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Learning Mathematics for Using its Language in a Universal Way

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ABSTRACT: *Mathematics is the only discipline whose contents are more or less the same in all the countries of the world in which it is taught, depending on the age of the students. Recently Unesco has published a long document outlining the mathematical knowledge that is necessary for future citizenship. We all tend to emphasize that Mathematics does not merely have practical applications, but that its extraordinary importance lies in the language that it is able to develop and that this is one of the principal objectives of its complex process of teaching / learning. We must enable future citizens to use mathematical language to interpret all natural phenomena and the disciplines that humanity is able to develop. Among these are the Arts and in particular Music and the Plastic Arts. For decades now, many art critics use mathematical language to interpret the phenomenon of artistic creation and to describe the work of artists who often are not even aware of the mathematics they are using. The descriptive and rational power of mathematical language here reveals all its extraordinary effectiveness. In this sense, it is ever more important to study better and in more depth the Mathematics Education in order to understand the dynamics of "learning situations". Mathematics Education is an autonomous science that has assumed enormous importance in recent decades; the research continues to enrich its contents, thanks also to the contribution of other domains of human knowledge.*

Key words: *Mathematics education, Mathematical language, Art and mathematics.*

SOME BASES OF MATHEMATICS EDUCATION

A certain amount of confusion between Pedagogy and Didactics still exists today. The term Pedagogy can be interpreted in various ways, and it is often seen as a specific aspect of Philosophy that examines fundamental terms such as ethics, education, the relationship between learner and teacher, the role of the school and so on. Didactics, on the other hand, places emphasis on concepts like learning, general features of cognitive construction, the individual and society in education, the relationship between the learner and Knowledge or the teacher and Knowledge, and so on.

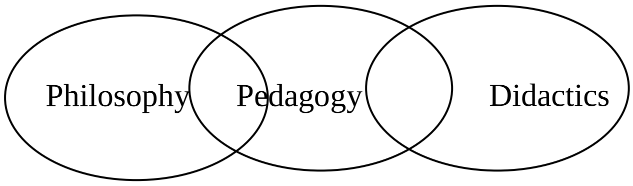


Figure 1. Philosophy – Pedagogy – Didactis in relations.

Today we can identify at least three different specific areas of Didactics: General Didactics, concerned with the broadest aspects of this discipline; Special Didactics, concerned with non-normal aspects of teaching and learning, with individual needs and particular teaching and learning situations; and Disciplinary Didactics, concerned with the specificity of the teaching and learning of given disciplines.

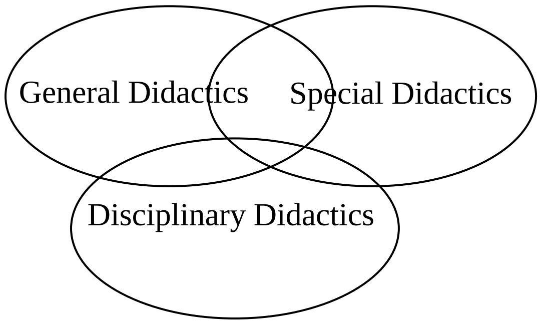


Figure 2. Relations between General, Special, and Disciplinary Didactics.

The pedagogues of the 18th and 19th centuries already focused on the difficulties of learning without distinguishing among the specificities of the disciplines. Today, it is clear that learning Mathematics is different from learning to swim or the History of Art. A clear demonstration of this is the fact that there are learners who have difficulties only in learning Mathematics. During the 1980s, the specificity of different forms of learning, together with our increasing understanding of given subjects, led to the idea of *episteme* and consequently to an Epistemology of specific learning, for example of the Mathematics learning, in order to highlight the characteristics that distinguish it from that of other subjects.

Today, we have a specific theory that within a plurality of disciplinary Didactics it enables us to focus on a given learning. In our case, there is a Didactics of Mathematics (or Mathematics Education), which may have certain characteristics in common with the Didactics of Modern Languages within the field of general Didactics, but which is a part of the specific areas of interest of Mathematics rather than of Pedagogy. In general, those who concern themselves with the Mathematics Education are Mathematicians.

