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GENERALIZED SAMUEL MULTIPLICITIES AND APPLICATIONS

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. In this note we survey and discuss the main results on the multiplicity sequence we introduced in former papers as a generalization of Samuel's multiplicity. We relate this new multiplicity to other numbers introduced in different contexts, for example the Segre numbers of Gaffney and Gassler and the Hilbert coefficients defined by Ciupercă. Discussing some examples we underline the usefulness of the multiplicity sequence for concrete calculations in algebraic geometry using computer algebra systems.

1. Introduction

Intersection theory and singularity theory stimulated the development of a theory of multiplicities in local rings.

For example, the Samuel multiplicity of the maximal ideal $\mathfrak{m}$ of a local ring $(A, \mathfrak{m})$ measures the singularity of a variety at some point or along some subvarieties, and the Samuel multiplicity of an $\mathfrak{m}$-primary ideal was introduced in order to define the intersection number of an irreducible component of the intersection of two varieties X and Y.

If X and Y intersect improperly, one must assign intersection numbers to certain embedded components of $X \cap Y$, see for example [18] and [16]. In Fulton and MacPherson's approach this is done without a preliminary study of intersection multiplicity, since their construction gives a well defined cycle, whose coefficients are the intersection multiplicities, which coincide with Samuel's multiplicities in the case of proper intersections. In any case the theory developed by Fulton and MacPherson gives a motivation to define algebraic multiplicities, which extend Samuel's one and can provide intersection numbers.

In Stückrad and Vogel's approach (see [16]) some components of the intersection cycle are defined over the base field k and are called k-rational (see Definition 1, Section 3.2), and in [2] it was proved that they correspond to ideals of maximal analytic spread. In [3] it was defined a multiplicity $j(I, A)$ for such ideals I which generalizes Samuel's multiplicity. Later in [4] this construction was extended to arbitrary ideals in a local ring. The new algebraic multiplicity for an ideal I of a d-dimensional local ring $(A, \mathfrak{m})$ is a sequence of non-negative integers $c(I) := (c_0(I, A), \dots, c_d(I, A))$ which can be used to describe the degrees of the intersection cycle in the sense of [16]. The first number $c_0(I, A)$ coincides with the multiplicity $j(I, A)$ defined in [3].

In this note we survey and discuss the main results on the multiplicity sequence $c(I)$ and relate this sequence to other numbers introduced in different contexts, for example the Segre numbers introduced by T. Gaffney and R. Gassler [19] and the Hilbert coefficients defined by C. Ciupercă [10]. Furthermore we give some examples which show the usefulness of the multiplicity sequence for concrete calculations in algebraic geometry using computer algebra systems.

Under certain hypothesis the j-multiplicity can be expressed as a length multiplicity (see 3.3) and can be computed by an intersection algorithm using generic filter-regular sequences called "super-reductions". In 3.4 the precise meaning of "generic" will be given.

Flenner and Manaresi [14] used the j-multiplicity to give a numerical characterization of reduction ideals, which generalizes a classical theorem of E. Böger (see 3.5). Later C. Ciupercă generalized this result using the j-multiplicity and second highest Hilbert coefficients, giving a numerical characterization of the S_2-closure of the extended Rees algebra, see Theorem 9.

The multiplicity sequence and the j-multiplicity can be calculated using intersection algorithms defined in [3] and in [4], which will be presented in Section 3.6. These algorithms have been used by several authors, who sometimes produced interesting modifications of them. Here we will refer only to the original versions.

In 1999, T. Gaffney and R. Gassler introduced *Segre numbers* of an ideal in a local ring of an analytic space that turned out to be very useful in the study of the equisingularity of families of hypersurfaces with non-isolated singularities. The Segre numbers of the Jacobian ideal are simply the generic Lê numbers of D. B. Massey [30]. In analytic intersection theory, P. Tworzewski [47] (see also Remark 5) defined the extended index of intersection, which, in the case of improper intersections, replaces the classical intersection number of a proper intersection component by a set of numbers. In 3.7 we will show that all these new invariants can be considered as generalizations of Samuel's multiplicity of an $\mathfrak{m}$-primary ideal in a noetherian local ring $(A, \mathfrak{m})$ to an arbitrary ideal.

In 3.8 we present an application of generalized Samuel multiplicities to singularity theory, more precisely to Whitney stratification of surfaces (see [5]), and discuss an example.

The paper is divided into two parts: in Section 2 we review some classical results on Samuel's multiplicity whose analogues for the generalized Samuel multiplicity will be presented in Section 3, which is devoted to this new multiplicity, its properties and its relations to other important invariants.

Notation. In this paper all rings are assumed to be noetherian and the dimension of a ring means its Krull dimension. A (noetherian) local ring $(A, \mathfrak{m})$ is *formally equidimensional* (or, in Nagata's terminology, *quasi-unmixed*) if each minimal prime ideal $\mathfrak{p}$ in the $\mathfrak{m}$-adic completion $\hat{A}$ satisfies $\dim(\hat{A}/\mathfrak{p}) = \dim(\hat{A})$. For the properties of formally equidimensional local rings we refer to [23], (18.17). In particular we recall that if A is a formally equidimensional local ring and I is an ideal of A, then the associated graded ring $G_I(A)$ of A with respect to I is formally equidimensional, see[23], (18.24).

We denote the n-dimensional projective space over a field K by $\mathbb{P}^n_K$ and the affine space by $\mathbb{A}^n_K$, respectively. If not explicitly stated the contrary, our base field will always be algebraically closed and we simply write $\mathbb{P}^n$ or $\mathbb{A}^n$. By a *variety* or *subvariety* of $\mathbb{P}^n$ we mean a closed reduced (but possibly reducible) equidimensional subscheme of $\mathbb{P}^n$ without embedded components. A *surface* is a 2-dimensional variety, a *hypersurface* is an $(n-1)$-dimensional subvariety. Sometimes, for simplicity, we will state the results for varieties, but most of them hold for algebraic schemes too.

2. Some classical results on Samuel's multiplicity

Following [39], in this section we review some classical results on Samuel's multiplicity whose analogues for the generalized Samuel multiplicity will be presented in the next section.

2.1. The Samuel multiplicity

Let $(A, \mathfrak{m})$ be a d-dimensional local ring and let I be an $\mathfrak{m}$-primary ideal.

For each non negative integer j let

$$H_I^{(0)}(j) := \text{length}(I^j/I^{j+1})$$

and

$$H_I^{(1)}(j) := \sum_{k=0}^{j} H_I^{(0)}(j) = \text{length}(A/I^{j+1})\,.$$

Note that these lengths are finite since I is $\mathfrak{m}$-primary. It is well known that for all sufficiently large j the function $H_I^{(1)}$ becomes a polynomial, the so called Hilbert-Samuel polynomial, which can be written in the form

$$e_0\binom{j+d}{d} - e_1\binom{j+d-1}{d-1} + \cdots + (-1)^d e_d\,,$$

where $e_0, e_1, \ldots, e_d$ are integers and $e_0 \geq 1$. The positive integer

$$e(I, A) := e_0$$

is called *Samuel multiplicity* of I in A. Sometimes, when the ring is clear from the context, the multiplicity $e(I, A)$ will be denoted by $e(I)$. In the case $I = \mathfrak{m}$ we write simply

$$e(A) := e(\mathfrak{m}, A).$$

It is immediate that for each couple of $\mathfrak{m}$-primary ideals $J \subset I \subset A$ one has $e(J) \geq e(I)$.

2.2. Samuel multiplicity as an intersection number

Samuel's multilplicity can be applied in the following situation.

Let X, Y be subvarieties of $\mathbb{A}^n$ (or $\mathbb{P}^n$). Assume that Y is a complete intersection defined by the ideal $I(Y)$ and let C be an irreducible component of $X \cap Y$. Let $A := \mathcal{O}_{X\cap Y,C}$ and let $I := I(Y)A$. Under these assumptions the ideal I is primary with respect to the maximal ideal of the local ring A and $e(I, A)$ is the *intersection number* $i(X, Y; C)$ of X and Y along C, see [40].

We remark that Samuel proposed this definition of intersection number without any assumption on the dimension of C. We recall that it is always

$$\dim C \geq \dim X + \dim Y - n$$

and C is called a *proper component* of $X \cap Y$ if equality holds, an *improper component* otherwise. Samuel only assumed C to be an irreducible (isolated) component of $X \cap Y$ and in [42] Stückrad and Vogel could prove a theorem of Bézout for improper intersections counting irreducible components with Samuel's multiplicities.

2.3. Samuel multiplicity as a length

Let $(A, \mathfrak{m})$ be local ring, $I \subset A$ an $\mathfrak{m}$-primary ideal generated by a system of parameters. One has

$$\text{length}(A/I) \geq e(I, A) \tag{1}$$

and equality holds if and only if A is the Cohen–Macaulay (see, for example, [43], Theorem 1.2 and Lemma 1.3).

D. Buchsbaum conjectured that the difference

$$\text{length}(A/I) - e(I, A)$$

was an invariant of A, that is, independent of the choice of I, but it turned out that the conjecture holds only for the so called Buchsbaum rings, which form a class of local rings containing the Cohen–Macaulay rings (see [43]).

In Cohen–Macaulay rings the inverse of (1) holds, that is

$$e(I, A) \geq \text{length}(A/I)\,, \tag{2}$$

and, if $A/\mathfrak{m}$ is infinite, equality holds if and only if I can be generated by $d = \dim A$ elements, that is, by a system of parameters (see for example [18], Example 4.3.5.).

We remark that, without the Cohen–Macaulay assumption on A, when the ideal I is generated by a system of parameters, then one can construct an ideal J containing I such that

$$\text{length}(A/J) = e(I, A).$$

The ideal J is obtained from I by the following *intersection algorithm* (see [7]). Let $a_1, \dots, a_d$ be a system of parameters generating I (observe that an $\mathfrak{m}$-primary ideal I needs at least d generators) and, for an ideal K of A, let $U(K)$ denote the intersection of all primary ideals Q associated to A/K such that $\dim A/Q = \dim A/K$. Then set

$$\begin{aligned} I_0 &:= (0) \qquad \text{and} \\ I_k &:= a_k A + U(I_{k-1}) \qquad \text{for} \quad k = 1, \dots, d. \end{aligned}$$

The last ideal $I_d \supseteq I$ is the desired ideal J such that

$$e(I, A) = \text{length}(A/I_d),$$

see [7], Proposition 1.

2.4. Reduction ideals and Samuel multiplicity

Let $(A, \mathfrak{m})$ be a d-dimensional local ring and let $I \subset A$ be an ideal. An ideal $J \subset I$ is called a *reduction* of I if

$$JI^n = I^{n+1} \quad \text{for at least one positive integer } n.$$

J is called a *minimal reduction of I* if no ideal strictly contained in J is a reduction of I.

