the presumable appearance of the figural concepts (point, line).
As Fischbein states, the results showed a relatively systematic evolution of the answers from a concrete representation to a conceptual-abstract one. But are we really sure that the conceptual interpretation is exclusively the abstract one, or does this depend on the context? It has been with certainty acknowledged that in the mathematical field the conceptualisation of point takes place when the subject is able to make abstractions and conceive the point as a dimensionless entity. However, in the question the mathematical point was not mentioned therefore the attention could be focused on any kind of point: as meant by the land surveyor, the painter, the Aboriginal, the designer, the musician, the geographer, ... In our point of view a surveyor of "the old guard" who is used to draw with pen-nibs but is not able to distinguish the different sizes of a point, for instance a point obtained using a pen-nib 0.2 or 0.8, has not succeeded in conceptualising in her/his own field. Hence, conceptualisation depends on context, and therefore it seems essential to us that Fischbein's question should evidently clarify the reference context.
It is legitimate to wonder: is it true all the time that the graphic perception is less conceptual than the abstract or does this depend on the context of reference? In some specific contexts, the graphic for example, could the figural aspect be considered more

conceptual than the abstract one? In our opinion, to note the different dimensions of two points requires a particular sensitivity, some keenness and a certain degree of "conceptualisation" which proves fundamental in certain fields. It clearly emerges therefore the necessity for teachers to be aware of the reference field when posing questions to students and even introducing a particular context to them, in order to make sure that the students' unexpected and unhoped-for answers, are not a consequence of the interviewee referring to a different context from that envisaged by the interviewer. In some respects it would be as if you expected to obtain solutions for an equation of a certain set without explicating to which set should the solutions belong to.

In this perspective it could be risky if what Fischbein hopes for (1993) is generalised to every field. Namely that the point is to become disconnected from its context so as to prepare the concept of geometrical point. As matter of fact, we believe as fundamental that students are aware of the context in which they move and that they have a conception that is consistent with the related context. At the same time, we do hope that students are also able to distinguish and therefore employ the different "uses" within the same context and even within all the different contexts.

Returning to infinity. When the researcher, even if considered an expert of mathematics, posed teachers the following question not providing the context:

R.: «What is infinity for you?» s/he collected the following answers, all pertaining to other fields different from the mathematical one:

A.: « Leopardi's Infinity» (primary school teacher)

F.: «The space or better still, the universe surrounding us» (primary school teacher)

C.: «Your e-mail address»[48] (primary school teacher)

G.: «The infinite love I feel for my daughter» (lower secondary school teacher)

A.: «The trust in God I feel inside» (lower secondary school teacher).

It is definitely true that, when dealing with infinity even when the context is explicitly provided to teachers: i.e. the mathematical one, the collected answers are not consistent with the context taken into account (see 3.7.1). Nevertheless in most cases, some generically and completely unexpected answers like those mentioned above are averted.

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[48] Translator's note: www.infinito.it is an Italian webmail provider.

The considerations regarding the point as seen in different contexts can be also valid when introducing to students the concept of infinity during their educational career, making clear reference to the use of this term in different contexts: philosophical, religious, mystic, linguistic, mathematical, …

Back to the point. As a consequence of these considerations concerning the point used in different contexts several activities have been planned. At the moment the teachers from Milan are experimenting these activities with primary school students. The experiment in progress would successively turn into a proper research activity as the future intention is to collect the results gathered during these years of study in this specific field, observing the didactical repercussion originating from this "new" didactical transposition. Our attempt would therefore be to survey the transformation teachers' and students' misconceptions will undergo, what kinds of images of the different mathematical concepts they will possess (in particular of the mathematical infinity and of the fundamental entities of geometry) and finally at that "point" what their idea of mathematics will be like. The specific treatment of these activities is outside the scope of the present work, our focus is on the "use" of the word point in different contexts: in music, language, geography, arts, drawing, mathematics, …, analysing in depth the characteristics for each context. For example when talking about painting, we highlight the aspect that we are dealing with a point with some peculiar characteristics such as size, shape, weight, colour, …, all depending on the drawing tools; the point in question has a different meaning according to what the artist intends to express. In the world of mathematics instead, the focus is on Euclid's choice of considering the point as dimensionless. Euclid assumed this principle as "starting point", the "primary rule" of the great game of mathematics which children are invited to play. But every game in order to be called a real game and to allow active participation needs the acceptance and attendance of some of its "rules", which in this specific case will lead participants "to see with the mind's eyes". The ability of accepting, respecting and sharing the others' choices and to make explicit the characteristics pertaining to different contexts, are in our opinion fundamental elements. These elements allow children to detach themselves from the physicality of the points that they normally draw, in order to accept a different world, that of mathematics with its own "rules" different from everyday rules.

