Seminario di Analisi Matematica

Dipartimento di Matematica
dell'Università di Bologna

Anno Accademico 2008-09

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Stime dell'energia e simmetria 1-dimensionale per equazioni frazionarie [1]

16 aprile 2009

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Abstract

We establish energy estimates for solutions of the fractional nonlinear equation $(-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^n$ where $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1,\beta}$ function for some $0 < \beta < 1$. As a consequence we deduce 1-d symmetry for monotone solutions in dimension 3.

1. Introduction

We establish energy estimates for solutions of the fractional nonlinear equation

$$(1) \qquad (-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^n$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^{1,\beta}$ function for some $0 < \beta < 1$.
The fractional powers of the Laplacian, which are called fractional Laplacians appear in anomalous diffusion phenomena in physics, biology as well as other areas. They occur in flame propagation, chemical reaction in liquids, population dynamics. Moreover, fractional Laplacians play an important role in the study of the quasi-geostrophic equations in geophysical fluid dynamics. Recently the fractional Laplacians attract much interest in nonlinear analysis. Caffarelli and Silvestre gave a new formulation of the fractional Laplacians through Dirichlet-Neumann maps.
We define

$$G(u) = \int_u^1 f.$$

In our results we assume some, or all, of the following conditions on f. For G defined as above:

$$(2) \qquad \begin{aligned} &(H1) \quad f \ \text{ is odd;} \\ &(H2) \quad G \geq 0 = G(\pm 1) \text{ in } \mathbb{R}, \text{ and } G > 0 \text{ in } (-1,1); \\ &(H3) \quad f' \ \text{ is decreasing in } (0,1). \end{aligned}$$

Note that, if (H1) and (H2) hold, then $f(0) = f(\pm 1) = 0$. Conversely, if f is odd in $\mathbb{R}$, positive with f' decreasing in $(0,1)$ and negative in $(1,\infty)$ then f satisfies (H1), (H2) and (H3). Hence, the nonlinearities f that we consider are of "balanced bistable type", while the potentials G are of "double well type". Our three assumptions (H1),(H2),(H3) are satisfied for the Allen-Cahn type equation

$$(3) \qquad (-\Delta)^{1/2}u = u - u^3.$$

In this case we have that $G(u) = (1/4)(1-u^2)^2$ and (H1),(H2),(H3) hold. The three hypothesis also hold for the equation $(-\Delta)^{1/2}u = \sin(\pi u)$, for which

$G(u) = (1/\pi)(1 + \cos(\pi u))$. We call equation (3) of Allen-Cahn type by the analogy with the corresponding equation involving the Laplacian instead of the half-Laplacian:

$$-\Delta u = u - u^3. \tag{4}$$

In 1978 De Giorgi conjectured that the level sets of every bounded, monotone in one direction, solutions of equation (4) in $\mathbb{R}^n$, must be hyperplanes, at least if $n \leq 8$, i.e. the solution u only depends on one variable. The conjecture has been proved to be true in dimension $n = 2$ by Ghoussoub and Gui [10], in dimension $n = 3$ by Ambrosio and Cabré [1]. For $4 \leq n \leq 8$ and assuming the additional limiting condition on u

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1,$$

it has been established by Savin [13]. Recently a counterexample to the conjecture for $n \geq 9$ has been announced by Del Pino, Kowalczyk and Wei.

In Theorem 1.3 we prove 1-d symmetry for monotone solutions u of the equation (1) in dimension $n = 3$, that is the analog of the Conjecture of De Giorgi for the half-Laplacian in dimension 3.

We recall that 1-d symmetry for stable solutions for the equation (1) in dimension $n = 2$ has been proved by Cabré and Solà-Morales in [6]; the same result in dimension 2 for the other fractional powers of the Laplacian, i.e. for the equation

$$(-\Delta)^s u = f(u) \ \text{ with } \ 0 < s < 1,$$

has been proved by Cabré and Sire in [5] and by Sire and Valdinoci in [14]; in the second work the authors used a different method which involves a geometric inequality of Poincaré type.

To study the non local problem (1), we will realize it in a local problem in $\mathbb{R}^{n+1}_+$ with a nonlinear Neumann condition.

