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Anno Accademico 2008-09

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Maxwell's equations in the Heisenberg group

4 giugno 2009

Abstract

In this note we present a geometric formulation of Maxwell's equations in Carnot groups in the setting of the intrinsic complex of differential forms defined by M. Rumin. Restricting ourselves to the first Heisenberg group $\mathbb{H}^1$, we show that these equations are invariant under the action of suitably defined Lorentz transformations, and we prove the equivalence of these equations with different equations "in coordinates". Moreover, we analyze the notion of "vector potential", and we show that it satisfies a new class of 4th order evolution differential equations.

This is joint work with Bruno Franchi, see [9], [10].

1. Multilinear algebra in Carnot groups

Let $(\mathbb{G}, \cdot)$ be a *Carnot group of step* κ identified to $\mathbb{R}^n$ through exponential coordinates (see [4] for details). By definition, the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$\text{(1)}\quad \mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Choose a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ adapted to the stratification, i.e. such that

$$e_1, \dots, e_{m_1} \text{ is a basis of } V_1$$

and, accordling,

$$e_{m_{j-1}+1}, \dots, e_{m_j} \text{ is a basis of } V_j \text{ for each } j = 2, \dots, \kappa.$$

Let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that that is also an orthonormal basis of $V_1 \equiv R^{m_1}$ at the origin, $X_i(0) = e_i$. The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle \cdot, \cdot \rangle$, making $\{X_1, \dots, X_n\}$ an orthonormal basis.

We can write the elements of $\mathbb{G}$ in *exponential coordinates*, identifying p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula. If $p \in \mathbb{G}$ and $i = 1, \dots, \kappa$, we put $p^i = (p_{h_{i-1}+1}, \dots, p_{h_i}) \in \mathbb{R}^{m_i}$, so that we can also identify p with $(p^1, \dots, p^\kappa) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_\kappa} = \mathbb{R}^n$.

Two important families of automorphism of $\mathbb{G}$ are given by left translations $p \mapsto \tau_q p := q \cdot p$ group dilations δ_λ for $\lambda > 0$. For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\text{(2)}\qquad \delta_\lambda(x_1, ..., x_n) = (\lambda^{d_1} x_1, ..., \lambda^{d_n} x_n),$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [7] Chapter 1) and is defined as

$$\text{(3)}\qquad d_j = i \quad \text{whenever } h_{i-1} + 1 \leq j \leq h_i,$$

hence $1 = d_1 = ... = d_{m_1} < d_{m_1+1} = 2 \leq ... \leq d_n = \kappa$.

As customary, we fix a smooth homogeneous norm $|\cdot|$ in $\mathbb{G}$ such that the gauge distance $d(x, y) := |y^{-1}x|$ is a left-invariant true distance, equivalent to the Carnot-Carathéodory distance in $\mathbb{G}$ (see [15], p.638). We set $B(p, r) = \{q \in \mathbb{G};\ d(p, q) < r\}$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is $\mathcal{L}$-measurable, we write also $|A| := \mathcal{L}(A)$.

We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(V_i).$$

Since for any $x \in \mathbb{G}$ $|B(x,r)| = |B(e,r)| = r^Q |B(e,1)|$, Q is the Hausdorff dimension of the metric space $(\mathbb{G}, d)$.

By (1), the subset $X_1, \dots, X_{m_1}$ generates by commutations all the other vector fields. Therefore, the subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by $X_1, \dots, X_{m_1}$ plays a particularly important role in the theory, and it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span}\,\{X_1(x), \dots, X_{m_1}(x)\}, \qquad x \in \mathbb{G}.$$

From now on, for sake of simplicity, sometimes we set $m := m_1$.

A subriemannian structure is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle \cdot, \cdot \rangle_x$ making the basis $X_1(x), \dots, X_m(x)$ an orthonormal basis. The sections of $H\mathbb{G}$ are called *horizontal sections*, and a vector of $H\mathbb{G}_x$ is an *horizontal vector*.

The Euclidean space $\mathbb{R}^n$ endowed with the usual (commutative) sum of vectors provides the simplest example of Carnot group. It is a trivial example, since in this case the stratification of the algebra consists of only one layer, i.e. the all the Lie algebra reduces to the horizontal layer.

On the other hand, Heisenberg groups $\mathbb{H}^n$ provide the simplest example of noncommutative Carnot groups. In this note, we deal mainly with the first Heisenberg group $\mathbb{H}^1$, with variables (x, y, t). Set $X := \partial_x + 2y\partial_t$, $Y := \partial_y - 2x\partial_t$, $T := \partial_t$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X, Y\}$ and $V_2 = \text{span}\,\{T\}$. More generally, if $n > 1$ in $\mathbb{H}^n$ we denote again by (x, y, t) the variables, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. A basis of the horizontal layer of the Lie algebra is then provided by the vector fields $X_j := \partial_{x_j} + 2y_j\partial_t$ and $Y_j := \partial_{y_j} - 2x_j\partial_t$, $j = 1, \dots, n$.

