DIFFERENTIAL FORMS IN CARNOT GROUPS: A VARIATIONAL APPROACH

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Abstract. Carnot groups (connected simply connected nilpotent stratified Lie groups) can be endowed with a complex (E_0^*, d_c) of "intrinsic" differential forms. In this paper we want to provide an evidence of the intrinsic character of Rumin's complex, in the spirit of the Riemannian approximation, like in [?] and [?]. More precisely, we want to show that the intrinsic differential d_c is a limit of suitably weighted usual first order de Rham differentials d_ε. As an application, we prove that the L^2-energies associated to classical Maxwell's equations in $\mathbb{R}^n$ Γ-converges to the L^2-energies associated tp an "intrinsic" Maxwell's equation in a free Carnot group.

1. Introduction

Classical Maxwell's equations for time-harmonic vector fields

$$e^{i\omega s}\vec{E}, e^{i\omega s}\vec{H}, e^{i\omega s}\vec{B}, e^{i\omega s}\vec{D}$$

in $\mathbb{R} \times \mathbb{R}^3$ read as follows:

$$\text{curl}\,\vec{H} - \frac{i\omega}{c}\vec{D} = \frac{4\pi}{c}\vec{J},$$

$$\text{curl}\,\vec{E} + \frac{i\omega}{c}\vec{B} = 0,$$

$$\text{div}\,\vec{D} = 4\pi\rho, \quad \text{div}\,\vec{B} = 0,$$

together with the constitutive relations

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H},$$

where $\varepsilon, \mu : \mathbb{R}^3 \to \mathbb{R}^3$ are linear maps and the constant c is the speed light. We denote by $[\varepsilon], [\mu]$ the matrices of ε and μ with respect to the Euclidean canonical basis. Usually, $[\varepsilon], [\mu]$ are called respectively the dielectric permittivity and the magnetic permeability.

Suppose for sake of simplicity $\vec{J} \equiv 0$ and $\rho \equiv 0$. Therefore

$$\text{curl}\,\varepsilon^{-1}\vec{D} = -\frac{i\omega}{c}\mu\vec{H},$$

and then

$$\text{curl}\,\mu^{-1}\text{curl}\,\varepsilon^{-1}\vec{D} = \frac{\omega^2}{c^2}\vec{D}.$$

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The results presented in this note are obtained in two joint works with Bruno Franchi, [?] and [?].

If (Ω^*, d) is the de Rham complex of differential forms in $\mathbb{R}^3$, classical Maxwell's equations can be formulated in their simplest form as follows. We fix the standard volume form dV in $\mathbb{R}^3$. If $\alpha, \beta \in \Omega^h$ we denote by $\langle \alpha, \beta \rangle_{\text{Euc}}$ their Euclidean scalar product (i.e. the scalar product making the basis $\{e_1, e_2, e_3\}$ and $\{dx_1, dx_2, dx_3\}$ be orthonormal); we denote by $*$ the Hodge duality operator. Consider now the 2-form $D := -(*\vec{D})^\natural$ (recall that if v is a vector field in $\mathbb{R}^n$, then its dual form $v^\natural$ acts as $v^\natural(w) = \langle v, w \rangle_{\text{Euc}}$, for all $w \in \mathbb{R}^n$). We have

$$\text{curl}\, \mu^{-1} \text{curl}\, \varepsilon^{-1} (*D)^\natural - \frac{\omega^2}{c^2} (*D)^\natural = 0.$$

We remind the following definition (see e.g. [**?**], Section 2.1).

Definition 1.1. If V, W are finite dimensional linear vector spaces and $L : V \to W$ is a linear map, we define

$$\Lambda_h L : \bigwedge\nolimits_h V \to \bigwedge\nolimits_h W$$

as the linear map defined by

$$(\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) = L(v_1) \wedge \cdots \wedge L(v_h)$$

for any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$, and

$$\Lambda^h L : \bigwedge\nolimits^h W \to \bigwedge\nolimits^h V$$

as the linear map defined by

$$\langle (\Lambda^h L)(\alpha) | v_1 \wedge \cdots \wedge v_h \rangle = \langle \alpha | (\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) \rangle$$

for any $\alpha \in \bigwedge^h W$ and any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$.

As it is proved in [**?**], we obtain that, eventually, $*D$ satisfies the differential equation

$$\delta M d N \alpha + \frac{\omega^2 \cdot \det \mu}{c^2} \alpha = 0, \tag{1}$$

where $M := \Lambda^2 \mu$, $N := \Lambda^1 (\varepsilon^t)^{-1}$ (see also Section 5 for more details).

It is well known that (31) makes perfectly sense in $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Therefore $*D$ is a stationary point of the functional

$$J^{\mu,\varepsilon}(\alpha) := \int_{\mathbb{R}^n} \langle M d N \alpha, d N \alpha \rangle_{\text{Euc}}\, dV + \frac{\omega^2 \cdot \det \mu}{c^2} \int_{\mathbb{R}^n} \langle N \alpha, \alpha \rangle_{\text{Euc}}\, dV \tag{2}$$

where $\alpha \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$.

In this note we want to show that a Γ-limit of functionals of the previous type is related to an intinsic Maxwell equations in Carnot groups.

A Carnot group $\mathbb{G}$ (see below for precise definition and [**?**] for a general survey), can be thought, roughly speaking, as the Lie group $(\mathbb{R}^n, \cdot)$, where $\cdot$ is a (non-commutative) multiplication such that its Lie algebra $\mathfrak{g}$ is nilpotent and admits a *step κ stratification.* This means that there exist linear subspaces $V_1, ..., V_\kappa$ (the layers of the stratification) such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. We refer to the first layer V_1 as to the *horizontal* *layer*, which plays a key role in our theory, since it generates the all of $\mathfrak{g}$ by commutation.

The stratification of the Lie algebra induces a family of anisotropic dilations δ_λ ($\lambda > 0$) on $\mathfrak{g}$ and therefore, through exponential map, on $\mathbb{G}$.

It is well known that the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ can be identified with the tangent space at the origin e of $\mathbb{G}$, and hence the horizontal layer of $\mathfrak{g}$ can be identified with a subspace $H\mathbb{G}_e$ of $T\mathbb{G}_e$. By left translation, $H\mathbb{G}_e$ generates a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ and eventually a sub-Riemannian structure on $\mathbb{R}^n$.

From now on, we use the word "intrinsic" when we want to stress a privileged role played by the horizontal layer and by group translations and dilations.

Starting from de Rham complex (Ω^*, d) of differential forms in $\mathbb{R}^n$, it is possible to define a complex of differential forms that has to be "intrinsic" for $\mathbb{G}$ in our sense. Such a complex, denoted by (E_0^*, d_c) , with $E_0^* \subset \Omega^*$, has been defined and studied by M. Rumin in [?] and [?] ([?] for contact structures). Rumin's theory needs a quite technical introduction that is sketched in Section 3 to make this note self-consistent. For a more exhaustive presentation, we refer to original Rumin's papers, as well as to the presentation in [?]. The main properties of (E_0^*, d_c) can be summarized in the following points:

i) Intrinsic 1-forms are horizontal 1–forms, i.e. forms that are dual of horizontal vector fields, where by duality we mean that, if v is a vector field in $\mathbb{R}^n$, then its dual form $v^\natural$ acts as $v^\natural(w) = \langle v, w\rangle$, for all $w \in \mathbb{R}^n$.
ii) The "intrinsic" exterior differential d_c on a smooth function is its horizontal differential (that is dual operator of the gradient along a basis of the horizontal bundle).
iii) The complex (E_0^*, d_c) is exact and self-dual under Hodge $*$-duality.