To enter the world of mathematics and to accept the "rules" of the game represent one of the main goals we try to achieve during our mathematics training courses addressed to teachers belonging to any educational level. The aim is to make the topic of infinity, so distant from the finite one, more accessible. What has resulted from the research study conducted in these years is a somewhat wider scope that focuses more generally on the basic idea that teachers have of mathematics before starting to work on their misconceptions related to infinity.

G.:«Now I understood what mathematics is, nothing can surprise me anymore» (primary school teacher).

Since this work is addressed to teachers, our intention is to point out how fundamental it has turned out to be for teachers themselves to cooperate with us in the planning of students' activities, because this served as an opportunity for them to reflect on their teaching method. Here we report two extracts from a "logbook" some of the teachers wrote during this experience that well express the feelings of two primary school teachers:

L.: «We primary school teachers are skilled designers of techniques and didactical material that are even regarded with admiration for their ingenuity. But sometimes we fall too deeply in love with our "expedients" and we use them with too much confidence. It's true, to talk about mathematics we need some models, we have very young students, and therefore we always think it necessary to materialise concepts for them. In this way we do not realise the unintentional trickery in which we get our students involved in: through our material didactics they get convinced that mathematical objects are real objects and that it is the way they should be treated. Me too in my primary school teaching history I happened to turn to material didactics and I spent time and energy to make it even more effective. But at a certain point of my teaching career, I realised that manipulating, doing, building, do not necessarily lead to mathematical knowledge. (…) I finally understood that the concept does not exist in the mind, unless you are not able to imagine things; the manipulating hands are of no help in building mathematical concepts. (…) I was *satisfied and it seemed to me that everything was working well. But it was just an impression due to my mathematical ignorance, I don't blame myself, I'm just stating something.*

Two years ago at a meeting in Castel San Pietro I heard Silvia Sbaragli talking about mathematical infinity: potential and actual infinity: I entered a new world. Silvia Sbaragli was talking and I was reevaluating myself as a maths teacher: too much confusion and many things taken for granted.

I ploughed my didactical soil well and I was able to immediately grasp the message contained in the speaker's words: a fruitful and fertile thought at the right moment.

That moment was crucial for my teaching experience: that was the start of a cooperation that enlightened and is still enlightening my profession (…) helping me a great deal to organise my still fragmented, confused and incomplete ideas.

I finally understood for good and all what is the right approach to do mathematics. I tried to make my children see the light in the same way as I did: I think I made it. I'm happy with the transformation that allowed me to peep into the world of mathematics with the correct outlook that washes away my ignorance in one go. This approach won't let me make big mistakes and omissions with my children and it will allow me and my students to enjoy such a rich knowledge in which and with which the human mind can play and have fun».

C.: «I have to admit that, many times, with the intention of facilitating the learning and of fulfilling the students' need for clarity and concreteness, we would rather favour our need for confidence trying to find contacts with or evidence in the real world. All this reassures us, we have a major control over it, it's there, its' visible, you can't make mistakes. To face the world of the non-sensible is scary and in every way we try to transfer the objects and rules of Mathematics to the real world, for even mathematical concepts need to find a real justification to exist. It was very challenging to work on representations, to understand that they are useful, to give shape to something which is not concrete. However, these representations can also be weak because to present concrete models in mathematics does not guarantee correct learning all the time, but this method rather conditions and even hinders it. I *remember Elena who after having reasoned, talked and reflected once again on the mathematical point told me: "Yes I don't see it, but I do understand it"».*

The only way to comment on these two so meaningful reflections is through a short, though effective Japanese proverb: *"To teach is to learn"*. This proverb is also valid for us researchers every time we come into contact with the fruitful world of didactics.
Our aim seems to be at least partly achieved: to substantially affect teachers' convictions in order to successively and indirectly affect the students' convictions.

At this point, it should be legitimate to ask oneself the reason why a thesis centred on primary school teachers' convictions on mathematical infinity has focused this chapter almost entirely on the point and its didactical transposition. As a matter of fact, though bearing always in mind that the main subject matter is that of infinity, these were the aspects we encountered along our pathway over the years. It has been proved impossible to deal with the mathematical infinity in the geometrical field without making any reference to the primitive geometrical entities. The majority of the wrong beliefs concerning infinity originate from some misconceptions related to these mathematical objects. Furthermore, we are convinced that these proposals constitute a new way of working in class, more flexible, closer to "our" idea of mathematics, capable of getting both teachers and students closer to the concept of infinity.

4.5 A further fundamental aspect: different representations of the point in mathematics

Another fundamental and delicate aspect connected to the preceding treatment concerns the different representations of the point in mathematics. The choice was once again to investigate the point but the following reflections can obviously be applied also to all of the other primitive entities of geometry, and not only to them. Where do the following considerations come from? In these chapters we have repeatedly shown that the majority of difficulties and misconceptions especially those regarding mathematical infinity mostly depend on the visual representation provided for geometry primitive

entities. But what kind of representation is this provided by teachers and accepted by the *noosphere*? As for the primitive entities of geometry, is there a tendency to provide one single representation or rather several, even adopting different semiotic[49] registers? How does the representation provided by teachers influence students' convictions?
Before answering these questions, let us first analyse in depth the reference theoretical framework.

