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Forme differenziali nei gruppi di Carnot e compattezza per compensazione (Parte II)

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Abstract

In this Part II, the results of Part I are applied, as in a joint paper with Annalisa Baldi, Bruno Franchi and Nicoletta Tchou, to prove a compensated compactness theorem in Carnot groups.

In the sequel, we follow the notations of the first part of this seminar.

1. Function spaces

Let $\{X_1,\dots,X_m\}$ be the fixed basis of the horizontal layer $\mathfrak{g}_1$ of $\mathfrak{g}$. We denote by $\Delta_{\mathbb{G}}$ the nonnegative horizontal sublaplacian

$$\Delta_{\mathbb{G}} := -\sum_{j=1}^{m} X_j^2.$$

If $1<s<\infty$ and $a\in\mathbb{C}$, we define $\Delta_{\mathbb{G}}^{a}$ in $L^s(\mathbb{G})$ following [5]. If in addition $m\ge 0$, again as in [5], we denote by $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ the domain of the realization of $\Delta_{\mathbb{G}}^{m/2}$ in $L^s(\mathbb{G})$ endowed with the graph norm. In fact, since $s\in(1,\infty)$ is fixed through all the paper, to avoid cumbersome notations, we do not stress the explicit dependence on s of the fractional powers $\Delta_{\mathbb{G}}^{m/2}$ and of its domain.

We remind that

Proposizione 1.1 ([5], Corollary 4.13)**.** *If $1<s<\infty$ and $m\in\mathbb{N}$, then the space $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ coincides with the space of all $u\in L^s(\mathbb{G})$ such that*

$$X^I u\in L^s(\mathbb{G}) \quad \textit{for all multi-index } I \textit{ with } d(I)=m,$$

endowed with the natural norm.

Proposizione 1.2 ([5], Corollary 4.14)**.** *If $1<s<\infty$ and $m\ge 0$, then the space $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ is independent of the choice of $X_1,\dots,X_m$.*

Proposizione 1.3. *If $1<s<\infty$ and $m\ge 0$, then $\mathcal{S}(\mathbb{G})$ and $\mathcal{D}(\mathbb{G})$ are dense subspaces of $W^{m,s}_{\mathbb{G}}(\mathbb{G})$.*

Definizione 1.1. *Let $m\ge 0$, $1<s<\infty$ be fixed indices. Let $\Omega\subset\mathbb{G}$ be a given open set with $\mathcal{L}^n(\partial\Omega)=0$ (from now on, even if not explicitly stated, we shall assume this regularity property whenever an open set is meant to localize a statement). We denote by $\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega)$ the completion in $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ of $\mathcal{D}(\Omega)$. More precisely, denote by $v\to r_\Omega v$ the restriction operator to Ω; we say that u belongs to $\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega)$ if there exists a sequence of test functions $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ and $U\in W^{m,s}_{\mathbb{G}}(\mathbb{G})$, such that $u_k\to U$ in $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ and $u=r_\Omega U$. On the other hand, since in particular $u_k\to U$ in $L^s(\mathbb{G})$, necessarily $U\equiv 0$*

outside of Ω*. Therefore, if* $u = r_\Omega U_1 = r_\Omega U_2$ *with* U_1, U_2 *both belonging to the completion in* $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ *of* $\mathcal{D}(\Omega)$*, then* $U_1 \equiv U_2$*, so that, without loss of generality, we can set*

$$\|u\|_{\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega)} := \|p_0(u)\|_{W^{m,s}_{\mathbb{G}}(\mathbb{G})},$$

where $p_0(u)$ *denotes the continuation of* u *by zero outside of* Ω*.*

It is well known that $W^{1,s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ is continuously imbedded in $W^{1/(\kappa+1)}_{\mathrm{loc}}(\mathbb{G})$; thus, by classical Rellich theorem and interpolation arguments, we have:

Lemma 1.1. *Let* $\Omega \subset \mathbb{G}$ *be a bounded open set. If* $s > 1$*, and* $m > 0$*, then*

$$\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega) \quad \textit{is compactly embedded in} \quad L^s(\Omega).$$

Proposizione 1.4. *If* $m \geq 0$*,* $1 < s < \infty$ *and* $\Omega \subset \mathbb{G}$ *is a bounded open set, then*

$$\|u\|_{\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega)} \approx \|\Delta_{\mathbb{G}}^{m/2} p_0(u)\|_{L^s(\mathbb{G})}$$

when $u \in \overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega)$ *and* $p_0(u)$ *denotes its continuation by zero outside of* Ω*.*

To keep the seminar as much self-contained as possible, we remind some basic definitions and results taken from [3] on pseudodifferential operators on homogeneous groups.

We set

$$\mathcal{S}_0 := \{u \in \mathcal{S} \ : \ \int_{\mathbb{G}} x^\alpha u(x)\, dx = 0\}$$

for all monomials x^α.

If $\alpha \in \mathbb{R}$ and $\alpha \notin \mathbb{Z}^+ := \mathbb{N} \cup \{0\}$, then we denote by $\mathbf{K}^\alpha$ the set of the distributions in $\mathbb{G}$ that are smooth away from the origin and homogeneous of degree α, whereas, if $\alpha \in \mathbb{Z}^+$, we say that $K \in \mathcal{D}'(\mathbb{G})$ belongs to $\mathbf{K}^\alpha$ if has the form

$$K = \tilde{K} + p(x) \ln |x|,$$

where $\tilde{K}$ is smooth away from the origin and homogeneous of degree α, and p is a homogeneous polynomial of degree α.

Kernels of type α according to Folland [5] belong to $\mathbf{K}^{\alpha - Q}$. In particular, if $0 < \alpha < Q$, and $h(t,x)$ is the heat kernel associated with the sub-Laplacian $\Delta_{\mathbb{G}}$, then ([5], Proposition

3.17) the kernel $R_\alpha \in L^1_{\rm loc}(\mathbb{G})$ defined by

$$R_\alpha(x) := \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{(\alpha/2)-1} h(x,t)\, dt$$

belongs to $\mathbf{K}^{\alpha-Q}$.

If $K \in \mathbf{K}^\alpha$, we denote by $\mathcal{O}_0(K)$ the operator defined on $\mathcal{S}_0$ by $\mathcal{O}_0(K)u := u * K$.

Proposizione 1.5 ([3], Proposition 2.2)**.** $\mathcal{O}_0(K) : \mathcal{S}_0 \to \mathcal{S}_0$.

Teorema 1.1 (see [7], [8])**.** *If $K \in \mathbf{K}^{-Q}$, then $\mathcal{O}_0(K) : L^2(\mathbb{G}) \to L^2(\mathbb{G})$.*

Teorema 1.2 (see [3], Theorem 5.11)**.** *If $K \in \mathbf{K}^{-Q}$, and let the following Rockland condition hold: for every nontrivial irreducible unitary representation π of $\mathbb{G}$, the operator $\overline{\pi_K}$ is injective on $\mathbf{C}^\infty(\pi)$, the space of smooth vectors of the representation π. Then the operator $\mathcal{O}_0(K) : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is left invertible.*

Obviously, if $\mathcal{O}_0(K)$ is formally self-adjoint, i.e. if $K = {}^{\mathrm{v}}K$, then $\mathcal{O}_0(K)$ is also right invertible.

Proposizione 1.6 ([3], Proposition 2.3)**.** *If $K_i \in \mathbf{K}^{\alpha_i}$, $i = 1, 2$, then there exists at least one $K \in \mathbf{K}^{\alpha_1+\alpha_2+Q}$ such that*

$$\mathcal{O}_0(K_2) \circ \mathcal{O}_0(K_1) = \mathcal{O}_0(K).$$

It is possible to provide a standard procedure yielding such a K (see [3], p.42). Following [3], we write $K = K_2 \underline{*} K_1$.