We want to show the intimate connection between the complex and the Carnot group. More precisely, we want to show that the intrinsic differential d_c is a limit of suitably weighted usual first order de Rham differentials d_ε.

For this purpose, we need to introduce the notion of *weight* of vectors in $\mathfrak{g}$ and, by duality, of covectors. Elements of the j-th layer of $\mathfrak{g}$ are said to have (pure) weight $w = j$; by duality, a 1-covector that is dual of a vector of (pure) weight $w = j$ will be said to have (pure) weight $w = j$.

This procedure can be extended to h-forms.

Then, the usual exterior differential d acting on a form α of pure weight splits as

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha,$$

where $d_0\alpha$ does not increase the weight, $d_1\alpha$ increases the weight by 1, and, more generally, $d_i\alpha$ increases the weight by i when $i = 0, 1, \ldots, \kappa$. Then, we define a ε-differential that weights the different terms of d according to their different actions with respect to the stratification of the Lie algebra $\mathfrak{g}$. Therefore we set

$$d_\varepsilon = d_0 + \varepsilon d_1 + \cdot + \varepsilon^\kappa d_\kappa.$$

The issue now is to specify in what sense the d_ε (that is a first order operator) converges to d, that is, in general, a higher order differential operator (see Theorem 3.7-vii) below).

The natural approach relies in the use of De Giorgi's Γ-convergence ([**?**], [**?**], and see also Section 4 below for precise definitions in our setting) for variational functionals. Indeed, we are able to prove that the L^2-energies associated with $\varepsilon^{-\kappa}d_\varepsilon$ on 1-forms Γ-converge, as $\varepsilon \to 0$, to the energy associated with d_c.

The main theorem of this note reads as follows. If we denote by $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ the space of differential 1-forms on $\mathbb{G}$ with coefficients belonging to the Folland-Stein space $W^{\kappa,2}(\mathbb{G})$ (see Definition 2.2), we have:

Theorem 1.2. *Let $\mathbb{G}$ be a free Carnot group of step κ. If $\omega \in W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$, we set*

$$F_\varepsilon(\omega) = \frac{1}{\varepsilon^{2\kappa}} \int_{\mathbb{G}} |d_\varepsilon \omega|^2 \, dV,$$

where

$$d_\varepsilon = d_0 + \varepsilon d_1 + \cdot + \varepsilon^\kappa d_\kappa.$$

Then F_ε sequentially Γ-coverges to F in the weak topology $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$, as $\varepsilon \to 0$, where

$$F(\omega) = \begin{cases} \displaystyle\int_{\mathbb{G}} |d_c\omega|^2 \, dV & \text{if } \omega \in W^{\kappa,2}(\mathbb{G}, E_0^1) \\ +\infty & \text{otherwise.} \end{cases}$$

We remind that the group $\mathbb{G}$ is said to be free if its Lie algebra is free, i.e. the commutators satisfy no linear relationships other than antisymmetry and the Jacobi identity. This is a large and relevant class of Carnot groups. We remind also that Carnot groups can always be "lifted" to free groups (see [**?**] and [**?**], Chapter 17). For our purposes, the main property of free Carnot groups relies on the fact that intrinsic 1-forms and 2-forms on free groups have all the same weight (see Theorem 3.9). This helps at several steps of the proofs (unfortunately, the same assertion fails to hold for higher order forms (see Remark 3.11 in [**?**])).

As is proved in Section 5, the previous result can be applied to functionals of the type of the equation (2) to derive a Γ-convergence result to the L^2-energy associated with intrinsic Maxwell's equations.

2. Carnot groups

Let $(\mathbb{G}, \cdot)$ be a *Carnot group of step κ* identified to $\mathbb{R}^n$ through exponential coordinates (see [**?**] for details). By definition, the Lie algebra $\mathfrak{g}$ has dimension n, and admits a *step κ stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ (the layers of the stratification) such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa, \tag{3}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Set $m_i = \dim(V_i)$, for $i = 1, \dots, \kappa$ and $h_i = m_1 + \cdots + m_i$ with $h_0 = 0$. Clearly, $h_\kappa = n$. Choose now a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ adapted to the stratification, i.e. such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \text{ is a basis of } V_j \text{ for each } j = 1, \dots, \kappa.$$

We refer to the first layer V_1 as to the *horizontal layer*. It plays a key role in our theory, since it generates the all of $\mathfrak{g}$ by commutation.

Let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$. Given (3), the subset $X_1, \dots, X_{m_1}$ generates by commutations all the other vector fields; we will refer to $X_1, \dots, X_{m_1}$ as to the *generating vector fields* of the algebra, or as to the *horizontal derivatives* of the group.

The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle \cdot, \cdot \rangle$, making $\{X_1, \dots, X_n\}$ be an orthonormal basis.

We can write the elements of $\mathbb{G}$ in *exponential coordinates*, identifying p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{d_1} x_1, ..., \lambda^{d_n} x_n), \tag{4}$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [**?**] Chapter 1) and is defined as

$$d_j = i \quad \text{whenever } h_{i-1} + 1 \le j \le h_i, \tag{5}$$

hence $1 = d_1 = ... = d_{m_1} < d_{m_1+1} = 2 \le ... \le d_n = \kappa$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$.

We denote also by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(V_i).$$

The Euclidean space $\mathbb{R}^n$ endowed with the usual (commutative) sum of vectors provides the simplest example of Carnot group. It is a trivial example, since in this case the stratification of the algebra consists of only one layer, i.e. the Lie algebra reduces to the horizontal layer.

Definition 2.1. Let $m \ge 2$ and $\kappa \ge 1$ be fixed integers. We say that $\mathfrak{f}_{m,\kappa}$ is the free Lie algebra with m generators $x_1, \dots, x_m$ and nilpotent of step κ if:

i) $\mathfrak{f}_{m,\kappa}$ is a Lie algebra generated by its elements $x_1, \dots, x_m$, i.e. $\mathfrak{f}_{m,\kappa} = \mathrm{Lie}(x_1, \dots, x_m)$;
ii) $\mathfrak{f}_{m,\kappa}$ is nilpotent of step κ;
iii) for every Lie algebra $\mathfrak{n}$ nilpotent of step κ and for every map ϕ from the set $\{x_1, \dots, x_m\}$ to $\mathfrak{n}$, there exists a (unique) homomorphism of Lie algebras Φ from $\mathfrak{f}_{m,\kappa}$ to $\mathfrak{n}$ which extends ϕ.

The Carnot group $\mathbb{G}$ is said free if its Lie algebra $\mathfrak{g}$ is isomorphic to a free Lie algebra.

When $\mathbb{G}$ is a free group, we can assume $\{X_1, \dots, X_n\}$ a Grayson-Grossman-Hall basis of $\mathfrak{g}$ (see [**?**], [**?**], Theorem 14.1.10). This makes several computations much simpler. In particular, $\{[X_i, X_j], X_i, X_j \in V_1, i < j\}$ provides an orthonormal basis of V_2.

From now on, following [**?**], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi–index, we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$. By the Poincaré–Birkhoff–Witt theorem (see, e.g. [**?**], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1 + \cdots + i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations. From the Poincaré–Birkhoff–Witt theorem, it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators X^I of the special form above.