FUTURE TRENDS IN MATHEMATICS

Mathematics is the only discipline whose contents are more or less the same in all the countries of the world in which it is taught, depending on the age of the students rather than the different geographical areas or social conditions. Recently, Unesco has published a long document outlining the mathematical knowledge that is necessary for future citizenship (Artigue, 2011). A particularly interesting aspect of this document is the distinction between basic mathematical knowledge for every citizen and advanced knowledge necessary for a significant use of Mathematics in a critical and analytic way, both from a personal and professional perspective.

As proposed in this document, we all tend to emphasize that Mathematics does not merely have practical applications, but that its extraordinary importance lies in the language that it is able to develop and that this is one of the principal objectives of its complex process of teaching-learning. We must enable future citizens to use mathematical language to interpret all natural phenomena and the disciplines that humanity is able to develop. Among these are the Arts and in particular Music and the Plastic Arts. A researcher in Mathematics Education may decide to devote all his work to this social duty.

PERSONAL COMMITMENTS IN MATHEMATICS EDUCATION

I began my research in Mathematics Education as a young Mathematician and I would never have imagined that I would have to abandon pure mathematical research to dedicate myself entirely to that applied to teaching and learning. In 1971, two years after my degree in Mathematics, I answered to a call on the part of a journal for an analysis of some didactic proposals by a group of researchers, but only five years later I decided with the conviction of a personal decision to dedicate my work to Mathematics Education. In this new field, I began the study of the interference between different types of language at school, for example the conflict between formal language and everyday language in the practice in school.

My research concentrated on the error, widespread in the 1980s, of trying to transform the resolution of a problem into an algorithm, a tendency based on an ingenuous interpretation of the books of George Polya. Moreover, I demonstrated the impossibility of this kind of reductionism and began investigating the linguistic causes of difficulties encountered in dealing with problems by students at all school levels. I examined at length the work environment “laboratory” as a learning environment in which learning by doing transforms the problem into a concrete need, interpreting student behaviour in terms of the theory of situations and providing tools for positive and negative analysis of the phenomenon.

With Martha Isabel Fandiño Pinilla, I studied certain specific aspects of learning such as the number zero, the relations between area and perimeter of bi-dimensional figures, the idea of mathematical infinity, which led me to become Chief Organizer of the Topic Group 14: Infinite processes throughout the curriculum, at the 8th ICME, Seville, July

14-21 in 1996. On this occasion, a member of the advisory panel was Raymond Duval. I have worked on the theme of the convictions of teachers on mathematical infinity in a doctoral thesis conducted in Italy and discussed it in Slovakia.

I have dedicated much energy in promoting the need of the study of the learning of Mathematics in young learners, even at nursery school, since this kind of situation affords insights into "ingenuous" learning applicable at other levels of schooling. Together with Francesco Speranza, I have dedicated years of research and experimentation to classroom practice with regard to the history of Mathematics and this has led to an ongoing production of articles on the epistemology of Mathematics, in progress today.

In 1986, I founded an annual national (soon to become international) conference which every year in November attracts thousands of participants and to which some of the most famous international researchers have contributed and which reaches its 27th edition this year. At the same time, I founded and directed for over 20 years a journal on Mathematics Education which was closed when I moved from Italy to Colombia and which has published research by the major international scholars and reached a B international classification.

My research has also contained elements of Ethnomathematics and analysis of key terms such as "competence", together with Martha Isabel Fandiño Pinilla and Juan Godino. A particularly surprising result came from a long-term study of spontaneous demonstrations produced by 9th and 10th grade students from which emerged that some demonstrations considered unacceptable by teachers were so far the simple reason that their logic was not Aristotelian but rather that of the Indian *nyaya.* This logic was much more concrete and closer to the basic needs of the students who tried to anchor their reasoning to example and thesis, considered as point of departure (elements common to *nyaya*) rather than to logical deduction (in turn, based on material implication).