Following [7], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi–index, we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$. By the Poincaré–Birkhoff–Witt theorem (see, e.g. [5], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1 + \cdots + i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations. From the Poincaré–Birkhoff–Witt theorem, it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators X^I of the special form above.

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1, \cdots, X_n$, is the family of covectors $\{\theta_1, \cdots, \theta_n\}$. We indicate by $\langle\cdot,\cdot\rangle$ also the inner product in $\bigwedge^1\mathfrak{g}$ that makes $\theta_1,\cdots,\theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1,\cdots,\theta_n$ may have polynomial (hence variable) coefficients.

Following Federer (see [6] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1\mathfrak{g}$ are the graded algebras indicated as $\bigwedge_*\mathfrak{g} = \bigoplus_{h=0}^{n}\bigwedge_h\mathfrak{g}$ and $\bigwedge^*\mathfrak{g} = \bigoplus_{h=0}^{n}\bigwedge^h\mathfrak{g}$ where $\bigwedge_0\mathfrak{g} = \bigwedge^0\mathfrak{g} = \mathbb{R}$ and, for $1\le h\le n$,

$$\begin{aligned}\bigwedge\nolimits_h\mathfrak{g} &:= \mathrm{span}\{X_{i_1}\wedge\cdots\wedge X_{i_h} : 1\le i_1<\cdots<i_h\le n\},\\ \bigwedge\nolimits^h\mathfrak{g} &:= \mathrm{span}\{\theta_{i_1}\wedge\cdots\wedge\theta_{i_h} : 1\le i_1<\cdots<i_h\le n\}.\end{aligned}$$

The elements of $\bigwedge_h\mathfrak{g}$ and $\bigwedge^h\mathfrak{g}$ are called *h-vectors* and *h-covectors*.

We denote by Θ^h the basis $\{\theta_{i_1}\wedge\cdots\wedge\theta_{i_h} : 1\le i_1<\cdots<i_h\le n\}$ of $\bigwedge^h\mathfrak{g}$. We remind that

$$\dim\bigwedge\nolimits^h\mathfrak{g} = \dim\bigwedge\nolimits_h\mathfrak{g} = \binom{h}{n}.$$

The dual space $\bigwedge^1(\bigwedge_h\mathfrak{g})$ of $\bigwedge_h\mathfrak{g}$ can be naturally identified with $\bigwedge^h\mathfrak{g}$. The action of a h-covector φ on a h-vector v is denoted as $\langle\varphi|v\rangle$.

The inner product $\langle\cdot,\cdot\rangle$ extends canonically to $\bigwedge_h\mathfrak{g}$ and to $\bigwedge^h\mathfrak{g}$ making the bases $X_{i_1}\wedge\cdots\wedge X_{i_h}$ and $\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}$ orthonormal.

Definition 1.1. We define linear isomorphisms (Hodge duality: see [6] 1.7.8)

$$*:\bigwedge\nolimits_h\mathfrak{g}\longleftrightarrow\bigwedge\nolimits_{n-h}\mathfrak{g}\quad\text{and}\quad *:\bigwedge\nolimits^h\mathfrak{g}\longleftrightarrow\bigwedge\nolimits^{n-h}\mathfrak{g},$$

for $1\le h\le n$, putting, for $v=\sum_I v_I X_I$ and $\varphi=\sum_I\varphi_I\theta_I$,

$$*v := \sum\nolimits_I v_I(*X_I)\quad\text{and}\quad *\varphi := \sum\nolimits_I\varphi_I(*\theta_I)$$

where

$$*X_I := (-1)^{\sigma(I)}X_{I^*}\quad\text{and}\quad *\theta_I := (-1)^{\sigma(I)}\theta_{I^*}$$

with $I=\{i_1,\cdots,i_h\}$, $1\le i_1<\cdots<i_h\le n$, $X_I = X_{i_1}\wedge\cdots\wedge X_{i_h}$, $\theta_I=\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}$, $I^*=\{i_1^*<\cdots<i_{n-h}^*\}=\{1,\cdots,n\}\setminus I$ and $\sigma(I)$ is the number of couples (i_h,i_ℓ^*) with $i_h>i_\ell^*$.

The following properties of the $*$ operator follow readily from the definition: $\forall v,w\in\bigwedge_h\mathfrak{g}$ and $\forall\varphi,\psi\in\bigwedge^h\mathfrak{g}$

$$\begin{aligned}&**v = (-1)^{h(n-h)}v = v, && **\varphi = (-1)^{h(n-h)}\varphi=\varphi,\\ &v\wedge *w = \langle v,w\rangle X_{\{1,\cdots,n\}}, && \varphi\wedge*\psi = \langle\varphi,\psi\rangle\theta_{\{1,\cdots,n\}},\\ &\langle *\varphi|*v\rangle = \langle\varphi|v\rangle.\end{aligned}\tag{4}$$

Notice that, if $v = v_1 \wedge \cdots \wedge v_h$ is a simple h-vector, then $*v$ is a simple $(n-h)$-vector.