We can give now a (simplified) definition of pseudodifferential operator on $\mathbb{G}$, following [3], Definition 2.4.

Definizione 1.2. *If $\alpha \in \mathbb{R}$, we say that $\mathcal{K}$ is a pseudodifferential operator of order α on $\mathbb{G}$ with core K if*

1) *$K \in \mathcal{D}'(\mathbb{G} \times \mathbb{G})$.*
2) *Let $\beta := -Q - \alpha$. There exist $K^m = K^m_x \in \mathbf{K}^{\beta+m}$ depending smoothly on $x \in \mathbb{G}$ such that for each $N \in \mathbb{N}$ there exists $M \in \mathbb{Z}^+$ such that, if we set*

$$K_x - \sum_{m=0}^{M} K^m_x := E_M(x, \cdot),$$

then $E_M \in \mathbf{C}^N(\mathbb{G}\times\mathbb{G})$.

3) *For some finite* $R \geq 0$, $\operatorname{supp} K_x \subset B(e,R)$ *for all* $x \in \mathbb{G}$.

4) *If* $u \in \mathcal{D}(\mathbb{G})$ *and* $x \in \mathbb{G}$, *then*

$$\mathcal{K}u(x) = (u * K_x)(x).$$

We write $K \sim \sum_m K^m$, $\mathcal{K} = \mathcal{O}(K)$, *and* $r(K) = r(\mathcal{K}) = \inf\{R > 0$ *such that 3) holds*$\}$.

We let

$$\mathcal{OC}^\alpha(\mathbb{G}) := \{\textit{pseudodifferential operators of order } \alpha \textit{ on } \mathbb{G}\}.$$

Clearly, if $\mathcal{K} \in \mathcal{OC}^\alpha(\mathbb{G})$, *then* $\mathcal{K} : \mathcal{D}(\mathbb{G}) \to \mathcal{E}(\mathbb{G})$. *Moreover,* $\mathcal{K}$ *can be extended to an operator* $\mathcal{K} : \mathcal{E}'(\mathbb{G}) \to \mathcal{D}'(\mathbb{G})$.

Lemma 1.2. *If* $\operatorname{supp} u \subset B(e,\rho)$, *then* $\operatorname{supp} \mathcal{K}u \subset B(e, \rho + r(\mathcal{K}))$.

If $\gamma = (\gamma_1, \dots, \gamma_n) \in (\mathbb{Z}^+)^n$, for any $f \in \mathcal{D}'(\mathbb{G})$ we set

$$M_\gamma f = x^\gamma f,$$

and, if $X = (X_1, \dots, X_n)$ is our fixed basis of $\mathfrak{g}$, we denote by $\sigma_\gamma(X)$ the coefficient of x^γ in the expansion of $(\gamma!/|\gamma|!)(x \cdot X)^{d(\gamma)}$.

Teorema 1.3 ([3], Theorem 2.5)**.** *We have:*

(a) *If* $\mathcal{K} := \mathcal{O}(K) \in \mathcal{OC}^\alpha(\mathbb{G})$, *then there exists a core* K^* *such that* $\mathcal{O}(K^*) \in \mathcal{OC}^\alpha(\mathbb{G})$ *and*

$$\langle v, \mathcal{K}u\rangle_{L^2(\mathbb{G})} = \langle \mathcal{O}(K^*)v, u\rangle_{L^2(\mathbb{G})}$$

for all $u, v \in \mathcal{D}(\mathbb{G})$.

(b) *If* $\mathcal{K} \in \mathcal{OC}^\alpha(\mathbb{G})$, $V \subset \mathbb{G}$ *is an open set, and* $u \in \mathcal{E}'(\mathbb{G})$ *is smooth on* V, *then* $\mathcal{K}u$ *is smooth on* V.

(c) *If* $\mathcal{K}_i \in \mathcal{OC}_i^\alpha(\mathbb{G})$, $K_i \sim \sum_m K_i^m$, $i = 1, 2$, *then* $\mathcal{K} := \mathcal{K}_2 \circ \mathcal{K}_1$ *(that is well defined by Lemma 1.2) belongs to* $\mathcal{OC}^{\alpha_1+\alpha_2}(\mathbb{G})$. *Moreover* $K \sim \sum_m K^m$, *where*

$$K_x^m = \sum_{d(\gamma)+j+\ell=m} \frac{1}{\gamma!}\, [(-M)^\gamma (K_2^\ell)_x] \,\underline{*}\, [\sigma_\gamma(X)(K_1^j)_x],$$

where $\sigma_\gamma(X)$ *acts in the* x*-variable.*

Teorema 1.4 (see [3], p.63 (3)). *If $\mathcal{K} \in \mathcal{OC}^0(\mathbb{G})$, then $\mathcal{O}(K) : L^p_{\mathrm{loc}}(\mathbb{G}) \to L^p_{\mathrm{loc}}(\mathbb{G})$ is continuous. In particular, by Lemma 1.2, $\mathcal{O}(K) : L^p(\mathbb{G}) \cap \mathcal{E}'(B(e,\rho)) \to L^p(\mathbb{G})$ continuously.*

We say that a convolution operator $u \to u * E(x,\cdot)$ from $\mathcal{E}'$ to $\mathcal{D}'$ belongs to $\mathcal{OC}^{-\infty}(\mathbb{G})$ if E is smooth on $\mathbb{G}\times\mathbb{G}$. We notice that, properly speaking, $\mathcal{OC}^{-\infty}(\mathbb{G})$ is not contained in $\mathcal{OC}^{\alpha}(\mathbb{G})$ for $\alpha \in \mathbb{R}$, since $E(x,\cdot)$ is not assumed to be compactly supported.

If $\mathcal{T},\mathcal{S} \in \mathcal{OC}^{\ell}(\mathbb{G})$, we say that $\mathcal{S} = \mathcal{T} \bmod \mathcal{OC}^{-\infty}$ if $\mathcal{S} - \mathcal{T} \in \mathcal{OC}^{-\infty}(\mathbb{G})$.

A straightforward computation proves the following result

Lemma 1.3. *If $\mathcal{S} \in \mathcal{OC}^{-\infty}(\mathbb{G})$, $\varphi \in \mathcal{D}(\mathbb{G})$, and $\mathcal{O}(K) \in \mathcal{OC}^m(\mathbb{G})$ for $m \in \mathbb{R}$, then both $(\varphi\mathcal{S}) \circ \mathcal{O}(K)$ and $\mathcal{O}(K) \circ (\varphi\mathcal{S})$ belong to $\mathcal{OC}^{-\infty}(\mathbb{G})$.*

Lemma 1.4. *If $\Omega \subset \mathbb{G}$ is a bounded open set, $m, m' \in \mathbb{R}$, $1 < s < \infty$, and $\mathcal{T} \in \mathcal{OC}^{-\infty}(\mathbb{G})$, then, if $\varphi \in \mathcal{D}(\mathbb{G})$, the map*

$$\varphi T : W^{m,s}_{\mathbb{G}}(\mathbb{G}) \cap \mathcal{E}'(\Omega) \to W^{m',s}_{\mathbb{G}}(\mathbb{G})$$

is compact.

From now on, let $\psi \in \mathcal{D}(\mathbb{G})$ be a fixed nonnegative function such that

$$\operatorname{supp}\psi \subset B(e,1) \quad \text{and} \quad \psi \equiv 1 \text{ on } B(e,\frac{1}{2}).$$

We set

$$\psi_R := \psi \circ \delta_{1/R}.$$

If $K \in \mathbf{K}^m$, then $K_R := \psi_R K$ is a core satisfying 1), 2), 3) of Definition 1.2. In addition, $K_R \sim K$, since we can write $K_R = K + (\psi_R - 1)K$, with $(\psi_R - 1)K \in \mathcal{E}(\mathbb{G})$. Thus $\mathcal{O}(K_R) \in \mathcal{OC}^{-m-Q}(\mathbb{G})$.