4.5.1 Reference theoretical framework

As for this treatment we referred to Duval who highlighted the fact that in Mathematics the conceptual acquisition of a piece of knowledge should necessary firstly go through the acquisition of one or more semiotic representations. The issue of registers was introduced in the famous articles of 1988 (a, b, c); and in the following work of 1993. Therefore borrowing one of Duval's affirmations: there is no ***noethics*** (conceptual acquisition of an object) without ***semiotics*** (representation by means of signs), we made the following considerations.

As D'Amore stated in his book of 2003:

- every mathematical concept has references to "non-objects"; therefore conceptualisation is not and cannot be based on concrete reality-based meanings; in other words in Mathematics broad references are not possible;
- every mathematical concept is forced to make use of representations, as there are no "objects" to put in their place or to recall them; therefore conceptualisation should go through representative registers that according to various reasons and especially if having a linguistic nature, cannot be univocal;
- in Mathematics we usually talk more of "mathematical objects" rather than "mathematical concepts" as Mathematics would study the objects rather than concepts; *«The notion of object is a notion that we are forced to use right from the moment in which you question yourself on nature, on conditions of validity or on the value of knowledge»* [our translation] (Duval, 1998).

[49] Taking as a starting point the framework of Duval that we are going to describe when talking about a "register of semiotic representation", we refer to a system of signs enabling us to fulfil the functions of communication, treatment, conversion and objectivation.

As revealed by this latter point, to Duval the notion of concept assumes a secondary importance, whereas priority is attributed by the Author to the couple (sign, object). Vygotskij quotes the importance of the sign also in a passage from 1962, mentioned by Duval (1996) where it is stated that there is no concept without sign. If we assume this as true, the didactical consequence is to pay special attention to the sign choice, or better still to the sign system representing the mathematical object selected to be taught to students. The above-mentioned attention is often underestimated or taken for granted. D'Amore (2003) reported Duval's thought stating that there is a group of didacticians that tend to reduce the *sign* to the *conventional symbols* that identify directly and singularly some objects but that can turn into misconceptions since they become the unique representatives of a given register. We believe this is exactly what happens with geometric primitive entities. The point is perceived as and referred as the unique representation that is commonly provided by the *noosphere*: a dot on the blackboard; the straight line as a continuous line, of variable thickness, straight and formed of three initial dots and three final. No one dares to take the distance from these representations. Teachers and indirectly also students perceive them as the only plausible and possible representations. As a consequence, the point is associated with the unique image provided for it: a "round" sign left on a sheet of paper, of variable diameter and with a certain dimension.

A.: *«I don't think there are other ways of representing the point other than that of gently touching a sheet with a pen»* (primary school teacher)

R.: «Can't you think of anything else? What do you do with your students?»

A.: *«If you ask me how I represent it, in order to produce it I make a little sign on the blackboard but if you mean what I say when describe it, I usually tell them to think of a grain of sand or of salt».*

Among the models selected by teachers to represent the point they are all the time "round like" images because misconceptions concern also the idea that the shape of a point is "spherical":

R.: «According to you, is it legitimate to represent a point as a star?»

A.: *«As a star? Of course not, what kind of question is this? A point is not in the least a star!»*

R.: «Why, is the point this: •?»

A.: «Yes, the point is spherical, it's not definitely star-shaped».

4.5.2 A particular case of Duval's paradox: the primitive entities

Let us analyse the famous Duval's paradox (1993) (quoted in D'Amore 1999 and 2003): *«(…) On the one hand, the learning of mathematical concepts cannot be other than a kind of conceptual learning and, on the other hand, it is only by means of semiotic representations that an activity on mathematical objects can be carried out. This paradox could represent a real vicious circle to learning. How is it possible then that learners should not confuse mathematical objects with the related semiotic representations if the only representations they come in contact with are the semiotic ones? The impossibility of a direct access to mathematical objects, beyond every semiotic representation, makes confusion almost inevitable»* [our translation]. This confusion is magnified in the case of primitive entities, as these are most of the time simply left to an act of intuition. Furthermore, the learning of these mathematical objects is made more complicated by the decision of providing the students only with some vain and univocal conventional representations, which are therefore blindly accepted because of the *didactical contract* constituted in class (see 2.1) and of the phenomenon of *scholarisation* (see 2.4).