I have also worked at length on the difficulties encountered by students in the cognitive construction of mathematical objects, identifying tools for interpreting, describing, and evaluating errors (e.g., work on TEP together with Hermann Maier). I have participated with seminars and papers at international conferences in Europe, America, and Asia. I have particular pleasure in recalling numerous occasions working with my colleague and friend Athanasios Gagatsis, first in Thesaloniki and then in Nicosia, with whom I share a deep interest in the field of representations and semiotics.

I have always been against the aberrant notion that the emergence of a new theory announces the death of previous theories. In my opinion, although significant and profound theories have more recently developed (to which I have also contributed, including the EOS of Juan Godino), the foundational theories remain the basis of the Mathematics Education. First and foremost is the theory of situations by Guy Rousseau, a historical foundation of our discipline. In this respect, I have always sought to give new value to the foundational disciplines through a demonstration of their reciprocal coherence and moreover, necessity, as regards specific problems of interpretation of classroom situations, the concept that has been central to my 42 years of research.

I have constantly studied the convictions that students have about Mathematics and about their way of operating in Mathematics, immediately understanding the need to place at the centre of the interpretation of classroom situations the convictions of teachers, an area to which I have dedicated many years of research, together with Martha Isabel Fandiño Pinilla, and for which I have developed tools of analysis. The convictions of teachers determine the nature of mathematical work in the classroom and exert massive influence on the convictions of students.

Finally, fascinated by the studies first by Raymond Duval and then by Luis Radford, I began to dedicate my work to the multiform presence of semiotics within the process of teaching and learning in Mathematics. I searched the history to study the evolution of Mathematics for examples and stimuli regarding the definition of the various mathematical objects in order to analyse them first from an epistemological and then from a didactic point of view, trying to identify opportune definitions of "mathematical object". I have long studied, together with Martha Isabel Fandiño Pinilla, the variations in meaning that teachers and learners attribute to different representations obtained through transformations of treatment carried out. In this field, we have published various articles, participated in international conferences and supervised two doctoral theses, in Italy and in Colombia, while still retaining that the question remains partially open.

At this point, I was ready to examine my deepest philosophical convictions, moving from a modern but ingenuous realism to a mature pragmatism in which today I firmly believe and which is embedded in my having placed anthropological theories at the base of my description of didactic phenomena. I have always devoted time to the diffusion and popularizing of Mathematics, trying to reach students and adults who do not appreciate Mathematics simply because they do not know it. In this field, in recent years together with Martha Isabel Fandiño Pinilla and other colleagues, I have written numerous books and also received prizes.

I have also always been fascinated by the language of Mathematics and how it may be understood as a basis, driven not just by erudite motives, such as the unity of human culture and the refusal of the attempt, to divide learning into "two cultures", but also by the fascination that poetry and figurative arts have had for me from an early age.

AN EXAMPLE: DANTE AND THE MATHEMATICS

Thus, I have devoted quite a few of my studies to the presence of Mathematics in the works of Dante Alighieri and in particular in his monumental jewel of universal poetry, the "Comedìa" (the Divine Comedy). This has led to the publication of numerous articles and books as well as to the presentation of papers at international conferences. Seven hundred years after the writing of this monument that enhances human knowledge, I provided guidance on the interpretation of certain parts that had remained hidden, due to the mathematical ignorance on behalf of critics and historians of literature. Today, many of my interpretations have been accepted by experts on Dante and I have been able to use these results to reiterate the groundlessness of the division of cultures. In the Middle Ages, Dante was able to use the (elementary) Mathematics of the time in describing nature and human feelings, theology and logic, to use metaphors and to narrate in a way far more profound than the mathematical pseudo-culture of certain writers allows today. Thus, this is another reason to understand and to use Mathematics especially on the part of the Humanists.

Figure 3. The cover of an old edition of the Divine Comedy (Alighieri, 1907).

