Computation of Segre numbers: an application to Whitney stratifications

Rüdiger Achilles (Bologna, Italy)

achilles@dm.unibo.it
http://www.dm.unibo.it/~achilles

Commutative and Non-Commutative Algebraic Geometry

Chisinau, June 6–11, 2004

with Mirella Manaresi (Bologna) and Davide Aliffi (Bologna)

1. R. Achilles and D. Aliffi, Segre: a script for the REDUCE package CALI. Bologna, 1999-2001. Available at `http://www.dm.unibo.it/~achilles/segre/`.

2. R. Achilles and M. Manaresi, Self-intersections of surfaces and Whitney stratifications. *Proc. Edinburgh Math. Soc.*(2) **46** (2003), 545–559.

1. Stückrad-Vogel intersection cycle

Stückrad - Vogel, 1982; Flenner - O'Carroll - Vogel, Joins and Intersections, Springer, 1999.

2. Generalized Samuel multiplicities and Segre numbers

Achilles - Manaresi, *Math. Ann.* **309** (1997), 573–591; Gaffney - Gassler, *J. Algeb. Geom.* **8** (1999), 695–736.

3. Polar multiplicities and Whitney stratifications

Teissier, SLN **961** (1982), 314–491.

4. Result and examples

Stückrad-Vogel intersection cycle

$X, Y \subset \mathbf{P}^n$, equidimensional

$$v(X,Y) = v_0 + v_1 + \dots$$

if $v_k \neq 0$ then v_k is of dimension k $(k = 0, 1, \dots)$

Example 1

$$\begin{aligned} X &: x^2 - yz = 0 && \textit{cone} \\ Y &: x = y = 0 && \textit{line on the cone} \end{aligned}$$

cut out Y by generic planes

$$\begin{aligned} u_{11}x + u_{12}y &= 0, \\ u_{21}x + u_{22}y &= 0; \end{aligned}$$

and intersect X step by step with these generic planes components lying in $X \cap Y$ are collected in the cycle $v(X, Y)$ and the rest is intersected with the next generic plane:

First step

$$(x^2 - yz, u_{11}x + u_{12}y) = (x, y) \cap (u_{12}x + u_{11}z, u_{11}x + u_{12}y)$$

Second step

$$(u_{12}x + u_{11}z, u_{11}x + u_{12}y, u_{21}x + u_{22}y) = (x, y, z).$$

$\Rightarrow v(X, Y) = Y + O.$

Example 2 *In* $\mathbb{P}^3_K$

$$X \;:\; xy - zt = 0 \qquad \textit{non-singular quadric}$$
$$Y \;:\; x = z = 0 \qquad \textit{line on it}$$

First step

$$(xy - zt, u_{11}x + u_{12}z) = (x, z) \cap (u_{12}y + u_{11}t, u_{11}x + u_{12}z)$$

Second step

$$(u_{12}y + u_{11}t, u_{11}x + u_{12}z, u_{21}x + u_{22}z) = (x, z, u_{12}y + u_{11}t).$$

$\Rightarrow v(X, Y) = Y + P$, *where* P *is a non* K*-rational (or a movable) point*

Theorem 1 (Flenner-Manaresi, 1997) $X, Y \subset \mathrm{P}^n_K$ *varieties,* $e := \dim X + \dim Y - n$. *Let* $k \in \mathbf{Z}$ *such that* $e - 1 \leq k \leq \dim X \cap Y - 1$, $L := K(\mathbf{u})$,

$$p\colon X_L \cap Y_L \to \mathbf{P}_L^{n+e-k-1}$$

the generic linear projection and $R(p) :=$ *ramification locus of* p. *Then* $\dim R(p) \leq k$, *and the associated* k*-cycle* $[R(p)]_k$ *is just* $v_k(X, Y)$ *on* $\{x \in X \cap Y \mid X, Y, X \cap Y$ *smooth at* $x\}$.