The paradox continues as follows: *«And, on the contrary, how could they* (learners) *acquire the mastery of mathematical treatments, necessarily bound to semiotic representations, if they do not already possess a conceptual learning related to the represented objects? This paradox is even stronger if both the mathematical and conceptual activity are considered as one single thing and if semiotic representations are considered to be of minor importance or extrinsic»* (Duval, 1993). [our translation]

We take into account the latter paradox with reference to the mathematical point: we wish as teachers that students would conceive the mathematical point conceptually, considering it as a dimensionless object, although it is only by means of semiotic representations that an activity on mathematical objects can be carried out. The learners will surely tend to confuse mathematical objects with their semiotic representations, but this phenomenon may take place especially when the provided representations are almost exclusively univocal and conventional. For instance in the cases of the point and the straight line and when teachers do not perform a mediation activity between the "personal object" and the "institutional object" (Godino and Batanero, 1994, Chevallard, 1991). So when dealing with primary school children, what is the right strategy to talk about the point without drawing it in only one way on the blackboard? How is it possible to be free from this representation that is fixed and stable transforming itself into an erroneous model for both teachers and students? How can students possibly acquire some mastery of mathematical *treatments*[50] and *conversions*[51] linked to semiotic representations when what is provided for geometrical entities is basically one and only one conventional representation?

Difficulties are not only due to the impossibility for students of having from the beginning a conceptual knowledge of mathematical objects but are nevertheless magnified by the revelation that most of the times even teachers do not possess this conceptual knowledge. Therefore they tend to confuse the mathematical object they intend to explain to their students with its representation (see chapter 3).

The constant and continuous cooperation over the years with teachers has quite often revealed that some of them tend to attribute the existence of a mathematical object to its possibility of being represented by means of images or concrete objects:

S.:«To me infinity doesn't exist, you cannot in the least see it» (primary school teacher)

R.: «Why can you see the number 3?»

S.: «Of course, you just need to show 3 fingers, write 3, show 3 objects. But how can you do it with infinity?»

R.: «So according to your way of thinking, you just need to write this: ∞»[52]

S.: «No, that's different you can't even touch it with your fingers. The 3 exists to me and infinity not».

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[50] By the word *treatment* we refer to a cognitive activity, typical of semiotics, which consists in the passing from a representation to another within the same semiotic register.

[51] By the term *conversion* we mean a cognitive activity, typical of semiotics, which consists in the passing from a representation to another, in different semiotic registers.

[52] Rucker (1991) presents a curious observation: the symbol ∞ first appeared in 1656 in a treatise by John Wallis on conical sections, *Arithmetica Infinitorum* (see: Scott, 1938). It was soon spread everywhere as the symbol for infinity or eternity in the most diverse contexts. In the 18th century, for example, the symbol for infinity appeared on the tarot card of the Fool. It is interesting to note that the cabalist symbol associated to this card is the Jewish alphabet letter aleph ℵ .

These affirmations underline once again the false convictions teachers have of mathematical objects and more in general of mathematics itself.

As Fischbein (1993) affirmed it is important to underline that: *«In empirical sciences the concept tends to approximate the corresponding existing reality, whereas as far as mathematics is concerned it is the concept that, through its definition, dictates the properties of the corresponding figures. This will lead to a fundamental consequence. The entire investigating process of the mathematician can be mentally carried out, in compliance with a specific axiomatic system, whereas the empirical scientist has to, sooner or later, return to the empirical sources. To a mathematician, reality can be source of inspiration but in no case a research object leading to mathematical truths and by no chance a final example to prove a mathematical truth. The mathematician, like the physician or the biologist, makes use of observations, experiments, inductions, comparisons, generalisations, though her/his research objects are purely mental. Her/his laboratory is, on the whole, confined to her/his mind. Her/his pieces of evidence are never empirical by nature, but exclusively logical»* [our translation].

Duval's paradox is even more evident if teachers let the concept coincide with its related representation and if they have never reflected on the topic and structure the didactical transposition taking into account the meaning and importance of semiotic representations.