Thus, if K is a Folland kernel of type $\alpha \in \mathbb{R}$, then K_R is a core of a pseudodifferential operator $\mathcal{O}(K_R) \in \mathcal{OC}^{-\alpha}(\mathbb{G})$. In particular, if $0 < \alpha < Q$, then $\mathcal{O}((R_\alpha)_R)$ belongs to $\mathcal{OC}^{-\alpha}(\mathbb{G})$ (see [5], Proposition 3.17).

Lemma 1.5. *If $K \in \mathbf{K}^m$, and X^I is a left invariant homogeneous differential operator, then*

$$X^I \mathcal{O}(K_R) \in \mathcal{OC}^{-m+d(I)-Q}(\mathbb{G}).$$

Moreover, the core $K_{R,I}$ of $X^I \mathcal{O}(K_R)$ satisfies

$$K_{R,I} \sim X^I K,$$

and

$$X^I \mathcal{O}(K_R) = \mathcal{O}((X^I K)_R) \mod \mathcal{OC}^{-\infty}.$$

Lemma 1.6. *If $u \in \mathcal{E}'(\mathbb{G})$ and* supp $u \subset B(0,\rho)$ *then* supp $\mathcal{O}(K_R)u \subset B(0, R+\rho))$. *Moreover, if $\rho = R$, then*

$$\mathcal{O}(K_{4R})u \equiv u * K \quad \textit{on } B(0,R).$$

Proposizione 1.7. *Let $K_i \in \mathbf{K}^i$ be given cores for $i = 1,2$, and let $R > 0$ be fixed. Then*

$$\mathcal{O}((K_2 \underline{*} K_1)_R) = \mathcal{O}((K_1)_R) \circ \mathcal{O}((K_2)_R) \mod \mathcal{OC}^{-\infty}.$$

In particular, $\mathcal{O}((K_1)_R) \circ \mathcal{O}((K_2)_R) = \mathcal{O}(K)$ for a suitable core K with $K \sim K_2 \underline{*} K_1$.

Osservazione 1.1. *As in Remark 5 at p. 63 of* [3], *the previous calculus can be formulated for matrix-valued operators and hence, once left invariant bases $\{\xi_j^h\}$ of E_0^h are chosen, we obtain pseudodifferential operators acting on h-forms and h-currents, together with the related calculus.*

In particular, let $K := \big(K_{ij}\big)_{\substack{i=1,\dots,N\\ j=1,\dots,M}}$ a $M \times N$ matrix whose entries K_{ij} belong to $\mathbf{K}^{m_{ij}}$. Then K acts between $\mathcal{S}_0(\mathbb{G})^N$ and $\mathcal{S}_0(\mathbb{G})^M$ as follows: if $T = (T_1, \dots, T_M)$, then

$$\mathcal{O}_0(K)T := T * K := (\sum_j T_j * K_{1j}, \dots, \sum_j T_j * K_{Mj}).$$

When $K_{ij} \in \mathbf{K}^m$ for all i,j, we write shortly that $K \in \mathbf{K}^m$.

If $K := \big(K_{ij}\big)_{\substack{i=1,\dots,N\\ j=1,\dots,M'}}$ and $K' := \big(K'_{ij}\big)_{\substack{i=1,\dots,M'\\ j=1,\dots,M}}$, we write

$$K' \underline{*} K := \big(\sum_\ell K'_{i\ell} \underline{*} K_{\ell j}\big).$$

Notice that

$$\mathcal{O}_0(K') \circ \mathcal{O}_0(K) = O_0(K' \underline{*} K). \tag{1}$$

Finally, we prove that the fractional powers of $\Delta_{\mathbb{G}}$, when acting on suitable function spaces, can be written as suitable convolution operators. This is more or less know (see for instance [3], Section 6), though not explicitly stated in the form we need. Because of that, we prefer to provide full proofs.

Teorema 1.5. *If $m \in \mathbb{R}$ and $1 < s < \infty$, then $\mathcal{S}_0(\mathbb{G}) \subset \mathrm{Dom}\,(\Delta_{\mathbb{G}}^{m/2})$, and there exists $P_m \in \mathbf{K}^{-m-Q}$ such that*

$$\Delta_{\mathbb{G}}^{m/2} u = u * P_m \quad \text{for all } u \in \mathcal{S}_0(\mathbb{G}).$$

Moreover, if $R > 0$ then

$$\mathcal{O}((P_m))_R \in \mathcal{OC}^m(\mathbb{G}). \tag{2}$$

Coherently, in the sequel we shall write

$$\Delta_{\mathbb{G},R}^{m/2} := \mathcal{O}((P_m))_R. \tag{3}$$

Lemma 1.7. *We have*

$$\Delta_{\mathbb{G},R}^{m/2} \circ \Delta_{\mathbb{G},R}^{-m/2} = Id \mod \mathcal{OC}^{-\infty},$$

and

$$\Delta_{\mathbb{G},R}^{-m/2} \circ \Delta_{\mathbb{G},R}^{m/2} = Id \mod \mathcal{OC}^{-\infty}.$$

Proposizione 1.8. *If $\Omega \subset \mathbb{G}$ is a bounded open set, $m, \alpha \in \mathbb{R}$, $1 < s < \infty$, and $\mathcal{T} \in \mathcal{OC}^{\alpha}(\mathbb{G})$, then*

$$\mathcal{T} : W_{\mathbb{G}}^{m+\alpha,s}(\mathbb{G}) \cap \mathcal{E}'(\Omega) \to W_{\mathbb{G}}^{m,s}(\mathbb{G})$$

continuously.

Lemma 1.8. *If $m > 0$ let $P_m \in \mathbf{K}^{-m-Q}$ be the kernel defined in Theorem 1.5. If $\Omega \subset\subset \mathbb{G}$ is an open set, $R > R_0(s, \mathbb{G}, m, \Omega)$ is sufficiently large, and $u \in \mathcal{D}(\Omega)$, then*

$$\|u\|_{W_{\mathbb{G}}^{m,s}(\mathbb{G})} \approx \|\mathcal{O}((P_m)_R)u\|_{L^s(\mathbb{G})} = \|\Delta_{\mathbb{G},R}^{m/2} u\|_{L^s(\mathbb{G})},$$

with equivalence constants depending on $s, \mathbb{G}, m, \Omega$.

Definizione 1.3. *Let $\Omega \subset \mathbb{G}$ be an open set. If $m \geq 0$ and $1 < s < \infty$, $W^{-m,s}_{\mathbb{G}}(\Omega)$ is the dual space of $\overset{\circ}{W}{}^{k,s'}_{\mathbb{G}}(\Omega)$, where $1/s + 1/s' = 1$. It is well known that, if $m \in \mathbb{N}$ and Ω is bounded, then*

$$W^{-m,s}_{\mathbb{G}}(\Omega) = \{ \sum_{d(I)=k} X^I f_I,\ f_I \in L^s(\Omega) \text{ for any } I \text{ such that } d(I) = k\},$$

and

$$\|u\|_{W^{-m,s}_{\mathbb{G}}(\Omega)} \approx \inf\{\sum_I \|f_I\|_{L^s(\Omega)}\,;\, d(I) = k, \sum_{d(I)=k} X^I f_I = u\}.$$

Proposizione 1.9. *If $1 < s < \infty$ and $m, m' \geq 0$, $m' < m$, then*

$$W^{m,s}_{\mathbb{G}}(\mathbb{G}) \hookrightarrow W^{m',s}_{\mathbb{G}}(\mathbb{G}) \quad \text{and} \quad W^{-m',s}_{\mathbb{G}}(\mathbb{G}) \hookrightarrow W^{-m,s}_{\mathbb{G}}(\mathbb{G})$$

algebraically and topologically.