More precisely, if $u = u(x)$ is a function defined on $\mathbb{R}^n$, consider the harmonic extension $v(x, \lambda)$ defined on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$. Then, u is a solution of (1) if and only if v satisfies the problem

$$
(5) \qquad \begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v(\cdot,0) = u & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+, \\ -\partial_\lambda v = f(v) & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+. \end{cases}
$$

The formulation (5) allows to introduce the notions of *energy*, *stability* and *global minimality* for a solution u of problem (1).

Let $\Omega \subset \mathbb{R}^{n+1}_+$ be a bounded domain. We denote

$$
\widetilde{B}^+_r = \{(x,\lambda) \in \mathbb{R}^{n+1}_+ : \lambda > 0,\ |(x,\lambda)| < r\}
$$

and by $\widetilde{B}^+_r(x,\lambda) = (x,\lambda) + \widetilde{B}^+_r$. We define the following subsets of $\partial\Omega$:

$$
(6) \qquad \partial^0\Omega := \{(x,0) \in \mathbb{R}^{n+1}_+ : \widetilde{B}^+_\varepsilon(x,0) \subset \Omega \text{ for some } \varepsilon > 0\}
$$

and

$$
(7) \qquad \partial^+\Omega := \overline{\partial\Omega \cap \mathbb{R}^{n+1}_+}.
$$

Let v be a $C^1(\overline{\Omega})$ function with $|v| \le 1$. We consider the energy functional

$$
(8) \qquad \mathcal{E}_\Omega(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{\partial^0\Omega} G(v) dx.
$$

Definition 1. We say that a bounded solution v of (5) is *stable* if the second variation of energy $\delta^2\mathcal{E}/\delta^2\xi$ with respect to perturbations ξ compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, is nonnegative. That is, if

$$
(9) \qquad Q_v(\xi) := \int_{\mathbb{R}^{n+1}_+} |\nabla\xi|^2 - \int_{\partial^0\mathbb{R}^{n+1}_+} f'(v)\xi^2 \ge 0
$$

for every $\xi \in C^\infty_0(\overline{\mathbb{R}^{n+1}_+})$.

We say that v is *unstable* if and only if v is not stable.

We use this notion of stability for a solution v of the problem (5) to define the stability for a solution u of the non local equation $(-\Delta)^{1/2}u = f(u)$.

Definition 2. We say that a bounded solution $u(x)$ of (1) in $\mathbb{R}^n$ is *stable* (*unstable*) if its harmonic extension $v(x,\lambda)$ is a stable (unstable) solution for the problem (5).

An important notion related to the energy functional $\mathcal{E}$ is the one of global minimality.

Definition 3. We say that a bounded $C^1(\overline{\mathbb{R}^{n+1}_+})$ function v in $\mathbb{R}^{n+1}_+$ is a *global minimizer* of (5) if

$$\mathcal{E}_\Omega(v) \leq \mathcal{E}_\Omega(v+\xi),$$

for every bounded domain $\Omega \subset \overline{\mathbb{R}^{n+1}_+}$ and every C^∞ function ξ with compact support in $\Omega \cup \partial^0\Omega$.

Definition 4. We say that a bounded C^1 function u in $\mathbb{R}^n$ is a *global minimizer* of (1) if its harmonic extension v is a global minimizer of (5).

Definition 5. We call *layer solutions* for the problem (1) bounded solutions that are monotone increasing in one of the x-variables; say

$$\partial_{x_n} u > 0 \quad \text{and} \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \forall\ x' \in \mathbb{R}^{n-1}.$$

Remark 1. We remind that every layer solution is a global minimizer in $\widetilde{B}_R^+$ (see Theorem 1.4 in [6]).

Our main result is the following energy estimate for bounded global minimizers of problem (1).

Let $C_R \subset \mathbb{R}^{n+1}$ be the cylinder of radius R and height R

$$C_R = B_R \times (0, R),$$

where B_R is the ball of centre 0 and radius R in $\mathbb{R}^n$.
Set $c_u = min\{G(s) : \inf v \leq s \leq \sup v\}$.