Figure 4. Dante e it re Regni, (Di Michelino, 1465).

Let us take a couple of examples of how Mathematics can help understand certain lines of the "Divine Comedy", previously obscure to literary critics who were not fond of this discipline. As a first example, we can consider an arithmetic reference in Paradiso XXVIII, lines 91 to 93:

> ...And every spark behind its fire did speed;
> Thousands there were beyond the numbering
> To which the doubled chessboard squares will lead...

The great number referred to is that of the angels who are born in succession, instant by instant, witnesses to the glory of God, counted not by doubling the number, but thousand by thousand. How great is this number of angels? Dante states that their growth by thousands is beyond the doubling of the chessboard squares. This is clearly a reference to the famous legend of Sissa Nassir, the inventor of chess. As a reward from his enthusiastic Sovereign, he asked for something apparently modest: on a chessboard eight squares by eight, he asked for a grain of rice on the first square, doubled on the second square to make two, doubled on the third square to make four, doubled on the fourth square to make eight, and so on until the last and sixty-fourth square. The calculation today is relatively easy, especially with the use of a calculator, but with the Roman system becomes, to say the least, arduous, and the grains due to Sissa Nassir are 18 446 744 073 709 551 615, a number almost unreadable. Today, with a more compact method of writing, we use the so-called scientific notation: $1.8447 \cdot 10^{19}$.

In order to have an idea of the enormity of this number we can imagine to distribute the grains over the whole surface of the Earth, the measure of which, expressed in current terms (and not in those of Dante's age) including seas, oceans, deserts, mountains, glaciers, etc. is around $5.0995 \cdot 10^{18}$ cm^2. As we distribute them, we find 3,62 grains (let us say, approximately three and a half) for every cm^2 of the Earth. (Thus it is clearly the reason why the Sovereign felt himself teased and rather than award Sissa Nassir his prize he cut off his head instead thereby obtaining a considerable saving by withholding the gift).

But the number of angels, rather than doubling, increased by thousands. If we proceed with this calculation (1 grain on the first square, 1000 on the second, 1000000 on the third, 1000000000 on the fourth, and so on), we have an immense, but still finite, number: 10^{189} (in other words, $2 \cdot 10^{170}$ angels for each cm^2 of the Earth. We may indeed be thankful to the angels for being immaterial!). Yet, beyond this jovial aspect of the story, there are two elements of great interest.

The first is that Dante could have said that the angels born in succession, instant by instant, witnesses to the glory of God, are infinite. By comparison to the infinite, even an immense number like 10^{189} is a drop in the ocean. The choice of a very great number has more impact than the adjective "infinite". Paradoxically, the number provokes thought more than the infinite.

The second is that many authors assert that Dante was not aware of the Arab-Indian numbers that had begun to circulate in Europe but were known only among a few scholars. But to get an idea of the immense value of those angels, it is necessary to use a positional system, not like the Roman system, which lacked that characteristic. A simple research in the libraries of Florence shows that the second son of Dante, Jacopo, was a student at one of the three Florentine public schools, Santa Trinita, where he studied Mathematics under the guidance of the prestigious scholar Paolo dell'Abbaco, who undoubtedly taught the obligatory Roman system, but also offered to his brighter students the basics of the new Arab system that was circulating precisely in Tuscany. Today we can't by no means be so sure that Dante was unaware of the positional arithmetic system!

The second example is certainly one of the most famous mathematical references in Dante (Paradiso XXXIII, lines 133 to 138):

> ...As the geometer in thought will strain
> To measure out the circle, nor can tell
> The principle he lacks, so toils in vain,
>
> Such was I at this new seen miracle;
> I longed to see how image and circle blend
> And how the image comes therein to dwell...

The "new seen" referred to here is the direct contact between Dante and God, through sight. The "Divine Comedy" is almost over and we are at the final lines. The Poet has travelled through "Inferno, Purgatorio and Paradiso". Shortly his journey will be over and he will return to Earth. The final part is the great fortune for him to have had visual contact with God.

