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Bruno Franchi

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Abstract

In this seminar we introduce the theory of differential forms in Carnot groups that is basically due to M. Rumin, providing several examples in particular situations. The aim of this presentation is prepare a Part II where these results will be applied, as in a joint paper with Annalisa Baldi, Nicoletta Tchou and Maria Carla Tesi, to prove a compensated compactness theorem in Carnot groups, with applications to homogenization of elliptic equations.

In this seminar we introduce the theory of differential forms in Carnot groups that is basically due to M. Rumin, providing several examples in particular situations. The aim of this presentation is prepare a Part II where these results will be applied, as in a joint paper with Annalisa Baldi, Nicoletta Tchou and Maria Carla Tesi ([1]), to prove a compensated compactness theorem in Carnot groups, with applications to homogenization of elliptic equations.

To fix our notations, let us remind some definition. For more exhaustive presentations, we refer to [2], [6], [7]. A *Carnot group* $\mathbb{G}$ *of step* κ is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa, \tag{1}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Let $m_i = \dim(V_i)$, for $i = 1, \dots, \kappa$ and $h_i = m_1 + \cdots + m_i$ with $h_0 = 0$ and, clearly, $h_\kappa = n$. Choose a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ adapted to the stratification, that is such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \text{ is a base of } V_j \ \text{ for each } j = 1, \dots, k.$$

Let $W = W_1, \dots, W_n$ be the family of left invariant vector fields such that $W_i(0) = e_i$. Given (1), the subset $W_1, \dots, W_{m_1}$ generates by commutations all the other vector fields; we will refer to $W_1, \dots, W_{m_1}$ as *generating vector fields* of the group. The exponential map is a one to one map from $\mathfrak{g}$ onto $\mathbb{G}$, i.e. any $p \in \mathbb{G}$ can be written in a unique way as $p = \exp(p_1 W_1 + \cdots + p_n W_n)$. Using these *exponential coordinates*, we identify p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$ where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula. If $p \in \mathbb{G}$ and $i = 1, \dots, \kappa$, we put $p^i = (p_{h_{i-1}+1}, \dots, p_{h_i}) \in \mathbb{R}^{m_i}$, so that we can also identify p with $[p^1, \dots, p^\kappa] \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_\kappa} = \mathbb{R}^n$.

The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $W_1, \dots, W_{m_1}$ plays a particularly important role in the theory, it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span } \{W_1(x), \dots, W_{m_1}(x)\}, \qquad x \in \mathbb{G}.$$

From now on, for sake of simplicity, sometimes we set $m := m_1$.

A subriemannian structure is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle\cdot,\cdot\rangle_x$ and with a norm $|\cdot|_x$ that make the basis $W_1(x),\dots,W_m(x)$ an orthonormal basis. That is if $v=\sum_{i=1}^m v_iW_i(x)=(v_1,\dots,v_m)$ and $w=\sum_{i=1}^m w_iW_i(x)=(w_1,\dots,w_m)$ are in $H\mathbb{G}_x$, then $\langle v,w\rangle_x := \sum_{j=1}^m v_jw_j$ and $|v|_x^2 := \langle v,v\rangle_x$.
The sections of $H\mathbb{G}$ are called *horizontal sections*, and a vector of $H\mathbb{G}_x$ is an *horizontal vector.*

Two important families of automorphism of $\mathbb{G}$ are the group translations and the group dilations of $\mathbb{G}$. For any $x\in\mathbb{G}$, the *(left) translation* $\tau_x:\mathbb{G}\to\mathbb{G}$ is defined as

$$z\mapsto \tau_x z := x\cdot z.$$

For any $\lambda>0$, the *dilation* $\delta_\lambda:\mathbb{G}\to\mathbb{G}$, is defined as

$$\delta_\lambda(x_1,...,x_n)=(\lambda^{d_1}x_1,...,\lambda^{d_n}x_n), \tag{2}$$

where $d_i\in\mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [4] Chapter 1) and is defined as

$$d_j=i \quad \text{whenever } h_{i-1}+1\le j\le h_i, \tag{3}$$

hence $1=d_1=...=d_{m_1}<d_{m_1+1}=2\le...\le d_n=\kappa$.

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1\mathfrak{g}$. The basis of $\bigwedge^1\mathfrak{g}$, dual to the basis $W_1,\cdots,W_n$, is the family of covectors $\{\theta_1,\cdots,\theta_n,\}$. We indicate as $\langle\cdot,\cdot\rangle$ also the inner product in $\bigwedge^1\mathfrak{g}$ that makes $\theta_1,\cdots,\theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1,\cdots,\theta_n$ may have polynomial (hence variable) coefficients.

Following Federer (see [3] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1\mathfrak{g}$ are the graded algebras indicated as $\bigwedge_*\mathfrak{g}=\bigoplus_{k=0}^n\bigwedge_k\mathfrak{g}$ and $\bigwedge^*\mathfrak{g}=\bigoplus_{k=0}^n\bigwedge^k\mathfrak{g}$ where $\bigwedge_0\mathfrak{g}=\bigwedge^0\mathfrak{g}=\mathbb{R}$ and, for $1\le k\le n$,

$$\bigwedge_k\mathfrak{g} := \operatorname{span}\{W_{i_1}\wedge\cdots\wedge W_{i_k} : 1\le i_1<\cdots<i_k\le n\},$$
$$\bigwedge^k\mathfrak{g} := \operatorname{span}\{\theta_{i_1}\wedge\cdots\wedge \theta_{i_k} : 1\le i_1<\cdots<i_k\le n\}.$$

The elements of $\bigwedge_k\mathfrak{g}$ and $\bigwedge^k\mathfrak{g}$ are called *k-vectors* and *k-covectors.*

We remind that

$$\dim \bigwedge^h \mathfrak{g} = \dim \bigwedge_h \mathfrak{g} = \binom{h}{n} := d_h.$$

The dual space $\bigwedge^1(\bigwedge_k \mathfrak{g})$ of $\bigwedge_k \mathfrak{g}$ can be naturally identified with $\bigwedge^k \mathfrak{g}$. The action of a k-covector φ on a k-vector v is denoted as $\langle \varphi | v\rangle$.

The inner product $\langle \cdot, \cdot\rangle$ extends canonically to $\bigwedge_k \mathfrak{g}$ and to $\bigwedge^k \mathfrak{g}$ making the bases $W_{i_1} \wedge \cdots \wedge W_{i_k}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ orthonormal.

Definition 0.1. *We define linear isomorphisms (see* [3] *1.7.8)*

$$* : \bigwedge_k \mathfrak{g} \longleftrightarrow \bigwedge_{2n+1-k} \mathfrak{g} \quad \textit{and} \quad * : \bigwedge^k \mathfrak{g} \longleftrightarrow \bigwedge^{2n+1-k} \mathfrak{g},$$

for $1 \le k \le n$, putting, for $v = \sum_I v_I W_I$ and $\varphi = \sum_I \varphi_I \theta_I$,

$$*v := \sum\nolimits_I v_I(*W_I) \quad \textit{and} \quad * \varphi := \sum\nolimits_I \varphi_I(*\theta_I)$$

where

$$*W_I := (-1)^{\sigma(I)} W_{I^*} \quad \textit{and} \quad * \theta_I := (-1)^{\sigma(I)} \theta_{I^*}$$

with $I = \{i_1, \cdots, i_k\}$, $1 \le i_1 < \cdots < i_k \le n$, $W_I = W_{i_1} \wedge \cdots \wedge W_{i_k}$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$, $I^* = \{i_1^* < \cdots < i_{2n+1-k}^*\} = \{1, \cdots, n\} \setminus I$ and $\sigma(I)$ is the number of couples (i_h, i_ℓ^*) with $i_h > i_\ell^*$.