Since here we are dealing only with integer order Folland-Stein function spaces, we can give this simpler definition (for a general presentation, see e.g. [**?**]).

Definition 2.2. If $1 < s < \infty$ and $m \in \mathbb{N}$, then the space $W^{m,s}(\mathbb{G})$ is the space of all $u \in L^s(\mathbb{G})$ such that

$$X^I u \in L^s(\mathbb{G}) \quad \text{for all multi-index } I \text{ with } d(I) = m,$$

endowed with the natural norm.

We remind that

Proposition 2.3 ([**?**], Corollary 4.14)**.** *If $1 < s < \infty$ and $m \geq 0$, then the space $W^{m,s}(\mathbb{G})$ is independent of the choice of $X_1, \dots, X_{m_1}$.*

Proposition 2.4. *If $1 < s < \infty$ and $m \geq 0$, then $\mathcal{S}(\mathbb{G})$ and $\mathcal{D}(\mathbb{G})$ are dense subspaces of $W^{m,s}(\mathbb{G})$.*

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1, \cdots, X_n$, is the family of covectors $\{\theta_1, \cdots, \theta_n\}$. We indicate by $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\theta_1, \cdots, \theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1, \cdots, \theta_n$ may have polynomial (hence variable) coefficients.

Following Federer (see [**?**] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge_h \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge^h \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \leq h \leq n$,

$$\bigwedge_h \mathfrak{g} := \text{span}\{X_{i_1} \wedge \cdots \wedge X_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\},$$
$$\bigwedge^h \mathfrak{g} := \text{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}.$$

The elements of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ are called *h-vectors* and *h-covectors*.

We denote by Θ^h the basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}$ of $\bigwedge^h \mathfrak{g}$. We remind that $\dim \bigwedge^h \mathfrak{g} = \dim \bigwedge_h \mathfrak{g} = \binom{n}{h}$.

The dual space $\bigwedge^1(\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$. The action of a h-covector φ on a h-vector v is denoted as $\langle \varphi | v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_h \mathfrak{g}$ and to $\bigwedge^h \mathfrak{g}$ making the bases $X_{i_1} \wedge \cdots \wedge X_{i_h}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ orthonormal.

We set also $X_{\{1,\cdots,n\}} := X_1 \wedge \cdots \wedge X_n$ and $\theta_{\{1,\cdots,n\}} := \theta_1 \wedge \cdots \wedge \theta_n$.

Starting from $\bigwedge_* \mathfrak{g}$ and $\bigwedge^* \mathfrak{g}$, by left translation, we can define now two families of vector bundles (still denoted by $\bigwedge_* \mathfrak{g}$ and $\bigwedge^* \mathfrak{g}$) over $\mathbb{G}$ (see [**?**] for details). Sections of these vector bundles are said respectively vector fields and differential forms.

Definition 2.5. If $0 \le h \le n$, $1 \le s \le \infty$ and $m \ge 0$, we denote by $W^{m,s}(\mathbb{G}, \bigwedge^h \mathfrak{g})$ the space of all sections of $\bigwedge^h \mathfrak{g}$ such that their components with respect to the basis Θ^h belong to $W^{m,s}(\mathbb{G})$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself.

Sobolev spaces of vector fields are defined in the same way.

We conclude this section recalling the classical definition of Hodge duality: see [**?**] 1.7.8.

Definition 2.6. We define linear isomorphisms

$$* : \bigwedge_h \mathfrak{g} \longleftrightarrow \bigwedge_{n-h} \mathfrak{g} \quad \text{and} \quad * : \bigwedge^h \mathfrak{g} \longleftrightarrow \bigwedge^{n-h} \mathfrak{g},$$

for $1 \le h \le n$, putting, for $v = \sum_I v_I X_I$ and $\varphi = \sum_I \varphi_I \theta_I$,

$$*v := \sum_I v_I (*X_I) \quad \text{and} \quad *\,\varphi := \sum_I \varphi_I (*\theta_I)$$

where

$$*X_I := (-1)^{\sigma(I)} X_{I^*} \quad \text{and} \quad *\,\theta_I := (-1)^{\sigma(I)} \theta_{I^*}$$

with $I = \{i_1, \cdots, i_h\}$, $1 \le i_1 < \cdots < i_h \le n$, $X_I = X_{i_1} \wedge \cdots \wedge X_{i_h}$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$, $I^* = \{i_1^* < \cdots < i_{n-h}^*\} = \{1, \cdots, n\} \setminus I$ and $\sigma(I)$ is the number of couples (i_h, i_ℓ^*) with $i_h > i_\ell^*$.

We refer to $dV := \theta_{\{1,\cdots,n\}}$ as to the canonical volume form of $\mathbb{G}$.

If $v \in \bigwedge_h \mathfrak{g}$ we define $v^\natural \in \bigwedge^h \mathfrak{g}$ by the identity $\langle v^\natural | w \rangle := \langle v, w \rangle$, for all $w \in \bigwedge_h \mathfrak{g}$, and analogously we define $\varphi^\natural \in \bigwedge_h \mathfrak{g}$ for $\varphi \in \bigwedge^h \mathfrak{g}$.

3. Differential forms in Carnot groups

The notion of intrinsic form in Carnot groups is due to M. Rumin ([**?**], [**?**]). A more extended presentation of the results of this section can be found in [**?**], [**?**].

The notion of weight of a differential form plays a key role.

Definition 3.1. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight* p, and we write $w(\alpha) = p$, if $\alpha^\natural \in V_p$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight p if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = p$.

In particular, the canonical volume form dV has weight Q (the homogeneous dimension of the group).

Remark 3.2. If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$.

If we denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{g}$, by prevoius remark it follows that

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{6}$$

Definition 3.3. Let now $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, \theta_i^h \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. Then we can write

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$$

does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{m_1} (X_j\alpha_i)\theta_j \wedge \theta_i^h$$

increases the weight of 1, and, more generally,

$$d_i\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{X_j \in V_i} (X_j\alpha_i)\theta_j \wedge \theta_i^h,$$

when $i = 0, 1, \ldots, \kappa$. In particular, d_0 is an algebraic operator.

Definition 3.4 (M. Rumin)**.** If $0 \le h \le n$ we set

$$E_0^h := \ker d_0 \cap \ker \delta_0 = \ker d_0 \cap (\mathrm{Im}\, d_0)^\perp \subset \Omega^h$$

In the sequel, we refer to the elements of E_0^h as to *intrinsic h-forms on* $\mathbb{G}$. Since the construction of E_0^h is left invariant, this space of forms can be seen as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{g}$, generated by left translation and still denoted by E_0^h. In particular E_0^h inherits from $\bigwedge^h \mathfrak{g}$ the scalar product on the fibers.

Moreover, there exists a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h that is adapted to the filtration (**??**).

Since it is easy to see that $E_0^1 = \mathrm{span}\,\{\theta_1, \ldots, \theta_m\}$, where the θ_i's are dual of the elements of the basis of V_1, without loss of generality, we can take $\xi_j = \theta_j$ for $j = 1, \ldots, m$.

Finally, we denote by $N_h^{\min}$ and $N_h^{\max}$ respectively the lowest and highest weight of forms in E_0^h.

Definition 3.5. If $0 \le h \le n$, $1 \le s \le \infty$ and $m \ge 0$, we denote by $W^{m,s}(\mathbb{G}, E_0^h)$ the space of all sections of E_0^h such that their components with respect to the basis Ξ_0^h belong to $W^{m,s}(\mathbb{G})$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself.