In addition, if Ω is a bounded open set, $1 < s < \infty$ and $m, m' \geq 0$, $m' < m$, then

$$\overset{\circ}{W}{}^{m,s}_{\mathbb{G}}(\Omega) \quad \text{is compactly embedded in } W^{m',s}_{\mathbb{G}}(\Omega)$$

and

$$W^{-m',s}_{\mathbb{G}}(\Omega) \quad \text{is compactly embedded in } W^{-m,s}_{\mathbb{G}}(\Omega).$$

We need a few definitions. For all our notations related to Rumin's complex, we refer to Part I of this seminar. We set

$$\mathcal{I}^h_0 := \{p\,;\, I^h_{0,p} \neq \emptyset\} \quad \text{and} \quad |\mathcal{I}^h_0| = \operatorname{card} \mathcal{I}^h_0. \tag{4}$$

Let

$$\underline{m} = (m_{N^{\min}_h}, \dots, m_{N^{\max}_h})$$

be a $|\mathcal{I}^h_0|$–dimensional vector where the components are indexed by the elements of $\mathcal{I}^h_0$ (i.e. by the possible weights) taken in increasing order. We stress that, since weights p such that $I^h_{0,p} = \emptyset$ can exist, then some consecutive indices in $\underline{m}$ can be missed. In the sequel we shall say that $\underline{m}$ is a *h–vector weight*. We say that $\underline{m} \geq 0$ if $m_p \geq 0$ for $p \in \mathcal{I}^h_0$, and that $\underline{m} \geq \underline{n}$ if $m_p \geq n_p$ for all $p \in \mathcal{I}^h_0$. We say also that $\underline{m} > \underline{n}$ if $m_p > n_p$ *for all* $p \in \mathcal{I}^h_0$. Finally, if m_0 is a real number, we identify m_0 with the h–vector weight $m_0 = (m_0, \dots, m_0)$. In particular, we set $\underline{m} - m_0 := (m_{N^{\min}_h} - m_0, \dots, m_{N^{\max}_h} - m_0)$.

Definizione 1.4. *A special h–vector weight that we shall use in the sequel is the h-vector weight* $\underline{N}_h = (m_{N_h^{\min}}, \dots, m_{N_h^{\max}})$ *with*

$$m_p = p \quad \textit{for all } p \in I_0^h.$$

If all h-forms have *pure weight* N_h, i.e. if $N_h^{\min} = N_h^{\max} := N_h$, then a h-vector weight has only one component, i.e. $\underline{m} = (m_{N_h})$.

Definizione 1.5. *If* $\underline{m} \geq 0$ *is a h-vector weight,* $0 \leq h \leq n$, *and* $s > 1$, *we say that a measurable section* α *of* E_0^h, $\alpha := \sum_p \sum_{j \in I_{0,p}^h} \alpha_j \xi_j^h$ *belongs to* $W_{\mathbb{G}}^{\underline{m},s}(\mathbb{G}, E_0^h)$ *if, for all* $p \in \mathcal{I}_0^h$, *i.e. for all* p, $N_h^{\min} \leq p \leq N_h^{\max}$, *such that* $I_{0,p}^h \neq \emptyset$,

$$\alpha_j \in W_{\mathbb{G}}^{m_p,s}(\mathbb{G})$$

for all $j \in I_{0,p}^h$, *endowed with the natural norm.*

The spaces $W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$, where Ω is an open set in $\mathbb{G}$, as well as the local spaces $W_{\mathbb{G},\mathrm{loc}}^{\underline{m},s}(\Omega, E_0^h)$ are defined in the obvious way.

Since

$$W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h) \quad \text{is isometric to} \prod_{p \in \mathcal{I}_0^h} \left(W_{\mathbb{G}}^{m_p,s}(\mathbb{G})\right)^{\mathrm{card}\, \mathcal{I}_{0,p}^h},$$

then

- $W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$ is a reflexive Banach space (remember $s > 1$);
- $\mathbf{C}^\infty(\Omega, E_0^h) \cap W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$ is dense in $W_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$.

The spaces $\overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m},s}(\Omega, E_0^h)$ are defined in the obvious way.

We can define and characterize the dual spaces of Sobolev spaces of forms.

Proposizione 1.10. *If* $1 < s < \infty$, $1/s + 1/s' = 1$, $0 \leq h \leq n$, $\underline{m}$ *is a h–vector weight, and* $\Omega \subset \mathbb{G}$ *is a bounded open set, then the dual space* $\left(\overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m}.s'}(\Omega, E_0^h)\right)^*$ *coincides with the set of all currents* $T \in D'(\Omega, E_0^h)$ *of the form*

$$T = \sum_p \sum_{j \in I_{0,p}^h} \tilde{T}_j \llcorner (*\xi_j^h) \tag{5}$$

with $T_j \in W_{\mathbb{G}}^{-m_p,s}(\Omega)$ *for all* $j \in I_{0,p}^h$ *and for* $p \in \mathcal{I}_0^h$. *The action of* T *on the form* $\alpha = \sum_p \sum_{j\in I_{0,p}^h} \alpha_j \xi_j^h \in \overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h)$ *is given by the identity*

$$T(\alpha) = \sum_p \sum_{j\in I_{0,p}^h} \langle T_j|\alpha_j\rangle. \tag{6}$$

In particular, it is natural to set

$$W_{\mathbb{G}}^{-\underline{m},s}(\Omega, E_0^h) := \big(\overset{\circ}{W}{}_{\mathbb{G}}^{\underline{m},s'}(\Omega, E_0^h)\big)^*.$$

Moreover, if T *is as in (5)*

$$\|T\|_{W_{\mathbb{G}}^{-\underline{m},s}(\Omega,E_0^h)} \approx \sum_p \sum_{j\in I_{0,p}^h} \|T_j\|_{W_{\mathbb{G}}^{-m_p,s}(\Omega)}.$$

Definizione 1.6. *Let* $T \in \mathcal{E}'(\mathbb{G}, E_0^h)$ *be a compactly supported* h*-current on* $\mathbb{G}$ *of the form*

$$T = \sum_p \sum_{j\in I_{0,p}^h} \tilde{T}_j \llcorner (*\xi_j^h) \quad \text{with } T_j \in \mathcal{E}'(\mathbb{G}) \text{ for } j = 1, \dots, \dim E_0^h.$$

Let $\underline{m}$ *be a* h*–vector weight, and let* $R > 0$ *be fixed. We set (with the notation of* (3)*)*

$$\Delta_{\mathbb{G},R}^{\underline{m}/2} T := \sum_p \sum_{j\in I_{0,p}^h} (\widetilde{\Delta_{\mathbb{G},R}^{m_p/2} T_j}) \llcorner (*\xi_j^h).$$

In particular, if T *can be identified with a compactly supported* h*-form* $\alpha = \sum_p \sum_{j\in I_{0,p}^h} \alpha_j \xi_j^h$, *then our previous definition becomes*

$$\Delta_{\mathbb{G},R}^{\underline{m}/2} \alpha = \sum_p \sum_{j\in I_{0,p}^h} (\alpha_j * (P_{m_p})_R)\xi_j^h.$$