Theorem 1.1. *Let f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0,1)$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded global minimizer of (1). Let v be the harmonic extension of u in $\mathbb{R}^{n+1}_+$.*
Then, for all $R > 2$,

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq C R^{n-1} \log R, \tag{10}$$

where C is a constant depending only on n and on $||u||_{L^\infty(\mathbb{R}^n)}$. In particular we have that

$$\int_{C_R} \frac{1}{2}|\nabla v|^2 dx d\lambda \leq C R^{n-1} \log R. \tag{11}$$

As a consequence (10) and (11) also hold for layer solutions u of (1); later we will give an alternative and simpler proof of this result for layers and G of double well type.

In dimension $n = 3$, energy estimate (10) holds also for monotone solutions without limit assumptions $\lim_{x_3 \to \pm\infty} u = \pm 1$.

Theorem 1.2. *Let $n = 3$, f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0,1)$ and u be a bounded solution of (1) such that $\partial_{x_n} u > 0$ in $\mathbb{R}^3$. Let v be its harmonic extension in $\mathbb{R}^4_+$.*

Then, for all $R > 2$,

$$\int_{C_R} \frac{1}{2} |\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq C R^2 \log R, \tag{12}$$

where C is a constant depending only on n and on $||u||_{L^\infty(\mathbb{R}^n)}$.

We remind that in dimension $n = 1$ the energy estimate (10), for layer solutions, has been proved by Cabré and Solà-Morales [6]

In dimension $n = 3$, Theorem 1.1 and Theorem 1.2 lead to 1-d simmetry of global minimizers and of monotone solutions of problem (1).

Theorem 1.3. *Let $n = 3$ and f be any $C^{1,\beta}$ nonlinearity with $\beta \in (0,1)$. Let u be either a bounded global minimizer of (1), or a bounded solution monotone in the direction x_n. Then, u depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$, or equivalently the level sets of u are planes.*

In dimension $n = 2$ the 1-d symmetry of stable solutions for problem (1) was proved in [6] (Theorem 1.5 b)) using a Liouville result (Lemma 2.6 in [6]), which is the analog for boundary reaction of a Liouville theorem from [2], used to prove a conjecture of De Giorgi for reaction in the interior. These results requires an estimate for kinetic energy of the type:

$$\int_{\widetilde{B}_R^+} |\nabla v|^2 dx d\lambda \leq C R^2.$$

Here we used an improvement of this Liouville result given by Moschini [12], which requires the weaker estimate

$$\int_{\widetilde{B}_R^+} |\nabla v|^2 \leq C R^2 \log R,$$

see Proposition 5.1 in section 5.

2. Energy estimate for monotone solutions of Allen-Cahn type equation

In this section we give an alternative and simpler proof of our energy estimates for layer solutions of the Allen-Cahn type equation.

Theorem 2.1. *Let f be a $C^{1,\beta}$ function, $0 < \beta < 1$, that satisfies (H2). Let u be a bounded solution of problem (1) in $\mathbb{R}^n$, $|u| < 1$, and v the harmonic extension of u in $\mathbb{R}^{n+1}_+$. Assume that*

$$u_{x_n} > 0 \ \ in \ \mathbb{R}^n. \tag{13}$$

If $n \geq 4$, assume also

$$\lim_{x_n \to +\infty} u(x', x_n) = 1 \ \ for\ all \ \ x' \in \mathbb{R}^{n-1}. \tag{14}$$

Then, for every $R > 2$

$$\mathcal{E}_{C_R}(v) \leq C R^{n-1} \log R,$$

for some constant C independent of R.

Proof of Theorem 2.1. We follow an argument used by Ambrosio and Cabré to prove an energy estimate for layer solutions of the analog problem $-\Delta u = f(u)$; it is based on sliding the solution u in the direction x_n. Consider the function

$$v^t(x, \lambda) := v(x', x_n + t, \lambda)$$

defined for $(x', x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For each t we have

$$\begin{cases} \Delta v^t = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\partial_\lambda v^t = f(v^t) & \text{on } \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+. \end{cases} \tag{15}$$

and

$$|v^t| < C \quad \text{and} \quad |\nabla v^t| \leq \frac{C}{1+\lambda}.$$

Throughout the proof, C will denote different positive constants independent of R and t. Moreover in Lemma 2.4 of [6], gradient bounds lead to

$$\lim_{t\to+\infty} v^t(x,\lambda) = 1$$

for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ (not only for $\lambda = 0$!). Denoting the derivative of $v^t(x,\lambda)$ with respect to t by $\partial_t v^t(x,\lambda)$, we have

$$\partial_t v^t(x,\lambda) = v_{x_n}(x', x_n + t, \lambda) > 0 \quad \text{for all } \ x \in \mathbb{R}^n,\ \lambda > 0.$$