Moreover, as in Proposition 2.4, $\mathcal{D}(\mathbb{G}, E_0^h)$ and $\mathcal{S}(\mathbb{G}, E_0^h)$ are dense in $W^{m,s}(\mathbb{G})$.

Lemma 3.6 ([?], Lemma 2.11)**.** *If $\beta \in \bigwedge^{h+1} \mathfrak{g}$, then there exists a unique $\alpha \in \bigwedge^h \mathfrak{g} \cap (\ker d_0)^\perp$ such that*

$$d_0^* d_0 \alpha = d_0^* \beta. \quad \textit{We set} \quad \alpha := d_0^{-1}\beta.$$

Here $d_0^* : \bigwedge^{h+1} \mathfrak{g} \to \bigwedge^h \mathfrak{g}$ is the adjoint of d_0 with respect to our fixed scalar product. In particular

$$\alpha = d_0^{-1}\beta \quad \textit{if and only if} \quad d_0\alpha - \beta \in \mathcal{R}(d_0)^\perp.$$

Moreover

i) $(\ker d_0)^\perp = \mathcal{R}(d_0^{-1})$;

ii) $d_0^{-1} d_0 = Id$ *on* $(\ker d_0)^\perp$;

iii) $d_0 d_0^{-1} - Id : \bigwedge^{h+1} \mathfrak{g} \to \mathcal{R}(d_0)^\perp$.

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [**?**] and [**?**], Section 2) .

Theorem 3.7. *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1} d) \quad \text{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(d d_0^{-1}).$$

We have:

i) *Let Π_E be the projection on E along F (that is not an orthogonal projection). Then for any $\alpha \in E_0^{h,p}$, if we denote by $(\Pi_E \alpha)_j$ the component of $\Pi_E \alpha$ of weight j, then*

$$(7) \qquad \begin{aligned} (\Pi_E \alpha)_p &= \alpha \\ (\Pi_E \alpha)_{p+k+1} &= -d_0^{-1}\big(\sum_{1 \le \ell \le k+1} d_\ell (\Pi_E \alpha)_{p+k+1-\ell} \big). \end{aligned}$$

Notice that $\alpha \to (\Pi_E \alpha)_{p+k+1}$ is an homogeneous differential operator of order $k+1$ in the horizontal derivatives.

ii) *Π_E is a chain map, i.e.*

$$d \Pi_E = \Pi_E d.$$

iii) *Let Π_{E_0} be the orthogonal projection from Ω^* on E_0^*, then*

$$(8) \qquad \Pi_{E_0} = Id - d_0^{-1} d_0 - d_0 d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1} d_0 + d_0 d_0^{-1}.$$

Notice that, since d_0 and d_0^{-1} are algebraic, then formulas (8) *hold also for covectors.*

iv) *$\Pi_{E_0} \Pi_E \Pi_{E_0} = \Pi_{E_0}$ and $\Pi_E \Pi_{E_0} \Pi_E = \Pi_E$.*

Set now

$$d_c = \Pi_{E_0}\, d\, \Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \dots, n-1.$$

We have:

v) *$d_c^2 = 0$;*

vi) *the complex $E_0 := (E_0^*, d_c)$ is exact;*

vii) *with respect to the bases Ξ_0^* the intrinsic differential d_c can be seen as a matrix-valued operator such that, if α has weight p, then the component of weight q of $d_c \alpha$ is given by an homogeneous differential operator in the horizontal derivatives of order $q - p \ge 1$, acting on the components of α.*

Remark 3.8. Let us give a gist of the construction of E. The map $d_0^{-1} d$ induces an isomorphism from $\mathcal{R}(d_0^{-1})$ to itself. Thus, since $d_0^{-1} d_0 = Id$ on $\mathcal{R}(d_0^{-1})$, we can write $d_0^{-1} d = Id + D$, where D is a differential operator that increases the weight. Clearly, $D : \mathcal{R}(d_0^{-1}) \to \mathcal{R}(d_0^{-1})$. As a consequence of the nilpotency of $\mathbb{G}$, $D^k = 0$ for k large enough, and therefore the Neumann series of $d_0^{-1} d$ reduces to a finite sum on $\mathcal{R}(d_0^{-1})$. Hence there exist a differential operator

$$P = \sum_{k=1}^{N} (-1)^k D^k, \quad N \in \mathbb{N} \text{ suitable},$$

such that

$$P d_0^{-1} d = d_0^{-1} d P = \mathrm{Id}_{\mathcal{R}(d_0^{-1})}.$$

We set $Q := P d_0^{-1}$. Then Π_E is given by

$$\Pi_E = Id - Qd - dQ.$$

From now on, we restrict ourselves to assume $\mathbb{G}$ is a free group of step κ (see Definition 2.1 above). The technical reason for this choice relies in the following property.

Theorem 3.9 ([?], Theorem 5.9)**.** *Let $\mathbb{G}$ be a free group of step κ. Then all forms in E_0^1 have weight 1 and all forms in E_0^2 have weight $\kappa + 1$.*

In particular, the differential $d_c : E_0^1 \to E_0^2$ can be identified, with respect to the adapted bases Ξ_0^1 and Ξ_0^2, with a homogeneous matrix-valued differential operator of degree κ in the horizontal derivatives.

Moreover, if $\xi \in \bigwedge^{2,p} \mathfrak{g}$ with $p \neq \kappa + 1$, then $\Pi_{E_0}\xi = 0$. Indeed, $\Pi_{E_0}\xi$ has weight p, and therefore has to be zero, since $\Pi_{E_0}\xi \in \bigwedge^{2,\kappa+1} \mathfrak{g}$.

Lemma 3.10. *If $\mathbb{G}$ is a free group of step $\kappa \geq 2$, then*

$$\bigwedge\nolimits^{2,2} \mathfrak{g} \subset d_0(\bigwedge\nolimits^{1,2} \mathfrak{g}) \subset \mathcal{R}(d_0),$$

or, equivalently,

$$\Omega^{2,2} \subset d_0(\Omega^{1,2}) \subset d_0(\Omega^1).$$

Proof. See Lemma 3.12 in [?].

□

4. Intrinsic differential as a Γ-limit

Definition 4.1. Let X be a separated topological space, and let

$$F_\varepsilon, \ F : \ X \longrightarrow [-\infty, +\infty]$$

with $\varepsilon > 0$ be functionals on X. We say that $\{F_\varepsilon\}_{\varepsilon>0}$ sequentially Γ-converges to F on X as ε goes to zero if the following two conditions hold:

1) for every $u \in X$ and for every sequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$, which converges to u in X, there holds

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \geq F(u); \tag{9}$$

2) for every $u \in X$ and for every sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ there exists a subsequence (still denoted by $\{\varepsilon_k\}_{k\in\mathbb{N}}$) such that $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ converges to u in X and

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \leq F(u) \tag{10}$$

For a deep and detailed survey on Γ-convergence, we refer to the monograph [**?**].

We recall the following reduction Lemma. The proof is only a minor variant of the one given in [**?**], Lemma *IV* (see also [**?**]), hence we shall omit such a proof.