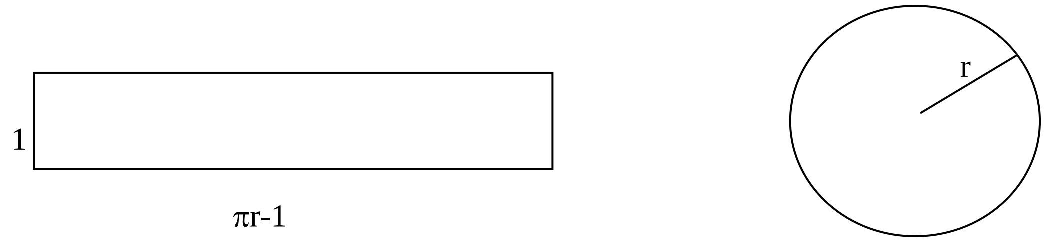
He must find a metaphor that enables him to explain the greatness of what is happening and to correlate the "new seen" with something capable of rendering the idea. And so he turns to geometry, to that "to measure out the circle".

The metaphor is by no means simple, but has been erroneously interpreted for centuries. One of the most common critical texts offers the following exegesis: "like the geometer who concentrates all his mental faculties on the *insoluble problem* of squaring the circle" (emphasis added), "such was I before that extraordinary vision, that in vain" (Par, 33, 133-138).

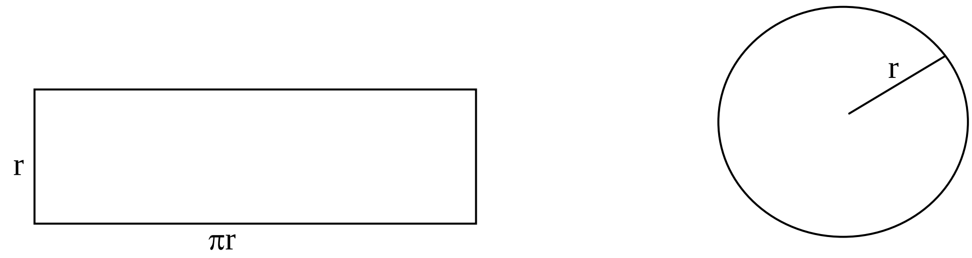
What exactly is the problem of the squaring of the circle? The question can be posed in two complementary ways: (1) given a circumference, find a square or a rectangle whose perimeter has the same length as the circumference; and (2) given a circle, find a square or a rectangle that has the same area as the circle.

This problem was brilliantly solved in Ancient Greece, for example, by Dinostratus in the fifth century B.C. (but not only by him). It was something well known among educated people, not only Mathematicians as it was also explained by Plato.

From a more modest scholastic point of view, everyone, towards the end of the primary school, can remember having learned that a circumference with radius r measures $2\pi r$. Therefore, if we take a rectangle with sides 1 and $\pi r-1$ (with $r > 1/\pi$), the lengths of the circumference and the perimeter are the same. In this way, as any ten-year old child knows, the area of a circle with radius r is πr^2 and so a rectangle with sides πr and r will have an area equal to that of the circle.



perimeter of the rectangle: $2\pi r$ measure of the circumference: $2\pi r$



area of the rectangle: πr^2 area of the circle: πr^2

Where, then, lies the impossibility of the problem? In Dante, something is implicit. It has always been well known that the Greek Mathematicians preferred ruler-and-compass solutions (a way of expressing something that goes beyond the mere reference to the two instruments, but which we can take here as a meaning working indeed with an (unscaled) ruler and a compass). The solution offered by Dinostratus and those of other Greek scholars are correct but they were not obtained by using ruler and compass.

For many centuries, unsuccessfully, the Greeks and subsequently all other Mathematicians tried to square the circle with these instruments. Today, we know that this is impossible, as demonstrated by Carl Louis Ferdinand von Lindemann only in 1882. The Greeks must have supposed it, even though implicitly. It is by no accident that the three most loved and studied problems (the “three classical problems of Greek geometry”, quoted by Plato) were perennially examined: the quadrature of the circle, the duplication of the cube, and the trisection of an arbitrary angle.