The considerations collected so far are once again strictly connected with the issue of *figural concepts* as illustrated by Fischbein (1993): *«A square is not an image drawn on a sheet of paper; it is a shape controlled by its definitions (even if it can be inspired by a real object). (…). A geometrical figure can be thus described as bearing some intrinsic conceptual features. Nevertheless a geometrical figure is not a pure concept. It is an image, a visual image. It possesses some characteristics not belonging to usual concepts, that is to say it includes the mental representation of spatial properties. (…). All geometrical figures represent mental constructions that simultaneously possess both conceptual and figural properties. (…). In geometrical reasoning the objects of study and manipulation are therefore mental entities which we call figural concepts and that mirror spatial properties (shape, position, size) but that also possess, at the same time, some conceptual properties such as: ideality, abstractness, generality, perfection. (…). We need some intellectual effort in order to understand that the logic-mathematical* *operations manipulate only a purified version of the image, the spatial-figural content of the image. (…). Ideally, it is the conceptual system that should completely control the figures' meanings, relationships and properties. (…). But in general the evolution of a figural concept is not a natural process. As a consequence, one of the main tasks of the didactics of mathematics (in the field of geometry) is to create some didactical situations that would systematically require a close cooperation between the two aspects, up until their fusion into unitary mental objects»* [our translation]. It is right on the basis of these considerations that we are working together with three teachers from Milan on the creation of suitable activities. These experiences are aimed at valorising and highlighting, as far as the primitive geometrical entities are concerned, several semiotic representations of different registers, letting students' imagination free and helping them to detach themselves from some false stereotypes. Such stereotypes are already accepted as conventional and in so doing they reach the *knowledge institutionalisation* that leads to the institutionalised knowledge of the various mathematical objects. It is exactly letting both teachers and students get rid of given and stable representations that constitute "wrong models" (see chapter 3) that it is possible to build an idea closer to the mathematical object to learn. Our aim is to try to inculcate first in teachers and then in students, the idea that the nature of a concept is independent of the kind of representation selected in order to represent it. The issue of infinity would be consequently easier to accept.

4.5.3 Some activity proposals

The activities structured together with the primary school teachers from Milan are intended to make students perceive the "weakness" characterising the mathematical representations, so as to make students grasp what lies beyond a specific concrete model (not only figural) and attribute a conceptual meaning, from a mathematical perspective, to the different images. In this way students will be able to see with "the mind's eyes" and find out the right connection between all different aspects by means of the use of various language codes: verbal, sign, figural, mental, ... In particular, we maintained it necessary to structure activities targeted at the formation of figural concepts as intended by Fischbein (1993).

At the same time, our aim is to enable students to "dare" to invent different representations for the same concept. This will allow students to perform *treatments* that is to say to pass from one representation to another within the same semiotic register for the same concept, as well as to perform *conversions* between one representation and another using different semiotic registers. *«Further more: knowledge "is" the intervention and use of signs. So, the mechanism of production and use, subjective and inter-subjective, of these signs and of the representation of the "objects" of the conceptual acquisition is crucial to knowledge»* [our translation] (D'Amore, 2003).
In order to do this, students should be able to validate[53] and socialise their choices defending their opinions with the appropriate argumentation but they should be even able to accept the other's motivations, so as to create some shared and conventional representations within the class. These representations will be at a later stage compared to those selected by the noosphere. As D'Amore affirms (2003): *«During the learning process of Mathematics, students are faced with a new conceptual and symbolic world (representative in particular). This world is not the result of a solitary construction but the outcome of a real and complex interaction with the members of the micro society which the learning subjects belong to: their classmates and teachers (and the noosphere, at times in the background, at times in the foreground)* (Chevallard, 1992). *It is thanks to a constant social debate that the learner becomes aware of the conflict existing between "spontaneous concepts" and "scientific concepts". Teaching is not a mere attempt to generalise, magnify, and develop in a more critical way the students' "common sense", teaching is about a much more complex action, ... Therefore learning seems to be a kind of construction subordinate to the need for "socialising". The socialising activity takes place thanks to a means of communication (language for instance) and that in Mathematics will be influenced by the symbolic mediator's choice, i.e. the representation chosen register (or imposed, by several factors or even simply by the circumstances)»* [our translation].

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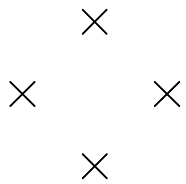
[53] *Validation* is the process adopted and followed to reach the conviction that a specific obtained result (or the construction of an idea) responds exactly to the requisites explicitly brought into play. This can happen when a student proposes her/his conceptual construction to the others, explicitly in a communicative situation, focusing her/his attention on the transformation of a piece of personal and private knowledge into a communicative product and defending her/his opinion (or solution) from sceptics (that is to say validating her/his reasoning).

Why is the point in mathematics represented only as a "round" sign? Does its "round shape" constitute one of its specific mathematical properties?
A point in mathematics should be an a-dimensional entity, therefore its representation, necessary to refer to this concept, can be of any kind since it should not stick to any specific characteristic but the one of not being represented. In our opinion, the varieties of representations allow students to purify the object from those features that are not proper to it: size, weight, colour, dimension of its diameter, … From a didactical point of view, it is sufficient to establish a position in the space to identify a point whereas as for the representation of this position, it will be the children's task to use their imagination and according to their wish and taste, they might represent the mathematical point as the end point of an arrow, the intersection of a cross, the centre of a little star, …
The creation of the wrong model deriving from the point's univocal given representation constitutes an analogous situation to what happens in nursery school when the teacher tries to make the pupil learn to recognise the square shape always providing the same model for it: red, made of wood, with a specific extension and thickness, … The child would believe that the square's characteristics are exactly those of being: red, made of wood, with that specific dimension. In order to purify the concept provided for the square from features that do not characterise it, students should be given the opportunity to "see" different images acting in different contexts which will allow them to attain those characteristics of ideality, abstractness, generality, perfection.[54]
«The point exists only in my mind, it's like a little ghost. A little ghost can pass through infinite little ghosts» (Luca, third year of primary school).