The following properties of the $*$ operator follow readily from the definition: $\forall v, w \in \bigwedge_k \mathfrak{g}$ and $\forall \varphi, \psi \in \bigwedge^k \mathfrak{g}$

$$\begin{aligned} &* * v = (-1)^{k(n-k)} v = v, && * * \varphi = (-1)^{k(n-k)} \varphi = \varphi, \\ &v \wedge *w = \langle v, w\rangle W_{\{1,\cdots,2n+1\}}, && \varphi \wedge *\psi = \langle \varphi, \psi\rangle \theta_{\{1,\cdots,2n+1\}}, \\ &\langle *\varphi | *v\rangle = \langle \varphi | v\rangle. \end{aligned} \tag{4}$$

Notice that, if $v = v_1 \wedge \cdots \wedge v_k$ is a simple k-vector, then $*v$ is a simple $(n-k)$-vector. If $v \in \bigwedge_k \mathfrak{g}$ we define $v^\natural \in \bigwedge^k \mathfrak{g}$ by the identity $\langle v^\natural | w\rangle := \langle v, w\rangle$, and analogously we define $\varphi^\natural \in \bigwedge_k \mathfrak{g}$ for $\varphi \in \bigwedge^k \mathfrak{g}$.

Definition 0.2. *For any $q, q' \in \mathbb{G}$ and for any linear map $L : T\mathbb{G}_q \to T\mathbb{G}_{q'}$,*

$$\Lambda_k L : \bigwedge_k T\mathbb{G}_q \to \bigwedge_k T\mathbb{G}_{q'}$$

is the linear map defined by

$$(\Lambda_k L)(v_1 \wedge \cdots \wedge v_k) = L(v_1) \wedge \cdots \wedge L(v_k).$$

Analogously, we can define

$$_H{\bigwedge}^k_p := (\Lambda^k d\tau_{p^{-1}})(_H{\bigwedge}^k_e)$$

for any $p \in \mathbb{G}$, where for any linear map $f : T\mathbb{G}_q \to T\mathbb{G}_{q'}$

$$\Lambda^k f : {\bigwedge}^k T\mathbb{G}_{q'} \to {\bigwedge}^k T\mathbb{G}_q$$

is the linear map defined by

$$\langle (\Lambda^k f)(\alpha) | v_1 \wedge \cdots \wedge v_k \rangle = \langle \alpha | (\Lambda_k f)(v_1 \wedge \cdots \wedge v_k) \rangle$$

for any $\alpha \in \bigwedge^k T\mathbb{G}_{q'}$ and any simple k-vector $v_1 \wedge \cdots \wedge v_k \in \bigwedge_k T\mathbb{G}_q$.

As customary, if $f : \mathbb{G} \to \mathbb{G}$ is an isomorphism, then the pull-back $f^\# \omega$ of a form $\omega \in \Omega^k$ is defined by

$$f^\# \omega(x) := \big(\Lambda^k(df_x)\big)\omega(f(x)).$$

It is easy to see that $(f^{-1})^\#(f^\# \omega) = \omega$.

If $\alpha \in \bigwedge^1 \mathfrak{g}$, we say that α has *pure weight* k and we write $w(\alpha) = k$ if $\alpha^\natural \in V_k$. Obviously,

$$w(\alpha) = k \quad \text{if and only if} \quad \alpha = \sum_{j=h_{k-1}+1}^{h_k} \alpha_j \theta_j,$$

with $\alpha_{h_{k-1}+1}, \dots, \alpha_{h_k} \in \mathbb{R}$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

Remark 0.1. *If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$. Indeed, it is enough to notice that, if $w(\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}) \neq w(\theta_{j_1} \wedge \cdots \wedge \theta_{j_h})$, with $i_1 < i_2 < \cdots < i_h$ and $j_1 < j_2 < \cdots < j_h$, then for at least one of the indices $\ell = 1, \dots, h$, $i_\ell \neq j_\ell$, and hence $\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}, \theta_{j_1} \wedge \cdots \wedge \theta_{j_h} \rangle = 0$.*

We have

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=N_h^{\min}}^{N_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{5}$$

where $\bigwedge^{h,p}\mathfrak{g}$ is the linear space of the h–covectors of weight p. We set $N_h := \dim \bigwedge^h \mathfrak{g}$ and $N_{h,p} := \dim \bigwedge^{h,p}\mathfrak{g}$.

From now on, if $1 \le h \le n$, we assume that the multi–indices $I = (i_1, \dots, i_h)$ are ordered once for all in increasing way with respect to the weight of the corresponding element of the basis that we call Θ^h.

$$\Theta^h = \bigcup_{p=N_h^{\min}}^{N_h^{\max}} \Theta^{h,p}, \tag{6}$$

where $\Theta^{h,p} = \Theta^h \cap \bigwedge^{h,p}\mathfrak{g}$ is an orthonormal basis of $\bigwedge^{h,p}\mathfrak{g}$.

Thus, we can write the canonical coordinates of a h–covector in $\bigwedge^h\mathfrak{g}$ with respect to the orthonormal basis

$$\Theta^h = \{\theta_j^h\} \quad \text{of} \quad \bigwedge^h \mathfrak{g} \tag{7}$$

as the union of a finite number of blocks (possibly empty) of size $N_{h,N_h^{\min}}, N_{h,N_h^{\min}+1}, \dots$ corresponding to the element of the basis in $\Theta^{h,N_h^{\min}}, \Theta^{h,N_h^{\min}+1}, \dots, \Theta^{N_h^{\max}}$, of weight $N_h^{\min}, N_h^{\min}+1, \dots, N_h^{\max}$, respectively, with $N_{h,1} + N_{h,2} + \cdots + N_h^{\max} = d_h$. We keep the notiation θ_j for θ_j^1.

Correspondingly, the set of indices $\{1, 2, \dots, d_h\}$ can be written as the union of finite sets of indices

$$\{1, 2, \dots, d_h\} = \bigcup_{p=N_h^{\min}}^{N_h^{\max}} I_p^h,$$

where

$$\theta_j^h \in \Theta^{h,p} \quad \text{if and only if} \quad j \in I_p^h.$$

As pointed out in Remark 0.1, the decomposition in (5) is orthogonal. We denote by $\Pi^{h,p}$ the orthogonal projection of $\bigwedge^h\mathfrak{g}$ on $\bigwedge^{h,p}\mathfrak{g}$.

Starting from $\bigwedge^h\mathfrak{g}$, we can define by left translation a fiber bundle over $\mathbb{G}$ that we can still denote by $\bigwedge^h\mathfrak{g}$. To do this, we identify $\bigwedge^h\mathfrak{g}$ with the fiber $\bigwedge_e^h\mathfrak{g}$ over the origin, and we define the fiber over $x \in \mathbb{G}$ pushing forward $\bigwedge_e^h\mathfrak{g}$ by the left translation τ_x, i.e. defining the fiber over x as $\bigwedge_x^h\mathfrak{g} := \Lambda^k(d\tau_{x^{-1}})\bigwedge_e^h\mathfrak{g}$.