Osservazione 1.2. *If* $\underline{m}$ *is a* h*-vector weight, we define the operator*

$$\mathcal{O}_0(P_{\underline{m}}) : \mathcal{S}_0(\mathbb{G}, E_0^h) \to \mathcal{S}_0(\mathbb{G}, E_0^h)$$

as follows: if $\alpha = \sum_p \sum_{j\in I_{0,p}^h} \alpha_j \xi_j^h$ *with* $\alpha_j \in \mathcal{S}_0(\mathbb{G})$, *then*

$$\mathcal{O}_0(P_{\underline{m}})\alpha := \sum_p \sum_{j\in I_{0,p}^h} (\alpha_j * P_{m_p})\xi_j^h.$$

In other words, $P_{\underline{m}}$ *can be identified with the matrix* $\big((P_{\underline{m}})_{ij}\big)$, *where*

$$(P_{\underline{m}})_{ij} = 0 \text{ if } i \neq j \text{ and } (P_{\underline{m}})_{jj} = m_p \text{ if } j \in I_{0,p}^h.$$

We can write

$$\Delta_{\mathbb{G},R}^{m/2} \sim P_{\underline{m}}.$$

2. Hodge decomposition

In this section we state and we prove our main results, i.e. a Hodge decomposition theorem for forms in E_0^* and – as a consequence – our compensated compactness theorem in E_0^*. Through this section, we assume that h, the degree of the forms we are dealing with, is fixed once and for all, $1 \le h \le n$, even if it is not mentioned explicitly in the statements.

From now on, we always assume that an ortonormal left invariant basis $\{\xi_j^\ell\}$ of E_0^ℓ has been fixed for all $\ell = 1, \dots, n$, and therefore pseudodifferential operators acting on intrinsic forms or current and matrix-valued pseudodifferential operators can be identified. We use this identification without referring explicitly to it.

Teorema 2.1. *Let $s > 1$ and $h = 1, \dots, n$ be fixed, and suppose h-forms have pure weight N_h. Let $\Omega \subset\subset \mathbb{G}$ a given open set, and let $\alpha^\varepsilon \in L^s(\mathbb{G}, E_0^h) \cap \mathcal{E}'(\Omega, E_0^h)$ be compactly supported differential h-forms such that*

$$\alpha^\varepsilon \rightharpoonup \alpha \quad \textit{as } \varepsilon \to 0 \quad \textit{weakly in } L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$$

and

$$\{d_c\alpha^\varepsilon\} \quad \textit{is pre-compact in } W^{-(\underline{N}_{h+1}-N_h),s}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^h).$$

Then there exist h–forms $\omega^\varepsilon \in E_0^h$ and $(h-1)$–forms $\psi^\varepsilon \in E_0^{h-1}$ such that

i) $\omega^\varepsilon \to \omega$ *strongly in* $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^h)$ *;*

ii) $\psi^\varepsilon \to \psi$ *strongly in* $L^s_{\mathrm{loc}}(\mathbb{G}, E_0^{h-1})$ *;*

iii) $\alpha^\varepsilon = \omega^\varepsilon + d_c\psi^\varepsilon$.

In addition, we can choose ω^ε and ψ^ε supported in a fixed suitable neigborhood of Ω, which are smooth forms if the α^ε are also smooth.

Osservazione 2.1. *We stress that $d_c : L^s(\mathbb{G}, E_0^h) \to W_{\mathbb{G}}^{-(\underline{N}_{h+1}-N_h),s}(\mathbb{G}, E_0^h)$. Indeed, if $\alpha = \sum_{j\in I^h_{0,N_h}} \alpha_j\xi_j^h \in L^s(\mathbb{G}, E_0^h)$ and $(d_c\alpha)_i$ is a component of weight q of $d_c\alpha$, then (keeping in mind that h-forms have pure weight N_h) $(d_c\alpha)_i = \sum_j L^h_{i,j}\alpha_j$, where $L^h_{i,j}$ is a* *homogeneous differential operator in the horizontal vector fields of order* $q - N_h \geq 1$*, so that* $(d_c\alpha)_i \in W_{\mathbb{G}}^{-(q-N_h),s}(\mathbb{G})$*. On the other hand* $(\underline{N}_{h+1} - N_h)_q = q - N_h$*, and the assertion follows.*

The proof of Theorem 2.1 entails several preliminary statements.

Definizione 2.1. *Let* $R > 0$ *be fixed. If* $0 \leq h \leq n$*, following Rumin we define the "0-order differential" acting on compactly supported* h*-currents belonging to* $\mathcal{E}'(B(e,R), E_0^h)$ *by*

$$\tilde{d}_c := \Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\, d_c\, \Delta_{\mathbb{G},R}^{\underline{N}_h/2},$$

where $\underline{N}_h$ *is defined in Definition 1.4. By Lemma 1.6, the definition is well posed, and*

$$\tilde{d}_c : \mathcal{E}'(B(e,R), E_0^h) \to \mathcal{E}'(B(e,3R), E_0^h).$$

Analogously, we define the following "0-order codifferential" acting on compactly supported $(h+1)$*-currents belonging to* $\mathcal{E}'(B(e,R), E_0^{h+1})$*:*

$$\widetilde{\delta_c} := \Delta_{\mathbb{G},R}^{\underline{N}_h/2}\, \delta_c\, \Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}.$$

Again the definition is well posed, and

$$\widetilde{\delta_c} : \mathcal{E}'(B(e,R), E_0^{h+1}) \to \mathcal{E}'(B(e,3R), E_0^h).$$

By Theorem 1.3(a),

$$\widetilde{\delta_c} = (\tilde{d}_c)^*.$$

Notice also that

$$\tilde{d}_c^2 = 0, \quad \tilde{\delta}_c^2 = 0 \pmod{\mathcal{OC}^{-\infty}}.$$

Let now $T = \sum_p \sum_{j \in I_{0,p}^h} \tilde{T}_j \llcorner (*\xi_j^h) \in \mathcal{E}'_{\mathbb{G},h}(B(e,R))$ *be given.*

The differential d_c acting on h-forms can be identified with a matrix-valued differential operator $L^h := \left(L_{i,j}^h\right)$, where the $L_{i,j}^h$'s are homogeneous left invariant differential operator of order $q - p$ if $j \in I_{0,p}^h$ and $i \in I_{0,q}^{h+1}$. Thus, by Definition 1.6, we have

$$\tilde{d}_c\, T = \sum_q \sum_{i \in I_{0,q}^{h+1}} \sum_{p<q} \sum_{j \in I_{0,p}^h} (\widetilde{\Delta_{\mathbb{G},R}^{-q/2}\, L_{i,j}^h\, \Delta_{\mathbb{G},R}^{p/2}} T_j) \llcorner (*\xi_i^{h+1}).$$

Analogously, if $T = \sum_p \sum_{j\in I^{h+1}_{0,p}} \tilde{T}_j \llcorner (*\xi^{h+1}_j) \in \mathcal{E}'(B(e,R), E^{h+1}_0)$, then

$$\widetilde{\delta_c}\, T = \sum_q \sum_{i\in I^h_{0,q}} \sum_{q<p} \sum_{j\in I^{h+1}_{0,p}} (\Delta^{q/2}_{\mathbb{G},R}\, \widetilde{{}^t L^h_{j,i}\, \Delta^{-p/2}_{\mathbb{G},R}} T_j) \llcorner (*\xi^h_i).$$

Proposizione 2.1. *Both $\tilde{d}_c$ and $\widetilde{\delta_c}$ are matrix-valued pseudodifferential operators of the CGGP-calculus, acting respectively on $\mathcal{E}'(\mathbb{G}, E^h_0)$ and $\mathcal{E}'(\mathbb{G}, E^{h+1}_0)$. Moreover $\tilde{d}_c \sim P^h := \big(P^h_{ij}\big)$, where*

$$P^h_{ij} = P_{-q} \underline{*} (L^h_{i,j} P_p) \quad \textit{if } i \in I^{h+1}_{0,q} \textit{ and } j \in I^h_{0,p}, \tag{7}$$

and $\widetilde{\delta_c} \sim Q^h := \big(Q^h_{ij}\big)$, where

$$Q^h_{ij} = P_q \underline{*} ({}^t L^h_{j,i} P_{-p}) \quad \textit{if } i \in I^h_{0,q} \textit{ and } j \in I^{h+1}_{0,p}. \tag{8}$$

Osservazione 2.2. *With Rumin's notations (see* [9]*), when acting on $\mathcal{S}_0(\mathbb{G}, E^h_0)$,*

$$\mathcal{O}_0(P^h) \equiv d^\nabla_c.$$

An analogous assertion hold for $\mathcal{O}_0(Q^h)$.