If $v \in \bigwedge_h \mathfrak{g}$ we define $v^\natural \in \bigwedge^h \mathfrak{g}$ by the identity $\langle v^\natural | w \rangle := \langle v, w \rangle$, and analogously we define $\varphi^\natural \in \bigwedge_h \mathfrak{g}$ for $\varphi \in \bigwedge^h \mathfrak{g}$.

To fix our notations, we remind the following definition (see e.g. [11], Section 2.1).

Definition 1.2. If V, W are finite dimensional linear vector spaces and $L : V \to W$ is a linear map, we define

$$\Lambda_h L : \bigwedge_h V \to \bigwedge_h W$$

as the linear map defined by

$$(\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) = L(v_1) \wedge \cdots \wedge L(v_h)$$

for any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$ By duality, we define

$$\Lambda^h L : \bigwedge^h W \to \bigwedge^h V$$

as the linear map defined by

$$\langle (\Lambda^h L)(\alpha) | v_1 \wedge \cdots \wedge v_h \rangle = \langle \alpha | (\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) \rangle$$

for any $\alpha \in \bigwedge^h W$ and any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$.

Proposition 1.3. *If V, W are finite dimensional linear vector spaces endowed with a scalar product and $L : V \to W$ is a linear map, then*

i) *if $v \in \bigwedge_1 V$ and $\alpha \in \bigwedge^1 W$, then $\Lambda_1 v = Lv$ and $((\Lambda^1 L)\alpha)^\natural = {}^t L(\alpha^\natural)$;*
ii) *if $\alpha \in \bigwedge^k W$ and $\beta \in \bigwedge^h W$, then $(\Lambda^{k+h} L)(\alpha \wedge \beta) = (\Lambda^k L)\alpha \wedge (\Lambda^k L)\beta$;*
iii) *if $v \in \bigwedge_k V$ and $w \in \bigwedge_h V$, then $(\Lambda_{k+h} L)(v \wedge w = (\Lambda_k L) v \wedge (\Lambda_k L) w$;*
iv) *${}^t\Lambda_h L = \Lambda_h({}^t L)$ and ${}^t\Lambda^h L = \Lambda^h({}^t L)$;*
v) *if H is another finite dimensional linear vector spaces and $G : H \to V$ is a linear map, then $\Lambda_h(L \circ G) = (\Lambda_h L) \circ (\Lambda_h G)$ and $\Lambda^h(L \circ G) = (\Lambda^h L) \circ (\Lambda^h G)$;*
vi) *if $L : V \to V$ is a unitary linear operator, then $\Lambda_h L$ and $\Lambda^h L$ are linear isometries.*

We can define now two families of vector bundles (still denoted by $\bigwedge_* \mathfrak{g}$ and $\bigwedge^* \mathfrak{g}$ over $\mathbb{G}$), by putting

$$\bigwedge_{h,p} \mathfrak{h} := (\Lambda_h d\tau_p)(\bigwedge_{h,e} \mathfrak{g})$$

and, respectively,

$$\bigwedge_p^h \mathfrak{h} := (\Lambda^h d\tau_{p^{-1}})(\bigwedge_e^h \mathfrak{g})$$

for any $p \in \mathbb{G}$ and $h = 1, \ldots, n$, where we have chosen

$$\bigwedge_{h,e} \mathfrak{g} \equiv \bigwedge_h \mathfrak{g} \quad \text{and} \quad \bigwedge_e^h \mathfrak{g} \equiv \bigwedge^h \mathfrak{g}.$$

The inner products $\langle\cdot,\cdot\rangle$ on $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ induce inner products on each fiber $\bigwedge_{h,p} \mathfrak{g}$ and $\bigwedge^h_p \mathfrak{g}$ by the identity

$$\langle \Lambda_h d\tau_p(v), \Lambda_h d\tau_p(w)\rangle_p := \langle v, w\rangle$$

and

$$\langle \Lambda^h d\tau_{p^{-1}}(\alpha), \Lambda^h d\tau_{p^{-1}}(\beta)\rangle_p := \langle \alpha, \beta\rangle.$$

Lemma 1.4. *If $p, q \in \mathbb{G}$, then*

$$\Lambda_h d\tau_q : \bigwedge\nolimits_{h,p} \mathfrak{g} \to \bigwedge\nolimits_{h,qp} \mathfrak{g}$$

and

$$\Lambda^h d\tau_{q^{-1}} : \bigwedge\nolimits^h_p \mathfrak{g} \to \bigwedge\nolimits^h_{qp} \mathfrak{g}$$

are isometries onto.

In general, a subbuondle $\mathcal{N}$ of $\bigwedge_h \mathfrak{g}$ is said to be left-invariant if

$$\mathcal{N}_p = (\Lambda_h d\tau_p)(\mathcal{N}_e)$$

for all $p \in \mathbb{G}$. Analogously, a subbundle $\mathcal{N}$ of $\bigwedge^h \mathfrak{g}$ is said to be left-invariant if

$$\mathcal{N}_p := (\Lambda^h d\tau_{p^{-1}})(\mathcal{N}_e)$$

for all $p \in \mathbb{G}$.