Lemma 4.2. *Let X be a separated topological space, let $F_h,\ F:\ M \longrightarrow [-\infty,+\infty]$ with $h\in\mathbb{N}$; consider $D\subset M$ and $x\in M$. Let us suppose that*

1) *for every $y\in D$ there exists a sequence $(y_h)_{h\in\mathbb{N}}\subset M$ such that $y_h\to y$ in M and $\limsup_{h\to\infty} F_h(y_h)\le F(y)$;*
2) *there exists a sequence $(x_h)_{h\in\mathbb{N}}\subset D$ such that $x_h\to x$ and $\limsup_{h\to\infty} F(x_h)\le F(x)$;*

then there exists a sequence $(\overline{x}_h)_{h\in\mathbb{N}}\subset M$ such that $\limsup_{h\to\infty} F_h(\overline{x}_h)\le F(x)$.

To avoid cumbersome notations, from now on we write systematically $\lim_{\varepsilon\to} 0$ to mean a limit with $\varepsilon=\varepsilon_k$, where $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is any sequence with $\varepsilon_k\to 0$ as $k\to\infty$.

Let $\varepsilon>0$ be given. If $\omega\in W^{\kappa,2}(\mathbb{G},\bigwedge^1\mathfrak{g})$, we set

$$F_\varepsilon(\omega)=\frac{1}{\varepsilon^{2\kappa}}\int_{\mathbb{G}}|d_\varepsilon\omega|^2\,dV,$$

where

$$d_\varepsilon=d_0+\varepsilon d_1+\cdot+\varepsilon^\kappa d_\kappa.$$

We stress that $F_\varepsilon(\omega)$ is always finite, since the coefficients of $d_i\omega$ contain horizontal derivatives of order $i\le\kappa$ of the coefficients of ω.

Theorem 4.3. *Let $\mathbb{G}$ be a free Carnot group of step κ. Then*

F_ε sequentially Γ-coverges to F in the weak topology $W^{\kappa,2}(\mathbb{G},\bigwedge^1\mathfrak{g})$,

as $\varepsilon\to 0$, where

$$F(\omega)=\begin{cases}\displaystyle\int_{\mathbb{G}}|d_c\omega|^2\,dV & \text{if } \omega\in W^{\kappa,2}(\mathbb{G},E_0^1)\\ +\infty & \text{otherwise.}\end{cases}$$

Proof. A detailed proof of this theorem can be found in [**?**]. Here we repeat only the main steps of that proof.

Let $\omega^\varepsilon\to\omega$ as $\varepsilon\to 0$ weakly in $W^{\kappa,2}(\mathbb{G},\bigwedge^1\mathfrak{g})$. We want to show that

$$F(\omega)\le\liminf_{\varepsilon\to 0}F_\varepsilon(\omega^\varepsilon). \tag{11}$$

In particular, it follows that $\omega\in W^{\kappa,2}(\mathbb{G},E_0^1)$ provided $\liminf_{\varepsilon\to 0}F_\varepsilon(\omega^\varepsilon)<\infty$.

Keeping in mind (6), we write

$$\omega^\varepsilon=\omega_1^\varepsilon+\cdots+\omega_\kappa^\varepsilon,$$

with $\omega_i^\varepsilon \in \Omega^{1,i}$, $i = 1, \dots, \kappa$. Reordering the terms of $d_\varepsilon \omega_\varepsilon$ according to their weights, as in (6), we have the following orthogonal decomposition:

$$
\begin{aligned}
d_\varepsilon \omega^\varepsilon = & \sum_{2\le p\le\kappa} \sum_{i=0}^{p-1} \varepsilon^i d_i \omega_{p-i}^\varepsilon \\
& + \left(\varepsilon d_1 \omega_\kappa^\varepsilon + \cdots + \varepsilon^\kappa d_\kappa \omega_1^\varepsilon\right) \\
& + \sum_{\kappa+2\le p\le 2\kappa} \sum_{i=p-\kappa}^{\kappa} \varepsilon^i d_i \omega_{p-i}^\varepsilon .
\end{aligned}
\tag{12}
$$

Therefore we can write

$$
\begin{aligned}
F_\varepsilon(\omega_\varepsilon) = & \varepsilon^{-2\kappa} \sum_{2\le p\le\kappa} \int_{\mathbb{G}} \Big\| \sum_{i=0}^{p-1} \varepsilon^i d_i \omega_{p-i}^\varepsilon \Big\|^2 dV \\
& + \varepsilon^{2(1-\kappa)} \int_{\mathbb{G}} \left\| d_1 \omega_\kappa^\varepsilon + \cdots + \varepsilon^{\kappa-1} d_\kappa \omega_1^\varepsilon \right\|^2 dV \\
& + \sum_{\kappa+2\le p\le 2\kappa} \varepsilon^{2(p-2\kappa)} \int_{\mathbb{G}} \Big\| \sum_{i=p-\kappa}^{\kappa} \varepsilon^{i-p+\kappa} d_i \omega_{p-i}^\varepsilon \Big\|^2 dV.
\end{aligned}
\tag{13}
$$

Without loss of generality, we may assume $\liminf_{\varepsilon\to 0} F_\varepsilon(\omega_\varepsilon) < \infty$. This implies that, if $2 \le p \le \kappa$, then, if $\varepsilon \in (0,1)$,

$$
\varepsilon^{-\kappa} \sum_{i=0}^{p-1} \varepsilon^i d_i \omega_{p-i}^\varepsilon \quad \text{is uniformly bounded in } L^2(\mathbb{G}, {\textstyle\bigwedge^2} \mathfrak{g})). \tag{14}
$$

In particular,

$$
\sum_{i=0}^{p-1} \varepsilon^i d_i \omega_{p-i}^\varepsilon \longrightarrow 0 \quad \text{in } L^2(\mathbb{G}, {\textstyle\bigwedge^2} \mathfrak{g})) \tag{15}
$$

as $\varepsilon \to 0$, since we can write (15) as

$$
d_0 \omega_p^\varepsilon + \varepsilon \sum_{i=1}^{p-1} \varepsilon^{i-1} d_i \omega_{p-i}^\varepsilon \longrightarrow 0 \tag{16}
$$

as $\varepsilon \to 0$. On the other hand, we know that $\omega_p^\varepsilon \to \omega_p$ weakly in $L^2(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ for $p \ge 1$, and therefore

$$
d_0 \omega_p^\varepsilon \to d_0 \omega_p \quad \text{in } L^2(\mathbb{G}, {\textstyle\bigwedge^2} \mathfrak{g}), \tag{17}
$$

since d_0 is algebraic.

Combining (16) with the boundedness of $\{\omega^\varepsilon\}$ in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ and with (17), it follows that

$$
d_0 \omega_p = 0 \quad \text{for } p = 2, \dots, \kappa \tag{18}
$$

(obviously, (18) holds also for $p = 1$ since $d_0(\bigwedge^{1,1} \mathfrak{g}) = \{0\}$). Hence $\omega \in \ker d_0 = E_0^1$, and therefore $\omega = \omega_1$.