The identification of $\bigwedge^h \mathfrak{g}$ and $\bigwedge_e^h \mathfrak{g}$ yields a corresponding identification of the basis Θ^h of $\bigwedge^h \mathfrak{g}$ and Θ_e^h of $\bigwedge_e^h \mathfrak{g}$. Then $\Theta_x^h := (\tau_x)_\# \Theta^h$ is a basis of $\bigwedge_x^h \mathfrak{g}$. Notice that, because of the left invariance of the h-covectors in $\bigwedge^h \mathfrak{g}$, the elements of Θ_x^h can be identified with the elements of Θ^h evaluated at the point x.

Keeping in mind the decomposition (5), we can define in the same way several fiber bundles over $\mathbb{G}$ (that we still denote with the same symbol $\bigwedge^{h,p} \mathfrak{g}$), by setting $\bigwedge_e^{h,p} \mathfrak{g} := \bigwedge^{h,p} \mathfrak{g}$ and $\bigwedge_x^{h,p} \mathfrak{g} := \Lambda^k(d\tau_{x^{-1}})\bigwedge_e^{h,p} \mathfrak{g}$. Clearly, all previous arguments related to the basis Θ^h can be repeated for the basis $\Theta^{h,p}$. Finally, we say that a section of $\bigwedge^{h,p} \mathfrak{g}$ is a h-form of (pure) weight p.

Lemma 0.1. *The fiber $\bigwedge_x^h \mathfrak{g}$ (and hence the fiber $\bigwedge_x^{h,p} \mathfrak{g}$) can be endowed with a natural scalar product $\langle \cdot, \cdot \rangle_x$ by the identity*

$$\langle \alpha, \beta \rangle_x := \langle \Lambda^h d\tau_x(\alpha), \Lambda^h d\tau_x(\beta) \rangle_e.$$

If $x, y \in \mathbb{G}$, then

$$\Lambda^h d\tau_{y^{-1}} : \bigwedge_x^h \mathfrak{g} \to \bigwedge_{yx}^h \mathfrak{g}$$

is an isometry onto.

If we denote by Ω^h the vector space of all smooth h–forms in $\mathbb{G}$, and by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, then again

$$\Omega^h = \bigoplus_{p=N_h^{\min}}^{N_h^{\max}} \Omega^{h,p}. \tag{8}$$

Lemma 0.2. *If $\alpha \in \bigwedge^h \mathfrak{g}$ is left invariant of weight k, then $w(d\alpha) = w(\alpha)$.*

Proof. See [5]. □

Let now $\alpha \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. We can write

$$\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, \theta_i^h, \quad \text{with } \alpha_i \in \mathcal{E}(\mathbb{G}).$$

Then

$$d\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_j (W_j \alpha_i) \theta_j \wedge \theta_i^h + \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h.$$

Hence we can write

$$d = d_0 + d_1 + \cdots + d_\kappa,$$

where $d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$ does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{m_1} (W_j\alpha_i)\theta_j \wedge \theta_i^h$$

increases the weight of 1, and, more generally,

$$d_k\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{w(\theta_j)=k} (W_j\alpha_i)\theta_j \wedge \theta_i^h \quad k = 1, \dots, \kappa.$$

In particular, d_0 is an algebraic operator, in the sense that its action can be identified at any point with the action of an operator on $\bigwedge^h \mathfrak{g}$ (that we denote again by d_0) through the formula

$$(d_0\alpha)(x) = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i(x) d\theta_i^h.$$

Using the canonical orthonormal system Θ^h, we have a canonical isomorphism i_Θ^h from $\bigwedge^h \mathfrak{g}$ onto $\mathbb{R}^{N_h}$ The map $M_h : \mathbb{R}^{N_h} \to \mathbb{R}^{N_{h+1}}$ makes the following diagram commutative

$$\begin{array}{ccc} \mathbb{R}^{N_h} & \xrightarrow{M_h} & R^{N_{h+1}} \\ {\scriptstyle i_\Theta^h}\uparrow & & \downarrow{\scriptstyle (i_\Theta^{h+1})^{-1}} \\ \bigwedge^h \mathfrak{g} & \xrightarrow[d_0]{} & \bigwedge^{h+1} \mathfrak{g}. \end{array}$$

Because of our choice of the order of the elements of Θ^h, the matrix associated with M_h (that we still denote by M_h) is a block matrix, as well as its transposed. More precisely, the entries of M_h are all 0 except for those that belong to groups of rows and columns "of the same weight".

This construction has a counterpart when we look at $\bigwedge^h \mathfrak{g}$ as a fiber boundle. In this case, if $x \in \mathbb{G}$ is fixed, using the canonical orthonormal system Θ_x^h, we have a canonical isomorphism $i_{\Theta_x}^h$ from $\bigwedge_x^h \mathfrak{g}$ onto $\mathbb{R}^{N_h}$ associating with $\xi \in \bigwedge_x^h \mathfrak{g}$ its canonical coordinates. Because of the identification of $\bigwedge^h \mathfrak{g}$ and $\bigwedge_e^h \mathfrak{g}$, we can analogously identify i_Θ^h and $i_{\Theta_e}^h$.

Hence, for any $x \in \mathbb{G}$, we have

$$(d_0\alpha)(x) = (i_{\Theta_x^{h+1}})^{-1} M_h i_{\Theta_x^h} \alpha(x). \tag{9}$$

Analogously, δ_0, the L^2–adjoint of d_0 in Ω^* defined by

$$\int \langle d_0\alpha, \beta\rangle \, dV = \int \langle \alpha, \delta_0\beta\rangle \, dV$$

for all compactly supported smooth forms $\alpha \in \Omega^h$ and $\beta \in \Omega^{h+1}$, is again an algebraic operator preserving the weight. Indeed, it can be written as

$$(\delta_0\beta)(x) = (i_{\Theta_x^h})^{-1}({}^tM_h) i_{\Theta_x^{h+1}}\beta(x). \tag{10}$$

Again, its matrix ${}^tM_h$ is a block matrix.

Definition 0.3. *If* $0 \le h \le n$ *we set*

$$E_0^h := \ker d_0 \cap \ker \delta_0,$$

or, in coordinates,

$$E_0^h = \{\alpha \in \Omega^h \ ; \ i_{\Theta_x^h}\alpha(x) \in \ker M_h \cap \ker^t M_{h-1} \ \textit{for all } x \in \mathbb{G}\}.$$

Proposition 0.1. *If* $0 \le h \le n$ *and* $*$ *denotes the Hodge duality, then*

$$*E_0^h = E_0^{n-h}.$$

If we set $E_0^{h,p} := E_0^h \cap \Omega^{h,p}$, then

$$E_0^h = \bigoplus_{p=N_h^{\min}}^{N_h^{\max}} E_0^{h,p}.$$

Indeed, if $\alpha \in E_0^h$, by (8), we can write

$$\alpha = \sum_{p=N_h^{\min}}^{N_h^{\max}} \alpha_p,$$

with $\alpha_p \in \Omega^{h,p}$ for all p. By definition,

$$0 = d_0\alpha = \sum_{p=N_h^{\min}}^{N_h^{\max}} d_0\alpha_p.$$

But $w(d_0\alpha_p) \ne w(d_0\alpha_q)$ for $p \ne q$, and hence the $d_0\alpha_p$'s are linear independent and therefore they are all 0. Analogously, $\delta_0\alpha_p = 0$ for all p, and the assertion follows.

We denote by $\Pi_{E_0}^{h,p}$ the orthogonal projection of Ω^h on $E_0^{h,p}$.

We notice that the space of forms $E_0^{h,p}$ (as well as the "full" space E_0^h) can be seen as the space of smooth sections of a suitable fiber bundle generated by left translations, that we still denote by $E_0^{h,p}$ (by E_0^h, respectively).