We consider the energy of v^t in the cylinder $C_R = B_R \times (0,R)$:

$$\mathcal{E}_{C_R}(v^t) = \int_{C_R} \frac{1}{2}|\nabla v^t|^2 dx d\lambda + \int_{B_R} G(v^t) dx.$$

Note that, by hypothesis (13) and using the dominated convergence theorem, we have

$$\lim_{t\to+\infty} \mathcal{E}_{C_R}(v^t) = 0.$$

Next, we bound the derivative of $\mathcal{E}_{C_R}(v^t)$ with respect to t. We use that v^t is a solution of problem (5), the bounds for $|v^t|$ and $|\nabla v^t|$ and the crucial fact that $\partial_t v^t > 0$. Let ν denote the exterior normal to the boundary ∂B_R; we find that

$$\begin{aligned}
\partial_t \mathcal{E}_{C_R}(v^t) &= \int_0^R d\lambda \int_{B_R} \nabla v^t \cdot \nabla(\partial_t v^t) dx + \int_{B_R} G'(v^t)\partial_t v^t dx \\
&= \int_0^R d\lambda \int_{\partial B_R} \frac{\partial v^t}{\partial \nu} \partial_t v^t(x,\lambda) dx + \int_{B_R\times\{\lambda=R\}} \frac{\partial v^t}{\partial \lambda} \partial_t v^t(x,R) dx \\
&\geq -C \int_0^R \frac{d\lambda}{1+\lambda} \int_{\partial B_R} \partial_t v^t d\sigma - C \int_{B_R\times\{\lambda=R\}} \frac{1}{R} \partial_t v^t(x,R) dx.
\end{aligned}$$

Hence, for every $T > 0$, we have

$$\begin{aligned}
\mathcal{E}_{C_R}(v) &= \mathcal{E}_{C_R}(v^T) - \int_0^T \partial_t \mathcal{E}_{C_R}(v^t) dt \\
&\leq \mathcal{E}_{C_R}(v^T) + C\int_0^T dt \int_0^R \frac{d\lambda}{1+\lambda} \int_{\partial B_R} \partial_t v^t d\sigma + C \int_0^T \int_{B_R\times\{\lambda=R\}} \frac{1}{R} \partial_t v^t(x,R) dx \\
&= \mathcal{E}_{C_R}(v^T) + C\int_0^R \frac{d\lambda}{1+\lambda} \int_{\partial B_R} d\sigma \int_0^T \partial_t v^t dt + C \int_{B_R\times\{\lambda=R\}} \frac{1}{R} \int_0^T \partial_t v^t dt \\
&\leq \mathcal{E}_{C_R}(v^T) + CR^{n-1}\log R + CR^{n-1}.
\end{aligned}$$

Letting $T \to +\infty$, we obtain the desired estimate.

We give now a sketch of the proof in the case $n = 3$ but without limit assumptions (14). The only difficulty arise when trying to show that

$$\lim_{t \to +\infty} \mathcal{E}_{C_R}(v^t) = 0,$$

since we no longer assume that $\lim_{x_3 \to +\infty} v(x', x_3, \lambda) = 1$. Hence we consider the function

$$\overline{v}(x', \lambda) = \lim_{x_3 \to +\infty} v(x', x_3, \lambda),$$

which is a solution of the same problem 5, but now in $\mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}_+$. Using a method developed by Berestycki, Caffarelli and Nirenberg [3], we establish a stability property for $\overline{v}$ which will imply that $\overline{v}$ is actually a solution depending only on two variables (Theorem 1.5, point b) in [6]). As a consequence, using the energy estimates in [6] for solutions in $\mathbb{R}^2_+$, we deduce that

$$\lim_{t + \infty} \mathcal{E}_{C_R}(v^t) \leq CR^2 \log R.$$

Proceeding exactly as before, this estimate will suffice to establish $\mathcal{E}_{C_R}(v) \leq CR^2 \log R$. $\square$