Moreover, again if $\varepsilon \in (0,1)$,

$$
\varepsilon^{1-\kappa} \big(d_1 \omega_\kappa^\varepsilon + \cdots + \varepsilon^{\kappa-1} d_\kappa \omega_1^\varepsilon \big) \quad \text{is uniformly bounded in } L^2(\mathbb{G}, {\textstyle\bigwedge^2} \mathfrak{g})). \tag{19}
$$

Recall now that, by definition, $d_c\omega = \Pi_{E_0} d\Pi_E\omega$. But, by Theorem 3.9, Π_{E_0} vanishes on all 2-forms of weight $p \neq \kappa+1$. Therefore, the full expression of $d_c\omega$ reduces to

$$d_c(\omega) = \Pi_{E_0}\Big(\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell}\Big). \tag{20}$$

As is proved in [?], it holds that

$$d_j(\Pi_E\omega)_{\kappa+1-\ell} = \lim_{\varepsilon\to 0} \varepsilon^{\ell-\kappa} d_j\omega^\varepsilon_{\kappa+1-\ell}, \tag{21}$$

in the sense of distributions for $\ell = 1,\dots,\kappa$, $j = 0,\dots,\ell$, i.e.

$$\int \langle \varepsilon^{\ell-\kappa} d_j\omega^\varepsilon_{\kappa+1-\ell}, \varphi\rangle\, dV \to \int \langle d_j(\Pi_E\omega)_{\kappa+1-\ell}, \varphi\rangle\, dV \tag{22}$$

for $\ell = 1,\dots,\kappa$, $j = 0,\dots,\ell$, and for any $\varphi \in \mathcal{D}(\mathbb{G}, \bigwedge^2 \mathfrak{g})$.

So far, we have used the equiboundedness of the first sum in (13) for ε close to zero. We proceed now to estimate the lim inf of the second term in (13).

To this end, we take $j = \ell$ in (21) and we sum up for $\ell = 1,\dots,\kappa$. We obtain

$$\frac{1}{\varepsilon^{\kappa-1}}\Big(d_1\omega^\varepsilon_\kappa + \dots + \varepsilon^{\kappa-1} d_\kappa\omega^\varepsilon_1\Big) \longrightarrow \sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell} \tag{23}$$

as $\varepsilon \to 0$ in the sense of distributions. On the other hand, the limit $\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell}$ belongs to $L^2(\mathbb{G}, \bigwedge^2 \mathfrak{g})$ (since $d_\ell(\Pi_E\omega)_{\kappa+1-\ell}$ is an homogeneous differential operator in the horizontal derivatives of order κ, by Theorem 3.7, i) and Definition 3.3), and

$$\Big\{\frac{1}{\varepsilon^{\kappa-1}}\big(d_1\omega^\varepsilon_\kappa + \dots + \varepsilon^{\kappa-1} d_\kappa\omega^\varepsilon_1\big)\Big\}_{\varepsilon>0} \quad \text{is equibounded in } L^2(\mathbb{G}, \textstyle\bigwedge^2 \mathfrak{g}), \tag{24}$$

as $\varepsilon \to 0$, by (19). Combining (24) and (23) we obtain that the limit in (23) is in fact a weak limit in $L^2(\mathbb{G}, \bigwedge^2 \mathfrak{g})$ (see, e.g., [?], Ch. V, Theorem 3). Thus, by (20), (13) and taking into account that Π_{E_0} is an orthogonal projection, we obtain eventually

$$\begin{aligned} F(\omega) &= \int_{\mathbb{G}} \|\Pi_{E_0}\big(\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell}\big)\|^2\, dV \\ &\le \int_{\mathbb{G}} \|\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell}\|^2\, dV \\ &\le \liminf_{\varepsilon\to 0} \varepsilon^{2(1-\kappa)} \int_{\mathbb{G}} \big\|d_1\omega^\varepsilon_\kappa + \dots + \varepsilon^{\kappa-1} d_\kappa\omega^\varepsilon_1\big\|^2\, dV \le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega_\varepsilon). \end{aligned}$$

This proves (11).

We prove now that, if $\omega \in W^{\kappa,2}(\mathbb{G}, E_0^1)$, then there exists a sequence $(\omega_\varepsilon)_{\varepsilon>0}$ in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ such that

i) $\omega_\varepsilon \to \omega$ weakly in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$;
ii) $F_\varepsilon(\omega_\varepsilon) \to F(\omega)$ as $\varepsilon \to 0$.

By Lemma 4.2, without loss of generality we may assume $\omega \in \mathcal{D}(\mathbb{G}, E_0^1)$.

We choose

$$\omega_\varepsilon = \omega + \varepsilon(\Pi_E\omega)_2 + \cdots + \varepsilon^{\kappa-1}(\Pi_E\omega)_\kappa. \tag{25}$$

If we write the identity $d^2 = 0$ gathering all terms of the same weight, we get

$$0 = \sum_{p=0}^{\kappa}\sum_{j=0}^{p} d_{p-j}d_j.$$

and therefore

$$\sum_{j=0}^{p} d_{p-j}d_j = 0 \quad \text{for } p = 0, \dots, \kappa, \tag{26}$$

since these terms are mutually orthogonal when applied to a form of pure weight. In particular,

$$d_0^2 = 0, \quad d_0d_1 = -d_1d_0, \quad d_0d_2 = -d_2d_0 - d_1^2, \quad \cdots \tag{27}$$

Thus,

$$\begin{aligned} F_\varepsilon(\omega_\varepsilon) &= \frac{1}{\varepsilon^{2\kappa}}\int_{\mathbb{G}} \|d_\varepsilon\big(\sum_{i=1}^{\kappa}\varepsilon^{i-1}(\Pi_E\omega)_i\big)\|^2\,dV \\ &= \frac{1}{\varepsilon^{2\kappa}}\left(\int_{\mathbb{G}} \|\Pi_{E_0}\big(d_\varepsilon\big(\sum_{i=1}^{\kappa}\varepsilon^{i-1}(\Pi_E\omega)_i\big)\big)\|^2\,dV\right. \\ &\qquad \left. + \int_{\mathbb{G}} \|\Pi_{E_0^\perp}\big(d_\varepsilon\big(\sum_{i=1}^{\kappa}\varepsilon^{i-1}(\Pi_E\omega)_i\big)\big)\|^2\,dV\right). \end{aligned} \tag{28}$$

Arguing as in (12), we can write

$$\begin{aligned} d_\varepsilon\big(&\sum_{i=1}^{\kappa}\varepsilon^{i-1}(\Pi_E\omega)_i\big) \\ &= \sum_{2\le p\le\kappa}\varepsilon^{p-1}\sum_{i=0}^{p-1} d_i(\Pi_E\omega)_{p-i} \\ &\quad + \varepsilon^\kappa\big(d_1(\Pi_E\omega)_\kappa + \cdots + d_\kappa(\Pi_E\omega)_1\big) \\ &\quad + \sum_{\kappa+2\le p\le 2\kappa}\varepsilon^{p-1}\sum_{i=p-\kappa}^{\kappa} d_i(\Pi_E\omega)_{p-i} \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Now, by Theorem 3.9,

$$\Pi_{E_0}I_1 = 0.$$

As it is proved in [**?**]

$$\sum_{i=0}^{p-1} d_i(\Pi_E\omega)_{p-i} \in E_0^2 \quad \text{for } 2 \le p \le \kappa + 1. \tag{29}$$

and hence

$$\Pi_{E_0}^\perp I_1 = \Pi_{E_0}^\perp I_2 = 0. \tag{30}$$

Recalling (28) we get,

$$
\begin{aligned}
F_\varepsilon(\omega_\varepsilon) &= \frac{1}{\varepsilon^{2\kappa}}\int_{\mathbb{G}} \|\Pi_{E_0} I_2\|^2\, dV + \frac{1}{\varepsilon^{2\kappa}}\int_{\mathbb{G}} \|\ I_3\|^2\, dV \\
&= \int_{\mathbb{G}} \|\Pi_{E_0}\big(d_1(\Pi_E\omega)_\kappa + \cdots + d_\kappa(\Pi_E\omega)_1\big)\|^2\, dV \\
&+ \frac{1}{\varepsilon^{2\kappa}}\int_{\mathbb{G}} \|\sum_{\kappa+2\le p\le 2\kappa} \varepsilon^{p-1}\sum_{i=p-\kappa}^{\kappa} d_i(\Pi_E\omega)_{p-i}\|^2\, dV;
\end{aligned}
$$

observing that the second term in previous expression goes to zero as $\varepsilon \to 0$, we get $\lim_{\varepsilon\to 0} F_\varepsilon(\omega_\varepsilon) = F(\omega)$ in $\mathcal{D}(\mathbb{G}, E_0^1)$. This achieves the proof of the theorem.