Since both $E_0^{h,p}$ and E_0^h are left invariant as $\bigwedge^h \mathfrak{g}$, they are subbundles of $\bigwedge^h \mathfrak{g}$ and inherit the scalar product on the fibers.

In particular, we can obtain a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h such that

$$\Xi_0^h = \bigcup_{p=M_h^{\min}}^{M_h^{\max}} \Xi_0^{h,p}, \tag{11}$$

where $\Xi_0^{h,p} := \Xi^h \cap \bigwedge^{h,p} \mathfrak{g}$ is a left invariant orthonormal basis of $E_0^{h,p}$. As above, the indices j of $\Xi_0^h = \{\xi_j\}$ are ordered once for all in increasing way with respect to the weight of the corresponding element of the basis.

Correspondingly, the set of indices $\{1, 2, \dots, \dim E_0^h\}$ can be written as the union of finite sets (possibly empty) of indices

$$\{1, 2, \dots, \dim E_0^h\} = \bigcup_{p=M_h^{\min}}^{M_h^{\max}} I_{0,p}^h,$$

where

$$\xi_j^h \in \Xi^{h,p} \quad \text{if and only if} \quad j \in I_{0,p}^h.$$

Proposition 0.2. *The equation*

$${}^t M_h M_h \alpha = {}^t M_h \beta$$

has one and only one solution $\alpha \in (\ker M_h)^\perp$ *for any* $\beta \in \mathbb{R}^{N_{h+1}}$.

Proof. Uniqueness: if ${}^t M_h M_h \alpha = 0$, then $0 = \langle M_h\alpha, M_h\alpha\rangle = \|M_h\alpha\|^2$, and hence $\alpha \in \ker M_h \cap (\ker M_h)^\perp = \{0\}$, achieving the proof of the uniqueness.

Existence: Since

$$\mathbb{R}^{N_{h+1}} = \mathcal{R}(M_h) \oplus \mathcal{R}(M_h)^\perp = \mathcal{R}(M_h) \oplus \ker {}^t M_h,$$

we can write $\beta = M_h\alpha + \beta_1$, with ${}^tM_h\beta_1 = 0$. Without loss of generality, we can always assume $\alpha \in (\ker M_h)^\perp$, because $\alpha = \alpha' + \alpha''$, with $\alpha' \in (\ker M_h)^\perp$, $\alpha'' \in \ker M_h$, and $M_h\alpha' = M_h\alpha$.

Thus we have ${}^tM_h\beta ={}^t M_hM_h\alpha + {}^tM_h\beta_1 ={}^t M_hM_h\alpha$, achieving the proof of the proposition. □

Corollary 0.1. *If $\beta \in \Omega^{h+1}$, then there exists a unique $\alpha \in \Omega^h \cap (\ker d_0)^\perp$ such that*

$$\delta_0 d_0\alpha = \delta_0\beta. \quad \textit{We set} \quad \alpha := d_0^{-1}\beta.$$

In particular

$$\alpha = d_0^{-1}\beta \quad \textit{if and only if} \quad d_0\alpha - \beta \in \ker \delta_0.$$

Lemma 0.3. *The map $d_0^{-1}d$ induces an isomorphism from $\mathcal{R}(d_0^{-1})$ to itself. In addition, there exist a differential operator P such that*

$$Pd_0^{-1}d = d_0^{-1}dP = \mathrm{Id}_{\mathcal{R}(d_0^{-1})}.$$

We set also $Q := Pd_0^{-1}$.

Proof. Clearly, $d_0^{-1}d$ maps $\mathcal{R}(d_0^{-1})$ into itself. Moreover, we can write

$$d_0^{-1}d = d_0^{-1}d_0 + d_0^{-1}(d - d_0) := Id + D,$$

where D is a differential operator that increases the weight of the forms and hence is nilpotent, i.e. there exists $N \in \mathbb{N}$ such that $D^{N+1} = 0$. Thus

$$P := \sum_{k=0}^{N} (-1)^k D^k$$

is the inverse of $d_0^{-1}d$. □

Remark 0.2. *If a for α has pure weight k, then $P\alpha$ is a sum of forms of pure weight greater or equal to k.*

Theorem 0.1. *The de Rham complex splits into the direct sum of two subcomplexes*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1}d) \quad \textit{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(d_0^{-1}d).$$

The projection Π_E *on* E *along* F *is a homotopical equivalence of the form* $\Pi_E = Id - Qd - dQ$*, where* Q *is the differential operator defined in Lemma 0.3. If we denote by* Π_{E_0} *the orthogonal projection on* E_0*, we have*

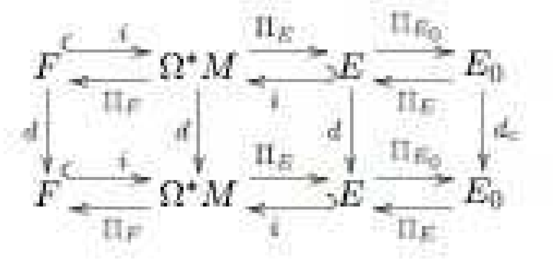
(12) $$\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0} \quad \Pi_E\Pi_{E_0}\Pi_E = \Pi_E.$$

Remark 0.3. *We stress explicitly that the exterior differential* d *maps* E^h *into* E^{h+1}*. In fact, if* $\alpha \in E^h$*, then* $d_0^{-1}(\alpha) = 0$ *and* $d_0^{-1}(d\alpha) = 0$*. Hence* $d_0^{-1}(d\alpha) = 0$ *and* $d_0^{-1}(d^2\alpha) = 0$*, so that* $d\alpha \in E^{h+1}$*.*

The fiber bundle E_0^* generate another complex (E_0, d_c) of differential forms by putting

(13) $$d_c = \Pi_{E_0} d\Pi_E.$$

From [5] we reproduce the following diagram:



Proposition 0.3. *The complexes* E *and* E_0 *are exact.*

Proof. The assertion for E follows trivially by homotopical equivalence. Keep in mind the identities (12). Let now $\alpha \in E_0$ be such that $d_c\alpha = 0$. By (13), $\Pi_{E_0} d\Pi_E\alpha = 0$. By Remark 0.3, $d\Pi_E\alpha \in E$, and hence, by (12),

$$0 = \Pi_E\Pi_{E_0} d\Pi_E\alpha = \Pi_E\Pi_{E_0}\Pi_E d\Pi_E\alpha = \Pi_E d\Pi_E\alpha = d\Pi_E\alpha.$$

Thus $\Pi_E\alpha$ (that belongs to E) is closed, and hence there exists $\gamma \in E$ such that $d\gamma = \Pi_E\alpha$. Set now $\gamma_0 := \Pi_{E_0}\gamma \in E_0$. We have (by (12), keeping in mind that $\gamma = \Pi_E\gamma$)

$$d_c\gamma_0 = \Pi_{E_0} d\Pi_E\Pi_{E_0}\gamma = \Pi_{E_0} d\Pi_E\Pi_{E_0}\Pi_E\gamma = \Pi_{E_0} d\Pi_E\gamma = \Pi_{E_0} d\gamma = \Pi_{E_0}\Pi_E\alpha.$$

Since $\alpha = \Pi_{E_0}\alpha$, by (12), we have $\Pi_{E_0}\Pi_E\alpha = \alpha$. Therefore $d_c\gamma_0 = \alpha$ and we are done. □

The equation $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ says that Π_E restricted to E_0 is $Id +$ a $E_0^\perp$ part. In other words, E is a particular space of liftings (extensions) of E_0.

