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[Robert Adams "Calcolo differenziale. Funzioni di una variabile reale 1

[Demidovich Esercizi e problemi di analisi matematica

[Enrico Giusti: Analisi Matematica 1 (e 2)
BORINGHIERI (2ª edizioni)

Notazioni:

Insieme: $\{1,2,3\} =$

$= \{2,1,3\}$

$\{x : x$ tali che $P(x)$ sia vera $\}$

$\{x : x$ è un numero pari $\}$

Insiemi numerici:

Conteggio → numeri naturali

$1, 2, 3, 4, \dots$

$\mathbb{N}$ = insieme dei numeri naturali

Se n è un numero naturale allora $n+1$ è un numero nat.

$$\left(n \underset{\substack{\uparrow \\ \text{appartiene}}}{\in} \mathbb{N} \overset{\text{implica}}{\Longrightarrow} n+1 \in \mathbb{N} \right)$$

Procedimento di induzione:

Esempio: $2^n \geq n$?

$2^1 \overset{?}{\geq} 1$ $2^2 = 4 \overset{?}{\geq} 2$

$2^3 = 8 \overset{?}{\geq} 3$

I passo: verificare l'affermaz.

per $n = 1$

$$2^1 = 2 \geq 1 \quad \text{vera}$$

II passo:

$$2^n \geq n \Longrightarrow 2^{n+1} \geq n+1$$

I e II $\Rightarrow$ aff. vale $\overset{\forall\ \text{per ogni}}{\forall} n \in \mathbb{N}$

So $2^n \ge n$ $\qquad \left| \underline{2^{n+1} \ge n+1} \right.$

$$\underset{\overset{\shortparallel}{2^{n+1}}}{2\,2^n} \ge 2n$$

se e solo se

$$2n \overset{?}{\ge} n+1 \iff \underline{n \ge 1}$$

quindi $2^{n+1} \ge n+1$

#

Esempio:

$$1+2+3+4+\cdots+n = ?$$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & n \\ + & + & + & + & & + \\ n & (n-1) & (n-2) & (n-3) & \cdots & 1 \\ \parallel & \parallel & & & & \parallel \\ n+1 & n+1 & & \cdots & & n+1 \end{array}$$

$$\rightarrow n(n+1) = 2(1+2+\cdots+n)$$

$$\boxed{1+\cdots+n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}}$$

Verifica per induzione

$\boxed{I}$

$$n=1 \qquad 1 \overset{?}{=} \frac{1(1+1)}{2} = 1$$

vero.

$\boxed{II}$

$$1+\dots+n = \frac{n(n+1)}{2}$$

$$\overset{?}{\Longrightarrow} 1+\dots+n+(n+1) = $$

$$= \frac{(n+1)(n+2)}{2} = (*)$$

$$1+2+\dots+n+(n+1) = \frac{n(n+1)}{2} +$$

$$+(n+1) = (n+1)\left(\frac{n}{2}+1\right) = (*)$$

Insiemi numerici:

$$1-2=-1 \notin \mathbb{N}$$

↑ non appartiene

$$\mathbb{Z} = \{ p : \; p \in \mathbb{N} \text{ oppure } p = 0 \;\; \underbrace{-p \in \mathbb{N}}_{} \}$$

$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

insieme dei numeri <u>interi</u>.

Divisione ⟶ razionali

$$\mathbb{Q} = \left\{ \frac{p}{q} : \quad p \in \mathbb{Z} \quad q \in \mathbb{N} \right\}$$

numeri razionali.

Problema:

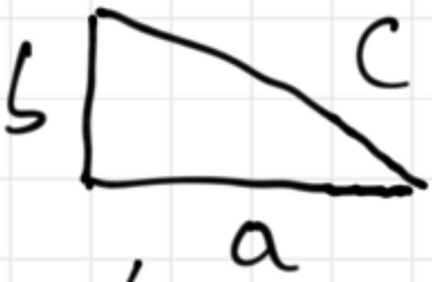
quadrato

lunghezza diagonale?