The question is, since Dante does not explicitly say “with ruler and compass”, did he fall into the same error of modern critics, or did he know the question well and imagined his readers also knew of it likewise so as to consider further erudition superfluous? We will never have an answer to this question. The geometrical knowledge widely disclosed by Dante in many parts of his works leads me to venture that we are in the presence of another example of the current defeat of the unity of culture. In Dante, the "two cultures" co-existed, something, alas, that is not like that for many readers today, not only non-Mathematicians but also anti-Mathematicians.

ANOTHER EXAMPLE

In the same way, I have devoted a part of my study to the relationships between Mathematics and figurative arts. I have studied this relationship between their languages by not only by seeking out authors and works that obviously lend themselves to mathematical interpretations, such as the decorative Arab tiles at the Alhambra in Granada but also the works of the Italian and German Renaissance, the work of Maurits Escher or Oscar Reutersvärd to name a few. My idea was to study the entire history of art and to go beyond the typical literary, philosophical or psychological critical interpretations and to build new ones which are rational, mathematical and formal.

Figure 5. A tile Arabic geometric medieval (D'Amore, 2015, p. 158).

Figure 6. A typical geometric design of Maurits Escher (D'Amore, 2015, p. 428).

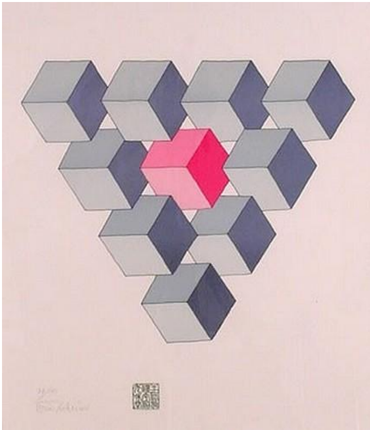


Figure 7. A typical geometric design of Oscar Reutersvärd (D'Amore, 2015, p. 454).

In this way, I was able to build a dialogue with critics and artists in order to influence a new way of practicing and writing art criticism, organizing international exhibitions in the field of conceptual art, offering a perspective known as "exact art" which was subsequently taken up by well-known artists. The name "exact art" evokes, within art, the fact that mathematics, within science, is known as "the exact science":

exact art : art = exact science : science

On this theme, I wrote hundreds of books and articles and my latest work is about to be published, even though its size (1000 pages and 1000 full-colour images) poses evident editorial problems.

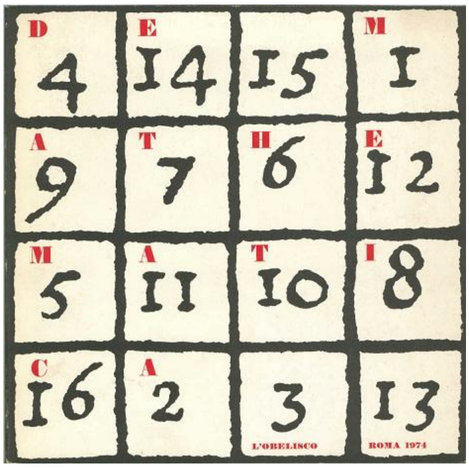


Figure 8. Cover of D'Amore and Menna (1974).

Even though many believe that the study of impossible perspectives has its origin in the middle of the nineteenth century this is quite incorrect. Human intelligence manifests in numerous ways one of which being the love of contradiction. After thousands of years of having searched for absolute, mathematical, formal, perfect rules of perspective representation and having found them it was then decided to contradict them for the pure cultural and intellectual love of a challenge. Thus, begins another story, in reverse, the attempt to represent in the plane, and so in the picture, impossible perspectives that surprise those who discover them and amuse those who analyze them.