Definition 1

$X, Y \subseteq Z$ closed subschemes of an algebraic K-scheme X, $f\colon Z \to N$ a morphism to an algebraic manifold with $\dim N \geq \dim X + \dim Y - \dim X \cap Y$. Then $R(f)$ is defined to be the degeneracy locus of

$$f^*(\Omega^1_N) \otimes \mathcal{O}_{X\cap Y} \to \Omega^1_{X\cup Y} \otimes \mathcal{O}_{X\cap Y}.$$

Generalized Samuel multiplicities

P. Samuel, 1951

$(A,\mathfrak{m})$ noetherian local ring I of dimension d an $\mathfrak{m}$-primary ideal (that is, an ideal of finite colength)

$$H_I^{(0)}(j) := \text{length}\,(I^j/I^{j+1})$$

Hilbert-Samuel function

$$\begin{aligned} H_I^{(1)}(j) \;&:=\; \sum_{k=0}^{j} H_I^{(0)}(j) = \text{length}\,(A/I^{j+1}) \quad \text{if } j \gg 1, \\ &=\; e_0\binom{j+d}{d} - e_1\binom{j+d-1}{d-1} + \ldots + (-1)^d e_d \end{aligned}$$

write $e(I,A)$ for e_0.

Achilles-Manaresi, 1997

I not necessarily $\mathfrak{m}$-primary
pass from the associated graded ring

$$G_I(A) := \oplus_{j\geq 0} I^j/I^{j+1}$$

to the bigraded ring

$$\begin{aligned} R &= \oplus_{i,j\geq 0} R_{ij} = G^i_{\mathfrak{m}}(G^j_I(A)) \\ &= \oplus_{i,j\geq 0} (\mathfrak{m}^i I^j + I^{j+1})/(\mathfrak{m}^{i+1} I^j + I^{j+1}) \end{aligned}$$

$R_{00} = A/\mathfrak{m}$, have Hilbert function $H^{(0,0)}(i,j) = \dim R_{ij}$, twofold sum transform $H^{(1,1)}(i,j) := \sum_{q=0}^{j} \sum_{p=0}^{i} H^{(0,0)}(p,q)$, for both $i,j \gg 1$ becomes a polynomial which can be written in the form

$$\sum_{k+l\leq d} a^{(1,1)}_{k,l} \binom{i+k}{k} \binom{j+l}{l},$$

Define the generalized Samuel multiplicity to be

$$(a_{0,d}^{(1,1)}, a_{1,d-1}^{(1,1)}, \dots, a_{d,0}^{(1,1)}) =: (c_0, c_1, \dots, c_d).$$

Theorem 2 *Set* $q := \dim(A/I)$, $G := G_I(A)$, $s := \dim G/\mathfrak{m}G$. *Then*

1. $c_k = 0$ *for* $k < d - s$ *and* $k > q$;
2. $c_{d-s} = \sum_P e(\mathfrak{m}G_P) \cdot e(G/P)$, *where* P *runs over all highest dimensional associated prime ideals of* $G/\mathfrak{m}G$ *such that* $\dim G/P + \dim G_P = \dim G$;

3. $c_q = \sum_{\mathfrak{p}} e(A_{\mathfrak{p}}) \cdot e(A/\mathfrak{p})$, *where* $\mathfrak{p}$ *runs over all highest dimensional associated prime ideals of* $A/\mathfrak{p}$ *such that* $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$;

4. *If* A *is the ring of the ruled join* $J := J(X, Y) \subset \mathbf{P}^{2n+1}$ *(localized at the irrelevant homogeneous maximal ideal),* I *the ideal of the 'diagonal'* $\Delta \subset \mathbf{P}^{2n+1}$ *and* $\pi\colon J \to XY$ *the canonical projection onto the embedded join* $XY \subset \mathbf{P}^n$*, then*

$$\begin{aligned} c_0 &= \deg \sum_{\Gamma \subseteq J} \deg(\Gamma/XY) \deg(\pi(\Gamma)), \\ c_k &= \deg v_{k-1}, \quad k = 1, \ldots, d-1 = \dim J, \\ c_d &= 0. \end{aligned}$$

Here the sum is taken over all irreducible components Γ of J with the induced scheme structure and $\pi(\Gamma)$ denotes the closure of $\pi(\Gamma \setminus \Delta) \subseteq \mathbf{P}^n_K$ equipped with its reduced structure.
If K is algebraically closed and X, Y are irreducible (but not necessarily reduced), then

$$c_0 = \deg(J/XY)\deg(XY).$$

Segre numbers of Gaffney and Gassler, 1999

$A = \mathcal{O}_{X,0}$, $(X,0) \subset (\mathbf{C}^n, 0)$ a reduced closed analytic space of pure dimension d and I an ideal which defines a nowhere dense subspace of $(X,0)$. They considered the blowup of X along I