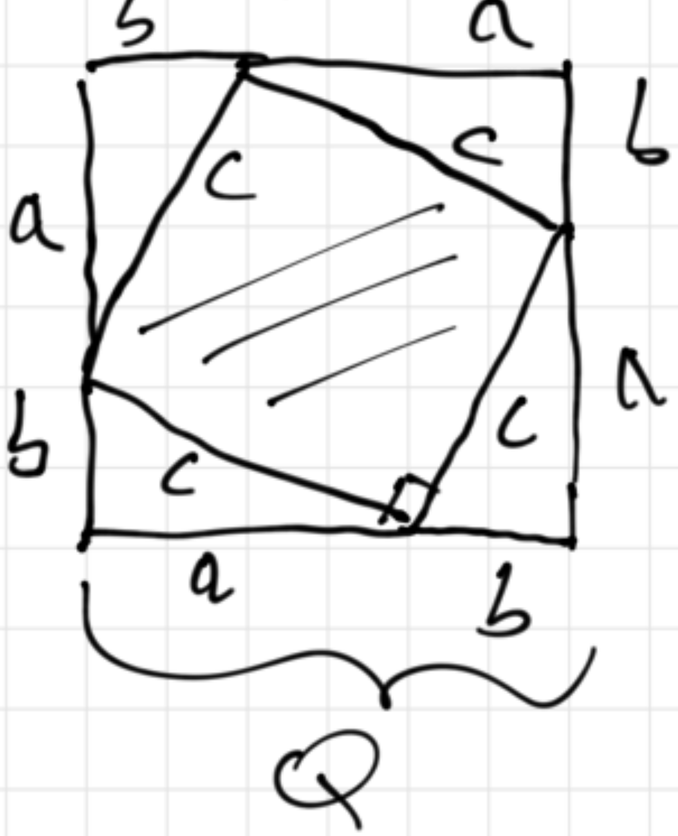
$1^2 + 1^2 = d^2$ (teorema di Pitagora)

$d^2 = 2$

T. di Pitagora :



$$a^2+b^2=c^2$$



$$A(Q)=(a+b)^2$$
$$=c^2+2ab$$

$$(a+b)^2 = c^2+2ab$$
$$\parallel$$
$$(a+b)(a+b)=a^2+b^2+2ab$$
$$\Rightarrow a^2+b^2=c^2$$

. #

$$\boxed{d^2 = 2} \Leftrightarrow \quad d = \pm\sqrt{2}$$

$d =$ diagonale $\Rightarrow \sqrt{2}$

$$\begin{array}{l|l}
\sqrt{2\qquad\qquad} & 1,4 \\
\underline{1^2} & 25\times5=125 \\
1\;0\;0 & 24\times4=96 \\
\underline{\;\;96} & \\
\quad\;\; 4 &
\end{array}$$

$a \in \mathbb{N} \qquad a^2 \leq 2 < (a+1)^2$

$1^2 = 1 < 2 < (1+1)^2 = 4$

$$\begin{array}{l|l} 40\boxed{0} & 1,414 \\ \hline 281 & 281 \times 1 = 281 \\ \hline 1190\boxed{0} & 2824 \times 4 = 11296 \\ 11296 & 2825 \times 5 > 11900 \\ \hline \vdots & \vdots \end{array}$$

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$\sqrt{2}$ $\qquad$ $x^2 = 2$

$$1^2 < 2 < 2^2 = 4$$

$$\left(1+\frac{b}{10}\right)^2 \text{ "=" } 2$$

$$b=0,\dots,9$$

$$1+\frac{b^2}{100}+2\,\frac{b}{10}=2$$

$$b^2+2\,b\,10=100$$

$$\shortparallel$$

$$b\,(2\cdot 10+b) \qquad b=4,\dots$$

$d = \sqrt{2} \in \mathbb{Q}$?

$\boxed{\sqrt{2} \notin \mathbb{Q}}$.

Dimostrazione per assurdo:

Sia $\sqrt{2} = \frac{p}{q}$ $p, q \in \mathbb{N}$

p e q primi tra loro

(esempio: $\frac{5}{7}$ $\frac{8}{6} = \frac{4 \cdot \cancel{2}}{3 \cdot \cancel{2}}$)

$$(\sqrt{2})^2 = \frac{p^2}{q^2} \iff \boxed{2q^2 = p^2}$$

$2q^2 = p^2 \Rightarrow p$ divisibile

per 2 $\Leftrightarrow$ $\boxed{p = 2p'}$ $(p' \in \mathbb{N})$

$2q^2 = (2p')^2 = 4(p')^2$

$2(p')^2 = q^2 \Rightarrow$

$\boxed{q = 2q'}$ $(q' \in \mathbb{N}) \Rightarrow$

p e q non _sono_ primi tra lo

$\Rightarrow \quad \sqrt{2} \notin \mathbb{Q}$

$\sqrt{2}$ è irrazionale.