Reductions can be described using the integral closure of ideals. Recall that if $I \subset A$ is an ideal, the *integral closure* $\overline{I}$ of I is the ideal of A defined by

$$\begin{aligned} \overline{I} = \{ & x \in A \mid \exists\, m \text{ positive integer and, for } i = 1, \dots, m, \text{ elements } a_i \in I^i \\ & \text{such that } x^m + a_1 x^{m-1} + \cdots + a_m = 0 \}. \end{aligned}$$

$J \subseteq I$ is a reduction of I if and only if $I \subseteq \overline{J}$ (see [32] p. 34 ex. 4 and [28] p. 112).

Denote by $G := G_I A := \bigoplus_{n \geq 0} I^n / I^{n+1}$ the graded ring of A with respect to I. The *analytic spread* of I can be defined as

$$s(I) := \dim(G/\mathfrak{m}G) = \dim(G \otimes_A k)$$

and one has

$$\text{height}(I) \leq \dim A - \dim A/I \leq s(I) \leq \dim A\,,$$

where height(I) denotes the height of the ideal I (see [34], p. 151).

The notions of reduction and analytic spread were introduced by Northcott and Rees in 1954 (see [34]), who proved that if k is infinite then minimal reductions exist and are minimally generated by $s(I)$ elements.

If I is $\mathfrak{m}$-primary, then $s(I) = \dim A$ and a minimal reduction J of I is generated by a system of parameters. In this case the algorithm of (2.3) gives

$$e(J, A) = \text{length}(A/I_d).$$

Moreover, if J is a reduction of I, then $e(I, A) = e(J, A)$. In fact, for large m the $\text{length}_A(A/I^m)$ is given by a polynomial of degree $d = \dim A$, which can be written in the form

$$h_I(m) = e(I)\frac{m^d}{d!} + \text{ terms of degree } < d\,.$$

If $JI^n = I^{n+1}$, for each integer $m \geq 1$ we have $J^m I^n = I^{n+m}$, hence $h_I(m+n) \geq h_J(m) \geq h_I(m)$ which implies $e(I) = e(J)$.

EXAMPLE 1. If $A = \mathbb{C}[[x, y]]$, $I = (x^3, y^2, x^2y)$, $J = (x^3, y^2)$, then $IJ = I^2$ and $e(I) = e(J) = 6$.

In 1961 D. Rees [38] proved that in formally equidimensional local rings the converse is also true.

THEOREM 1 (Rees, 1961). *Let* $(A, \mathfrak{m})$ *be a formally equidimensional local ring, let* $J \subseteq I$ *be* $\mathfrak{m}$*-primary ideals such that* $e(J) = e(I)$. *Then* J *is a reduction of* I.

For the geometric significance of this theorem we refer the reader to J. Lipman [28].

If $J \subseteq I$ are ideals in a local ring $(A, \mathfrak{m})$ such that J is a reduction of I, then $e(JA_\mathfrak{p}) = e(IA_\mathfrak{p})$ for each minimal prime ideal $\mathfrak{p}$ of I. In general the converse is not true, but, using the characterization of ideals with $\text{height}(I) = s(I)$ given by E. C. Dade in [12], E. Böger proved the following result.

THEOREM 2 (Böger [8], 1970). *Let* $J \subseteq I$ *be ideals in a formally equidimensional local ring* $(A, \mathfrak{m})$ *such that* $\sqrt{J} = \sqrt{I}$, $\text{height}(J) = s(J)$ *and* $e(JA_\mathfrak{p}) = e(IA_\mathfrak{p})$ *for each minimal prime ideal* $\mathfrak{p}$ *of* I. *Then* J *is a reduction of* I.

A further generalization of this result was given by B. Ulrich (see [16], (3.6.3)), who proved:

PROPOSITION 1. *Let* $(A, \mathfrak{m})$ *be a formally equidimensional local ring and* $J \subseteq I$ *be ideals of* I. *Then* either $\text{height}(JI^{n-1} : I^n) < s(J)$ *for all* $n \geq 1$ or J *is a reduction of* I.

COROLLARY 1. *Let* $(A, \mathfrak{m})$ *and* $J \subseteq I$ *as in the above proposition. Then* J *is a reduction of* I *if and only if* $J_\mathfrak{p}$ *is a reduction of* $I_\mathfrak{p}$ *for all prime ideals* $\mathfrak{p}$ *with* $\text{height}(\mathfrak{p}) = s(J_\mathfrak{p})$.

REMARK 1. If $J \subseteq \mathfrak{m}$ is an ideal of a formally equidimensional local ring $(A, \mathfrak{m})$, then $\{\mathfrak{p} \in \operatorname{Spec} A \mid s(J_\mathfrak{p}) = \text{height}(\mathfrak{p})\}$ is a finite set, precisely, it coincides with $\operatorname{Ass}_A(A/\overline{J^n})$ for sufficiently large n, where $\overline{J^n}$ denotes the integral closure of the ideal J^n. In fact,

$$\operatorname{Ass}_A(A/\overline{J}) \subseteq \operatorname{Ass}_A(A/\overline{J^2}) \subseteq \operatorname{Ass}_A(A/\overline{J^3}) \subseteq \ldots$$

and this sequence eventually stabilizes to a limit set

$$\bigcup_{k=1}^{\infty} \mathrm{Ass}_A(A/\overline{J^k}) = \mathrm{Ass}_A(A/\overline{J^n}) \quad (\text{for some } n > 0)\,,$$

which is called the set of *asymptotic primes* of J (see [31], (4.1), p. 26) and which will be denoted by $\mathrm{Asymp}(J)$. One can prove that

$$\begin{aligned} \mathrm{Asymp}(J) &= \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \exists\, \mathfrak{P} \in \mathrm{Ass}_{G_J(A)}(G_J A) \text{ such that } \mathfrak{p} = \mathfrak{P} \cap A\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}\, A \mid s(J_{\mathfrak{p}}) = \mathrm{height}(\mathfrak{p})\}\,. \end{aligned}$$

3. Generalized Samuel multiplicities

This section is devoted to a generalization of Samuel's multiplicity by a sequence of numbers, the so-called generalized Samuel multiplicity, which we have introduced and studied in several papers, partly in collaboration with H. Flenner. We also present the main properties of this new multiplicity and its relation to other important invariants of local rings.

3.1. Generalized Samuel multiplicities (see [4])

Let $(A, \mathfrak{m}, k)$ be a d-dimensional local ring and let $I \subset A$ be an arbitrary ideal (not necessarily $\mathfrak{m}$-primary).

Let $G_I(A) := \bigoplus_{j\geq 0} I^j/I^{j+1}$ be the associated graded ring of A with respect to I and let us consider the bigraded ring

$$R = \bigoplus_{i,j\geq 0} R_{i,j} = \bigoplus_{i,j\geq 0} G^i_{\mathfrak{m}}(G^j_I(A)) = \bigoplus_{i,j\geq 0} (\mathfrak{m}^i I^j + I^{j+1})/(\mathfrak{m}^{i+1} I^j + I^{j+1})\,,$$

where $R_{00} = A/\mathfrak{m} = k$ is a field.

Let $H^{(0,0)}(i,j) := \dim R_{ij}$ the Hilbert function of the bigraded ring R and let

$$H^{(1,1)}(i,j) := \sum_{q=0}^{j}\sum_{p=0}^{i} H^{(0,0)}(p,q)$$

its twofold sum transform. For both $i, j >> 1$ this function becomes a polynomial in (i, j), which can be written in the form

$$\sum_{k+l\leq d} a^{(1,1)}_{k,l} \binom{i+k}{k}\binom{j+l}{l}\,.$$

Following [4] define the *generalized Samuel multiplicity* to be

$$(a^{(1,1)}_{0,d}, a^{(1,1)}_{1,d-1}, \ldots, a^{(1,1)}_{d,0}) =: (c_0(I), c_1(I), \ldots, c_d(I)) =: c(I)\,.$$

It will turn out that the first coefficient $c_0(I)$ plays an important role as an intersection number and permits generalizations of results about Samuel's multiplicity. We will call it the *j-multiplicity* $j(I) = j(I, A)$.

In a geometric way the j-multiplicity can be described as follows. Let $X = \operatorname{Spec} A$, let $d = \dim X$, let Y be the subscheme of X defined by I and let $p\colon Z \to X$ be the blowing up of X along Y. Consider the union E of all the irreducible components of the exceptional set $p^{-1}(Y)$ contained in the special fiber $p^{-1}(\mathfrak{m})$. This is a projective scheme over $A/\mathfrak{m}^n$ for some n. The multiplicity $j(I, A)$ is the $(d-1)$-dimensional degree of E.

THEOREM 3 ([4], Prop. 2.3, 2.4, 2.5). *With the above notations set* $q := \dim(A/I)$, $G := G_I(A)$, $s := s(I) = \dim G/\mathfrak{m}G$. *Then*

1. $c_k = 0$ *for* $k < d - s$ *and* $k > q$, *that is,*

$$(c_0, c_1, \ldots, c_d) = (0, \ldots, 0, c_{d-s}, \ldots, c_q, 0, \ldots, 0);$$

2. $c_{d-s} = \sum_{\mathfrak{P}} e(\mathfrak{m}G_{\mathfrak{P}}) \cdot e(G_{\mathfrak{P}})$,
   where $\mathfrak{P}$ *runs over all highest dimensional associated prime ideals of* $G/\mathfrak{m}G$ *such that* $\dim G/\mathfrak{P} + \dim G_{\mathfrak{P}} = \dim G$*;*

3. $c_q = \sum_{\mathfrak{p}} e(IA_{\mathfrak{p}}) \cdot e(A/\mathfrak{p})$,
   where $\mathfrak{p}$ *runs over all highest dimensional associated prime ideals of* A/I *such that* $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$*;*

4. $e(G_I(A)) = \sum_{k=0}^{d} c_k(I, A)$*;*

5. *if* $\operatorname{height}(I) = s(I)$ *then*

$$(c_0, \ldots, c_{q-1}, c_q, c_{q+1}, \ldots, c_d) = (0, \ldots, 0, c_q, 0, \ldots, 0).$$

In particular, if I is $\mathfrak{m}$*-primary, then* $\operatorname{height}(I) = s(I) = d$, $q = 0$ *and*

$$(c_0, \ldots, c_d) = (e(I), 0, \ldots, 0)\,,$$

that is, the sequence $(c_0, \ldots, c_d)$ *generalizes the Samuel multiplicity to arbitrary ideals.*

REMARK 2. By Theorem 3, (1) and (2), if $G = G_I A$ is formally equidimensional, then

$$j(I) = c_0(I) \neq 0 \quad \text{if and only if} \quad s(I) = \dim A\,.$$

REMARK 3. With the notation of (3.1) N.V. Trung (see [46], Cor. 2.8) proved that if $(A, \mathfrak{m})$ and I are such that the ring $R = G_{\mathfrak{m}}(G_I A)$ is a domain or a Cohen–Macaulay ring, then $c_i(I) > 0$ for all $d - s \leq i \leq q$ where $d = \dim A$, $s = s(I)$ and $q = \dim(A/I)$.