We set

$$\Delta^{(0)}_{\mathbb{G},R} := \widetilde{\delta_c}\widetilde{d_c} + \widetilde{d_c}\widetilde{\delta_c}.$$

The following assertion is a straightforward consequence of Theorem 1.3 and Proposition 2.1.

Proposizione 2.2. *$\Delta^{(0)}_{\mathbb{G},R}$ is a matrix-valued 0-order pseudodifferential operator of the CGGP-calculus acting on $\mathcal{E}'(\mathbb{G}, E^h_0)$, and*

$$\Delta^{(0)}_{\mathbb{G},R} \sim \Delta^{(0)}_{\mathbb{G}} := \big(\Delta^{(0)}_{\mathbb{G},ij}\big),$$

where

$$\Delta^{(0)}_{\mathbb{G},ij} = \sum_\ell \big(Q^h_{i\ell} \underline{*} P^h_{\ell j} + P^{h-1}_{i\ell} \underline{*} Q^{h-1}_{\ell j}\big).$$

Osservazione 2.3. *As in Remark 2.2, with the notations of* [9]*, when acting on $\mathcal{S}_0(\mathbb{G}, E^h_0)$,*

$$\begin{aligned}\mathcal{O}_0(\Delta^{(0)}_{\mathbb{G}}) &= \mathcal{O}_0(Q^h) \circ d_c \mathcal{O}_0(P^h) + \mathcal{O}_0(P^{h-1}) \circ \delta_c \mathcal{O}_0(Q^{h-1}) \\ &= \delta^\nabla_c d^\nabla_c + d^\nabla_c \delta^\nabla_c = \Box_{d_c}.\end{aligned}$$

Teorema 2.2. *For any $R > 0$ there exists a (matrix-valued) CGGP-pseudodifferential operator $(\Delta^{(0)}_{\mathbb{G},R})^{-1}$ such that*

$$(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{(0)}_{\mathbb{G},R} = Id \quad on \quad \mathcal{E}'(\mathbb{G}, E^h_0) \quad (\mathrm{mod}\ \mathcal{OC}^{-\infty}), \tag{9}$$

and

$$\Delta^{(0)}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1} = Id \quad on \quad \mathcal{E}'(\mathbb{G}, E^h_0) \quad (\mathrm{mod}\ \mathcal{OC}^{-\infty}). \tag{10}$$

Osservazione 2.4. *If $\alpha \in \mathcal{E}'(B(e,r), E^h_0)$, then, by Lemma 1.6, both*

$$\mathrm{supp}\ (\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{(0)}_{\mathbb{G},R}\alpha \quad and \quad \mathrm{supp}\ (\Delta^{(0)}_{\mathbb{G},R}\Delta^{(0)}_{\mathbb{G},R})^{-1}\alpha$$

are contained in a fixed ball B depending only on r, R. Thus, we can multiply the identities (9) *and* (10) *by a suitable test function φ that is identically one on B, and then we can replace the smoothing operators S appearing in* (9) *and* (10) *by operators of the form φS, that maps $\mathcal{E}'(\mathbb{G}, E^h_0)$ in $\mathcal{D}(\mathbb{G}, E^h_0)$.*

Proposizione 2.3. *For any $R > 0$*

$$(\Delta^{(0)}_{\mathbb{G},R})^{-1}\tilde{d}_c = \tilde{d}_c(\Delta^{(0)}_{\mathbb{G},R})^{-1} \quad on \quad \mathcal{E}'(\mathbb{G}, E^h_0) \quad (\mathrm{mod}\ \mathcal{OC}^{-\infty}), \tag{11}$$

and

$$(\Delta^{(0)}_{\mathbb{G},R})^{-1}\widetilde{\delta_c} = \widetilde{\delta_c}(\Delta^{(0)}_{\mathbb{G},R})^{-1} \quad on \quad \mathcal{E}'(\mathbb{G}, E^h_0) \quad (\mathrm{mod}\ \mathcal{OC}^{-\infty}). \tag{12}$$

Proof of Theorem 2.1. In the sequel, S will always denote a smoothing operator belonging to $\mathcal{OC}^{-\infty}$ that may change from formula to formula, and, with the same convention, we shall denote by S_0 an operator of the form φS, with $S \in \mathcal{OC}^{-\infty}$ and $\varphi \in \mathcal{D}(\mathbb{G})$. Moreover, without loss of generality, we may assume $\alpha^\varepsilon \in \mathcal{D}(\Omega, E^h_0)$. Take now $R > 0$ such that $\Omega \subset B(e,R)$; by Lemma 1.6, $\Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon \in \mathcal{D}(B(e,2R), E^h_0)$ and therefore, by (10),

$$\Delta^{(0)}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon - \Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon = S\Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon, \tag{13}$$

with $S \in \mathcal{OC}^{-\infty}$. Since $\mathrm{supp}\ \Delta^{(0)}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon \subset B(e,4R)$, we can multiply the previous identity by a cut-off function $\varphi_1 \equiv 1$ on $B(e,4R)$ without affecting the left hand side of the identity. Thus, we can write (13) as

$$\Delta^{(0)}_{\mathbb{G},R}(\Delta^{(0)}_{\mathbb{G},R})^{-1}\Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon - \Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon = \varphi S\Delta^{-\underline{N}_h/2}_{\mathbb{G},R}\alpha^\varepsilon = S_0\alpha^\varepsilon, \tag{14}$$

by Lemma 1.3. From (14), it follows easily that

$$\Delta_{\mathbb{G},R}^{N_h/2}\Delta_{\mathbb{G},R}^{(0)}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon = \Delta_{\mathbb{G},R}^{N_h/2}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon + \Delta_{\mathbb{G},R}^{N_h/2}S_0\alpha^\varepsilon, \tag{15}$$

so that, arguing as above,

$$\Delta_{\mathbb{G},R}^{N_h/2}\Delta_{\mathbb{G},R}^{(0)}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon = \alpha^\varepsilon + S_0\alpha^\varepsilon. \tag{16}$$

If we write explicitly $\Delta_{\mathbb{G},R}^{(0)}$ in (16), we get

$$\begin{aligned}
\alpha^\varepsilon &= \Delta_{\mathbb{G},R}^{N_h/2}\Delta_{\mathbb{G},R}^{N_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_{h+1}/2}\Delta_{\mathbb{G},R}^{-N_{h+1}/2}\,d_c\,\Delta_{\mathbb{G},R}^{N_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon\\
&+ \Delta_{\mathbb{G},R}^{N_h/2}\Delta_{\mathbb{G},R}^{-N_h/2}\,d_c\,\Delta_{\mathbb{G},R}^{N_{h-1}/2}\Delta_{\mathbb{G},R}^{N_{h-1}/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon\\
&+ S_0\alpha^\varepsilon := I_1 + I_2 + S_0\alpha^\varepsilon.
\end{aligned} \tag{17}$$