Proposition 0.4 ([5], formula (7)). *For any $\alpha \in E_0^{h,p}$, if we denote by $(\Pi_E\alpha)_j$ the component of $\Pi_E\alpha$ of weight j (that is necessarily greater or equal than p, by Remark 0.2), then*

$$(14)\qquad \begin{aligned} (\Pi_E\alpha)_p &= \alpha \\ (\Pi_E\alpha)_{p+k+1} &= -d_0^{-1}\big(\sum_{\ell\le k+1} d_\ell(\Pi_E\alpha)_{p+k+1-\ell}\big). \end{aligned}$$

Remark 0.4. *In fact, we can notice that, if $\alpha \in E_0^{h,p}$, then $d_c\alpha$ has no components of weight $j = p$. Indeed,*

$$\Pi_E\alpha = \alpha + \textit{terms of weight greater than } p.$$

Thus

$$d\Pi_E\alpha = d_0\alpha + \textit{terms of weight greater than } p.$$

But $d_0\alpha = 0$ by the very definition of $E_0^{h,p}$, and the assertion follows.

Proposition 0.5. *The map $d_c : E_0^h \to E_0^{h+1}$ can be written in the form*

$$(15)\qquad \begin{aligned} \alpha = \sum_p \sum_{i\in E_{0,p}^h} \alpha_{p,i}\xi_i^h \longrightarrow \\ \longrightarrow \sum_p \sum_{i\in E_{0,p}^h} \sum_{q=\max\{p+1,M_{h+1}^{\min}\}}^{M_{h+1}^{\max}} \sum_{\ell\in I_{0,q}^{h+1}} \sum_{i\in I_p^h} \big(P_{p,q,i,\ell}^h\alpha_{p,i}\big)\xi_\ell^{h+1}, \end{aligned}$$

where the $P_{p,q,i,j}^h$'s are homogeneous polynomials of degree $q-p$ in the horizontal derivatives.

Example 0.1. Let $\mathbb{G} := \mathbb{H}^1 \times \mathbb{R}$, and denote by (x, y, t) the variables in $\mathbb{H}^1$ and by s the variable in $\mathbb{R}$. Set $X := \partial_x + 2y\partial_t$, $Y := \partial_x - 2x\partial_t$, $T := \partial_t$, $S := \partial_s$. We have $X^\natural = dx$, $Y^\natural = dy$, $S^\natural = ds$, $T^\natural = \theta$ (the contact form of $\mathbb{H}^1$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X, Y, S\}$ and $V_2 = \text{span}\,\{T\}$. In this case

$$E_0^1 = \text{span}\,\{dx, dy, ds\};$$
$$E_0^2 = \text{span}\,\{dx \wedge ds, dy \wedge ds, dx \wedge \theta, dy \wedge \theta\};$$
$$E_0^3 = \text{span}\,\{dx \wedge dy \wedge \theta, dx \wedge ds \wedge \theta, dy \wedge ds \wedge \theta\}.$$

Moreover

$$\begin{aligned} & d_c(\alpha_1 dx + \alpha_2 dy + \alpha_3 ds) \\ & \quad = \Pi_{E_0} d\big(\alpha_1 dx + \alpha_2 dy - \frac{1}{4}(X\alpha_2 - Y\alpha_1)\theta\big) + d\alpha_3 \wedge ds \\ & \quad = D(\alpha_1 dx + \alpha_2 dy) + (X\alpha_3 - S\alpha_1) dx \wedge ds + (Y\alpha_3 - S\alpha_2) dy \wedge ds, \end{aligned}$$

where D is the second order differential of horizontal 1-forms in $\mathbb{H}^1$ that has the form $D(\alpha_1 dx + \alpha_2 dy) = P_1(\alpha_1, \alpha_2) dx \wedge \theta + P_2(\alpha_1, \alpha_2) dy \wedge \theta$.

On the other hand, if

$$\alpha = \alpha_{13} dx \wedge ds + \alpha_{23} dy \wedge ds + \alpha_{14} dx \wedge \theta + \alpha_{24} dy \wedge \theta \in E_0^2,$$

then

$$\begin{aligned} d_c\alpha = & \, (X\alpha_{24} - Y\alpha_{14})\, dx \wedge dy \wedge \theta \\ & + \big(T\alpha_{13} - S\alpha_{14} + \frac{1}{4}(X^2\alpha_{23} - XY\alpha_{13})\big) dx \wedge ds \wedge \theta \\ & + \big(T\alpha_{23} - S\alpha_{24} + \frac{1}{4}(YX\alpha_{23} - Y^2\alpha_{13})\big) dy \wedge ds \wedge \theta. \end{aligned}$$

Example 0.2. Let now $\mathbb{G} := \mathbb{H}^2 \times \mathbb{R}$, and denote by (x_1, x_2, y_1, y_2, t) the variables in $\mathbb{H}^2$ and by s the variable in $\mathbb{R}$. Set $X_i := \partial_{x_i} + 2y_i\partial_t$, $Y_i := \partial_{x_i} - 2x_i\partial_t$, $i = 1, 2$, $T := \partial_t$, and $S := \partial_s$. We have $X_i^\natural = dx_i$, $Y_i^\natural = dy_i$, $i = 1, 2$, $S^\natural = ds$, $T^\natural = \theta$ (the contact form of $\mathbb{H}^2$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X_1, X_2, Y_1, Y_2, S\}$ and $V_2 = \text{span}\,\{T\}$.

Let us restrict ourselves to show the structure of the intrinsic differential on E_0^1, i.e on horizontal 1-forms. Using the notations of (5), (6) and (7), we can chose an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 1, 2, 3$ as follows:

$\mathbf{h = 1}$: $\Theta^{1,1} = (\theta_1^1, \dots, \theta_5^1) = (dx_1, dx_2, dy_1, dy_2, ds)$, and $\Theta^{1,2} = (\theta_6^1) = (\theta)$.

h = 2: $\Theta^{2,2} = (\theta_1^2, \dots, \theta_{10}^1) = (dx_1 \wedge dx_2, dy_1 \wedge dy_2, dx_1 \wedge dy_1, dx_1 \wedge dy_2, dx_2 \wedge dy_1, dx_2 \wedge dy_2, dx_1 \wedge ds, dx_2 \wedge ds, dy_1 \wedge ds, dy_2 \wedge ds)$, $\Theta^{2,3} = (\theta_{11}^2, \dots, \theta_{15}^2) = (dx_1 \wedge \theta, dx_2 \wedge \theta, dy_1 \wedge \theta, dy_2 \wedge \theta, ds \wedge \theta)$.

h = 3: $\Theta^{3,3} = (\theta_1^3, \dots, \theta_{10}^3) = (dx_1 \wedge dx_2 \wedge dy_1, dx_1 \wedge dx_2 \wedge dy_2, dx_1 \wedge dx_2 \wedge ds, dx_1 \wedge dy_1 \wedge dy_2, dx_1 \wedge dy_1 \wedge ds, dx_2 \wedge dy_1 \wedge dy_2, dy_1 \wedge dy_2 \wedge ds, dx_1 \wedge dy_2 \wedge ds, dx_2 \wedge dy_2 \wedge ds, dy_1 \wedge dy_2 \wedge ds)$. $\Theta^{3,4} = (\theta_{11}^3, \dots, \theta_{20}^3) = (dx_1 \wedge dx_2 \wedge \theta, dy_1 \wedge dy_2 \wedge \theta, dx_1 \wedge dy_1 \wedge \theta, dx_1 \wedge dy_2 \wedge \theta, dx_2 \wedge dy_1 \wedge \theta, dx_2 \wedge dy_2 \wedge \theta, dx_1 \wedge ds \wedge \theta, dx_2 \wedge ds \wedge \theta, dy_1 \wedge ds \wedge \theta, dy_2 \wedge ds \wedge \theta)$.