REMARK 4. If $J \subset I \subset A$ are two ideals of A with $\sqrt{I} = \sqrt{J}$ and $c(I) = (c_0(I), \dots, c_d(I)), c(J) = (c_0(J), \dots, c_d(J))$ are their generalized Samuel multiplicities, then from Theorem 3 (3) we have $c_q(I) \leq c_q(J)$, but we can say nothing about $c_{q-1}(I)$ and $c_{q-1}(J)$, as one can see from the two following examples.

EXAMPLE 2. (Erika Giorgi) Let $A = k[x, y, z]_{(x,y,z)}$ where K is a field and consider the ideals $J = (x, y)^3 \cap (x, z)^3 \cap (y, z)$ and $I = (x^2, y) \cap (x, z)^2 \cap (y, z)$. Obviously $J \subset I$ and $\sqrt{J} = \sqrt{I}$. By using [1] we get $c(J) = (4, 19, 0, 0)$ and $c(I) = (6, 7, 0, 0)$.

Let $L = (x^2, y^2) \cap (x^3, z^3) \cap (y, z)$ and $M = (x, y)^2 \cap (x, z)^3 \cap (y, z)$. Obviously $L \subset M$ and $\sqrt{L} = \sqrt{M}$. In this case we get $c(L) = (29, 14, 0, 0)$ and $c(M) = (0, 14, 0, 0)$.

The last example shows also the importance of the condition height$(L) = s(L)$ in Böger's Theorem 2. Here the condition $e(L_{\mathfrak{p}}, A_{\mathfrak{p}}) = e(M_{\mathfrak{p}}, A_{\mathfrak{p}})$ for all minimal primes $\mathfrak{p}$ of M is satisfied, but L is not a reduction of M. In fact, if L was a reduction of M then L and M would share a minimal reduction and then they would have the same analytic spread, but $s(L) = \dim A = 3$, while $s(M) = 2$.

3.2. Generalized Samuel multiplicities as intersection numbers

Let X, Y be equidimensional closed subschemes of $\mathbb{P}^n_K = \mathrm{Proj}(K[X_0, \dots, X_n])$, where K is an arbitrary field. For indeterminates U_{ij} $(0 \leq i, j \leq n)$ let L be the pure transcendental field extension $K(U_{ij})_{0 \leq i,j \leq n}$ and $X_L := X \otimes_K L$, etc. Proving a Bézout theorem for improper intersections, Stückrad and Vogel (see [16]) introduced a cycle $v(X, Y) = v^0 + \cdots + v^n$ on $X_L \cap Y_L$, which is obtained by an intersection algorithm on the ruled join variety

$$J := J(X_L, Y_L) \subset \mathbb{P}^{2n+1}_L = \mathrm{Proj}(L[X_0, \dots, X_n, Y_0, \dots, Y_n])$$

as follows.

Let Δ be the "diagonal" subspace of $\mathbb{P}^{2n+1}_L$ given by the equations

$$X_0 - Y_0 = \cdots = X_n - Y_n = 0\,,$$

let $H_i \subseteq J$ be the divisor given by the equation

$$\ell_i := \sum_{j=0}^{n} U_{ij}(X_j - Y_j) = 0,$$

and set $\underline{\ell} := (\ell_0, \dots, \ell_n)$. Then one defines inductively cycles β^k and v^k by setting $\beta^0 := [J]$. If β^k is already defined, decompose the intersection

$$\beta^k \cap H_k = v^{k+1} + \beta^{k+1} \qquad (0 \leq k \leq \dim J)\,,$$

where the support of v^{k+1} lies in Δ and where no component of β^{k+1} is contained in Δ (if $k = \dim J$, then the support of v^{k+1} is the empty set). It follows that v^k is a $(\dim J - k)$-cycle on $X_L \cap Y_L \cong J \cap \Delta$. The part of dimension k of the cycle $v(X, Y) := v(\underline{\ell}, J) := \sum v^k$ will be denoted by v_k, so that the upper index denotes the codimension in the ruled join and the lower one the dimension of the cycle. In general, the cycle $v(X, Y)$ is defined over L.

DEFINITION 1. *The cycle $v(X, Y)$ is called the v-*cycle *of the intersection of X and Y. An irreducible subvariety C of $X_L \cap Y_L$ is said to be a* characteristic subvariety *if C occurs in $v(X, Y)$. The coefficient of C in $v(X, Y)$ is denoted by $j(X, Y; C)$. Thus*

$$v(X, Y) = \sum_C j(X, Y; C)\,[C]\,,$$

where C runs through the characteristic subvarieties. The set of all these subvarieties is denoted by $\mathcal{C} = \mathcal{C}(X, Y)$. Moreover, the set of all elements of $\mathcal{C}$ which are defined over K is denoted by $\mathcal{C}_{rat} = \mathcal{C}_{rat}(X, Y)$, that is, $\mathcal{C}_{rat}$ is the set of K-rational *or distinguished or* fixed subvarieties *and $\mathcal{C} \setminus \mathcal{C}_{rat}$ is the set of the so-called* non K-rational *or* movable subvarieties *of the intersection of X and Y.*

By a result of van Gastel ([21], Prop. 3.9), a K-rational irreducible subvariety C of $X_L \cap Y_L$ occurs in $v(X, Y)$ if and only if C is a distinguished variety of the intersection of X and Y in the sense of Fulton ([18], p. 95), and this is equivalent to the maximality of the analytic spread (see [2]) or the maximality of the dimension of the so-called limit of join variety (see [17]).

For an arbitrary irreducible subvariety $Z \subseteq X_L \cap Y_L \subset \mathbb{P}^n_L$ we set $Z_\Delta := J(Z, Z) \cap \Delta$. By $\hat{J}$ and $\hat{Z}_\Delta$ we denote the affine cones of the ruled join $J := J(X_L, Y_L) \subset \mathbb{P}^{2n+1}_L$ and Z_Δ in the affine space $\mathbb{A}^{2n+2}_L$. Let $(A, \mathfrak{m})$ be the local ring $\mathcal{O}_{\hat{J}, \hat{Z}_\Delta}$ and $I \subset A$ be the ideal of the diagonal subspace Δ and let $G(X, Y; Z)$ denote the associated graded ring $G_I(A) = \oplus_{j=0}^{\infty} I^j/I^{j+1}$. If Z is the empty subvariety of $\mathbb{P}^n$, then A becomes the homogeneous ring of coordinates of the ruled join $J \subset \mathbb{P}^{2n+1}_L$ localized at the irrelevant maximal ideal; that is, we obtain a global picture of the intersection algorithm.

PROPOSITION 2 ([4], Section 4). *With the preceding notation,*

$$e(G(X, Y; Z)) = e(G_I(A)) = \sum_{k=0}^{d} c_k(I, A) = \sum_C j(X, Y; C) \cdot e(\mathcal{O}_{C,Z})\,,$$

where C runs through the characteristic subvarieties of X and Y with $C \supseteq Z$. In particular, if $Z \in \mathcal{C}_{rat}(X, Y)$, then $j(X, Y; Z) = j(I, A)$.

If $Z = \emptyset$, then $d = \dim A = \dim J + 1$ and

$$c_0 = j(X, Y; \emptyset),\ c_1 = \deg v_0,\ c_2 = \deg v_1, \ldots,\ c_d = \deg v_{d-1}\,;$$

moreover, if $k > \dim(X \cap Y) + 1$, then $c_k = 0$.

If $Z \neq \emptyset$ is K-rational, then $d = \dim A = \dim J - \dim Z$ and

$$c_k = \sum_C j(X, Y; C) \cdot e(\mathcal{O}_{C,Z}) \qquad (0 \leq k \leq d)\,,$$

where C runs through all varieties of $\mathcal{C}(X, Y)$ with $C \supseteq Z$ and $\operatorname{codim}_C Z = k$*. If* $k > \dim(X \cap Y) - \dim Z$*, then* $c_k = 0$.

We will illustrate the above proposition by recomputing the self-intersection of a monomial curve in $\mathbb{P}^3$ obtained by hand in [44], Example 2, p. 269, as a result of a heavy calculation.

EXAMPLE 3. Consider the curve X in $\mathbb{P}^3_K$ ($\mathrm{char}(K) \neq 2, 3$) given parametrically by $(s^6, s^4t^2, s^3t^3, t^6)$ and with defining ideal

$$I(X) = (x_0x_3 - x_2^2, x_0^2x_3 - x_1^3) \subset K[x_0, x_1, x_2, x_3]\,.$$

Using [1] (see the sample file Segre4.txt) and the above proposition in the case $Z = \emptyset$, we want to calculate its self-intersection cycle $v(X, X)$. Running the computer program, with the notation of the proposition, we get $c(I, A) = (c_0, c_1, c_2, 0, 0)$, where

$$\begin{aligned} c_0(I, A) &= j(X, X; \emptyset) = 12, \\ c_1(I, A) &= \deg(v_0(X, X)) = 18, \\ c_2(I, A) &= \deg(v_1(X, X)) = 6. \end{aligned}$$

Since X is a complete intersection of degree 6, it follows that $j(X, X; X) = 1$. In order to understand $v_1(X, X)$ we recall that by [2], Corollary 2.5, a point of X is a K-rational component of $v_1(X, X)$ if and only if it is a singular point of X. One checks that X has the two singular points $P = (0 : 0 : 0 : 1)$ and $Q = (1 : 0 : 0 : 0)$. One applies now Proposition 2 in the cases $Z = P$ and $Z = Q$, more precisely, in the previous calculation one substitutes $x_3 = 1$ for P obtaining $c = (8, 3, 0)$ and $x_0 = 1$ for Q obtaining $c = (3, 2, 0)$. This means that P is a point of multiplicity 3 in X and $j(X, X; P) = 8$ and Q is a double point of X and $j(X, X, Q) = 3$. The contribution of non K-rational points is therefore 7.

REMARK 5 (Analytic case). In the paper [47], Tworzewski has constructed an intersection cycle for complex analytic subsets X and Y of a manifold M which do not intersect necessarily properly. His construction is based on a pointwise defined intersection multiplicity $g(x) = g(X \times Y, \Delta_M, x)$ for a point $x \in \Delta_M$, where Δ_M is the diagonal of $M \times M$ and $g(x)$ is the sum of the coordinates of the so-called extended index of intersection $\tilde{g}(x)$ (see [47], Definition (4.2), p. 185).

Let $A = \mathcal{O}_{X\times Y,x}$ and $I = \mathcal{I}_{\Delta_M} \cdot \mathcal{O}_{X\times Y,x}$. K. Nowak [35], [36] (see also [6]) has proved that $g(x) = e(G_I(A))$ and that $\tilde{g}(X)$ is composed of the generalized Samuel multiplicities $c_0(I, A), \ldots, c_{\dim(X\cap Y)}(I, A)$ and of zeros.