In addition

$$\begin{aligned}
I_2 &= d_c\,\Delta_{\mathbb{G},R}^{N_{h-1}/2}\Delta_{\mathbb{G},R}^{N_{h-1}/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon + S_0\alpha^\varepsilon\\
&:= d_c\psi^\varepsilon + S_0\alpha^\varepsilon.
\end{aligned} \tag{18}$$

Thus (17) becomes

$$\begin{aligned}
\alpha^\varepsilon &= \Delta_{\mathbb{G},R}^{N_h/2}\Delta_{\mathbb{G},R}^{N_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-N_{h+1}/2}\Delta_{\mathbb{G},R}^{-N_{h+1}/2}\,d_c\,\Delta_{\mathbb{G},R}^{N_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon\\
&+ S_0\alpha^\varepsilon + d_c\psi^\varepsilon := \omega^\varepsilon + d_c\psi^\varepsilon.
\end{aligned} \tag{19}$$

We want to show that $(\psi^\varepsilon)_{\varepsilon>0}$ and $(\omega^\varepsilon)_{\varepsilon>0}$ converge strongly in $L^s_{\mathrm{loc}}(\mathbb{G},E_0^{h-1})$ and $L^s_{\mathrm{loc}}(\mathbb{G},E_0^h)$, respectively. As for $(\psi^\varepsilon)_{\varepsilon>0}$, $(\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon)_{\varepsilon>0}$ converges weakly in $W^{N_h,s}_{\mathbb{G}}(\mathbb{G},E_0^h)$. On the other hand, by Proposition 1.8, also $\big((\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon\big)_{\varepsilon>0}$ converges weakly in $W^{N_h,s}_{\mathbb{G}}(\mathbb{G},E_0^h)$. Thus, again by Proposition 1.8, also $\big(\Delta_{\mathbb{G},R}^{-N_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon\big)_{\varepsilon>0}$ converges weakly in $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G},E_0^h)$. We remind that all intrinsic h-forms have the same weight N_h, so that all the components of a form in E_0^h belonging to $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G},E_0^h)$ belong to the same Sobolev space $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G},E_0^h)$.

For sake of simplicity, denote now by β_j^ε, $j\in I^h_{0,N_h}$, a generic component of

$$\Delta_{\mathbb{G},R}^{-N_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-N_h/2}\alpha^\varepsilon$$

that converges weakly in $W^{2N_h,s}_{\mathbb{G}}(\mathbb{G},E_0^{h-1})$. If $i\in I^{h-1}_{0,q}$ ($q<N_h$), then the i-th component of $\delta_c\beta_j^\varepsilon$ is given by ${}^tL_{j,i}\beta_j^\varepsilon$. Keeping in mind that $L_{j,i}$ is a homogeneous differential

operator in the horizontal vector fields of order $N_h - q$, then $\left({}^t L_{j,i}\beta_j^\varepsilon\right)_{\varepsilon>0}$ converges weakly in $W_{\mathbb{G}}^{N_h+q,s}(\mathbb{G})$, so that, eventually, the i-th component of $(\psi^\varepsilon)_{\varepsilon>0}$ converges weakly in $W_{\mathbb{G}}^{N_h-q,s}(\mathbb{G})$. Then the assertion follows by Rellich theorem (Proposition 1.9), since supp ψ^ε is contained is a fixed neighborhood of Ω, and $q < N_h$.

Let us consider now $(\omega^\varepsilon)_{\varepsilon>0}$. By Lemma 1.4, we can forget the smoothing operator S_0. By Proposition 2.3, we can write

$$\begin{aligned}
&\Delta_{\mathbb{G},R}^{\underline{N}_h}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\Delta_{\mathbb{G},R}^{\underline{N}_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_h/2}\alpha^\varepsilon\\
&\quad=\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon+S_0\alpha^\varepsilon\\
&\quad=\Delta_{\mathbb{G},R}^{\underline{N}_h/2}(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\,\delta_c\,\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon+S_0\alpha^\varepsilon.
\end{aligned}\tag{20}$$

Moreover

$$\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon \quad \text{is pre-compact in } W_{\mathbb{G},\mathrm{loc}}^{\underline{N}_{h+1}+N_h,s}(\mathbb{G},E_0^h).$$

Arguing as above, denote now by β_j^ε, $j\in I_{0,p}^{h+1}$, a generic component of

$$\beta^\varepsilon := \Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon.$$

We know that β_j^ε is pre-compact in $W_{\mathbb{G},\mathrm{loc}}^{p+N_h,s}(\mathbb{G},E_0^{h+1})$. Moreover notice that $\delta_c\beta_\varepsilon$ is a h-form, and therefore, by assumption, has pure weight N_h. If $i\in I_{0,N_h}^h$ $(N_h<p)$, then the i-th component of $\delta_c\beta_j^\varepsilon$ is given by ${}^tL_{j,i}\beta_j^\varepsilon$. Keeping in mind that $L_{j,i}$ is a homogeneous differential operator in the horizontal vector fields of order $j-i=p-N_h$, then $(\delta_c\beta_j^\varepsilon)_i$ is pre-compact in $W_{\mathbb{G},\mathrm{loc}}^{2N_h,s}(\mathbb{G})$. Thus, $\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$ is pre-compact in $W_{\mathbb{G},\mathrm{loc}}^{2\underline{N}_h,s}(\mathbb{G},E_0^h)$. Again, $\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$ is pre-compact in $W_{\mathbb{G},\mathrm{loc}}^{\underline{N}_h,s}(\mathbb{G},E_0^h)$. As above, we can rely now on the fact that all components of $\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$ have the same weight and hence belong to the same Sobolev space, to conclude that

$$(\Delta_{\mathbb{G},R}^{(0)})^{-1}\Delta_{\mathbb{G},R}^{\underline{N}_h/2}\delta_c\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\Delta_{\mathbb{G},R}^{-\underline{N}_{h+1}/2}\,d_c\,\alpha^\varepsilon$$

is pre-compact in $W_{\mathbb{G},\mathrm{loc}}^{\underline{N}_h,s}(\mathbb{G},E_0^h)$. Then, the proof of the assertion follows.

Finally, the last statement follows by Lemma 1.6 and Theorem 1.3, (b).

□

3. Compensated compactness

Lemma 3.1. *If* $\alpha \in \mathcal{E}(\mathbb{G}, E_0^h)$ *with* $2 \le h \le n$ *and* $\beta \in \mathcal{E}(\mathbb{G}, E_0^{n-h-2})$, *then*

$$d\, d_c\alpha \wedge (\Pi_E\beta) = 0.$$

Teorema 3.1. *If* $1 < s_i < \infty$, $0 \le h_i \le n$ *for* $i = 1, 2$, *and* $0 < \varepsilon < 1$, *assume that* $\alpha_i^\varepsilon \in L^{s_i}_{\mathrm{loc}}(\mathbb{G}, E_0^{h_i})$ *for* $i = 1, 2$, *where* $\frac{1}{s_1} + \frac{1}{s_2} = 1$ *and* $h_1 + h_2 = n$. *Suppose* h_1*-forms have pure weight* N_{h_1} *(by Hodge duality, this implies that also* h_2*-forms have pure weight* N_{h_2}*). Assume that, for any open set* $\Omega_0 \subset\subset \mathbb{G}$,

$$\alpha_i^\varepsilon \to \alpha_i \text{ weakly in } L^{s_i}(\Omega_0, E_0^{h_i}), \tag{21}$$

and that

$$\{d_c\alpha_i^\varepsilon\} \quad \text{is pre-compact in } W^{-(\underline{N}_{h_i+1} - N_{h_i}), s_i}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^{h_i}) \tag{22}$$

for $i = 1, 2$.