3. Energy estimate for global minimizers

We give only a sketch of the proof of Theorem 1.1, which is based on a comparison argument. The proof can be resumed in 3 steps:

(1) Construct the comparison function $\overline{w}$, which takes the same value of v on $\partial^+ C_R$ and thus, by minimality of v

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(\overline{w}).$$

If s is such that $G(s) = c_u$ (which is the minimum for G in $[\inf v, \sup v]$), we define the function $\overline{w}(x, \lambda)$ in the following way. First we define $\overline{w}(x, 0)$ on the base of the cylinder C_R as a smooth function $g(x)$ which is identically equal to s in B_{R-1} and $g(x) = v(x, 0)$ for $|x| = R$; now $\overline{w}(x, \lambda)$ is the unique solution of the Dirichlet problem:

$$
(16) \qquad \begin{cases} \Delta \overline{w} = 0 & \text{in } C_R \\ \overline{w}(x,0) = g(x) & \text{on } \partial^0 C_R \\ \overline{w}(x,\lambda) = v(x,\lambda) & \text{on } \partial^+ C_R. \end{cases}
$$

(2) use the rescaled $H^{1/2}(\partial C_1) \to H^1(C_1)$ estimate in the cylinder of radius 1 and height 1:

$$
\int_{C_1} |\nabla \overline{w}|^2 dx d\lambda \le C \int_{\partial C_1} \int_{\partial C_1} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} d\sigma_x d\sigma_{\overline{x}} + C||w||^2_{L^2(\partial C_1)},
$$

where w is the trace of $\overline{w}$ on ∂C_1,

iii) give the key estimate

$$
||w||^2_{H^{1/2}(\partial C_R)} \le C R^{n-1} \log R.
$$

4. Energy estimate for monotone solutions in dimension 3

Set

$$
\underline{v}(x', \lambda) = \lim_{x_n \to -\infty} v(x', x_n, \lambda) \text{ and } \overline{v}(x', \lambda) = \lim_{x_n \to +\infty} v(x', x_n, \lambda).
$$

The proof of Theorem 1.2 is based on the following:

Proposition 4.1. *Let f be any $C^{1,\beta}$ nonlinearity, with $\beta \in (0,1)$. Let u be a bounded solution of (1) in $\mathbb{R}^n$, such that $u_{x_n} > 0$ and let v be its harmonic extension in $\mathbb{R}^{n+1}_+$. Then*

$$
\int_{C_R} \frac{1}{2} |\nabla v(x,\lambda)|^2 dx d\lambda + \int_{B_R} G(v(x,0)) dx \le \int_{C_R} \frac{1}{2} |\nabla w(x,\lambda)|^2 dx d\lambda + \int_{B_R} G(w(x,0)) dx,
$$

for every $w \in C^1$ such that $w = v$ on $\partial^+ C_R$ and $\underline{v} \le w \le \overline{v}$ in C_R.

Proof. This property of local minimality of monotone solutions follows from the following two results:

(1) uniqueness of the monotone solution v of the problem

$$
\begin{cases}
\Delta w = 0 & \text{in } C_R, \\
w = v & \text{on } \partial^+ C_R, \\
-\partial_\lambda w = f(w) & \text{on } \partial^0 C_R, \\
\underline{v} \leq w \leq \overline{v} & \text{in } C_R,
\end{cases}
\tag{17}
$$

which is the analog of Lemma 3.1 of [6];

(2) existence of an absolute minimizer for $\mathcal{E}_{C_R}$ in the set

$$
C_v = \{w \in H^1(C_R) | w \equiv v \text{ on } \partial^+ C_R,\ \underline{v} \leq w \leq \overline{v}\},
$$

which is the analog of Lemma 2.10 of [6].

Indeed, the monotone solution v, by uniqueness, must agree with the absolute minimizer in C_R.

To proof points 1) and 2), we proceed exactly as in [6], but in this case we have not the assumption on the limits as $x_n \to \pm\infty$. We have only to substitute $+1$ and -1, by $\underline{v}$ and $\overline{v}$ respectively, in the proofs of Lemma 3.1 and Lemma 2.10. We make a short comment about these proofs.