REMARK 6. Recall that a d-dimensional projective variety X is said to be *connected in dimension $d - 1$* if for every closed subvariety Z of X of dimension $< d - 1$ the set $X \setminus Z$ is connected.

Flenner, van Gastel and Vogel (see [13], Theorem 3.4) proved that if X and Y are pure dimensional projective varieties connected in $\dim X - 1$ and $\dim Y - 1$ respectively, A is the ring of coordinates of the ruled join of X and Y localized at the irrelevant maximal ideal and I is the ideal of "the diagonal" in the ring A, then we have $c_i(I) > 0$ for all $d - s \leq i \leq q$, where $d = \dim A$, $s = s(I)$ and $q = \dim(A/I)$.

The assumption of the theorem by Flenner–van Gastel–Vogel does not imply the assumption of Trung's Corollary 2.8 as one can see by the following example.

EXAMPLE 4. ([42]) Let $X \subset \mathbb{P}^3_K$ be the non-singular curve of F. S. Macaulay ([29], page 98) given parametrically by $\{(s^4, s^3t, st^3, t^4)\}$ and let $Y \subset \mathbb{P}^3$ be the line $x_0 = x_1 = 0$. Then X and Y are connected in dimension 0 and the theorem of Flenner–van Gastel–Vogel can be applied.

However, with the notation of Remark 6, the ring $R = G_{\mathfrak{m}}(G_I(A))$ is neither Cohen–Macaulay (since the coordinate ring of X is not Cohen–Macaulay) nor a domain (since the intersection cycle $v(X, Y)$ has two K-rational components), hence both the conditions of Trung [46], Corollary 2.8 are not fulfilled.

3.3. The j-multiplicity as a length

We have seen that if $(A, \mathfrak{m})$ is a local ring of dimension d and $I \subset A$ is an arbitrary ideal, the j-multiplicity $j(I) := c_0(I)$ is an important generalization of the classical Samuel multiplicity of an $\mathfrak{m}$-primary ideal since it measures the contribution of distinguished components of the intersection (see Proposition 2).

If the ring A is Cohen–Macaulay and the ideal I is $\mathfrak{m}$-primary, then $e(I, A)$ is given by

$$\operatorname{length}(A/(f_1, \ldots, f_d))$$

where $f_1, \ldots, f_d \in I$ are sufficiently generic elements, hence a minimal reduction of I. Using the theory of residual intersections due to Huneke (see [25], [26]) and others, under certain hypothesis on the pair (A, I) a similar formula can be proved for the j-multiplicity, (see [15], Theorem 3.4).

We recall the following definitions. Let A be a local ring, $I \subseteq A$ an ideal. As usual $H^*(I, A)$ will denote the Koszul cohomology of (I, A), i.e. $H^*(I, A)$ is the cohomology of the Koszul complex $K^\bullet(x_1, \ldots, x_k; A)$, where $x_1, \ldots, x_k \in I$ is a minimal set of generators for I. Following [16], the pair (A, I) is called *strongly Cohen–Macaulay* (SCM) if $H^p(I, A)$ is either zero or a Cohen–Macaulay module for all $p \geq 0$; note that this differs from the notion originally given in [25] as we also require the Cohen–Macaulayness of A. For basic properties of this concept we refer the reader to [25] and [16] (7.2). In particular we will need the following two facts.

If (A, I) is SCM and $H^p(I, A)$ is nonzero then it is automatically a Cohen–Macaulay module of dimension $\dim A/I$ over A/I, see [25] or [16], (7.2.7). Moreover, the pair (A, I) is SCM if and only if the Koszul cohomology $H^*(y_1, \ldots, y_e;\ A)$ is a Cohen–Macaulay module for an *arbitrary* generating set $y_1, \ldots, y_e$ of I.

Another important notion in the theory of residual intersections is the Artin–Nagata condition. An ideal I of a local ring A is said to satisfy the *Artin–Nagata condition G_s* if

$$(G_s)\quad \mu(I_{\mathfrak{p}}) \leq \operatorname{height} \mathfrak{p} \ \text{ for all primes } \mathfrak{p} \in V(I) \text{ with height } \mathfrak{p} < s.$$

Here $\mu(I_{\mathfrak{p}})$ denotes the minimal number of generators of the ideal $I_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$. The ideal I is said to satisfy G_∞ if G_s holds for all $s \geq 1$.

THEOREM 4 ([15], Theorem 3.4). *Let A be a d-dimensional local Cohen–Macaulay ring, $I \subseteq A$ an ideal. Assume that (A, I) is SCM and that I satisfies G_d. Then $j(I, A)$ is given by the length of*

$$I/(f_1 + \ldots + f_{d-1}) + f_d I,$$

where $f_1, \ldots, f_d$ are sufficiently generic elements of I.

REMARK 7. For the precise meaning of "sufficiently generic elements of I" see [15], (3.3). Let $I = (x_1, \ldots, x_n)$. Extending the ring A with new indeterminates $u_{11}, \ldots, u_{dn}$, that is passing from A to $A' = A[u_{11}, \ldots, u_{dn}]_{\mathfrak{m}A[u_{11},\ldots,u_{dn}]}$, the elements

$$f_i := \sum_{h=1}^{n} u_{ih} x_h \qquad \text{for } i = 1, \ldots, d$$

are sufficiently generic.

REMARK 8. If I is an $\mathfrak{m}$-primary ideal of a d-dimensional Cohen–Macaulay ring A, then (A, I) is automatically strongly Cohen–Macaulay and satisfies the Artin–Nagata condition G_d, so Theorem 4 generalizes the classical length formula.

REMARK 9. If one admits the assumption on the local ring A and I only satisfies the G_d-condition, then by the proof of [15], Theorem 3.4 one can see that the following inequality holds:

$$j(I, A) \leq \text{length}(I/(f_1, \ldots, f_{d-1}) + f_d I)\,,$$

which generalizes the classical inequality for the Samuel multiplicity of a system of parameters given in (2.3).

The result of Theorem 4 can be applied to give explicitly expressions for the j-multiplicity in many examples where the SCM condition is satisfied (see [15], Section 4). In particular one obtains a positive answer to the following problem posed by Ein, Lazarsfeld and Nakamaye in some special cases.

PROBLEM 1. Let $H \subseteq \mathbb{A}^n$ be a hypersurface and $C \subseteq H$ an irreducible subset of codimension c such that C is an irreducible component of the two equimultiplicity strata Σ_l and Σ_{l+m}, where $\Sigma_i := \{x \in H \mid e(\mathcal{O}_{H,x}) > i\}$.

Is then C a distinguished component of the $(c + 1)$-fold intersection H^{c+1}? Moreover, does C appear with a coefficient $\geq m^{c+1}$ in the intersection cycle?

The general problem is even open in the special case when C is a point where the multiplicity jumps (see [15], (4.4)). In this case the question becomes: assume that $H \subseteq \mathbb{A}^n$ is a hypersurface with a jump of multiplicity at 0, i.e. $e(\mathcal{O}_{H,0}) > e(\mathcal{O}_{H,x})$

for all $x \neq 0$ near 0. Is the point 0 then a distinguished component of the n-fold self-intersection of H? By [15], Prop. 4.1 in this situation we only know that $C = 0$ is a distinguished component of H^n when it is an irreducible component of the singular locus Sing H.

3.4. Intersection algorithms for filter-regular sequences and computation of the j-multiplicity

In [3] and [4] we introduced intersection algorithms in a local ring, which are counterparts of the construction of the Stückrad–Vogel cycle (see Definition 1), and compared them with analogous algorithms in the associated graded ring. These algorithms can be used to express multiplicities as lengthes and to generalize the last result of 2.3.

Again let $(A, \mathfrak{m})$ be a d-dimensional local ring, let $I \subseteq \mathfrak{m}$ be an ideal of A and let $G := G_I(A)$ be the associated graded ring. Consider a sequence $\underline{a} = (a_1, \ldots, a_t)$ of elements of I such that $\sqrt{\underline{a}A} = \sqrt{I}$ and the sequence $\underline{a}^* = (a_1^*, \ldots, a_t^*)$ of the initial forms of $a_1, \ldots, a_t$ in G is contained in $G^1 = A/I$ and is a *filter-regular sequence* with respect to the ideal $G^+ = \oplus_{n \geq 1} I^n/I^{n+1}$, that is

$$(a_1^*, \ldots, a_{k-1}^*)G :_G a_k^* \subseteq \bigcup_{n>0}((a_1^*, \ldots, a_{k-1}^*)G :_G (G^+)^n)$$

for $k = 1, \ldots, t$, or equivalently, $a_k^* \notin \mathfrak{P}$ for all relevant associated prime ideals $\mathfrak{P} \in \mathrm{Ass}_G(G/(a_1^*, \ldots, a_{k-1}^*)G)$ for $k = 1, \ldots, t$ (see, for example, [43], Def. 1, p. 252). In particular this implies that $\underline{a} = (a_1, \ldots, a_t)$ is a filter-regular sequence in A with respect to I, see, for example, [3], (2.2).

We define a cycle $\upsilon(\underline{a}, A)$ of A supported on $V(I) = V(\underline{a}A)$ by the following *intersection algorithm in* A. Set $\mathfrak{a}_{-1} := (0)$, $a_0 := 0$, $J := \underline{a}A$ and inductively

$$\mathfrak{a}_k := \bigcup_{n\geq 0}((\mathfrak{a}_{k-1} + a_k A) :_A J^n) \qquad (0 \leq k \leq t)\,.$$

Observe that $\mathfrak{a}_t = A$. Then

$$\upsilon^k(\underline{a}, A) := \sum_{\mathfrak{p}} \mathrm{length}(A/(\mathfrak{a}_{k-1} + a_k A))_{\mathfrak{p}}\,[\mathfrak{p}]\,,$$

where the sum is taken over all $(d-k)$-dimensional associated prime ideals $\mathfrak{p}$ of $A/(\mathfrak{a}_{k-1} + a_k A)$ that contain J and $[\mathfrak{p}]$ denotes the cycle associated with $\mathfrak{p}$. We define $\upsilon(\underline{a}, A) := \sum_{k=0}^{t} \upsilon^k(\underline{a}, A)$, and the *degree* of $\upsilon^k(\underline{a}, A)$ by

$$\deg \upsilon^k(\underline{a}, A) := \sum_{\mathfrak{p}} \mathrm{length}(A/(\mathfrak{a}_{k-1} + a_k A))_{\mathfrak{p}} \cdot e(A/\mathfrak{p})\,.$$

The cycle $\upsilon(\underline{a}, A)$ can also be constructed by the following *unmixed intersection algorithm in* A, which is more closely related to the approach of Stückrad and Vogel in

[42]. Recall that, given an ideal L, we denote by $U(L)$ the intersection of all highest dimensional primary ideals of L. Set $\mathfrak{a}'_{-1} := (0)$ and inductively

$$\mathfrak{a}'_k := \bigcup_{n\geq 0}(U(\mathfrak{a}'_{k-1} + a_k A)) :_A J^n) \qquad (0 \leq k \leq t)\,.$$

Then, if $\mathfrak{a}'_k \neq A$, it holds $\mathfrak{a}'_k = U(\mathfrak{a}_k)$ (see [3], proof of Proposition 3.2), hence

$$v^k(\underline{a}, A) = \sum_{\mathfrak{p}} \operatorname{length}(A/(\mathfrak{a}'_{k-1} + a_k A))_{\mathfrak{p}}\,[\mathfrak{p}]\,,$$

where the sum is taken over all $(d-k)$-dimensional associated prime ideals $\mathfrak{p}$ of $A/(\mathfrak{a}'_{k-1} + a_k A)$ that contain J.