One day entering a third year class where teachers were adopting the above-mentioned approach, children asked the researcher the following question:
«Try to understand what it is». And they drew on the blackboard the following image:

[54] In this respect Locke (1690) asserted: *«As for the general terms (common nouns), … the general and the universal clearly do not belong to the sphere of real things, but they are inventions and creatures of the intellect made up for its own purposes, and they just pertain to the signs, be they words or ideas»* [our translation].



The answer was: *«Square»*.[55]

Successively they posed the following question:
«What is this?»

And the researcher's answer: ***«Two points»,***
children reacted in this way: *«No, try again»*
R.: «Is it the segment that has those two points as end-points?»
B.: «No, try once again. C'mon you can get it!»
R.: «The straight line passing by those two points»
B.: «Well done Silvia, now we can draw it»

Children demonstrated to have chosen an alternative way also to represent the straight line. These proposals imply students' "personal risk", their commitment and their direct *involvement* in learning manifested with the breaking of the didactical contract (see 2.1):

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55 A propos of this, Speranza (1996) wrote: *«Let us go back to the Ancient Greece. In The Republic Plato wrote: «Those who deal with geometry… make use of… visible figures and they reflect on them but in fact they think of what they represent, reasoning on the square itself and on the diagonal rather than on the drawings…». (…). Plato spoke about "square itself", of which the drawings of the square are "images". This recalls the "myth of the cave": the true reality is that of general ideas, that exist by themselves: sensible things are just "images" we can see, they are like shadows of the real entities which are outside projected on the back of a cave»* [our translation].

«The need for such a break can be summed up by the following aphorism: believe me, says the teacher to his pupil, dare to use your knowledge and you will learn» (Sarrazy, 1995).

If it is true what Duval claims (1993) that the creation and development of new semiotic systems is the symbol (historical) for the progress of knowledge, we intend, by means of these activities, to activate such a progress within the classroom adequately, considering all three cognitive activities "proper to semiotics": *representation, treatment, conversion*. In particular, to conversion we attribute a major position, according to the grounds provided by D'Amore (2003) and even before by Duval (1993). Among these reasons we consider it fundamental that such a specific cognitive activity enables to define some independent variables concerning both the observation and the teaching activity. This will favour the "conceptualisation" which is actually activated, or even simply sketched, through the coordination of two distinct representation registers.

Other activities structured by teachers are focused on the main differences between the finite and the infinite field as to avert that the infinity concept is banally reduced to an extension of the finite. The treatment of the issues concerning infinity requires the development of different intuitive models at times even opposed to those used when dealing with the finite. According to our point of view, in order to avoid the creation of misconceptions regarding this topic, a proper education centred on the handling of infinity sets should be started already in primary schools. This approach would allow students to begin observing the principal difference between the two fields. The goals we have set when structuring these activities are mainly intended to enable students to: grasp the real essence and charm of mathematics; distance themselves from the everyday routine regarding the finite; transfer biunivocal correspondence from finite to infinity; sense the meaning of infinity both in the numerical and geometrical field.
In this respect, we reported some of the statements given by a lower secondary school teacher as the result of having introduced to her/his students some of the biunivocal correspondences between infinite sets (in particular, between the set of natural numbers

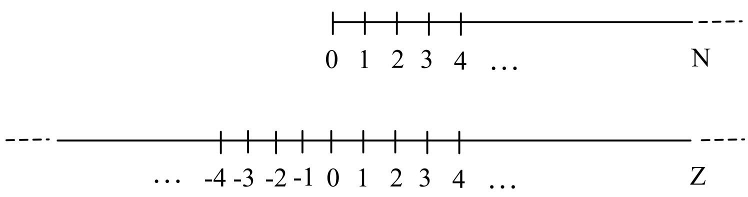
and that of even numbers, natural and odd, natural and whole) learned during the training course:

C.: «I feel a kind of relief since I discovered I can position numbers in different ways. The fact of knowing that it is possible to play with the order of numbers is amusing for me as well as for my students. School tends to be very rigid, you never depart from the norms».

Here the teacher in question was referring to the possibility of finding a different order from the "natural" one for infinite sets, such as the set of whole numbers. This allowed the teacher to show children the biunivocal correspondences between infinite sets.

C.: «After having shown this order: 0, + 1, -1, +2, -2, +3, -3… related to the set of whole numbers, a child spontaneously modified it in this way: 0, -1, +1, -2, +2, -3, +3… Children had found of some help the example of the game of bingo, where numbers are contained in a bag in a scattered order. This activity did not surprise them as it surprised me when attending the course I saw the biunivocal correspondence between N and Z: it seemed students had accepted it with no problem at all. I also tried not to make them concentrate on the common and so misleading graphic representation».