From now on, if $\mathcal{U} \subset \mathbb{G}$ is an open set and $h = 0, 1, \dots, n$ we denote by $\Omega_h(\mathcal{U})$ and $\Omega^h(\mathcal{U})$ the sets of all (smooth) sections of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, respectively. If $\mathcal{U} = \mathbb{G}$ we write only Ω_h and Ω^h. We refer to elements of Ω_h as to fields of h-vectors and to elements of Ω^h as to h-forms.

A h-form α on $\mathbb{G}$ is said left-invariant if $\tau_p^{\#}\alpha = \alpha$ for any $p \in \mathbb{G}$.

Notice that h-covectors can be identified with left-invariant h-forms.

If X is a vector field and α is a h-form, we denote by $i_X\alpha$ the contraction of α with X given by $(i_X\alpha)(v_1\wedge\cdots\wedge v_{h-1}) := \alpha(X\wedge v_1\wedge\cdots\wedge v_{h-1})$.

If d is the usual De Rham's exterior differential, we denote by $\delta = d^*$ its formal adjoint in $L^2(\mathbb{G}, \Omega^*)$.

As customary, if $f : \mathbb{G} \to \mathbb{G}$ is a continuously differentiable map, then the pull-back $f^{\#}\omega$ of a form $\omega \in \Omega^h$ is defined by

$$f^{\#}\omega(x) := \big(\Lambda^h(df_x)\big)\omega(f(x)).$$

Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups, and let $\mathfrak{g} = \oplus_{i=1}^{\kappa_1}\mathfrak{g}_i$ and $\mathfrak{m} = \oplus_{i=1}^{\kappa_2}\mathfrak{m}_i$ be their Lie algebras (respectively n-dimensional and N-dimensional).

We denote by $\hat{e}_1, \dots, \hat{e}_N$ an adapted basis of $\mathfrak{m}$, and by $\hat{X}_1, \dots, \hat{X}_N$ the corresponding family of vector fields.

Definition 1.5. A map $L : \mathbb{G} \to \mathbb{M}$ is said to H-linear (and we write $L \in HL(\mathbb{G}, \mathbb{M})$) if

i) is a group homeomorphism;
ii) is homogeneous, i.e. $\delta_r(Lx) = L(\delta_r x)$ for all $r > 0$.

A H-linear map induces an algebra homomorphism (that we still denote by L) between $\mathfrak{g}$ and $\mathfrak{m}$ by taking $\ln \circ L \circ \exp$. In particular the induced map L is linear.

Since we are using exponential coordinates in $\mathbb{G}$ and $\mathbb{M}$, the map L itself from $\mathbb{G}$ to $\mathbb{M}$ can be written as $N \times n$ real matrix, and we still denote by $HL(\mathbb{G}, \mathbb{M})$ the set of associated matrices.

Example 1.6. In $\mathbb{H}^1$, H-linear are associated with 3×3 real matrices of the form (see [13], [12])

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{44}, \end{pmatrix}, \quad \text{with} \quad a_{44} = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

with

$$a_{44} = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Example 1.7. Later on, we have to deal with a space-time group like $\mathbb{H}^1 \times \mathbb{R}$. In this case case, a H-linear map $L : \mathbb{H}^1 \times \mathbb{R} \to \mathbb{H}^1 \times \mathbb{R}$ has the two following possible structures:

i) either the associated matrix L has the form

$$L = \left(\begin{array}{ccc|c} & & & 0 \\ & L_\Sigma & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right),$$

where L_Σ is a 3×3 real matrix with the first two row linearly dependent,

ii) or the associated matrix L has the form

$$L = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.$$

Theorem 1.8. *Let $L : \mathbb{G} \to \mathbb{M}$ be a H-linear map. Then L enjoys the contact property*

$$L(\mathfrak{g}_i) \subset \mathfrak{m}_i \quad i = 1, \dots, \kappa_1. \tag{5}$$

2. Space-time Carnot groups

From now on, we denote by x a “space” point in the Carnot group $\mathbb{G}$, and by $s \in \mathbb{R}$ the “time”, and we choose in $\mathbb{R} \times \mathbb{G}$ the canonical volume form $ds \wedge dV$, where dV is the canonical volume form in $\mathbb{G}$. Moreover, we denote by $(E^*_{0,\mathbb{G}}, d_{c,\mathbb{G}})$ and $(E^*_{0,\mathbb{R}\times\mathbb{G}}, d_{c,\mathbb{R}\times\mathbb{G}})$ the intrinsic forms on $\mathbb{G}$ and on $\mathbb{R} \times \mathbb{G}$, respectively. For sake of brevity, we shall write

$$d_c := d_{c,\mathbb{G}} \quad \text{and} \quad \hat{d}_c := d_{c,\mathbb{R}\times\mathbb{G}}.$$