In the same way, replacing $\underline{a}$ by $\underline{a}^*$ and J by G^+, we define a cycle $v(\underline{a}^*, G)$ by an *intersection algorithm in* $G = G_I(A)$ with $\tilde{\mathfrak{a}}_{-1} := 0 \cdot G$, $a_0^* := 0$, and

$$\tilde{\mathfrak{a}}_k := (\tilde{\mathfrak{a}}_{k-1} + a_k^* G) :_G \langle G^+\rangle \qquad (0 \leq k \leq t)\,.$$

We put

$$v_k(\underline{a}^*, G) := \sum_{\mathfrak{P}} \operatorname{length}(G/(\tilde{\mathfrak{a}}_{k-1} + a_k^* G))_{\mathfrak{P}}\,[\mathfrak{P}]\,,$$

where the sum is over all $(d-k)$-dimensional associated prime ideals $\mathfrak{P}$ of $G/(\tilde{\mathfrak{a}}_{k-1} + a_k^* G)$ that contain G^+. Observe that the prime ideals of $v(\underline{a}; A)$ contain I and hence correspond to prime ideals in the ring A/I. On the other hand, the prime ideals of $v(\underline{a}^*, G)$ contain G^+ and correspond to their contraction ideals in $G_I^0(A) = A/I$. So both cycles $v(\underline{a}; A)$ and $v(\underline{a}^*, G)$ can be considered as cycles of A/I and we have the following theorem ([4], 3.3):

THEOREM 5 (Deformation to the normal cone).

$$v(\underline{a}, A) = v(\underline{a}^*, G) \quad \textit{as cycles of } A/I\,.$$

A natural deformation space for the deformation to the normal cone is given by the extended Rees ring of A with respect to I. L. O'Carroll and T. Pruschke used this to introduce an analogue of the previous algorithms in Rees rings, see [37].

In order to generalize the last result of 2.3, that is, to compute the j-multiplicity as a length, we must use in the above algorithms "generic elements" $a_1, \ldots, a_d$ of I. The precise meaning of "generic" is that the elements must be a "super-reduction" in the sense of [3], (2.7):

DEFINITION 2. *Let $(A, \mathfrak{m})$ be a local ring, let I be an ideal of A such that $s(I) = \dim A = d$. A sequence of elements $a_1, \ldots, a_d$ in I is called a* super-reduction *for I if:*

1. *their initial forms $a_1^*, \ldots, a_d^*$ in $G = G_I(A)$ are of degree one and form a filter-regular sequence for G with respect to $G^+ := \bigoplus_{i>0} I^i/I^{i+1}$, that is, $(a_1^*, \ldots, a_{i-1}^*)G : a_i^* \subseteq \bigcup_{n\geq 0}((a_1^*, \ldots, a_{i-1}^*)G : (G^+)^n)$;*

2. *for every relevant highest dimensional prime ideal* $\mathfrak{p}$ *of* $G = G_I A$ *and* $d(\mathfrak{p}) := \dim G/(\mathfrak{m}G + \mathfrak{p})$ *the initial forms* $a_1^*, \ldots, a_{d(\mathfrak{p})}^*$ *are a system of parameters for* $G/(\mathfrak{m}G + \mathfrak{p})$.

REMARK 10. If $A/\mathfrak{m}$ is infinite, then every ideal of maximal analytic spread has a super-reduction ([3], (2.9)).

If $(a_1, \ldots, a_d)$ is a super-reduction of I, then $a_1, \ldots, a_d$ form a minimal basis of a minimal reduction of I (see [3], (2.8)).

If I is $\mathfrak{m}$-primary, then the notion of super-reduction coincides with that of a *superficial system of parameters* in the sense of [39], p. 185.

Now let $(A, \mathfrak{m})$ be a d-dimensional local ring, let I be an ideal of A of maximal analytic spread $s(I) = \dim A = d > 0$ and let $\underline{a} = (a_1, \ldots, a_d)$ be a super-reduction of I. We set

$$\mathrm{Int}(\underline{a}, A) := \mathfrak{a}_{d-1} + a_d A \quad \text{and} \quad \text{U-Int}(\underline{a}, A) := \mathfrak{a}'_{d-1} + a_d A,$$

where $\mathfrak{a}_{d-1}$ and $\mathfrak{a}'_{d-1}$ are the ideals produced by the intersection algorithm and the unmixed intersection algorithm, respectively. Then the ideals $\mathrm{Int}(\underline{a}, A)$, $\text{U-Int}(\underline{a}, A)$, are equal and $\mathfrak{m}$-primary (see [3], 3.2). Analogously, the ideals $\mathrm{Int}(\underline{a}^*, G)$ and $\text{U-Int}(\underline{a}^*, G)$ are equal and primary with respect to the homogeneous maximal ideal of G (see [3], 3.3) and we have the following theorem ([3], Theorem 3.8):

THEOREM 6 (Computation of the j-multiplicity by super-reductions). *Let* $(A, \mathfrak{m})$ *be a* d*-dimensional local ring, let* I *be an ideal of* A *of maximal analytic spread* $s(I) = \dim A = d > 0$ *and let* $\underline{a} = (a_1, \ldots, a_d)$ *be a super-reduction of* I. *With the notation introduced before, one has*

$$\begin{aligned} j(I, A) &= \mathrm{length}_A(A/\mathrm{Int}(\underline{a}, A)) = \mathrm{length}_A(A/\text{U-Int}(\underline{a}, A)) \\ &= \mathrm{length}_G(G/\mathrm{Int}(\underline{a}^*, G)) = \mathrm{length}_G(G/\text{U-Int}(\underline{a}^*, G)) = j(G^+, G). \end{aligned}$$

3.5. Reduction ideals and j-multiplicity

Using the j-multiplicity it is possible to give a numerical characterization of reduction ideals, which generalizes Böger's theorem, see Theorem 2. The easy direction is given by the following proposition.

PROPOSITION 3 ([14], Proposition 2.10). *Let* $(A, \mathfrak{m})$ *be a local ring, let* $J \subseteq I \subseteq \mathfrak{m}$ *be ideals of* A. *If* J *is a reduction of* I, *then* $j(J, A) = j(I, A)$.

For the other direction we need to consider formally equidimensional local rings. Precisely we have the following result.

THEOREM 7 ([14], Theorem 3.3). *Let* $J \subseteq I \subseteq \mathfrak{m}$ *be ideals of an equidimensional local ring* $(A, \mathfrak{m})$. *Then* J *is a reduction of* I *if and only if* $j(J_{\mathfrak{p}}, A_{\mathfrak{p}}) = j(I_{\mathfrak{p}}, A_{\mathfrak{p}})$ *for all prime ideals* $\mathfrak{p} \in \operatorname{Spec} A$.

REMARK 11. Theorem 7 generalizes Böger's theorem (see Theorem 2), since if $\sqrt{I} = \sqrt{J}$ then the ideals I and J have the same minimal primes, that is $\mathrm{Min}(A/I) = \mathrm{Min}(A/J)$. Moreover $s(J) = \mathrm{height}(J)$ implies that

$$\{\mathfrak{p} \in \mathrm{Spec}\, A \mid \mathrm{height}(\mathfrak{p}) = s(J_\mathfrak{p})\} = \mathrm{Asymp}(J) = \mathrm{Min}(A/I)$$

by Lipman's Theorem 3, p. 116 and remark p. 117.

For each $\mathfrak{p} \in \mathrm{Min}(A/I)$ we have

$$j(J_\mathfrak{p}, A_\mathfrak{p}) = e(J_\mathfrak{p}, A_\mathfrak{p})$$

$$j(I_\mathfrak{p}, A_\mathfrak{p}) = e(I_\mathfrak{p}, A_\mathfrak{p})$$

hence by Corollary 1 of (2.4) and Rees' theorem (see Theorem 1) we can conclude that J is a reduction of I.

REMARK 12. The numerical condition of Theorem 7 must be checked only for a finite number of prime ideals, $\mathfrak{p} \in \mathrm{Spec}\, A$. In fact, by Remark 2 in (3.1) and Remark 1 in (2.4) the multiplicity $j(J_\mathfrak{p}, A_\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} \in \mathrm{Asymp}(J)$, hence it is sufficient to compare $j(J_\mathfrak{p}, A_\mathfrak{p})$ and $j(I_\mathfrak{p}, A_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Asymp}(J) \cup \mathrm{Asymp}(I)$ and this is a finite set.

REMARK 13. Theorem 7 says that $\overline{I}$ is the largest ideal N containing I such that $j(I_\mathfrak{p}, A_\mathfrak{p}) = j(N_\mathfrak{p}, A_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}\, A$, see (2.4).

The proof of Theorem 7 uses the generalized multiplicity $j(I, M)$ for a finite A-module M, which was introduced in [16], Section 6.1. This multiplicity coincides with $j(I, A)$ when $M = A$, has nice properties like additivity for exact sequences of A-modules (see [15], Lemma 3.1) and it is preserved under generic hyperplane sections (see [15], Proposition 3.2). These properties are used to prove Theorem 7.

In [10] C. Ciupercă generalizes Proposition 3 in the following way.

PROPOSITION 4 ([10], Proposition 2.7). *Let $(A, \mathfrak{m})$ be a d-dimensional local ring and let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of A. If J is a reduction of I, then J and I have the same generalized Samuel multiplicities, that is $c(J) = c(I)$.*

It would be interesting to have a converse of this proposition, which is known in the analytic case, see [19], Corollary 4.9 and our Thorem 11. This would avoid localization and would give a more useful numerical condition to test if an ideal J is a reduction of I by using computer algebra systems.