$$X \times \mathbf{P}^{\mu(I)-1} \supset Bl_I(X) \xrightarrow{b} X$$

with exceptional divisor E and defined the kth Segre number as

$$e_k(I,Y) := \mathrm{mult}_0(b_*(H_1 \cdots H_{k-1} \cdot E \cdot Bl_I X)),$$

where H_i is a generic hyperplane on $Bl_I X$ induced by one in $\mathbf{P}^{\mu(I)-1}$. Note that

$$e_k(I,Y) = c_{d-k}(A,I), \quad k = 1, \ldots, d,$$

that is, Segre numbers are a special case of the generalized Samuel multiplicity.

Our generalized Samuel multiplicities are also related to the degrees of Segre classes of cones and subvarieties:
$W \subset V$; let W, V be nonsingular, $N_W V$ the normal bundle with Chern classes $c_i(N_W V) \cap [W]$ and Segre classes $s_i(N_W V) \cap [W]$,

$$1 + c_1 + c_2 \cdots = (1 + s_1 + s_2 + \cdots)^{-1}.$$

In general there is only a normal cone, $C = C_W V$ of V along W. The total Segre class $s(W, V) \in A_* W$ is defined as follows: if $W = V$ then $s(W, V) = [V]$. Otherwise $\tilde{V} = Bl_W V$, $E = \mathbf{P}(C)$ exceptional divisor, $\eta\colon E \to W$ projection, $d := \dim V = \dim \tilde{V}$. The i-fold self intersections $E^i = E * \ldots * E$ are well defined classes in $A_{d-i}(E)$.

$$s(W, V) := \sum_{i \geq 1} (-1)^{i-1} \eta_*(E^i)$$

(B. Segre, 1953, sottovarietà covarianti)

Now $N_E\tilde{V} = \mathcal{O}_{\tilde{V}}(E)|_{E=\mathbf{P}(C)} = \mathcal{O}_C(-1)$ is the dual of the canonical line bundle $\mathcal{O}_C$ on $\mathbf{P}(C)$. It follows

$E^i = (-1)^{i-1}c_1((\mathcal{O}_C(1))^{i-1}) \cap [\mathbf{P}(C)]$, hence

$$s = s(C) = s(W,V) = \sum_{i\geq 0} \eta_*(c_1(\mathcal{O}_C(1))^{i-1} \cap [\mathbf{P}(C)])$$

which makes sense for every cone C on a scheme W under the assumption that there is no irreducible component C' of C with $\mathbf{P}(C') = \emptyset$.

By s_i we denote the part of s of dimension i, and by s^i the part of codimension i in V. Thus, if V is equidimensional (which we assume), then $s_i = s^{d-i}$.

Connection with Samuel multiplicity:

$W \subset V$ irreducible and reduced, $r := \text{codim}(W, V) > 0$, $q = d - r$.

$$\begin{aligned} e(\mathcal{O}_{V,W}) = e_W V[W] &= \eta_*(c_1(\mathcal{O}_C(1))^{r-1} \cap [\mathbf{P}(C)]) \\ &= (-1)^{q-1} \eta_*(E^{r-1}), \end{aligned}$$

that is, $\deg s^r(C_W V) = \deg s_q(C_W V) = c_q$. In the situation of Theorem 2, (4):

$$c_k = \sum_{i=k}^{q} \binom{d-k-1}{d-i-1} \deg s_i(C_{\Delta \cap J} J)$$

and

$$\deg s^k = \deg s_{d-k} = \sum_{i=0}^{k} \binom{k-1}{i-1} (-1)^{k-i} c_{d-i},$$

$k = 0, \ldots, d-1$. Convention: $\binom{m}{-1} := 0$ for $m \geq 0$, $\binom{-1}{-1} := 1$.

More general, if V is an equidimensional algebraic scheme over the base field K, $\mathcal{L}$ a line bundle of degree δ on V, $\sigma_1, \ldots \sigma_t \in H^0(V, \mathcal{L})$ and $W := V(\sigma_1) \cap \ldots \cap V(\sigma_t)$, then

$$c_k = \sum_{i=k}^{q} \binom{d-k-1}{d-i-1} \delta^{k-i} \deg s_i(C_{\Delta \cap J} J)$$

and

$$\deg s^k = \deg s_{d-k} = \sum_{i=0}^{k} \binom{k-1}{i-1} (-\delta)^{k-i} c_{d-i},$$

$k = 0, \ldots, d-1$.