(1) The proof of uniqueness is based on sliding the function $v(x, \lambda)$ in the direction x_n, we set

$$
v^t(x_1, ..., x_n, \lambda) = v(x_1, ..., x_n + t, \lambda), \text{ for every } (x, \lambda) \in \overline{C}_R.
$$

Let w be a weak solution of (17); we want to prove $w \equiv v$. We first prove $w \leq v$. Since $v^t \to \overline{v}$ as $t \to +\infty$ uniformly in $\overline{C}_R$ and $\underline{v} \leq w \leq \overline{v}$, then $w < v^t$ in $\overline{C}_R$, for t large enough. We want to prove that $w < v^t$ in $\overline{C}_R$, for every $t > 0$. Suppose that $s > 0$ is the infimum of those $t > 0$ such that $w < v^t$ in $\overline{C}_R$. We observe that $\underline{v}$ and $\overline{v}$ are solutions of (5), then by applying maximum principle and Hopf Lemma, we get a contradiction. To prove the reverse inequality $v \leq w$, we use the same sliding method, but now for $t < 0$.

(2) To prove the existence of an absolute minimizer for $\mathcal{E}_{C_R}$ in the convex set C_v, we proceed exactly as in the proof of Lemma 2.10 of [6], substituting $+1$ and -1 by $\underline{v}$ and $\overline{v}$, respectively.

□

We can give now a sketch of the proof of Theorem 1.2. As before, we want to make a comparison argument, using the function w defined as in (16). Now we don't assume v to be global minimizer, but by Proposition 4.1, we know that, since v is monotone, then it is a minimizer in the set $\{\underline{v} \leq w \leq \overline{v}\}$.We have only to check that our comparison function w belongs to this set. The proof can be resumed in the following points:

(1) $\underline{v}$ and $\overline{v}$ are **stable** solutions of problem (5) in $\mathbb{R}^3_+$;
(2) by a result due to Cabré-Solà Morales they depend only on λ and another variable;
(3) if s is such that $G(s) = c_u$, then $\underline{v} \leq s \leq \overline{v}$,
(4) we can apply Proposition 4.1 and make comparison argument.

5. Sketch of the proof of 1-d symmetry result in dimension 3

1-d symmetry of minimizers and of monotone solutions in dimension 3 follows by our energy estimate and the following Liouville type Theorem due to Moschini:

Proposition 5.1. *Let $\varphi \in L^\infty_{loc}(\mathbb{R}^{n+1}_+)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^{n+1}_+)$ satisfies*

$$\begin{cases} -\sigma\, div(\varphi^2 \nabla \sigma) \leq 0 & in\ \mathbb{R}^{n+1}_+ \\ -\sigma \partial_\lambda \sigma \leq 0 & in\ \partial\mathbb{R}^{n+1}_+ \end{cases} \tag{18}$$

in the weak sense. If

$$\int_{\widetilde{B}^+_R} (\varphi\sigma)^2 dx \leq CR^2 \log R$$

for some finite constant C independent of R, then σ is constant.

We give a sketch of the proof of 1-d symmetry result for monotone solutions in dimension 3.

Suppose $v_{x_3} > 0$; set $\varphi = v_{x_3}$ and for $i = 1, ..., n-1$ fixed, consider the function:

$$\sigma_i = \frac{v_{x_i}}{\varphi}.$$

We prove that σ_i is constant in $\mathbb{R}^{n+1}_+$, using the Liouville result due to Moschini and our energy estimate. We can resume the proof in the following steps:

- the function σ_i satisfies

$$\begin{cases} -\sigma_i \mathrm{div}(\varphi^2 \nabla \sigma_i) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\sigma_i \partial_\lambda \sigma_i = 0 & \text{in } \partial \mathbb{R}^{n+1}_+ \end{cases} \tag{19}$$

- by our energy estimate, we get

$$\int_{\widetilde{B}_R^+} (\varphi \sigma_i)^2 \leq \int_{\widetilde{B}_R^+} |\nabla v|^2 \leq C R^2 \log R$$

- by Proposition (5.1) we deduce $\sigma_i = c_i$ is constant then v depends only on λ and the variable parallel to the vector $(c_1, c_2, c_3, 0)$ and then $u(x) = v(x, 0)$ is 1-d.

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