The teacher was referring here to the following graphic representation:



This latter representation makes teachers as well as students believe that the number of whole numbers is twice as that of natural numbers (someone also makes a clarification: with the exception of the zero which is the neutral element).

In this school year, the mentioned activities and others more, constituted the core issue of many articles centred on didactical workshops and published in a widespread Italian journal called: *La Vita scolastica* (The school life) addressed to primary school teachers. This represents, in our opinion, a great result as the issues of geometrical primitive entities and mathematical infinity will have a much more influential didactical repercussion and at the same time will push teachers to approach these issues and all the related topics too. Further, it is meaningful that the articles in question are structured as workshops,[56] where students shave an active role *building,* even literally, objects that try to eradicate several misconceptions. Special relational mechanisms are therefore enhanced (teachers-students) as well as cognitive relationships (student-mathematics) of major theoretical interest (Caldelli and D'Amore, 1986; D'Amore, 1988, 1990-91, 2001b).

4.6 The "sense of infinity"

The last aspect to be pointed out in this work regards a research study still going on today and that involves 9 researchers working for the following organisations: NRD (Mathematical Didactics Research Group, Mathematics Department, University of Bologna, Italy), DSE (Education Sciences Department, Ministry for Education, Bellinzona, Switzerland), ASP (Pedagogical Specialised School of Canton Ticino, Locarno, Switzerland), Mescud (School Mathematics University of Distrital, Univ. Distrital "Francisco J. de Caldas", Bogotà, Colombia).

The idea developed by D'Amore proposes to investigate, in different contexts and involving a broad sample of participants, whether or not a "sense of infinity" exists. In order to understand what is meant by this expression a clarification of the concept of "estimate", as intended by Pellegrino (1999) is needed: *«The result of a process (conscious or unconscious) that aims at identifying the unknown value of a quantity or magnitude»*. It is therefore about to sense the essence of the cardinal of a collection. The necessary skills to be a "good connoisseur of estimates" as highlighted by Pellegrino (1999), regard different factors: psychological, metacognitive, emotional and mathematical. These are some of the most important questions we asked ourselves:

[56] As D'Amore (2001b) claims: *«The "Workshop" is an environment where objects are produced, where people concretely work, where they build something; the most peculiar feature of a workshop must be some sort of creative practice; in a Workshop there must be a tendency towards ideation, planning, creation of something which is not repetitive or banal, otherwise a factory… would be enough».*

what happens if this *unknown value* is infinite? Does a “sense of infinity” exist, as does a “sense of the finite number”? If it exists, how does it manifest? If not, why? Is it possible to convey an intuitive meaning to the difference between the denumerable infinity and the continuous infinity?

During the research carried out in 1996 with primary school children, we came across some statements, spontaneously reported, that showed certain confusion between finite and infinite numbers. A good example is provided by a conversation that took place with the researcher and two children after they had been shown two segments of different length and were asked the following question: ***«According to you are there more points in this segment or in this other one?»***

M.: «We studied that a line is a set of points»

I.: «The line, not the segment»

R.: «Do you know what a segment is?»

I.: «Yes, it’s a line which starts and ends with two points and the points have letters» (I.’s answers are inconsistent: the line is formed of points, the segment, though still being a line, is not formed of points)

R.: «And in a line?»

I.: «There are many points»

R.: «How many?»

M.: «Infinite points»

R.: «So also here in the segment there are infinite points»

I.: «No, it’s limited» (also in children emerges the idea of points as unlimited as already spotted in teachers’ convictions in 3.7.1)

M.: «There won’t be so many as in this one (indicating the longer segment)»

R.: «So, you think there are more here (indicating the longer segment) ***than here*** (pointing at the shorter segment)»

M.: «It depends on how large they are, if one is one km large and the other one mm, there can be two or one million» (this affirmation recalls those of the teachers reported in 3.7.2 and underlines also a lack of “sense of measure”)

R.: «And in the line?»

M.: «Billions».

M., although claiming that the number of points of a line are infinite, subsequently affirms that in a line there would be billions of points. Where does this inconsistency come from? Does it depend only on the misconceptions concerning the geometrical primitive entities pointed out in the preceding paragraphs or also, by any chance, on a total inability of creating an image for infinity in its actual meaning? Wouldn't it be, by any chance, also the complete impossibility of *estimating infinity*?

The research studies carried out by Arrigo and D'Amore (1999, 2002) with higher secondary school students mirrored the same kinds of misconceptions.

Let us once again turn to the aspect that we held as most important: teachers. The present work has already clearly shown (paragraph 3.7.1) that teachers provided some curious estimates about infinity. Some examples are reported here as follows.