Denote by S the vector field $\frac{\partial}{\partial s}$. The Lie group $\mathbb{R}\times\mathbb{G}$ is a Carnot group; its Lie algebra $\hat{\mathfrak{g}}$ admits the stratification

$$\hat{\mathfrak{g}} = \hat{V}_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \tag{6}$$

where $\hat{V}_1 = \operatorname{span}\{S, V_1\}$. Since the adapted basis $\{X_1, \dots, X_n\}$ has been already fixed once and for all, the associated fixed basis for $\hat{\mathfrak{g}}$ will be

$$\{S, X_1, \dots, X_{m_1}, \dots, X_n\} := \{X_0, \dots, X_n\},$$

where we have set $X_0 := S$. Coherently, we write also $\theta_0 := ds$. Consider the Lie derivative $\mathcal{L}_S$ along S. If $f\theta_{i_1}\wedge\cdots\theta_{i_h}$ is a h-form in $\mathbb{G}$, $1\le i_1<\cdots<i_h\le n$, we have $\mathcal{L}_S(f\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}) = (Sf)\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}$. Indeed

- if f is a scalar function, by definition $\mathcal{L}_S f = i_S df = \sum_{j=0}^n (X_j f)\theta_j(X_0) = Sf$;
- if $f\theta_{i_1}\wedge\cdots\theta_{i_h}$ is a h-form in $\mathbb{G}$, then $\mathcal{L}_S(f\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}) = (Sf)\theta_{i_1}\wedge\cdots\wedge\theta_{i_h} + f\mathcal{L}_S(\theta_{i_1}\wedge\cdots\wedge\theta_{i_h})$. But $\mathcal{L}_S(\theta_{i_1}\wedge\cdots\wedge\theta_{i_h})$ is a sum of terms of the form $\theta_{i_1}\wedge\cdots\wedge\mathcal{L}_S\theta_{i_\ell}\wedge\cdots\wedge\theta_{i_h}=0$, since $\mathcal{L}_S\theta_{i_\ell}=0$.

Thus, when acting on h-forms α in $\mathbb{G}$, without risk of misunderstandings, we write $S\alpha$ for $\mathcal{L}_S\alpha$.

We point out that S commutes with d, the exterior differential in $\mathbb{G}$. Indeed, if $\alpha = \sum_{j=1}^n \alpha_j\theta_j^h$, then

$$Sd\alpha = \sum_{j=1}^n\sum_{\ell=1}^n (SX_\ell\alpha_j)\theta_\ell\wedge\theta_j^h = \sum_{j=1}^n\sum_{\ell=1}^n (X_\ell S\alpha_j)\theta_\ell\wedge\theta_j^h = d(S\alpha).$$

Let us state preliminarily a structure lemma for intrinsic forms in $\mathbb{R}\times\mathbb{G}$ (see [3]).

Lemma 2.1. *If $1\le h\le n$, then a h-form α belongs to $E_{0,\mathbb{R}\times\mathbb{G}}^h$ if and only if it can be written as*

$$\alpha = ds\wedge\beta+\gamma, \tag{7}$$

where $\beta\in E_{0,\mathbb{G}}^{h-1}$ and $\gamma\in E_{0,\mathbb{G}}^h$ are respectively intrinsic $(h-1)$-forms and h-forms in $\mathbb{G}$ with coefficients depending on x and s.

Proposition 2.2 ([9]). *If $1\le h\le n$, and $\alpha = ds\wedge\beta+\gamma\in E_{0,\mathbb{R}\times\mathbb{G}}^h$, then*

$$\hat{d}_c\alpha = ds\wedge(S\gamma - d_c\beta) + d_c\gamma. \tag{8}$$

Definition 2.3. We denote by $HO_{\mathbb{G}}$ the group of all $(n+1)\times(n+1)$ matrices L such that ${}^tLGL = G$, where $G=(g_{ij})_{i,j=0,\dots,n}$ with $g_{ij}=0$ if $i\ne j$, $g_{ii}=1$ if $i>0$, $i\ne j$, $g_{00}=-1$.

$$G = \left(\begin{array}{cccc|c|c} -1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ & \ddots & & & 0 & 0 \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ \hline & & & & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \\ & 0 & & & & 0 \\ \hline & 0 & & & 0 & \ddots \end{array}\right)$$

We refer to $HO_{\mathbb{G}}$ as to the contact Lorentzian group of $\mathbb{G}$. If $L \in HO_{\mathbb{G}}$, then $\det L = \pm 1$.

Example 2.4. As in Example 1.7, consider the first Heisenberg group. A matrix as in i) does not belong to $HO_{\mathbb{G}}$, since it has zero determinant. Thus, a matrix L belongs to $HO_{\mathbb{G}}$ if and only if it has the form

$$L = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with

$$a_{11}a_{22} - a_{12}a_{21} = \pm 1.$$

If $1 \le \ell \le n$, by Lemma 2.1, keeping in mind again that forms in $E^{\ell}_{0,\mathbb{R}\times\mathbb{G}}$ are orthogonal to forms of the form $ds \wedge \sigma$ with $\sigma \in E^{\ell-1}_{0,\ \mathbb{G}}$, we can define a new (Minkowskian) scalar product $\langle \cdot, \cdot \rangle_M$ in $E^h_{0,\mathbb{R}\times\mathbb{G}}$ as

$$\langle ds \wedge \beta + \gamma, ds \wedge \beta' + \gamma' \rangle_M := \langle \gamma, \gamma' \rangle - \langle \beta, \beta' \rangle.$$

Notice that

$$\langle \alpha, \alpha' \rangle_M = \langle (\Lambda^h G)\alpha, \alpha' \rangle.$$

In particular, the bilinear form $\langle \cdot, \cdot \rangle_M$ is nondegenerate.