We have:

$$d_0\theta_i^1 = 0 \text{ when } i = 1, \dots, 5, \quad d_0\theta_6^1 = 4(\theta_3^2 + \theta_6^2);$$

$$d_0\theta_i^2 = 0 \text{ when } i = 1, \dots, 10 \quad d_0\theta_{11}^2 = 4\theta_2^3, \quad d_0\theta_{12}^2 = -4\theta_1^3,$$

$$d_0\theta_{13}^2 = -4\theta_6^3, \quad d_0\theta_{14}^2 = 4\theta_4^3, \quad d_0\theta_{15}^2 = 4(\theta_5^3 + \theta_{10}^3).$$

Thus

$$M_1 = \left(\begin{array}{ccc|c} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 4 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 4 \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ \hline 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{array}\right)$$

$$
M_2 = \left(\begin{array}{ccc|ccccc}
0 & \dots & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & \dots & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & \dots & 0 & 0 & 0 & -4 & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 4 \\
\hline
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

As usual, E_0^1 is the space of left invariant horizontal 1-forms, i.e. an orthonormal basis of E_0^1 is given by $\{dx_1, dx_2, dy_1, dy_2, ds\}$. Keeping into account that E_0^2 can be identified with $\ker M_2 \cap \ker^t M_1$, then the left invariant form $\alpha = \sum_j \alpha_j \theta_j^2$ belongs to E_0^2 if and only if

$$\alpha_6 = -\alpha_3$$

and

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 0.$$

Hence an orthonormal basis of E_0^2 is given by $\{\xi_1^2, \xi_2^2, \frac{1}{\sqrt{2}}(\xi_3^2 - \xi_6^2), \xi_4^2, \xi_5^2, \xi_7^2, \xi_8^2, \xi_9^2, \xi_{10}^2\} = \{dx_1 \wedge dx_2, dy_1 \wedge dy_2, \frac{1}{\sqrt{2}}(dx_1 \wedge dy_1 - dx_2 \wedge dy_2), dx_1 \wedge dy_2, dx_2 \wedge dy_1, dx_1 \wedge ds, dx_2 \wedge ds, dy_1 \wedge ds, dy_2 \wedge ds\}$. In particular, the orthogonal projection $\Pi_{E_0}\alpha$ of α on E_0 has the form

$$\Pi_{E_0}\alpha = \sum_{\substack{j=1 \\ j\neq 3,6}}^{10} \alpha_j \xi_j^2 + \frac{\alpha_3 - \alpha_6}{2}(\xi_3^2 - \xi_6^2). \tag{16}$$

We want now to write explicitly d_c acting on forms $\alpha = \alpha(x) = \sum_{j=1}^5 \alpha_j(x)\xi_j^1$. To this end, let us write first $\Pi_{E^1}\alpha$. Because of the structure of $\bigwedge^1 \mathfrak{g}$, by Proposition 0.4,

$$\Pi_{E^1}\alpha = \alpha + \gamma\theta,$$

for a smooth function γ, with $\gamma\theta = -d_0^{-1}(d_1\alpha)$, i.e.

$$d_0(\gamma\theta) + d_1\alpha \in \ker \delta_0, \tag{17}$$

by Corollary 0.1. We can write (17) in the form

$$\begin{aligned}
&4\gamma(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\
&\quad + (X_1\alpha_2 - X_2\alpha_1)dx_1 \wedge dx_2 + (Y_1\alpha_4 - Y_2\alpha_3)dy_1 \wedge dy_2 \\
&\quad + (X_1\alpha_3 - Y_1\alpha_1)dx_1 \wedge dy_1 + (X_1\alpha_4 - Y_2\alpha_1)dx_1 \wedge dy_2 \\
&\quad + (X_2\alpha_3 - Y_1\alpha_2)dx_2 \wedge dy_1 + (X_2\alpha_4 - Y_2\alpha_2)dx_2 \wedge dy_2 \\
&\quad + (X_1\alpha_5 - S\alpha_1)dx_1 \wedge ds + (X_2\alpha_5 - S\alpha_2)dx_2 \wedge ds, \\
&\quad + (Y_1\alpha_5 - S\alpha_3)dy_1 \wedge ds + (Y_2\alpha_5 - S\alpha_4)dy_2 \wedge ds \in \ker \delta_0.
\end{aligned} \tag{18}$$

Because of the form of ${}^tM_1$ above, this gives

$$8\gamma + X_1\alpha_3 - Y_1\alpha_1 + X_2\alpha_4 - Y_2\alpha_2 = 0,$$

i.e.

$$\gamma = -\frac{1}{8}(X_1\alpha_3 - Y_1\alpha_1 + X_2\alpha_4 - Y_2\alpha_2).$$

However, the explicit form of γ does not matter in the final expression of $d_c\alpha$. Indeed, keeping in mind that $d_0\alpha = 0$, and that $\Pi_{E_0}(d_1(\gamma\theta)) = \Pi_{E_0}(d\gamma \wedge \theta) = 0$, and $\Pi_{E_0}(d_2(\alpha + \gamma\theta)) = 0$, since Π_{E_0} vanishes on forms of weight 3, by our previous computation (18), we have

$$\begin{aligned}
d_c\alpha &= \Pi_{E_0}(d(\alpha+\gamma\theta)) \\
&= \Pi_{E_0}(d_0(\alpha+\gamma\theta)+d_1(\alpha+\gamma\theta)) + \Pi_{E_0}(d_2(\alpha+\gamma\theta)) \\
&= \Pi_{E_0}(d_0(\gamma\theta)+d_1\alpha) \\
&= \Pi_{E_0}\big((X_1\alpha_2 - X_2\alpha_1)dx_1\wedge dx_2 + (Y_1\alpha_4 - Y_2\alpha_3)dy_1\wedge dy_2 \\
&\qquad + (X_1\alpha_3 - Y_1\alpha_1 + 4\gamma)dx_1\wedge dy_1 + (X_1\alpha_4 - Y_2\alpha_1)dx_1\wedge dy_2 \\
&\qquad + (X_2\alpha_3 - Y_1\alpha_2)dx_2\wedge dy_1 + (X_2\alpha_4 - Y_2\alpha_2 + 4\gamma)dx_2\wedge dy_2 \\
&\qquad + (X_1\alpha_5 - S\alpha_1)dx_1\wedge ds + (X_2\alpha_5 - S\alpha_2)dx_2\wedge ds, \\
&\qquad + (Y_1\alpha_5 - S\alpha_3)dy_1\wedge ds + (Y_2\alpha_5 - S\alpha_4)dy_2\wedge ds\big) \\
&= (X_1\alpha_2 - X_2\alpha_1)dx_1\wedge dx_2 + (Y_1\alpha_4 - Y_2\alpha_3)dy_1\wedge dy_2 \\
&\qquad + (X_1\alpha_4 - Y_2\alpha_1)dx_1\wedge dy_2 + (X_2\alpha_3 - Y_1\alpha_2)dx_2\wedge dy_1 \\
&\qquad + (X_1\alpha_5 - S\alpha_1)dx_1\wedge ds + (X_2\alpha_5 - S\alpha_2)dx_2\wedge ds, \\
&\qquad + (Y_1\alpha_5 - S\alpha_3)dy_1\wedge ds + (Y_2\alpha_5 - S\alpha_4)dy_2\wedge ds \\
&\qquad + \frac{X_1\alpha_3 - Y_1\alpha_1 - X_2\alpha_4 + Y_2\alpha_2}{\sqrt{2}}\,\frac{1}{\sqrt{2}}(dx_1\wedge dy_1 - dx_2\wedge dy_2),
\end{aligned}$$

by (16).