□

5. Maxwell's equations

If $L : \bigwedge_h \mathfrak{g} \to \bigwedge_h \mathfrak{g}$ or $L : \bigwedge^h \mathfrak{g} \to \bigwedge^h \mathfrak{g}$, we denote by $[L]$ the matrix of L with respect to the Euclidean canonical basis $\{e_I\}$ and $\{dx^I\}$.

As pointed out in the introduction, classical Maxwell's equations for time-harmonic vector fields in $\mathbb{R}^\times\mathbb{R}^3$

$$
e^{i\omega t}\vec{E}, e^{i\omega t}\vec{H}, e^{i\omega t}\vec{B}, e^{i\omega t}\vec{D}\ ,
$$

where we denote by $s \in \mathbb{R}$ the time variable and by $x \in \mathbb{R}^3$ the space variable, read as follows:

$$
\begin{gathered}
\operatorname{curl} \vec{H} - \frac{i\omega}{c}\vec{D} = 0, \\
\operatorname{curl} \vec{E} + \frac{i\omega}{c}\vec{B} = 0, \\
\operatorname{div} \vec{D} = 0, \quad \operatorname{div} \vec{B} = 0,
\end{gathered}
$$

together with the constitutive relations

$$
\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H},
$$

where $\varepsilon = \varepsilon(x), \mu = \mu(x) : \mathbb{R}^3 \to \mathbb{R}^3$ are linear maps depending (say) smoothly on x.

Considering the 2-form $D := -(*\vec{D})^\natural$ the previous equations can be stated in terms of de Rham differential forms. As it is proved in [?] it holds the following Lemma.

Lemma 5.1. *If $\alpha \in \Omega^h$, then*

$$
*(\Lambda^h L)\alpha = (-1)^{h(n-h)} \cdot (\det L) \cdot (\Lambda^{n-h}(L^t)^{-1}) * \alpha.
$$

Hence

$$
(\Lambda^1(\mu^t)^{-1}) = * * (\det\mu)^{-1}(\Lambda^2\mu) = (\det\mu)^{-1}(\Lambda^2\mu).
$$

Thus, eventually, $*D$ satisfies the differential equation

$$
\delta M d N\alpha + \frac{\omega^2\cdot\det\mu}{c^2}\alpha = 0, \tag{31}
$$

where $M := \Lambda^2\mu$, $N := \Lambda^1(\varepsilon^t)^{-1}$. Obviously (31) makes perfectly sense in $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Hence $*D$ is a stationary point of the functional

$$J^{\mu,\varepsilon}(\alpha) := \int_{\mathbb{R}^n} \langle M dN\alpha, dN\alpha\rangle_{\rm Euc}\, dV + \frac{\omega^2 \cdot \det \mu}{c} \int_{\mathbb{R}^n} \langle N\alpha, \alpha\rangle_{\rm Euc}\, dV,$$

where $\alpha \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$.

The identification of $\bigwedge^h \mathfrak{g}$ and $\bigwedge^h_e \mathfrak{g}$ yields a corresponding identification of the basis Θ^h of $\bigwedge^h \mathfrak{g}$ and Θ^h_e of $\bigwedge^h_e \mathfrak{g}$. Then $\Theta^h_x := \Lambda^h(d\tau_{x^{-1}})\Theta^h_e$ is a basis of $\bigwedge^h_x \mathfrak{g}$. Analogously, if we replace the group translation $\tau_{x^{-1}}$ by the Euclidean translation $y \to y - x := t_{-x}(y)$, we obtain a new basis $\Theta^{h,\rm Euc}_x := \Lambda^h(dt_{-x})\Theta^h_e$ of $\bigwedge^h_x \mathfrak{g}$.

In particular,

$$\Theta^h_x := \Lambda^h(d\tau_{x^{-1}})\Lambda^h(dt_x)\Theta^{h,\rm Euc}_x.$$

i.e., the elements of Θ^h can be viewed as differential forms in the Euclidean coordinates. Thus, the elements of Θ^h_x can be identified with the elements of Θ^h evaluated at the point x. Through all this paper, we make systematic use of these identifications, interchanging the roles of left invariant vector fields and elements of $\bigwedge_1 \mathfrak{g}$.

Denote by $[T]_x = (T_{ij}(x))_{ij}$ the matrix of the identity map in $\mathbb{R}^n$ with respect to the bases $\{e_1, \dots, e_n\}$ and $\{X_1, \dots, X_n\}$, i.e.

$$X_i(x) = \sum_j T_{ij}(x) e_j, \quad j = 1, \dots, n.$$

Hence $[T]_x$ coincides with $[d\tau_{x^{-1}}]$, the matrix of $d\tau_{x^{-1}}$ with respect to the canonical bases of $\bigwedge_{1,x} \mathfrak{g}$ and $\bigwedge_{1,e} \mathfrak{g}$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we denote by $[\Lambda^h T]_x$ the matrix associated with the identity map from $\bigwedge^h \mathfrak{g}$ to $\bigwedge^h \mathfrak{g}$. We stress that $[\Lambda^h T]_x = [\Lambda^h d\tau_{x^{-1}}]$.

Remark 5.2. We keep in mind that the two bases $\{e_1, \dots, e_n\}$ and $\{X_1, \dots, X_n\}$ (as well as the associated bases for vectors and covectors) coincide at $x = e$. Therefore, there is no ambiguity if we identify, say, a h-covector at the origin with a h-covector of the Euclidean space $\mathbb{R}^n$.

Lemma 5.3. *If $\alpha, \beta \in \bigwedge^h_x \mathfrak{g}$, then*

$$\langle \alpha, \beta\rangle_x = \langle [\Lambda^h T^{-1}]\alpha, [\Lambda^h T^{-1}]\beta\rangle_{\bigwedge^h \mathbb{R}^n}.$$

Proof. Keeping in mind Remark 5.2, we have

$$\begin{aligned}\langle \alpha, \beta\rangle_x &= \langle (\Lambda^h d\tau_x)\alpha, (\Lambda^h d\tau_x)\beta\rangle_e = \langle [\Lambda^h d\tau_x]\alpha, [\Lambda^h d\tau_x]\beta\rangle_{\bigwedge^h \mathbb{R}^n} \\ &= \langle (\Lambda^h d\tau_x)\alpha, (\Lambda^h d\tau_x)\beta\rangle_{{\rm Euc},e} = \langle (\Lambda^h d\tau_x)\alpha, (\Lambda^h d\tau_x)\beta\rangle_{{\rm Euc},x}.\end{aligned}$$

□

If $I = (i_1, \dots, i_h)$, the I-component of $\alpha \in \bigwedge^h_x \mathfrak{g}$ with respect to Θ^h_x equal the I-component of $(\Lambda^h T)\alpha$ with respect to $\Theta^{h,\rm Euc}_x$. Indeed $\langle \alpha, \theta^I\rangle = \langle \alpha | X_I\rangle = \langle \alpha | (\Lambda_h T) X_I\rangle = \langle (\Lambda^h T)\alpha | e_I\rangle = \langle (\Lambda^h T)\alpha, dx^I\rangle_{\rm Euc}$.