3.6. Generalized Hilbert coefficients and Serre's property (S_2) for the Rees algebra

Let $(A, \mathfrak{m})$ be a formally equidimensional local ring of dimension d, let I be an ideal in A, let $R = A[It, t^{-1}]$ be the extended Rees algebra of A with respect to I and let $\overline{R} = \oplus_{n\in\mathbf{Z}} \overline{I^n} t^n$ be the integral closure of R. Then Flenner-Manaresi's numerical characterization of reduction ideals of Theorem 7 can be interpreted as follows:

$\overline{I^n}$ is equal to the largest ideal, say K, containing I^n such that $j(I^n A_\mathfrak{p}, A_\mathfrak{p}) = j(K_\mathfrak{p}, A_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} A$.

Let R be a noetherian integral domain. Then the well-known normality criterion of Krull-Serre states that R is integrally closed if and only if it satisfies the following two properties:

(R_1) for each prime P of codimension ≤ 1, R_P is regular;

(S_2) for each prime P of codimension ≥ 2, depth $R_P \geq 2$.

If R is lacking the S_2-property of Serre, one can try to construct the minimal extension $\tilde{R}$ of R which satisfies S_2. The ring $\tilde{R}$ is called the *S_2-closure* or the *S_2-ification* of R and it exists when R has a canonical module or when R is a universally catenary, analytically unramified domain (see [22], (5.11.2) and [24], (2.7)). It is a step in the construction of the integral closure of R.

If $I \subset A$ is an $\mathfrak{m}$-primary ideal, then K. Shah [41] proved the existence of unique largest ideals (the so-called *k-th coefficient ideals*) $I_{\{k\}}$ ($1 \leq k \leq d$) lying between I and $\overline{I}$ such that the first $k+1$ Hilbert coefficients $e_0, \ldots, e_k$ (see Section 2.1) of I and $I_{\{k\}}$ coincide. If A is a formally equidimensional, analytically unramified local domain with infinite residue field and if A has positive dimension and is (S_2), then C. Ciupercă [9] showed that the n-th graded piece of the S_2-closure of $R = A[It, t^{-1}]$ is precisely the first coefficient ideal $(I^n)_{\{1\}}$, that is, the largest ideal $K \supseteq I^n$ such that $e_0(K) = e_0(I^n)$ and $e_1(K) = e_1(I^n)$.

Using the generalized Hilbert coefficients $a_{k,l}^{(1,1)}$ (see Section 3.1), Ciupercă [10] has generalized this to not necessarily $\mathfrak{m}$-primary ideals I. In order to describe Ciupercă's result, we assume that $d = \dim A > 0$ and introduce the following notation:

$$j_0(I) := j(I) = c_0(I) = a_{0,d}^{(1,1)}(I)\,,$$
$$j_1(I) := (c_1(I), a_{0,d-1}^{(1,1)}(I)) = (a_{1,d-1}^{(1,1)}(I), a_{0,d-1}^{(1,1)}(I))\,.$$

Note that in the case of an $\mathfrak{m}$-primary ideal I one gets the first two classical Hilbert coefficients: $j_0(I) = e_0(I) = e(I)$ and $j_1(I) = (0, -e_1(I))$. Ciupercă ([10], Def. 3.1) extended the definition of the *first coefficient ideal* $I_{\{1\}}$ to not necessarily $\mathfrak{m}$-primary ideals I as follows: if $\dim A/I < \dim A$, she defined

$$I_{\{1\}} := \bigcup (I^{n+1} :_A a)\,,$$

where the union ranges over all $n \geq 1$ and all $a \in I^n \setminus I^{n+1}$ such that the initial form a^* of a in $G_I(A)$ is a part of a system of parameters of $G_I(A)$. If I is $\mathfrak{m}$-primary, this definition coincides with the one given by Shah. Indeed, by the structure theorem for the coefficient ideals proved by Shah ([41], Theorem 2), we have $I_{\{1\}} = \bigcup (I^{n+1} :_A a)$, where the union ranges over all $n \geq 1$ and all $a \in I^n$ extendable to some minimal reduction of I^n. Note that a is extendable to some minimal reduction of I^n if and only if the image of a^* in $G_I(A)/\mathfrak{m}G_I(A)$ is part of a system of parameters. But if the ideal I is $\mathfrak{m}$-primary this is equivalent to the fact that a^* is part of a system of parameters of $G_I(A)$, since the ideal $\mathfrak{m}G_I(A)$ is nilpotent.

With this new definition of the first coefficient ideal $I_{\{1\}}$, the above description of the S_2-closure of the extended Rees algebra can be generalized to not necessarily $\mathfrak{m}$-primary ideals I.

THEOREM 8 ([10], Theorem 3.4). *Let $(A, \mathfrak{m})$ be a formally equidimensional, analytically unramified local domain with infinite residue field and positive dimension, and let I be an arbitrary ideal of A. If $\tilde{R} = \oplus_{n\in\mathbb{Z}} I_n t^n$ is the S_2-ification of $R = A[It, t^{-1}]$, then*

$$I_n \cap A = (I^n)_{\{1\}} \quad \textit{for all } n \geq 1\,.$$

In particular, if A is (S_2), then $I_n = (I^n)_{\{1\}}$ for all $n \geq 1$.

Now the announced numerical characterization of the S_2-ification of the extended Rees algebra reduces to the problem of finding a numerical characterization of the generalized first coefficient ideals. This is the contents of the following theorem.

THEOREM 9 ([10], Theorem 4.5). *Let $(A, \mathfrak{m})$ be a formally equidimensional local ring and let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of positive height. Then the following conditions are equivalent:*

1. $I \subseteq J_{\{1\}}$;
2. $j_0(JA_\mathfrak{p}) = j_0(IA_\mathfrak{p})$ *and* $j_1(JA_\mathfrak{p}) = j_1(IA_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec} A$;
3. $j_0(JA_\mathfrak{p}) = j_0(IA_\mathfrak{p})$ *and* $a^{(1,1)}_{0,d-1}(IA_\mathfrak{p}) = a^{(1,1)}_{0,d-1}(JA_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec} A$.

REMARK 14. Condition 3 of the previous theorem is not contained in [10], Theorem 4.5. Obviously 2 implies 3. To see the converse, one can observe that by Theorem 7 from $j_0(JA_\mathfrak{p}) = j_0(IA_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} A$ it follows that J is a reduction of I, hence $J_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$, therefore by Proposition 4 one has $c_1(JA_\mathfrak{p}) = c_1(IA_\mathfrak{p})$.

In view of the above consideration it seems to be better to define $j_1(I) := a^{(1,1)}_{0,d-1}(I)$ instead of $j_1(I) := (c_1(I), a^{(1,1)}_{0,d-1}(I))$.

The previous theorem can be considered as a generalization of Flenner-Manaresi's numerical characterization of reduction ideals (Theorem 7), which can be reformulated as follows.

THEOREM 10. *Let $(A, \mathfrak{m})$ be a formally equidimensional local ring and let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of A. Then the following conditions are equivalent:*

1. $I \subseteq J_{\{0\}} := \overline{J}$, *that is, J is a reduction of I;*
2. $j_0(IA_\mathfrak{p}) = j_0(JA_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec} A$;

3. $j_0(IA_\mathfrak{p}) = j_0(JA_\mathfrak{p})$ *for all ideals* $\mathfrak{p} \in \operatorname{Asymp}(I) \cup \operatorname{Asymp}(J)$.

PROBLEM 2. In the previous theorem only a finite number of localizations has to be considered, see Remark 12 in (3.5). It is not clear if it is enough to test also condition 2 of Theorem 9 only for a finite number of prime ideals $\mathfrak{p} \in \operatorname{Spec} A$.

3.7. Generalized Samuel multiplicities and Segre numbers

T. Gaffney and R. Gassler [19] introduced and studied the so-called Segre numbers of an ideal in the following set-up.

Let $(X, 0) \subseteq (\mathbb{C}^n, 0)$ denote a germ of an analytic subset of pure dimension d, and let I be an ideal in $\mathcal{O}_{X,0}$ which defines a nowhere dense subspace of $(X, 0)$. Choose a minimal set $h_1, \dots, h_r$ of generators of I. The *polar varieties* $P_k(I, X)$ and *Segre cycles* $\Lambda_{k+1}(I, X)$ are defined inductively for $k = 0, \dots, d-1$ as follows (see [19] and [20], Section 2.1): $P_0(I, X) := X$, and for $k \geq 1$ the polar variety $P_k(I, X)$ is defined to be the closure of $V(\tilde{h}_k|_{P_{k-1}(I,X)}) \setminus V(I)$, where $\tilde{h}_k$ is a generic linear combination of $h_1, \dots, h_r$. The word "generic" means in particular that the subspace Y of $P_{k-1}(I, X)$ defined by the sheaf of ideals $(\tilde{h}_k)\mathcal{O}_{P_{k-1}(I,X)}$ has to be reduced outside $V(I)$ in a sufficiently small neighbourhood of the point 0. The k-th *Segre cycle* is defined as the difference of cycles

$$\Lambda_k(I, X) := [V(\tilde{h}_k|_{P_{k-1}(I,X)})] - [P_k(I, X)]\,.$$

We recall that the cycle $[V(\tilde{h}_k|_{P_{k-1}(I,X)}]$ is defined as $\sum m_W[W]$, where the W's run over all irreducible components of the set $V(\tilde{h}_k|_{P_{k-1}(I,X)})$, and the integer m_W equals by definition the length of the local ring $(\mathcal{O}_{Y,y})_{(W_y)_i}$, where $(W_y)_i$ is the prime ideal of a component of the germ of the set W at a point $y \in W$ (see [30], p. 9).

The k-th Segre cycle $\Lambda_k(I, X)$ can also be described by using the blowup of X along $V(I)$. Let

$$X \times \mathbb{P}^{r-1} \supset Bl_I(X) \stackrel{b}{\to} X\,,$$

E the exceptional divisor and $H_1, \dots, H_{k-1}$ generic hyperplanes on $Bl_I(X)$ induced by generic hyperplanes of $\mathbb{P}^{r-1}$. Then

$$\Lambda_k(I, X) = b_*(H_1 \cdots H_{k-1} \cdot E \cdot Bl_I X)\,.$$

The k-th *Segre number* is defined as

$$e_k(I, X) := \operatorname{mult}_0(\Lambda_k(I, X)) := e(\mathcal{O}_{\Lambda_k(I,X),0}), \qquad k = 1, \dots, d.$$

If s denotes the analytic spread of I, then one can easily see that the sequence $\underline{\tilde{h}} := (\tilde{h}_1, \dots, \tilde{h}_s)$ is filter-regular with respect to I, hence one can perform the intersection algorithm for the sequence $\underline{\tilde{h}}$ as in Section 3.6. By the definition of Segre cycles, one has

$$\begin{array}{rcll} \operatorname{mult}_0(\Lambda_k(I, X)) & = & \deg v^k(\underline{\tilde{h}}, \mathcal{O}_{X,0}) & \text{for } k = 1, \dots, s, \text{ and} \\ \operatorname{mult}_0(\Lambda_k(I, X)) & = & 0 & \text{for } k = s+1, \dots, d \end{array}$$

(see [6], proof of Theorem 1).