Insieme dei numeri reali:

$\sqrt{2}$

$A = \{ q \in \mathbb{Q} \ : \ q < \sqrt{2} \}$

$B = \{ q \in \mathbb{Q} \ : \ q > \sqrt{2} \}$

$\mathbb{Q} = A \cup B$ sezione

↑ unione (Dedekind)

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \boxed{\mathbb{R}}$$

Lezione 24/9

Esercizio 1:

Dimostrare usando l'induzione

$\forall n \in \mathbb{N}$: $2^n \geq 1+n$

Esercizio 2 (disuguaglianza di Bernoulli)

Sia $h > -1$

$\forall n \in \mathbb{N}$: $(1+h)^n \geq 1+nh$

Def.: $n \in \mathbb{N}$

$1! = 1 \qquad (n+1)! = (n+1)\, n!$

$0! = 1$ (per definizione)

esempio: $3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 2 \cdot 3 \cdot 4 = 24$

$5! = 120 \qquad \ldots$

Esempio 3 : $a, b \in \mathbb{R}$

$n \in \mathbb{N}$

$$(a+b)^n = \binom{n}{0} a^n \overbrace{b^0}^{\overset{1}{\shortparallel}} +$$

$$+ \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2$$

$$+ \cdots + \binom{n}{n-1} a^1 b^{n-1} +$$

$$+ \binom{n}{n} a^0 b^n$$

(formula di Newton)

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Esempi:

$$\binom{n}{0} = \frac{n!}{(n-0)!\;0!} = \frac{n!}{n!\;1}$$

$$= 1$$

$$\binom{n}{n} = \frac{n!}{(n-n)!\;n!} = 1$$

#

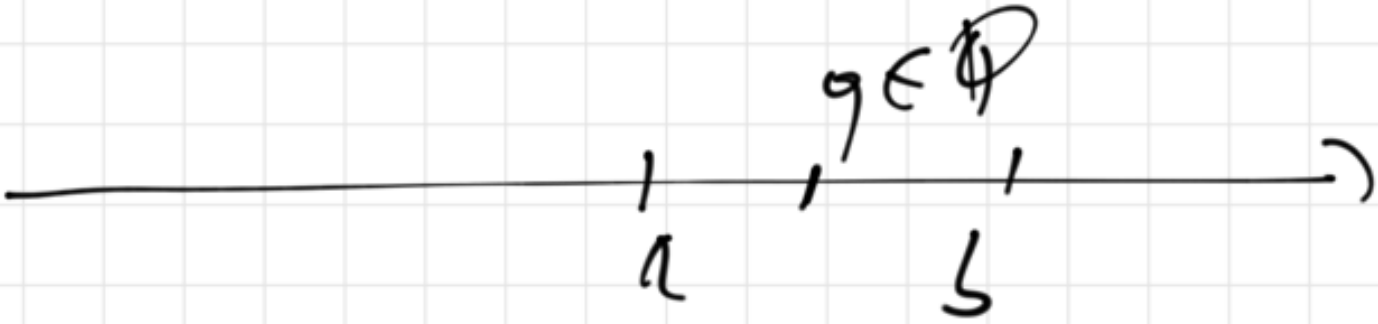
La retta reale ($\mathbb{R}$):

Def: $\mathbb{Q}$ (i razionali)