Finally, if $r > 0$, we denote by C_r the linear map on $\mathfrak{g}$ defined by

$$C_r(X_\ell) := r^j X_\ell \quad \text{if } X_\ell \in V_j. \tag{32}$$

Notice $(\Lambda^2 C_r)\theta_i \wedge \theta_j = r^{w(i)+w(j)}\theta_i \wedge \theta_j$ if $w(i), w(j)$ are the weights of the $\theta_i, \theta_j \in \bigwedge^1 \mathfrak{g}$, respectively.

Lemma 5.4. *If $r > 0$ and*

$$d_r = d_0 + rd_1 + \cdot + r^\kappa d_\kappa,$$

then

$$d_r\alpha = (\Lambda^2 C_r)d(\Lambda^1 C_r^{-1})\alpha$$

for any $\alpha \in \Omega^1$.

Proof. Choose $\alpha = \alpha_i\theta_i$. Then

$$d_r\alpha = \sum_{j=0}^{\kappa} r^j d_j(\alpha_i\theta_i) = \sum_{j=0}^{\kappa} r^{j+w(i)} d_j(r^{-w(i)}\alpha_i\theta_i) = (\Lambda^2 C_r)d(\Lambda^1 C_r^{-1})\alpha.$$

□

Proposition 5.5. *Let $r > 0$ be given. If we choose*

$$[\mu_r] = r^{(1+\kappa-2Q)/(n-1)}[(T^{-1}C_rT)^t T^{-1}C_rT]$$

and

$$[\varepsilon_r] = r^{(1+n\kappa-2Q)/(n-1)}[T^{-1}C_rT]^t,$$

then, if $\alpha \in W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^1 \mathfrak{g})\Big(\subset W^{1,2}_{\rm loc}(\mathbb{G}, \bigwedge^1 \mathfrak{g})\Big)$,

$$r^{-2\kappa}|d_r\alpha|^2 = \langle M_r dN_r\alpha, dN_r\alpha\rangle_{\rm Euc} \tag{33}$$

and

$$r^{-(1+n\kappa-2Q)/(n-1)}\langle C_r^{-1}\alpha, \alpha\rangle = \langle N_r\alpha, \alpha\rangle_{\rm Euc}, \tag{34}$$

where $M_r := \Lambda^2\mu_r$, $N_r := \Lambda^1(\varepsilon_r^t)^{-1}$.

Proof. We have:

$$\begin{aligned}|d_r\alpha|_x^2 &= \langle(\Lambda^2 C_r)d(\Lambda^1 C_r^{-1})\alpha, (\Lambda^2 C_r)d(\Lambda^1 C_r^{-1})\alpha\rangle_x \\ &= \langle(\Lambda^2(d\tau_x C_r))d(\Lambda^1 C_r^{-1})\alpha, (\Lambda^2(d\tau_x C_r))d(\Lambda^1 C_r^{-1})\alpha\rangle_{{\rm Euc},x} \\ &= \langle[\Lambda^2((T^{-1}C_rT)^t(T^{-1}C_rT))]d[\Lambda^1(T^{-1}C_r^{-1}T)]\alpha, d[\Lambda^1(T^{-1}C_r^{-1}T)]\alpha\rangle_{\bigwedge^2\mathbb{R}^n}.\end{aligned}$$

Then the assertion follows since $(\Lambda^2\sigma L) = \sigma^2(\Lambda^2 L)$ for any linear map L and for any $\sigma \in \mathbb{R}$, and

$$r^{2(1+\kappa-2Q)/(n-1)} \cdot r^{(2Q-n\kappa-1)/(n-1)} \cdot r^{(2Q-n\kappa-1)/(n-1)} = r^{-2\kappa}.$$

This proves (33). Identity (34) follows analogously. □

Theorem 5.6. *With the notations of Proposition 5.5, if $r > 0$, we denote by J_r the functional $J^{\mu_r,\varepsilon_r}(\alpha)$ in $W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ Then J_r sequentially Γ-coverges to J in the weak topology $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$, as $r \to 0$, where*

$$J(\alpha) = \begin{cases} \displaystyle\int_{\mathbb{G}} |d_c\alpha|^2\, dV + \frac{\omega^2}{c^2}\int_{\mathbb{G}} |\alpha|^2\, dV & \text{if } \alpha \in W^{\kappa,2}(\mathbb{G}, E_0^1) \\ +\infty \quad \text{otherwise.} \end{cases}$$

Proof. A proof of this theorem is given in [**?**]. Here we sketch briefly only the liminf part.

First of all, we notice that

$$\det[\mu_r] = r^{(n+n\kappa-2Q)/(n-1)}.$$

Keeping also in mind Proposition 5.5, if $\alpha \in W^{\kappa,2}_{\mathbb{G}}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ and $J_r(\alpha) < \infty$, writing α with respect to the basis $\theta_1, \dots, \theta_n$, we have

$$J_r(\alpha) = r^{-2\kappa} \int_{\mathbb{G}} |d_r\alpha|^2 \, dV + \frac{\omega^2 \, r}{c^2} \int_{\mathbb{G}} \langle C_r^{-1}\alpha, \alpha\rangle \, dV.$$

Let now $\alpha^r \to \alpha$ as $r \to 0$ weakly in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$. We want to show that

$$J(\alpha) \le \liminf_{r\to 0} J_r(\alpha^r). \tag{35}$$

As usual, without loss of generality, we may assume $\liminf_{r\to 0} J_r(\alpha^r) < \infty$. Thus, by Theorem 4.3, $\alpha \in W^{\kappa,2}(\mathbb{G}, E_0^1)$ and

$$\int_{\mathbb{G}} |d_c\alpha|^2 \, dV \le \liminf_{r\to 0} r^{-2\kappa} \int_{\mathbb{G}} |d_r\alpha|^2 \, dV. \tag{36}$$

On the other hand, if we split α^r gathering the terms by their weights (i.e. $\alpha^r = \alpha_1^r + \dots + \alpha_\kappa$, with $\alpha_j^r \in \Omega^{1,j}$), keeping in mind that $\alpha_1^r \to \alpha_1$ weakly in L^2, we have

$$\begin{aligned} \liminf_{r\to 0} \int_{\mathbb{G}} r\langle C_r^{-1}\alpha, \alpha\rangle \, dV &= \liminf_{r\to 0} \Big(\int_{\mathbb{G}} |\alpha_1^r|^2 \, dV + \sum_{j>1} r^{1-j} \int_{\mathbb{G}} |\alpha_j^r|^2 \, dV \Big) \\ &\ge \liminf_{r\to 0} \int_{\mathbb{G}} |\alpha_1^r|^2 \, dV \ge \int_{\mathbb{G}} |\alpha_1|^2 \, dV = \int_{\mathbb{G}} |\alpha|^2 \, dV, \end{aligned}$$

since $\alpha \in E_0^1$. Summing this inequality with inequality (36) we get the liminf inequality (35) $\square$