In 1754, a book was published by the English scholar of architectonic drawing John Joshua Kirby (1716 – 1774); its almost endless title (as was common at the time) was "Dr. Brook Taylor's Method of Perspective Made Easy both in Theory and Practice, Being an attempt to make the art of perspective easy and familiar to adapt it entirely to the arts of design; and to make it an entertaining study to any gentleman who shall choose so polite an amusement". The book was printed by W. Craighton at Ipswich, near London, and the publishers were J. Swan, F. Noble, and J. Noble. The book is itself odd in that, for example, the numbering of the pages is not always progressive. But what makes it most noteworthy is the fact that the illustrations are by the great painter, drawer and engraver, William Hogarth (1697 – 1764), author of many irreverent satirical prints that made a considerable impression at that time.

Particularly famous and often quoted is the figure found in the front, the piece entitled "Prospective Absurdities".

Figure 9. Prospective Absurdities (Hogarth, 1754).

Let us consider some of its features.

Figure 10. Some details of Prospective Absurdities.

Here the play is clear: near-far and in front-behind are inverted, also by means of a subtle play with proportions and measures. The artists most quoted in terms of impossible perspectives are certainly the Dutchman Maurits Cornelis Escher (1898 – 1972) and the Sweden Oscar Reutersvärd (1915 – 2002).

As regards perspectives and impossible drawings, the studies of the Penroses are always quoted: father (Lionel Sharples, Psychologist, 1898 – 1972) and son (Roger, born in 1931, Mathematician and Physicist, famous scholar of space-time and black holes, as well as notable narrator), in particular with reference to an article published in the British Journal of Psychology in 1958 (Penrose & Penrose, 1958) in which appears a celebrated impossible "triangle".

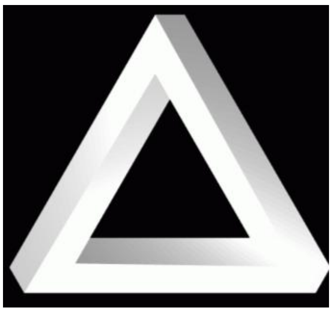


Figure 11. Tribar (Penrose & Penrose, 1958).

But the first impossible drawing by Reutersvärd dates from 1934, much before the triangle of the Penroses (1958). Among the famous optical illusions, one of the first (1832) was the Necker's "Cube", from the name of the Swiss crystallographer Louis Albert Necker (1786 – 1861), that appears in the Escher's "Belvedere".

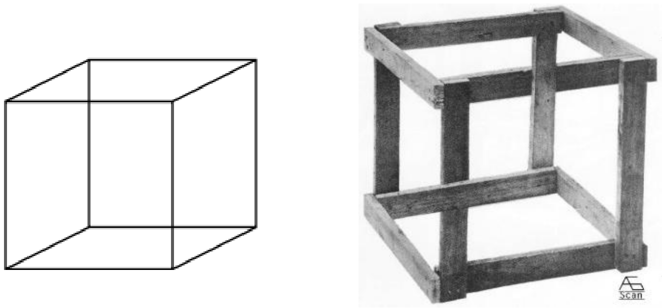


Figure 12. Albert Necker's cube (Necker, 1832).