Definition 2.5. We denote by $*_M$ the associated Hodge operator such that $\alpha \wedge *_M \beta = \langle \alpha, \beta \rangle_M ds \wedge dV$. If $\alpha = ds \wedge \beta + \gamma \in E^h_{0,\mathbb{R}\times\mathbb{G}}$, $1 \le h \le n$, we have ([3])

$$*_M \alpha = (-1)^h ds \wedge *\gamma + *\beta.$$

Lemma 2.6. *If $L \in HO_{\mathbb{G}}$ and $\alpha \in E^h_{0,\mathbb{R}\times\mathbb{G}}$, then*

$$*_M(L^{\#}\alpha) = L^{\#}(*_M \alpha) \quad \textit{if } \det L = 1$$

and

$$*_M(L^{\#}\alpha) = -L^{\#}(*_M \alpha) \quad \textit{if } \det L = -1.$$

Let now $\mathcal{J}$ be a fixed closed intrinsic n-form in $\mathbb{R} \times \mathbb{G}$ (a source form). By Lemma 2.1, $\mathcal{J} = *J \wedge ds - \rho$, where $\rho(\cdot, s) = \rho_0(\cdot, s)\, dV$ is a volume form on $\mathbb{G}$ for any fixed $s \in \mathbb{R}$.

If $F \in E^2_{0,\mathbb{R}\times\mathbb{G}}$, we call *Maxwell equations in* $\mathbb{G}$ the system in $E^*_{0,\mathbb{R}\times\mathbb{G}}$

$$\hat{d}_c F = 0 \quad \text{and} \quad \hat{d}_c(*_M F) = \mathcal{J} \tag{9}$$

(for sake of simplicity, we assume all "physical" constants to be 1). This this system corresponds to a particular choice of the so-called "constitutive relations".

Theorem 2.7 ([9]). *Equations* (9) *are invariant under the action of* $HO_{\mathbb{G}}$, *i.e., if* $L \in HO_{\mathbb{G}}$ *and* F *satisfies* (9), *then*

$$\hat{d}_c(L^\# F) = 0 \quad \textit{and} \quad \hat{d}_c(*_M L^\# F) = L^\# \mathcal{J}. \tag{10}$$

This is a consequence of the following result.

Theorem 2.8 ([9]). *If* $L \in HO_{\mathbb{G}}$ *and* $0 \le h \le n+1$, *then*

i) $L^\# : E^h_{0,\mathbb{R}\times\mathbb{G}} \to E^h_{0,\mathbb{R}\times\mathbb{G}}$;
ii) $\hat{d}_c L^\# = L^\# \hat{d}_c$;
iii) $*_M L^\# = (\det L) \cdot L^\#(*_M)$.

3. Maxwell equations in $\mathbb{H}^1$

Let us consider now the specific case $\mathbb{G} = \mathbb{H}^1 \equiv \mathbb{R}^3$, the first Heisenberg group, with variables x, y, t. For sake of simplicity, in some parts of this Section, we use the following customary notation: we set $X_1 : X = \partial_x + 2y\partial_t$, $X_2 : Y = \partial_y - 2x\partial_t$, $X_3 := T = \partial_t$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span}\,\{X, Y\}$ and $V_2 = \text{span}\,\{T\}$. We have $X^\natural = dx$, $Y^\natural = dy$, $T^\natural = \theta$ (the contact form of $\mathbb{H}^1$). In this case

$$E^1_{0,\mathbb{H}^1} = \text{span}\,\{dx, dy\};$$
$$E^2_{0,\mathbb{H}^1} = \text{span}\,\{dx \wedge \theta, dy \wedge \theta\};$$
$$E^3_{0,\mathbb{H}^1} = \text{span}\,\{dx \wedge dy \wedge \theta\}.$$

The action of d_c on $E^*_{0,\mathbb{H}^1}$ is given by the following indentities ([14], [8], [2]).