THEOREM 11 ([6], Theorem 2). *With the previous notation, the following equalities hold:*

$$e_k(I, X) = c_{d-k}(I, \mathcal{O}_{X,0}) \qquad \text{for } k = 1, \dots, d$$

and $c_d(I, \mathcal{O}_{X,0}) = 0$.

The generalized Samuel multiplicities are also related to the degrees of *Segre classes of cones and subvarieties* $Y \subset X$. For the theory of Segre classes we refer the reader to [18], Chapter 4.

The *total Segre class* $s(Y, X) \in A_*Y$ is defined as follows: if $Y = X$ then $s(Y, X) = [X]$, otherwise let $\tilde{X} = Bl_Y X$, $C := C_Y X$ the normal cone of X along Y, $E = \mathbb{P}(C)$ the exceptional divisor, $\eta\colon E \to Y$ the projection, and $d := \dim X = \dim \tilde{X}$. The i-fold self intersections $E^i = E * \cdots * E$ are well defined classes in $A_{d-i}(E)$ and one defines

$$s(Y, X) := \sum_{i \geq 1} (-1)^{i-1} \eta_*(E^i)\,.$$

This means that the total Segre class is constructed by blowing up X along Y, and pushing down various self-intersections of the exceptional divisor. It depends only on the normal cone to Y in X. One writes

$$s(Y, X) = s(C_Y X) := \sum_{i \geq 0} s_i(Y, X) = \sum_{i \geq 0} s^i(Y, X)\,,$$

where by s_i one denotes the part of s of dimension i, and by s^i the part of codimension i in X. Thus, if X is equidimensional (as we always assume), then $s_i = s^{\dim X - i}$.

If X and Y are nonsingular, then the normal cone is a bundle, the normal bundle $N_Y X$ of X along Y, with Chern classes $c_i = c_i(N_Y X)$, and the Segre classes $s_i(N_Y X)$ can be regarded as their formal inverse:

$$1 + c_1 + c_2 \cdots = (1 + s^1 + s^2 + \cdots)^{-1}\,,$$

i. e., $s^0 = 1$, $s^1 = -c_1$, $s_2 = c_1^2 - c_2$, ...

Turning back to the general case, that is, X and Y not necessarily nonsingular, for $E = \mathbb{P}(C)$ one has that $N_E \tilde{X} = \mathcal{O}_{\tilde{X}}(E)|_E = \mathcal{O}_C(-1)$ is the dual of the canonical line bundle $\mathcal{O}_C$ on $\mathbb{P}(C)$. It follows that

$$E^i = (-1)^{i-1} c_1((\mathcal{O}_C(1))^{i-1}) \cap [\mathbb{P}(C)]\,,$$

hence

$$s = s(C) = s(Y, X) = \sum_{i \geq 1} \eta_*(c_1(\mathcal{O}_C(1))^{i-1} \cap [\mathbb{P}(C)])\,.$$

If $Y \subset X \subseteq \mathbb{P}^n$ is an irreducible and reduced subscheme of X and $r := \operatorname{codim}_X Y > 0$, $q := \dim(X) + 1 - r$, then the degree of the Segre class $s^r(Y, X) =$

$s^r(C_Y X)$ is related to the Samuel multiplicity $e(\mathcal{O}_{X,Y}) = e_Y X$ of X along Y as follows (see [18], 4.3):

$$\begin{aligned} e(\mathcal{O}_{X,Y})[Y] = e_Y X[Y] &= s^r(Y,X) = \eta_*(c_1(\mathcal{O}_C(1))^{r-1} \cap [\mathbb{P}(C)]) \\ &= (-1)^{r-1}\eta_*(E^{r-1}), \end{aligned}$$

that is,

$$\deg s^r(C_Y X) = \deg s_{q-1}(C_Y X) = e(\mathcal{O}_{X,Y}) \cdot \deg Y = c_q(I, A)\,,$$

where A is the homogeneous ring of coordinates of X localized at the irrelevant maximal ideal and I is the ideal of Y in A.

In general, with the convention that $\binom{m}{-1} := 0$ for $m \geq 0$ and $\binom{-1}{-1} := 1$, one has the following proposition, which gives the relation between generalized Samuel multiplicities and degrees of Segre classes of cones.

PROPOSITION 5 ([4], Corollary 4.3). *Under the hypothesis of Proposition* 2, if $Z = \emptyset$ then $d = \dim A = \dim J + 1$, $q = \dim(J \cap \Delta) + 1$ and, for $k = -1, \ldots, d-1$,

$$c_{k+1}(I, A) = \sum_{i=k}^{q-1} \binom{d-k-2}{d-i-2} \deg s_i(C_{J\cap\Delta} J)$$

and

$$\deg s^k(C_{J\cap\Delta} J) = \deg s_{d-k-1}(C_{J\cap\Delta} J) = \sum_{i=0}^{k} \binom{k-1}{i-1} (-1)^{k-i} c_{d-i}(I, A)\,.$$

REMARK 15. More general, if X is an equidimensional algebraic scheme over the base field K, $\mathcal{L}$ a line bundle of degree δ on X, $\sigma_1, \ldots, \sigma_t \in H^0(X, \mathcal{L})$ and $Y := V(\sigma_1) \cap \cdots \cap V(\sigma_t)$, then

$$c_{k+1} = \sum_{i=k}^{q-1} \binom{d-k-2}{d-i-2} \delta^{i-k} \deg s_i(Y, X) \tag{3}$$

and

$$\deg s^k(Y, X) = \deg s_{d-k-1}(Y, X) = \sum_{i=0}^{k} \binom{k-1}{i-1} (-\delta)^{k-i} c_{d-i}, \tag{4}$$

$k = 0, \ldots, d-1$.

We want to illustrate the usefulness of the generalized Samuel multiplicities for the calculation of the degrees of Segre classes using computer algebra systems, discussing an example which can be easily checked by hand. The same method can be applied to much more complicated examples which cannot be calculated by hand.

EXAMPLE 5 ([11], Example 5.3). Let us consider the flat family X in $\mathbb{A}^3$ defined over the affine line T by the ideal $(x, z) \cap (y, z) \cap (x - y, z - tx)$. Note that X is the union of three lines X_1, X_2 and X_3 passing through the origin P (and lying in a plane if $t = 0$). Our aim is to calculate the degrees of the Segre classes of X diagonally embedded in $X \times X$.

The normal cone $C_X(X \times X) = \operatorname{Spec} K[x, y, z, u, v, w]/J$ can be calculated by a computer, getting

$$J = (t^2xy - z^2, z(tx - z), z(ty - z), tyw + tzv - 2zw, txw + tzu - 2zw,\\ t^2xv + t^2yu - 2zw, w(t^2uv - tuw - tvw + w^2)) .$$

Observe that J is a bigraded ideal with respect to the variables x, y, z and u, v, w respectively. Let A be the ring of coordinates of $X \times X$ localized at (x, y, z, u, v, w) and $I = (x - u, y - v, z - w)A$ the ideal of the diagonal in A. Then the bidegrees of J are the generalized Samuel multiplicities $c(I, A)$ of I in A and, by a computer calculation, $c(I, A) = (6, 3, 0, 0)$. By the previous proposition one gets the degrees of the Segre classes:

$$\deg s_0(X, X \times X) = 0, \qquad \deg s_1(X, X \times X) = 3 = \deg X.$$

The same results hold if $t = 0$.

We observe that, since X is the union of three lines, using the bilinearity of the intersection cycle $v(X, X)$ (see for example [16], Section 2.1), we have $v(X, X) = [X_1] + [X_2] + [X_3] + 6[P]$ (which holds also in the case $t = 0$). From this it follows immediately that $c(I, A) = (6, 3, 0, 0)$ and hence one obtains the degrees of the Segre classes as above.

3.8. Generalized Samuel multiplicities and Whitney stratifications

We recall the following definitions.

DEFINITION 3. *Let $X \subseteq \mathbb{P}^n$ be a d-dimensional complex projective variety, and let $Y \subset X$ be a non-singular subvariety. We say that the pair (X_{reg}, Y) satisfies the* Whitney conditions *at a point $x_0 \in Y$ if for each sequence (x_i) of points of X_{reg} and each sequence (y_i) of points of Y both converging to x_0 and such that the limits $\lim_{x_i \to x_0} T_{x_i} X$ and $\lim_{x_i, y_i \to x_0} \overline{x_i y_i}$ exist in the Grassmannians $G(d, n)$ and $G(1, n)$ respectively, one has:*

$$\text{(a)} \quad \lim_{x_i \to x_0} T_{x_i} X \supset T_{x_0} Y ,$$
$$\text{(b)} \quad \lim_{x_i \to x_0} T_{x_i} X \supset \lim_{x_i, y_i \to x_0} \overline{x_i y_i} .$$

We remark that (b) implies (a).

DEFINITION 4. *A* Whitney stratification *of X ($d = \dim X$) is given by a filtration of X by closed subsets F_i*

$$X = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{d+1} = \emptyset$$

such that

(i) $F_i \setminus F_{i+1}$ *is either empty or is a non-singular quasi-projective variety of pure codimension i (the connected components of* $F_i \setminus F_{i+1}$ *are called the* strata *of the stratification);*

(ii) *whenever* S_j *and* S_k *are connected components of* $F_i \setminus F_{i+1}$ *and* $F_l \setminus F_{l+1}$ *respectively, with* $S_j \subset \overline{S}_k$*, then the pair* (S_k, S_j) *satisfies the Whitney conditions (a) and (b).*

DEFINITION 5 (Polar varieties). *Let* $L_{(k)}$ *be an* $(n-d+k-2)$*-dimensional linear subspace of* $\mathbb{P}^n$, $1 \leq k \leq d = \dim X$. *The* k-th polar variety *(or* polar locus*) of* X *associated with* $L_{(k)}$ *is*

$$P(L_{(k)}, X) := \textit{closure of } \{x \in X_{reg} \mid \dim(T_x X \cap L_{(k)}) \geq k-1\}.$$

For $k = 0$ *we set* $P(L_{(0)}, X) := X$.

If $L_{(k)}$ is generic, we write $P_k(X) = P(L_{(k)}, X)$ since it is well known that $P(L_{(k)}, X)$ is empty or equidimensional of codimension k in X and its degree does not depend on $L_{(k)}$. If

$$L_{(0)} \subset L_{(1)} \subset \ldots \subset L_{(d)}$$

is a generic flag, then we have

$$X = P_0(X) \supset P_1(X) \supset \ldots \supset P_d(X)\,.$$

The polar varieties defined here are different from the polar varieties of Gaffney-Gassler defined in Section 3.7.

Let $x \in X$. Teissier showed that the sequence of multiplicities

$$m_0 = e_x(P_0(X)), \ldots, m_{d-1} = e_x(P_{d-1}(X))$$

does not depend upon the choice of the general flag. Moreover he proved the following result.