Example 0.3. Let $\mathbb{G}\equiv\mathbb{R}^6$ be the Carnot group associated with the vector fields

$$\begin{aligned}
X_1 &= \partial_1 \\
X_2 &= \partial_2 + x_1\partial_4 \\
X_3 &= \partial_3 + x_2\partial_5 + x_4\partial_6
\end{aligned}$$

and

$$\begin{aligned}
X_4 &= \partial_4 \\
X_5 &= \partial_5 + x_1\partial_6 \\
X_6 &= \partial_6.
\end{aligned}$$

Only non–trivial commutation rules are

$$[X_1, X_2] = X_4, \quad [X_2, X_3] = X_5, \quad [X_1, X_5] = X_6, \quad [X_4, X_3] = X_6.$$

The X_j's are left invariant and coincide with the elements of the canonical basis of $\mathbb{R}^6$ at the origin. The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits the stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3,$$

where $\mathfrak{g}_1 = \text{span}\,\{X_1, X_2, X_3\}$, $\mathfrak{g}_2 = \text{span}\,\{X_4, X_5\}$, and $\mathfrak{g}_3 = \text{span}\,\{X_6\}$. We set also

$$\begin{aligned} \theta_5 &= dx_5 - x_2 dx_3 \\ \theta_4 &= dx_4 - x_1 dx_2 \\ \theta_6 &= dx_6 - x_1 dx_5 + (x_1 x_2 - x_4) dx_3 \end{aligned}$$

and

$$\theta_1 = dx_1, \quad \theta_2 = dx_2, \quad \theta_3 = dx_3.$$

Proposition 0.6. *We have*

$$\theta_i = X_i^\natural \quad \textit{for } i, j = 1, ..., 6.$$

Moreover

$$d\theta_4 = -\theta_1 \wedge \theta_2, \quad d\theta_5 = -\theta_2 \wedge \theta_3, \quad d\theta_6 = \theta_3 \wedge \theta_4 - \theta_1 \wedge \theta_5.$$

As in Example 0.2, let us restrict ourselves to show the structure of the intrinsic differential on E_0^1, i.e on horizontal 1-forms. Using the notations of (5), (6) and (7), we can chose an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 1, 2, 3$ as follows:

h = 1: $\Theta^{1,1} = (\theta_1, \theta_2, \theta_3)$, $\Theta^{1,2} = (\theta_4, \theta_5)$, and $\Theta^{1,3} = (\theta_6)$.

h = 2: $\Theta^{2,2} = (\theta_1^2, \theta_2^2, \theta_3^2) = (\theta_1 \wedge \theta_2, \theta_1 \wedge \theta_3, \theta_2 \wedge \theta_3)$, $\Theta^{2,3} = (\theta_4^2, \dots, \theta_9^2) = (\theta_1 \wedge \theta_4, \theta_1 \wedge \theta_5, \theta_2 \wedge \theta_4, \theta_2 \wedge \theta_5, \theta_3 \wedge \theta_4, \theta_3 \wedge \theta_5)$, $\Theta^{2,4} = (\theta_{10}^2, \dots, \theta_{13}^2) = (\theta_1 \wedge \theta_6, \theta_2 \wedge \theta_6, \theta_3 \wedge \theta_6, \theta_4 \wedge \theta_5)$, $\Theta^{2,5} = (\theta_{14}^2, \theta_{15}^2) = (\theta_4 \wedge \theta_6, \theta_5 \wedge \theta_6)$

h = 3: $\Theta^{3,3} = (\theta_1^3) = (\theta_1 \wedge \theta_2 \wedge \theta_3)$. $\Theta^{3,4} = (\theta_2^3, \dots, \theta_7^3) = (\theta_1 \wedge \theta_2 \wedge \theta_4, \theta_1 \wedge \theta_2 \wedge \theta_5, \theta_1 \wedge \theta_3 \wedge \theta_4, \theta_1 \wedge \theta_3 \wedge \theta_5, \theta_2 \wedge \theta_3 \wedge \theta_4, \theta_2 \wedge \theta_3 \wedge \theta_5)$, $\Theta^{3,5} = (\theta_8^3, \dots, \theta_{13}^3) = (\theta_1 \wedge \theta_2 \wedge \theta_6, \theta_1 \wedge \theta_3 \wedge \theta_6, \theta_2 \wedge \theta_3 \wedge \theta_6, \theta_1 \wedge \theta_4 \wedge \theta_5, \theta_2 \wedge \theta_4 \wedge \theta_5, \theta_3 \wedge \theta_4 \wedge \theta_5)$, $\Theta^{3,6} = (\theta_{14}^3, \dots, \theta_{19}^3) = (\theta_1 \wedge \theta_4 \wedge \theta_6, \theta_1 \wedge \theta_5 \wedge \theta_6, \theta_2 \wedge \theta_4 \wedge \theta_6, \theta_2 \wedge \theta_5 \wedge \theta_6, \theta_3 \wedge \theta_4 \wedge \theta_6, \theta_3 \wedge \theta_5 \wedge \theta_6)$, $\Theta^{3,7} = (\theta_{20}^3) = (\theta_4 \wedge \theta_5 \wedge \theta_6)$.

We notice that an orthonormal basis of $\bigwedge^h \mathfrak{g}$, $h = 4, 5, 6$ can be obtained by Hodge duality.

By Proposition 0.6

$$M_1 = \left(\begin{array}{ccc|cc|c}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$$M_2 = \left(\begin{array}{ccc|cccccc|cccc|cc}
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

As usual, E_0^1 is the space of left invariant horizontal 1-forms, i.e. an orthonormal basis of E_0^1 is given by $\{\theta_1, \theta_2, \theta_3\}$. Keeping into account that E_0^2 can be identified with $\ker M_2 \cap \ker^t M_1$, then the left invariant form $\alpha = \sum_j \alpha_j \theta_j^2$ belongs to E_0^2 if and only if

$$\alpha_5 = -\alpha_8, \quad \alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 0$$

and

$$\alpha_5 = \alpha_8, \quad \alpha_3 = \alpha_1 = 0.$$

Therefore, an orthonormal basis $\{\xi_1^2, \dots, \xi_5^2\}$ of $E_0^2 = E_0^{2,2} \oplus E_0^{2,3}$ is given by

$$\{\theta_1 \wedge \theta_3\} \cup \{\theta_1 \wedge \theta_4, \theta_2 \wedge \theta_4, \theta_2 \wedge \theta_5, \theta_3 \wedge \theta_5\}.$$

In particular, the orthogonal projection $\Pi_{E_0}\alpha$ of $\alpha \in \bigwedge^2 \mathfrak{g}$ on E_0^2 has the form

$$\Pi_{E_0}\alpha = \alpha_2\, \theta_1 \wedge \theta_3 + \alpha_4\, \theta_1 \wedge \theta_4 + \alpha_6\, \theta_2 \wedge \theta_4 + \alpha_7\, \theta_2 \wedge \theta_5 + \alpha_9\, \theta_3 \wedge \theta_5. \tag{19}$$