Proposition 3.1. *If* $\alpha = \alpha_1 dx + \alpha_2 dy \in E^1_{0,\mathbb{H}^1}$, *then*

$$\begin{aligned} d_c\alpha &= -\frac{1}{4}(X^2\alpha_2 - 2XY\alpha_1 + YX\alpha_1)dx \wedge \theta \\ &\quad - \frac{1}{4}(2YX\alpha_2 - Y^2\alpha_1 - XY\alpha_2)dy \wedge \theta \\ &:= P_1(\alpha_1, \alpha_2)dx \wedge \theta + P_2(\alpha_1, \alpha_2)dy \wedge \theta. \end{aligned} \tag{11}$$

On the other hand, the action of d_c *on* $\alpha = \alpha_{13}dx\wedge\theta + \alpha_{23}dy\wedge\theta \in E^2_{0,\mathbb{H}^1}$, *is given by*

$$d_c(\alpha_{13}dx \wedge \theta + \alpha_{23}dy \wedge \theta) = (X\alpha_{23} - Y\alpha_{13})\, dx \wedge dy \wedge \theta. \tag{12}$$

Moreover,

$$\delta_c(\alpha_1 dx + \alpha_2 dy) = -X\alpha_1 - Y\alpha_2, \tag{13}$$

and

$$\delta_c(\alpha_{13} dx \wedge \theta + \alpha_{23} dy \wedge \theta) = P_2(\alpha_{23}, -\alpha_{13})dx - P_1(\alpha_{23}, -\alpha_{13})dy. \tag{14}$$

Coherently, if $\vec{V} = (V_1, V_2)$ is a horizontal vector field, following [8], [2], [1], we set

$$\mathrm{curl}_{\mathbb{H}}\, \vec{V} = (*d_c(\vec{V}^\natural))^\natural,$$

or, in explicit form

$$\mathrm{curl}_{\mathbb{H}}\, \vec{V} = (P_2(V_1, V_2), -P_1(V_1, V_2)).$$

As usual, we put also

$$\mathrm{div}_{\mathbb{H}} \vec{V} = XV_1 + YV_2.$$

The following identity follows from Proposition 2.2

Lemma 3.2. *If* $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 ds \in E^1_{0,\mathbb{R}\times\mathbb{H}^1}$, *then*

$$\begin{aligned}\hat{d}_c\alpha &= (S\alpha_1 - X\alpha_3)ds \wedge dx + (S\alpha_2 - Y\alpha_3)ds \wedge dy \\ &\quad + P_1(\alpha_1, \alpha_2)dx \wedge \theta + P_2(\alpha_1, \alpha_2)dy \wedge \theta.\end{aligned}$$

If

$$\alpha = \alpha_{13} dx \wedge ds + \alpha_{23} dy \wedge ds + \alpha_{14} dx \wedge \theta + \alpha_{24} dy \wedge \theta \in E^2_{0,\mathbb{R}\times\mathbb{H}^1},$$

then

$$\begin{aligned}\hat{d}_c\alpha &= (X\alpha_{24} - Y\alpha_{14})\, dx \wedge dy \wedge \theta \\ &\quad + (S\alpha_{14} - P_1(\alpha_{13}, \alpha_{23}))\, ds \wedge dx \wedge \theta \\ &\quad + (S\alpha_{24} - P_2(\alpha_{13}, \alpha_{23}))\, ds \wedge dy \wedge \theta.\end{aligned} \tag{15}$$

By Proposition 4.2 in [3], Maxwell system (9) in $\mathbb{H}^1$ can be written as follows.

Theorem 3.3. *Suppose* $\mathbb{G} = \mathbb{H}^1$. *If* $F = E \wedge ds + B \in E_{0,\mathbb{R}\times\mathbb{H}^1}$, *let us set*

$$E = E_1 dx + E_2 dy \quad \textit{and} \quad B = B_1 dy \wedge \theta - B_2 dx \wedge \theta$$

(in classical electrodynamics we refer to F *as to Faraday's form). In addition, let*

$$\mathcal{J} = J_1 dy \wedge \theta \wedge ds - J_2\, dx \wedge \theta \wedge ds - \rho\, dx \wedge dy \wedge \theta \ (\in E^3_{0,\mathbb{R}\times\mathbb{H}^1})$$

be a closed form. Put now $\vec{E} = E^\natural = (E_1, E_2)$, $\vec{B} = (*B)^\natural = (B_1, B_2)$, *and* $\vec{J} = (J_1, J_2)$, *Then the system (9) is equivalent to*

$$\frac{\partial \vec{B}}{\partial s} + \mathrm{curl}_{\mathbb{H}}\, \vec{E} = 0, \quad \mathrm{div}_{\mathbb{H}}\, \vec{B} = 0, \tag{16}$$

$$\frac{\partial \vec{E}}{\partial s} - \mathrm{curl}_{\mathbb{H}}\, \vec{B} = -\vec{J}, \quad \mathrm{div}_{\mathbb{H}}\, \vec{E} = \rho, \tag{17}$$

If, in particular, $\rho \equiv 0$ and $\vec{J} \equiv 0$, i.e. in absence of charges and currents, equations (16) and (17) become

$$\frac{\partial \vec{B}}{\partial s} + \mathrm{curl}_{\mathbb{H}}\, \vec{E} = 0, \quad \mathrm{div}_{\mathbb{H}}\, \vec{B} = 0, \tag{18}$$

$$\frac{\partial \vec{E}}{\partial s} - \mathrm{curl}_{\mathbb{H}}\, \vec{B} = 0, \quad \mathrm{div}_{\mathbb{H}}\, \vec{E} = 0. \tag{19}$$