Definition 2 $Y \subset X$ *complex projective varieties,* Y *non-singular.* (X_{reg}, Y) *satisfies the Whitney conditions at a point* $x_0 \in Y$ *if for each sequence* (x_i) *of points of* X_{reg} *and each sequence* (y_i) *of points of* Y *both converging to* x_0 *and such that the limits* $\lim_{x_i \to x_0} T_{x_i} X$ *and* $\lim_{x_i, y_i \to x_0} \overline{x_i y_i}$ *exist in the Grassmannians* $G(d, n)$ *and* $G(1, n)$ *respectively, one has:*

$$\text{(a)} \qquad \lim_{x_i \to x_0} T_{x_i} X \supset T_{x_0} Y \,,$$

$$\text{(b)} \qquad \lim_{x_i \to x_0} T_{x_i} X \supset \lim_{x_i, y_i \to x_0} \overline{x_i y_i} \,.$$

We remark that (b) implies (a).

Definition 3 *A Whitney stratification of X ($d = \dim X$) is given by a filtration of X by algebraic sets F_i*

$$X = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{d+1} = \emptyset$$

such that

(i) $F_i \setminus F_{i+1}$ is either empty or is a non-singular quasi-projective variety of pure codimension i. (The connected components of $F_i \setminus F_{i+1}$ are called the strata of the stratification.)

(ii) Whenever S_j and S_k are connected components of $F_i \setminus F_{i+1}$ and $F_l \setminus F_{l+1}$ respectively with $S_j \subset \overline{S}_k$, then the pair (S_k, S_j) satisfies the Whitney conditions (a) and (b).

Polar varieties

$L_{(k)}$ = $(n-d+k-2)$-dimensional linear subspace of $\mathbf{P}^n$, $1 \leq k \leq d = \dim X$. The kth polar variety (or *polar locus*) of X associated with $L_{(k)}$ is

$$P(L_k, X) := \text{ closure of } \{x \in X_{\mathrm{reg}} \mid \dim(T_x X \cap L_{(k)}) \geq k-1\}.$$

For $k = 0$ we set $P(L_{(0)}, X) := X$.

If $L_{(k)}$ is generic, we write $P_k(X) = P(L_{(k)}, X)$ and if

$$L_{(0)} \subset L_{(1)} \subset \ldots \subset L_{(d)}$$

is a generic flag, then we have

$$X = P_0(X) \supset P_1(X) \supset \ldots \supset P_d(X).$$

Let $x \in X$. Teissier showed that the sequence of multiplicities

$$m_0 = e_x(P_0(X)), \ldots, m_{d-1} = e_x(P_{d-1}(X))$$

does not depend upon the choice of the general flag.

Theorem 3 (Teissier, 1982) *The pair (X_{reg}, Y) satisfies the Whitney conditions in x_0 if and only if the sequence of polar multiplicities*

$$m_0 = e_y(X), m_1 = e_y(P_1(X)), \ldots, m_{d-1} = e_y(P_{d-1}(X))$$

is locally constant in Y around x_0.

We propose the following function g to measure the singularity of X in a point x of X:

$A := \mathcal{O}_{X\times X,(x,x)}$, $I :=$ diagonal ideal in A,

$$g(x) := \sum_{i=0}^{d} c_i(I, A) = e(G_I(A)).$$

Note that $\dim A = 2d, c_{d+1} = \cdots = c_{2d} = 0$ and that

$$(c_0(I, A), c_1(I, A), \ldots, c_d(I, A))$$

is a refinement of the multiplicity

$$c_d(I, A) = e_x X = e(\mathcal{O}_{X,x})$$

of X at x.

Theorem 4 (Achilles-Manaresi, 2003) *Let $X \subset \mathbf{P}^n$ be a (reduced) surface and $x \in X$ be a closed point. Then*

$$X_j := \{x \in X \mid g(x) \geq j\},\ j = 0, 1, \ldots$$

are closed subschemes of X or empty, and the connected components of

$$S_g(j) := g^{-1}(j) = X_j \setminus X_{j+1}$$

are the strata of a Whitney stratification of X (the coarsest one if $n = 3$).