To the question: *«What do you think mathematical infinity means?»*

A primary school teacher answered:

C.: «Something that you cannot say»

S.: «In what sense?»

C.: «You don't know how much it is».

Some other primary school teachers affirmed:

A.: «To me it's a large number, so large that you cannot say its exact value»

B.: «After a while, when you are tired of counting, you say infinity meaning an ever-increasing number»

M.: «Something that I cannot quantitatively measure»

D.: «It's something so big that, no matter how much you try, it's impossible to classify it thoroughly. Mathematics, with its discipline, attempts to study a part of it».

Whereas two lower secondary school teachers wrote:

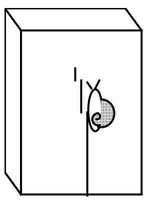
L.: «Mathematical infinity is when it never ends, it's a convention. When I cannot indicate the "beyond" I use this term: infinity and I indicate it with this sign: ∞»

F: «Mathematical infinity is a world constituted by elements that are impossible to think in their totality».

Could the fact of conceiving infinity as a large finite number, or as the unlimited, or also as a kind of process as reported in 3.7.1, be caused by an inability of *estimating infinity*?

The goal we are aiming at is to try to provide plausible answers to a number of questions that, as suggested by D'Amore, we chose to classify into two main groups: one of intuitive and linguistic nature and the other basically of more refined and technical nature. The first category concerns students not particularly skilled in mathematics or people with not a good knowledge of mathematics (students with not a solid educational background, primary school teachers, people with no connections with the world of school or the academic world, people that have a medium-high cultural level); the second one is mainly centred on evolved students or people with a good mathematical background [as for example, secondary school teachers, undergraduate students of mathematics (III and IV university years) and postgraduate students attending specialisation courses]. The TEPs (D'Amore and Maier, 2002) methodology together with the interview technique was once again preferred on the basis of what has been already described in 4.3.
We will not provide a detailed report on the research questions as well as the TEPs contents and the interview topics: they all had a shared and common beginning but then they developed differently, according to the interviewee's skills and educational level. Consequently, the results will not be reported in this work as they will soon be published but since we are dealing with teachers' convictions on mathematical infinity, we will conclude this thesis with two interviews.

After having showed the following TEP…



A snail wants to climb a wall.
In the first hour it gets up to the half of it.
In the second hour, the snail being tired, gets only up to the half of the space covered before.
In the third hour, even more tired, it performs the half of the distance covered in the preceding hour.
And so on …

I don't think it will ever get to the top

Of course it'll get there: if you consider that after two hours the snail has already walked along three quarters of the pathway …

What do you think?

Primary school teacher:

K.: «Is there a time limit?»

R.: «No, there are no limits»

K.: «So why shouldn't it get there, it will make it»

K.: «½ + ½ of ½ + How much is this sum? I can't say it really. It should result the length of the wall, but the time employed does not matter»

K.: «Let me think… the wall has an end, the height does not depend on the fact that it succeeds in getting there or not, it influences the number of hours. Yes, I think it'll make it»

R.: «In your opinion how much is the sum you told me: «½ + ½ of ½ + …?»

K.: «It can be the wall's height»

R.: «In what sense?»

K.: «The snail will get there»

R.: «And how much is this sum exactly? Could you tell me?»

K.: «Infinity? I don't' exclude it can be infinity. Yes, I think so… but also I don't think so, I don't know. I'm not really good at these kinds of things».

A lower secondary school teacher:

L.: «According to me the snail will never get there. I do not know anything about the series, but let's try it.

For each hour you have to put the covered pathway plus the half of it and so on the thing goes on to infinity, so it will not get there».

In the meantime the teacher was writing on a sheet of paper:

x =

1° hour= ½ x

2° hour= ½ x + ¼ x

3° hour=½ + ¼ x + ½(½x + ¼ x)

4° hour = ½ x + ¼ x + ½(½x + ¼ x) + ½…

L.: «The result is always a fraction of the whole pathway which is x and the more the time goes by the more it still remains a fraction. According to calculations it seems as if the snail can't make it: you have to add ever-smaller fractions of the pathway *but it will never get there. The stretches become smaller and smaller but you add them up. It will never get there».*

Does therefore a *sense of infinity* exist? Our surveys are still focused along this direction and the results of this curious research work will be soon published.

We have introduced in this chapter several research lines that we are following at present and that we intend to investigate in the near future. This complex outline demonstrated that as far as the infinity issue is concerned, there is always a new world to discover, study, analyse and investigate in depth. The feeling we receive is that year after year we are just at the beginning of such an "infinite" pathway: *«Infinity! No other problem has ever so deeply shaken the spirit of the humankind; no other idea has ever so profoundly stimulated their intellect; and nevertheless no other concept is so in need of* ***clarification*** *as infinity»* (Hilbert) [our translation].

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