Replacing (19) in (18) and then (18) in (19), we get

$$\frac{\partial^2 \vec{B}}{\partial s^2} = -\mathrm{curl}_{\mathbb{H}}\, (\mathrm{curl}_{\mathbb{H}}\, \vec{B}) \tag{20}$$

$$\frac{\partial^2 \vec{E}}{\partial s^2} = -\mathrm{curl}_{\mathbb{H}}\, (\mathrm{curl}_{\mathbb{H}}\, \vec{E}). \tag{21}$$

Keeping in mind that $0 = -\mathrm{div}_{\mathbb{H}}\vec{B} = \delta_c(\vec{B})^\natural$, we have

$$\begin{aligned}\big(\mathrm{curl}_{\mathbb{H}}\, (\mathrm{curl}_{\mathbb{H}}\, \vec{B})\big)^\natural &= *d_c(*d_c(\vec{B})^\natural) = \delta_c d_c(\vec{B})^\natural \\ &= \big(\delta_c d_c + \frac{1}{16}(d_c\delta_c)^2\big)(\vec{B})^\natural = \Delta_{\mathbb{H},1}(\vec{B})^\natural,\end{aligned} \tag{22}$$

where $\Delta_{\mathbb{H},1}$ is the (4th order) Rumin's Laplacian on intrinsic 1-forms (see Theorem **??** below). Accordingly, we can define $\Delta_{\mathbb{H},1}$ acting on horizontal vector fields as $\Delta_{\mathbb{H},1}\vec{B} := \big(\Delta_{\mathbb{H},1}(\vec{B})^\natural\big)^\natural$. We stress that this is a positive operator and that, unlike the usual Laplacian on forms, it is not diagonal. Notice the factor $\frac{1}{16}$ in (22) comes from the commutation rule $[X, Y] = -4T$ we adopt in this paper, unlike the commutation rule $[X, Y] = T$ adopted in [14]. Thus, equations (18) in (19) yield

$$\frac{\partial^2 \vec{B}}{\partial s^2} = -\Delta_{\mathbb{H},1}\vec{B} \tag{23}$$

$$\frac{\partial^2 \vec{E}}{\partial s^2} = -\Delta_{\mathbb{H},1}\vec{E}. \tag{24}$$

We want now to write our system (9) in terms of the "vector potential" A. By Lemma 2.1, we can write $A = A_\Sigma + \phi ds$, with $A_\Sigma \in E^1_{0,\mathbb{H}^1}$ and ϕ is a (say smooth) scalar function.

Theorem 3.4 ([9]). *Suppose $F \in E^2_{0,\mathbb{R}\times\mathbb{H}^1}$ satisfies (9). Then $F = \hat{d}_c A$ with $A = A_1 dx + A_2 dy + \phi\, ds := A_\Sigma + \phi\, ds \in E^1_{0,\mathbb{R}\times\mathbb{H}^1}$, where*

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\Delta_{\mathbb{H},1}A_\Sigma + J \tag{25}$$

$$\frac{\partial^2 \phi}{\partial s^2} = -\frac{1}{16}\Delta^2_{\mathbb{H}}\phi + \frac{1}{16}\Delta_{\mathbb{H}}\rho_0, \tag{26}$$

where $\Delta_{\mathbb{H}} := X^2 + Y^2 (= -\Delta_{\mathbb{H},0})$ is the usual subelliptic Laplacian in $\mathbb{H}^1$, provided the following gauge condition holds:

$$\frac{1}{16}d_c^* d_c d_c^* A_\Sigma + \frac{\partial \phi}{\partial s} = 0. \tag{27}$$

In addition, (27) *is always satisfied if we replace* A *by* $A+\hat{d}_c f$*, with* f *satisfying*

$$\frac{\partial^2 f}{\partial s^2} = -\frac{1}{16}\Delta^2_{\mathbb{H}} f - \Big(\frac{1}{16} d_c^* d_c d_c^* A_\Sigma + \frac{\partial \phi}{\partial s}\Big).$$

Theorem 3.5 ([9])**.** *Let* $A_\Sigma \in E^1_{0,\mathbb{H}^1}$ *and* ϕ *satisfy* (27)*,* (25) *and* (26) *and assume all their horizontal second derivatives are bounded (remember* S *is a horizontal derivative). Suppose also* ρ_0 *is bounded. Then there exist* $g_1, g_2 \in \mathbb{R}$ *such that, if we set* $G = G(z,s) := s(g_1 x dx + g_2 y dy)$*, then*

$$F := \hat{d}_c A := \hat{d}_c(A_\Sigma + G + \phi\, ds)$$

satisfies (9)*.*

In addition, if

$$\big(S d_c^* A_\Sigma + \Delta_{\mathbb{H}}\phi + \rho_0\big)(\bar{x},\bar{s}) = 0 \tag{28}$$

for some point $(\bar{x},\bar{s}) \in \mathbb{G}\times\mathbb{R}$*, then we can choose* $G=0$*.*

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