Generalized Samuel multiplicities (c_2, c_1, c_0) and polar multiplicities (m_0, m_1) (both ordered by codimension) for the surface in $\mathbf{C}^3$ (or in $\mathbf{P}^3$) defined by the equation

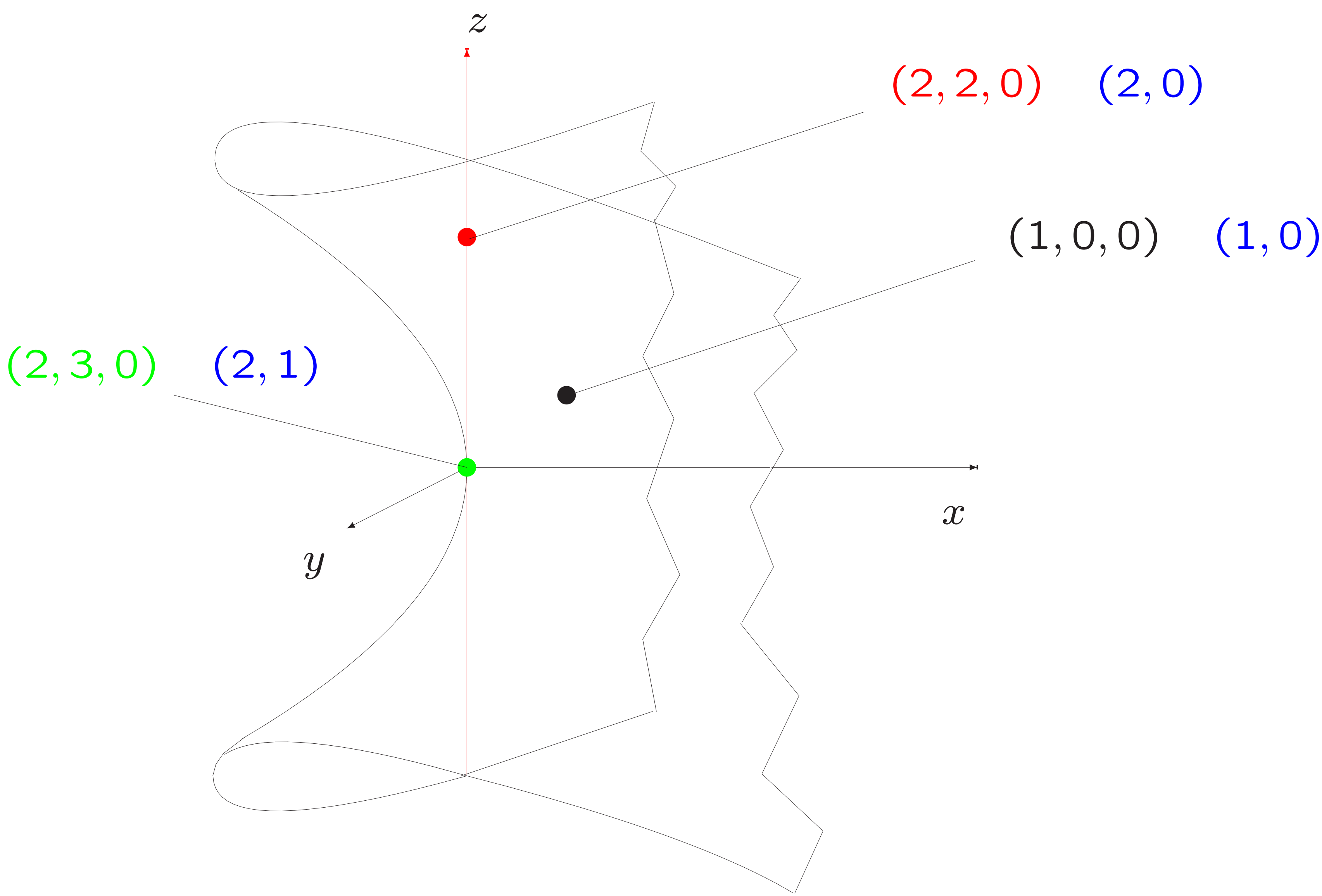
1. $x^4 + y^4 - xyz = 0$;

2. $y^2 - x^3 - x^2z^2 = 0$.

In both cases the Whitney stratification is given by

$$\text{surface} \supset z\text{-axis} \supset \text{origin} .$$

We illustrate the second example:



z
(2, 2, 0) (2, 0)
(1, 0, 0) (1, 0)
(2, 3, 0) (2, 1)
x
y