We want now to write explicitly d_c acting on forms $\alpha = \alpha(x) = \sum_{j=1}^3 \alpha_j(x)\theta_j$. To this end, let us write first $\Pi_{E^1}\alpha$. We have

$$\begin{aligned}\Pi_{E^1}\alpha &= (\Pi_{E^1}\alpha)_1 + (\Pi_{E^1}\alpha)_2 + (\Pi_{E^1}\alpha)_3\\ &= \alpha + (\Pi_{E^1}\alpha)_2 + (\Pi_{E^1}\alpha)_3\\ &:= \alpha + (\gamma_4\theta_4 + \gamma_5\theta_5) + \gamma_6\theta_6,\end{aligned}$$

with

$$\begin{aligned}\gamma_4\theta_4 + \gamma_5\theta_5 &= -d_0^{-1}(d_1(\alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3))\\ &= -d_0^{-1}((X_1\alpha_2 - X_2\alpha_1)\theta_1\wedge\theta_2 + (X_1\alpha_3 - X_3\alpha_1)\theta_1\wedge\theta_3\\ &\quad + (X_2\alpha_3 - X_3\alpha_2)\theta_2\wedge\theta_3),\end{aligned} \tag{20}$$

and

$$\gamma_6\theta_6 = -d_0^{-1}(d_1(\gamma_4\theta_4 + \gamma_5\theta_5) + d_2\alpha) \tag{21}$$

Now (20) is equivalent to

$$\begin{aligned}&d_0(\gamma_4\theta_4 + \gamma_5\theta_5) + (X_1\alpha_2 - X_2\alpha_1)\theta_1\wedge\theta_2 + (X_1\alpha_3 - X_3\alpha_1)\theta_1\wedge\theta_3\\ &\quad + (X_2\alpha_3 - X_3\alpha_2)\theta_2\wedge\theta_3 \in \ker {}^tM_1,\end{aligned} \tag{22}$$

i.e.

$$\begin{aligned}&(-\gamma_4 + X_1\alpha_2 - X_2\alpha_1)\theta_1\wedge\theta_2 + (X_1\alpha_3 - X_3\alpha_1)\theta_1\wedge\theta_3\\ &\quad + (-\gamma_5 + X_2\alpha_3 - X_3\alpha_2)\theta_2\wedge\theta_3 \in \ker {}^tM_1,\end{aligned} \tag{23}$$

that gives eventually

$$\gamma_4 = X_1\alpha_2 - X_2\alpha_1 \quad \text{and} \quad \gamma_5 = X_2\alpha_3 - X_3\alpha_2$$

Consider now (21), that is equivalent to

$$
\begin{aligned}
&d_0(\gamma_6\theta_6) + d_1((X_1\alpha_2 - X_2\alpha_1)\theta_4 + (X_2\alpha_3 - X_3\alpha_2)\theta_5 + d_2\alpha) \\
&\quad = \gamma_6(\theta_3\wedge\theta_4 - \theta_1\wedge\theta_5) + X_1(X_1\alpha_2 - X_2\alpha_1)\theta_1\wedge\theta_4 + X_2(X_1\alpha_2 - X_2\alpha_1)\theta_2\wedge\theta_4 \\
&\quad + X_3(X_1\alpha_2 - X_2\alpha_1)\theta_3\wedge\theta_4 + X_1(X_2\alpha_3 - X_3\alpha_2)\theta_1\wedge\theta_5 + X_2(X_2\alpha_3 - X_3\alpha_2)\theta_2\wedge\theta_5 \\
&\quad + X_3(X_2\alpha_3 - X_3\alpha_2)\theta_3\wedge\theta_5 - X_4\alpha_1\theta_1\wedge\theta_4 - X_4\alpha_2\theta_2\wedge\theta_4 - X_4\alpha_3\theta_3\wedge\theta_4 - X_5\alpha_1\theta_1\wedge\theta_5 \\
&\quad - X_5\alpha_2\theta_2\wedge\theta_5 - X_5\alpha_3\theta_3\wedge\theta_5 \\
&\quad = X_1(X_1\alpha_2 - X_2\alpha_1)\theta_1\wedge\theta_4 + X_2(X_1\alpha_2 - X_2\alpha_1)\theta_2\wedge\theta_4 \\
&\quad + (X_3(X_1\alpha_2 - X_2\alpha_1) + \gamma_6)\theta_3\wedge\theta_4 + (X_1(X_2\alpha_3 - X_3\alpha_2) - \gamma_6)\theta_1\wedge\theta_5 + X_2(X_2\alpha_3 - X_3\alpha_2)\theta_2\wedge\theta_5 \\
&\quad + X_3(X_2\alpha_3 - X_3\alpha_2)\theta_3\wedge\theta_5 - X_4\alpha_1\theta_1\wedge\theta_4 - X_4\alpha_2\theta_2\wedge\theta_4 - X_4\alpha_3\theta_3\wedge\theta_4 - X_5\alpha_1\theta_1\wedge\theta_5 \\
&\quad - X_5\alpha_2\theta_2\wedge\theta_5 - X_5\alpha_3\theta_3\wedge\theta_5 \\
&\quad = (X_1(X_1\alpha_2 - X_2\alpha_1) - X_4\alpha_1)\theta_4^2 + (X_1(X_2\alpha_3 - X_3\alpha_2) - \gamma_6 - X_5\alpha_1)\theta_5^2 \\
&\quad + (X_2(X_1\alpha_2 - X_2\alpha_1) - X_4\alpha_2)\theta_6^2 + (X_2(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_2)\theta_7^2 \\
&\quad + (X_3(X_1\alpha_2 - X_2\alpha_1) + \gamma_6 - X_4\alpha_3)\theta_8^2 + (X_3(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_3)\theta_9^2 \\
&\qquad \in \ker {}^tM_1,
\end{aligned}
$$

i.e. to

$$X_1(X_2\alpha_3 - X_3\alpha_2) - \gamma_6 - X_5\alpha_1 - (X_3(X_1\alpha_2 - X_2\alpha_1) + \gamma_6 - X_4\alpha_3) = 0$$

This yields

$$\gamma_6 = \frac{1}{2}\big(X_1(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_1 - X_3(X_1\alpha_2 - X_2\alpha_1) + X_4\alpha_3\big).$$

Thus

$$
\begin{aligned}
\Pi_{E^1}\alpha &= \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3 \\
&\quad + (X_1\alpha_2 - X_2\alpha_1)\theta_4 + (X_2\alpha_3 - X_3\alpha_2)\theta_5 \\
&\quad + \frac{1}{2}\big(X_1(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_1 - X_3(X_1\alpha_2 - X_2\alpha_1) + X_4\alpha_3\big)\theta_6.
\end{aligned}
$$

Then, by (19)

$$d_c\alpha = (X_1\alpha_3 - X_3\alpha_1)\theta_1 \wedge \theta_3 + (X_1(X_1\alpha_2 - X_2\alpha_1) - X_4\alpha_1)\theta_1 \wedge \theta_4$$
$$+ (X_2(X_1\alpha_2 - X_2\alpha_1) - X_4\alpha_2)\theta_2 \wedge \theta_4 + (X_2(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_2)\theta_2 \wedge \theta_5$$
$$+ (X_3(X_2\alpha_3 - X_3\alpha_2) - X_5\alpha_3)\theta_3 \wedge \theta_5.$$

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