THEOREM 12 (Teissier [45]). *The pair* (X_{reg}, Y) *satisfies the Whitney conditions in* x_0 *if and only if the sequence of polar multiplicities*

$$m_0 = e_y(X), m_1 = e_y(P_1(X)), \ldots, m_{d-1} = e_y(P_{d-1}(X))$$

is locally constant in Y *around* x_0.

DEFINITION 6 (The stratifying function g). *Let* $X \subseteq \mathbb{P}^n$ *be a* d*-dimensional complex projective variety and let* x *a point of* X.

Let $A := \mathcal{O}_{X \times X, (x,x)}$ *and let* I *be the diagonal ideal in* A. *We define*

$$g(x) := e(G_I(A)) = \sum_{i=0}^{d} c_i(I, A)\,.$$

Note that $\dim A = 2d$, that $c_{d+1} = \cdots = c_{2d} = 0$ and that

$$(c_0(I, A), c_1(I, A), \ldots, c_d(I, A))$$

is a refinement of the multiplicity $c_d(I, A) = e_x X = e(\mathcal{O}_{X,x})$ of X at x.

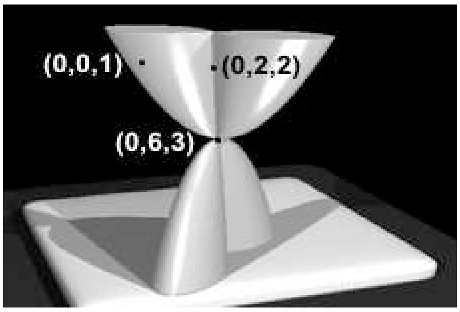

Figure 1: The g-stratification of the surface $x^4 + y^4 = xyz$.

THEOREM 13 ([5], Theorem 4.2). *Let $X \subset \mathbb{P}^n$ be a (reduced) surface and $x \in X$ be a closed point. Then*

$$X_j := \{x \in X \mid g(x) \geq j\}, \quad j = 0, 1, \ldots$$

are closed subschemes of X or empty, and the connected components of

$$S_g(j) := g^{-1}(j) = X_j \setminus X_{j+1}$$

are the strata of a Whitney stratification of X (the coarsest one if $n = 3$).

EXAMPLE 6. Consider the surface X in $\mathbb{C}^3$ (or in $\mathbb{P}^3$) defined by the equation $x^4 + y^4 - xyz = 0$, whose singular locus is the z-axis (see Figure 1). We want to determine the coarsest Whitney stratification. Using [1] we obtain for the generalized Samuel multiplicities (c_2, c_1, c_0) and the polar multiplicities (m_0, m_1) (both ordered by codimension) the following values:

Locus	$(\mathbf{c_2}, \mathbf{c_1}, \mathbf{c_0})$	**g**	$(\mathbf{m_0}, \mathbf{m_1})$
$X \setminus \operatorname{Sing} X$	$(1, 0, 0)$	1	$(1, 0)$
z-axis	$(2, 2, 0)$	4	$(2, 0)$
origin	$(3, 6, 0)$	9	$(3, 4)$

Hence the Whitney stratification is given by

$$\text{surface} \supset z\text{-axis} \supset \text{origin}.$$

References

[1] ACHILLES R. AND ALIFFI D., Segre: a script for the REDUCE package CALI (Bologna, 1999-2001), http://www.dm.unibo.it/∼achilles/segre/.

[2] ACHILLES R. AND MANARESI M., *An algebraic characterization of distinguished varieties of intersection*, Rev. Roumaine Math. Pures Appl. **38** (1993), 569–578.

[3] ACHILLES R. AND MANARESI M., *Multiplicity for ideals of maximal analytic spread and intersection theory*, J. Math. Kyoto Univ. **33** (1993), 1029–1046.

[4] ACHILLES R. AND MANARESI M., *Multiplicities of a bigraded ring and intersection theory*, Math. Ann. **309** (1997), 573–591.

[5] ACHILLES R. AND MANARESI M., *Self-intersections of surfaces and Whitney stratifications*, Proc. Edinb. Math. Soc. **46** 2 (2003), 545–559.

[6] ACHILLES R. AND RAMS S., *Intersection numbers, Segre numbers and generalized Samuel multiplicities*, Arch. Math. **77** (2001), 391–398.

[7] BODA E. AND VOGEL W., *On system of parameters, local intersection multiplicity and Bézout's theorem*, Proc. Amer. Math. Soc. **78** (1980), 1–7.

[8] BÖGER E., *Einige Bemerkungen zur Theorie der ganz-algebraischen Abhängigkeit von Idealen*, Math. Ann. **185** (1970), 303–308.

[9] CIUPERCĂ C., *First coefficient ideals and the S_2-ification of a Rees algebra*, J. Algebra **242** (2001), 782–794.

[10] CIUPERCĂ C., *A numerical characterization of the S_2-ification of a Rees algebra*, J. Pure Appl. Algebra **178** (2003), 25–48.

[11] COLLEY S.J. AND KENNEDY G., *Detecting flat normal cones using Segre classes*, J. Algebra **245** (2001),182–192.

[12] DADE E.C., *Multiplicity and monoidal transformations*, Ph.D. Thesis, Princeton University, Princeton 1960.

[13] FLENNER H., VAN GASTEL L.J. AND VOGEL W., *Joins and intersections*, Math. Ann. **291** (1991), 691–704.

[14] FLENNER H. AND MANARESI M., *A numerical characterization of reduction ideals*, Math. Z. **238** (2001), 205–214.

[15] FLENNER H. AND MANARESI M., *A length formula for the multiplicity of distinguished components of intersections*, J. Pure Appl. Algebra **165** (2001), 155–168.

[16] FLENNER H., O'CARROLL L. AND VOGEL W., *Joins and intersections*, Springer Monographs in Mathematics, Springer-Verlag, Berlin–Heidelberg–New York 1999.

[17] FLENNER H. AND VOGEL W., *Limits of joins and intersections*, in: "Higher Dimensional Complex Varieties, Trento 1994", Walter de Gruyter, Berlin–New York 1996, 209–220.

[18] FULTON W., *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 2, Springer-Verlag, Berlin–Heidelberg–New York, 1984.

[19] GAFFNEY T. AND GASSLER R., *Segre numbers and hypersurface singularities*, J. Algebraic Geom. **8** (1999), 695–736.

[20] GASSLER R., *Mixed Segre numbers and integral closures of ideals*, preprint 1997, http://xxx.lanl.gov/form/math.AG, alg-geom/9703018.

[21] VAN GASTEL L.J., *Excess intersections and a correspondence principle*, Invent. Math. **103** (1991), 197–222.

[22] GROTHENDIECK A., *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32**, 1967.

[23] HERRMANN M., IKEDA S. AND ORBANZ U., *Equimultiplicity and blowing up. An algebraic study. (With an appendix by B. Moonen)*, Springer-Verlag, Berlin 1988.

[24] HOCHSTER M. AND HUNEKE C., *Indecomposable canonical modules and connectedness*, in: "Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)", Contemp. Math. **159**, Amer. Math. Soc., Providence, RI 1994, 197–208.

[25] HUNEKE C., *Strongly Cohen-Macaulay schemes and residual intersections*, Trans. Amer. Math. Soc. **277** (1983), 739–763.

[26] HUNEKE C. AND ULRICH B., *Residual intersections*, J. Reine Angew. Math. **390** (1988), 1–20.

[27] IDÀ M. AND MANARESI M., *Some remarks on normal flatness and multiplicity in complex spaces*, in: "Commutative algebra, Trento 1981", Lecture Notes in Pure and Appl. Math. **84**, Dekker, New York 1983, 171–182.

[28] LIPMAN J., *Equimultiplicity, reduction, and blowing up*, in: "Commutative algebra, Fairfax Va. 1979", Lecture Notes in Pure and Appl. Math. **68**, Dekker, New York 1982, 111–147.

[29] MACAULAY F.S., *The algebraic theory of modular sysytems*, Cambridge University Press, Cambridge 1916.

[30] MASSEY D.B., *Lê cycles and hypersurface singularities*, Lecture Notes Math. **1615**, Springer-Verlag, Berlin-Heidelberg 1995.

[31] MCADAM S., *Asymptotic prime divisors*, Lecture Notes in Math. **1023**, Springer-Verlag Berlin 1983.

[32] NAGATA M., *Local rings*, Interscience Tracts in Pure and Appl. Math. **13**, Wiley, New York 1962.

[33] NOH S. AND VASCONCELOS W.V., *The S_2-closure of a Rees algebra*, Results Math. **23** (1993), 149–162.

[34] NORTHCOTT D.G. AND REES D., *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158.

[35] NOWAK K., *Analytic improper intersections. I. Deformation to the normal cone*, Bull. Polish Acad. Sci. Math. **48** (2000), 121–130.

[36] NOWAK K., *Analytic improper intersections. II. Deformation to an algebraic bicone and applications*, Bull. Polish Acad. Sci. Math. **48** (2000), 131–140.

[37] O'CARROLL L. AND PRUSCHKE T., *Deformation to the normal cone, rationality and the Stückrad-Vogel cycle*, Beiträge Algebra Geom. **40** (1999), 229–242.

[38] REES D., *$\mathfrak{a}$-transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8–17.

[39] SAMUEL P., *La notion de multiplicité en algèbre et en géométrie algébrique*, J. Math. Pures Appl. **30** (1951), 159–274.

[40] SAMUEL P., *Méthodes d'algèbre abstraite en géométrie algébrique*, Seconde édition, corrigée, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 4, Springer-Verlag, Berlin-New York 1967.

[41] SHAH K., *Coefficient ideals*, Trans. Amer. Math. Soc. **327** (1991), 373–384.

[42] STÜCKRAD J. AND VOGEL W., *An algebraic approach to the intersection theory*, in: "The Curves Seminar at Queen's, Vol. II", Queen's Papers in Pure and Appl. Math. 61, Kingston, Ont. 1982, 1–32.

[43] STÜCKRAD J. AND VOGEL W., *Buchsbaum Rings and Applications*, Springer-Verlag, Berlin–Heidelberg–New York 1986.

[44] STÜCKRAD J. AND VOGEL W., *An Euler-Poincaré characteristic for improper intersections*, Math. Ann. **274** (1986), 257–271.

[45] TEISSIER B., *Variétés polaires II. Multiplicités polaires, sections planes, et conditions de Whitney*, in: "Algebraic Geometry, La Rábida 1981", (Eds. J. M. Aroca, R. Buchweitz, M. Giusti and M. Merle), Lecture Notes in Math. **961**, Springer-Verlag, Berlin 1982, 314–491.

[46] TRUNG N.V., *Positivity of mixed multiplicities*, Math. Ann. **319** (2001), 33–63.

[47] TWORZEWSKI P., *Intersection theory in complex analytic geometry*, Ann. Polon. Math. **62** (1995), 177–191.

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