Then

$$\int_{\mathbb{G}} \varphi\, \alpha_1^\varepsilon \wedge \alpha_2^\varepsilon \to \int_{\mathbb{G}} \varphi\, \alpha_1 \wedge \alpha_2 \tag{23}$$

for any $\varphi \in \mathcal{D}(\mathbb{G})$.

Dimostrazione. By Remark 2.1, without loss of generality we can assume that both α_1^ε and α_2^ε are smooth forms. In addition, let us prove that, if Ω is an open neighborhood of supp φ, then

$$d_c(\varphi\alpha_1^\varepsilon) \quad \text{is pre-compact in } W^{-(\underline{N}_{h_i+1} - N_{h_i}), s_1}_{\mathbb{G},\mathrm{loc}}(\mathbb{G}, E_0^{h}). \tag{24}$$

An analogous argument can be repeat for $\psi\alpha_2^\varepsilon$, where $\psi \in \mathcal{D}(\Omega)$ is identically 1 on supp φ. Thus, without loss of generality, we could restrict ourselves to prove that

$$\int_{\mathbb{G}} \alpha_1^\varepsilon \wedge \alpha_2^\varepsilon \to \int_{\mathbb{G}} \alpha_1 \wedge \alpha_2 \tag{25}$$

when (21) and (22) hold and $\alpha_i \in \mathcal{D}(\Omega, E_0^{h_i})$ for $i = 1, 2$.

In order to prove (24), set $\beta^\varepsilon := d_c(\varphi\alpha_1^\varepsilon)$, with $\beta^\varepsilon = \sum_q \sum_{i \in I_{0,q}^{h_1+1}} \beta_i^\varepsilon \xi_i^{h+1}$.

If $\alpha_1^\varepsilon = \sum_p \sum_{j\in I_{0,p}^{h_1}} (\alpha_1^\varepsilon)_j \xi_j^h$, then, when $i \in I_{0,q}^{h_1+1}$, we have

$$\begin{aligned}\beta_i &= \sum_{p<q} \sum_{j\in I_{0,p}^h} (L_{i,j}^h(\varphi(\alpha_1^\varepsilon)_j)\\ &= \varphi \sum_{p<q} \sum_{j\in I_{0,p}^h} L_{i,j}^h(\alpha_1^\varepsilon)_j + \sum_{p<q} \sum_{j\in I_{0,p}^h} \sum_{1\le|\gamma|\le q-p} (P_\gamma\varphi)(Q_\gamma(\alpha_1^\varepsilon)_j)\\ &= \varphi(d_c(\alpha_1^\varepsilon))_i + \sum_{p<q} \sum_{j\in I_{0,p}^h} \sum_{1\le|\gamma|\le q-p} (P_\gamma\varphi)(Q_\gamma(\alpha_1^\varepsilon)_j),\end{aligned}$$

where P_γ and Q_γ are homogeneous left invariant differential operators of order $|\gamma|$ and $q-p-|\gamma|$, respectively, in the horizontal derivatives. By (22), $\varphi(d_c(\alpha_1^\varepsilon))_i$ is compact in $W_{\mathbb{G}}^{-(q-p),s}(\Omega)$. On the other hand $Q_\gamma(\alpha_1^\varepsilon)_j$ is bounded in $W_{\mathbb{G}}^{-(q-p-|\gamma|),s}(\Omega)$, and therefore compact in $W_{\mathbb{G}}^{-(q-p),s}(\Omega)$ by Proposition 1.9, since $|\gamma| > 0$. This proves (24).

We can proceed now to prove (25). By Theorem 2.1 we can write

$$\alpha_i^\varepsilon = d_c\psi_i^\varepsilon + \omega_i^\varepsilon, \quad i = 1, 2,$$

with ψ_i^ε and ω_i^ε supported in a suitable neighborhood Ω_0 of $\bar\Omega$ and converging strongly in $L^{s_i}(\Omega_0, E_0^{h_i})$. Thus the integral of $\alpha_1^\varepsilon \wedge \alpha_2^\varepsilon$ in (25) splits into the sum of 4 terms. Clearly, 3 of them are easy to deal with, since they are the integral of the wedge product of two sequences of forms, at least one of them converging strongly. Thus, we are left with the term

$$\int_{\mathbb{G}} d_c\psi_1^\varepsilon \wedge d_c\psi_2^\varepsilon,$$

with $\psi_i^\varepsilon \in \mathcal{D}(\Omega_0, E_0^{k_i})$ for $i = 1, 2$. We have

$$\begin{aligned}
\int_{\mathbb{G}} d_c\psi_1^\varepsilon \wedge d_c\psi_2^\varepsilon &= \int_{\mathbb{G}} (\Pi_{E_0}\, d\, \Pi_E\, \psi_1^\varepsilon) \wedge (d_c\psi_2^\varepsilon) \\
&= \int_{\mathbb{G}} (d\, \Pi_E\, \psi_1^\varepsilon) \wedge (d_c\psi_2^\varepsilon) \\
&= \int_{\mathbb{G}} d\left((\Pi_E\, \psi_1^\varepsilon) \wedge (d_c\psi_2^\varepsilon)\right) + (-1)^{h_1} \int_{\mathbb{G}} (\Pi_E\, \psi_1^\varepsilon) \wedge d(d_c\psi_2^\varepsilon) \\
&= (-1)^{h_1} \int_{\mathbb{G}} (\Pi_E\, \psi_1^\varepsilon) \wedge d(d_c\psi_2^\varepsilon) \quad \text{(by Stokes theorem)} \\
&= (-1)^{h_1} \int_{\mathbb{G}} \psi_1^\varepsilon \wedge (\Pi_E\, d(d_c\psi_2^\varepsilon)) \\
&= (-1)^{h_1} \int_{\mathbb{G}} \psi_1^\varepsilon \wedge (d\, \Pi_E(d_c\psi_2^\varepsilon)) \\
&= (-1)^{h_1} \int_{\mathbb{G}} \psi_1^\varepsilon \wedge (d_c(d_c\psi_2^\varepsilon)) \\
&= 0,
\end{aligned}$$

since $d_c^2 = 0$. This achieves the proof of the theorem. □

Riferimenti bibliografici

[1] A.Baldi, B.Franchi, N.Tchou & M.C.Tesi, *Compensated Compactness for Differential Forms in Carnot Groups and Applications*, preprint (2008), http://hal.archives-ouvertes.fr/hal-00298362/fr/.

[2] A.Bonfiglioli, E.Lanconelli & F.Uguzzoni. *Stratified Lie groups and potential theory for their sub-laplacians*, Springer, Berlin (2007).

[3] M. Christ, D. Geller, P. Głowacki and L. Polin, *Pseudodifferential operators on groups with dilations*, Duke Math. J., **68** (1992), 31–65.

[4] H.Federer, *Geometric Measure Theory*, Springer, (1969).

[5] G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat., **13** (1975), 161–207,

[6] G.B.Folland & E.M.Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton (1982).

[7] A.W. Knapp, and E.M. Stein, *Intertwining operators for semisimple groups*, Ann. of Math., **93** (1971), 489–578.

[8] A. Korányi and S. Vági, *Singular integrals on homogeneous spaces and some problems of classical analysis*, Ann. Scuola Norm. Sup. Pisa (3), **25** (1971), 575–648.

[9] M.Rumin, *Around heat decay on forms and relations of nilpotent Lie groups*, Sémin. Théor. Spectr. Géom., **19**, Univ. Grenoble I, Saint-Martin-d'Hères, 2001, 123–164.

[10] E.M.Stein, *Harmonic Analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton University Press, Princeton (1993).

[11] N.Th.Varopoulos, L.Saloff-Coste & T.Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, (1992).