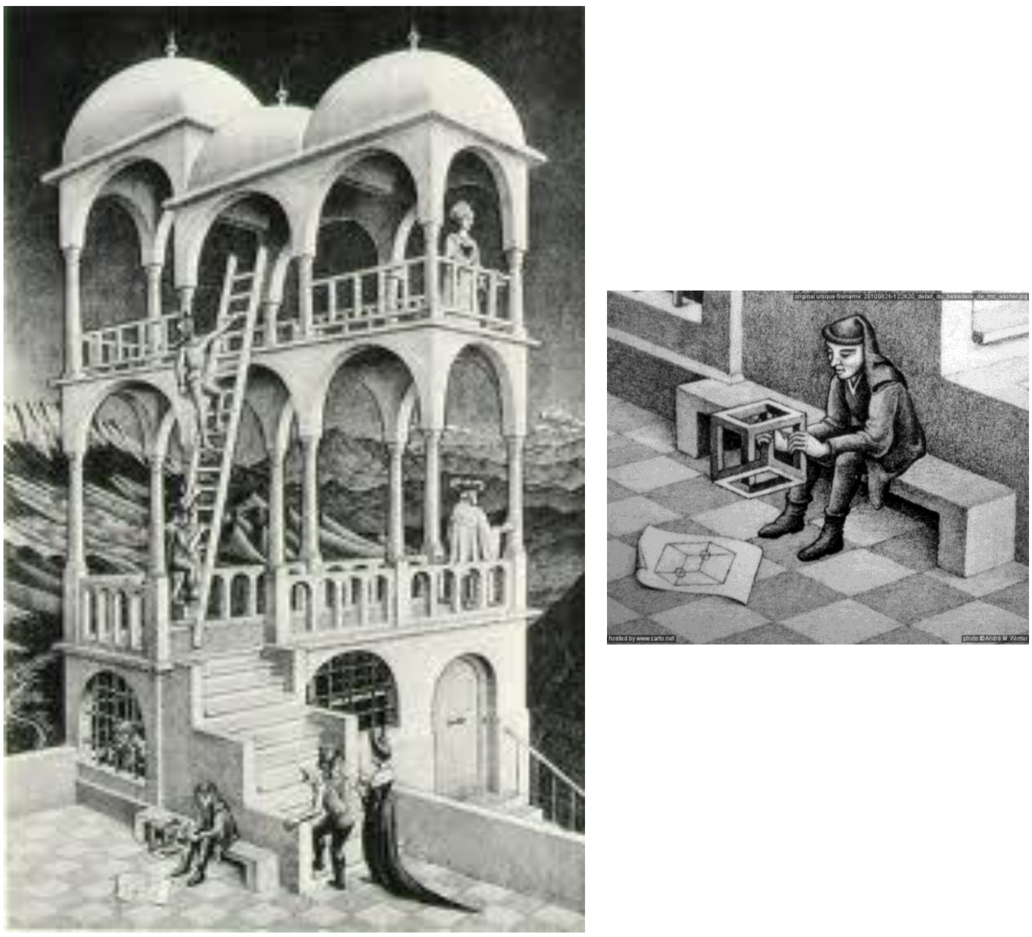


Figure 13. Belvedere, lithograph and a detail (Escher, 1956).

In the interesting book by Jan Gullberg (1936 – 1998), "Mathematics, from the birth of numbers", published in 1997, in the chapter dedicated to Geometry, there is a reference (p. 374) to "Phantasmagorical Geometries". Apart from a brief mention of the work of the Penroses, the discussion is based on the work of Oscar Reutersvärd.

I have also been closely bound by friendship to his family and by a work collaboration with Oscar throughout my life. My wife, Martha, and I possess a collection of several hundred original works of his.

CONCLUSION

For many decades art critics have used mathematical language to interpret the phenomenon of artistic creation and to describe the work of artists, many of whom are quite unaware of the Mathematics they use in their work. Here emerges all the descriptive and rational power of mathematical language. Beyond general motivations and concrete applications it becomes ever more important to study thoroughly and more deeply the Mathematics Education in order to understand the development of classroom situations, mathematical learning, and its multiple languages, the semiotic nuances that enable representation of objects hidden to the senses and that therefore can be communicated and made accessible only through semiotic representations in appropriate registers, via the two different semiotic transformations: treatment and conversion. The Mathematics Education is an autonomous science that in recent decades has assumed great importance. Scientific research produces ever richer results, also thanks to the contributions of other domains of human knowledge.

NOTE

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ACKNOWLEDGMENT

I want to dedicate this prestigious award to my wife Martha, companion, partner and accomplice, indefatigable collaborator in every adventure, both cultural and existential, someone of unmatchable human depth and extraordinary critical ability. Without her support and belief in me, all this would not have been possible. Today, as ever, any award given to me is in fact given to both of us.

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