è **denso** in $\mathbb{R}$ $\iff$

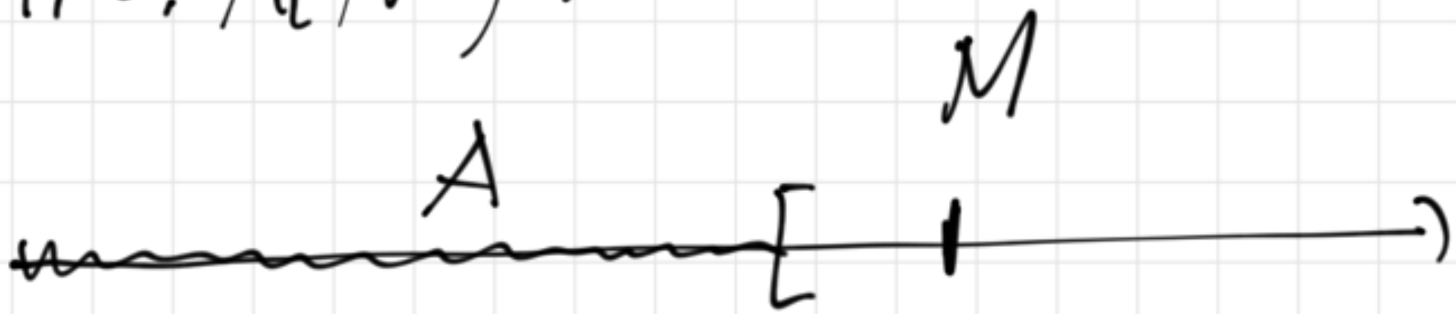
$\forall a, b \in \mathbb{R} \quad a < b$

allora $\exists\, q \in \mathbb{Q} : a < q < b$



Assioma di completezza:

$A \subseteq \mathbb{R}$ A superiormente limitato (inferiormente limitato):



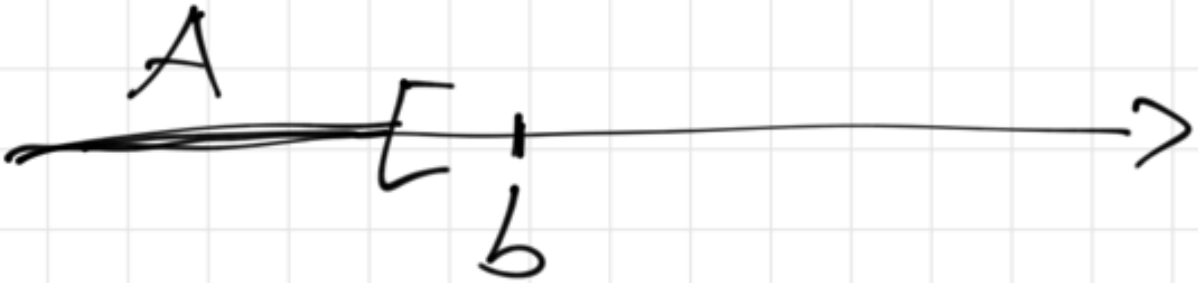
$\exists M \in \mathbb{R}, \forall a \in A: \; a \leq M.$

($\exists m \in \mathbb{R}, \forall a \in A: \; a \geq m$ inf. limitato).

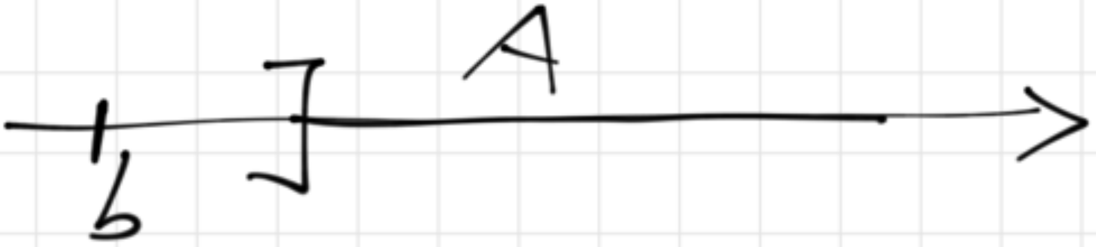
$A \subseteq \mathbb{R}$ maggiorante per A

(minorante per A)

$b \in \mathbb{R}$: $b \geq a \quad \forall a \in A$



($b \in \mathbb{R}$: $b \leq a \quad \forall a \in A$



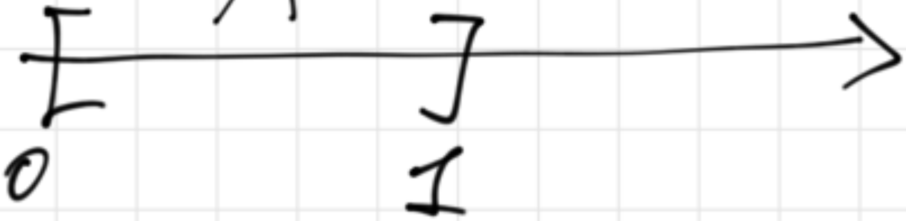
)

$A \subseteq \mathbb{R}$ massimo di A

$M \in \mathbb{R}$ (1) M maggiorante per A

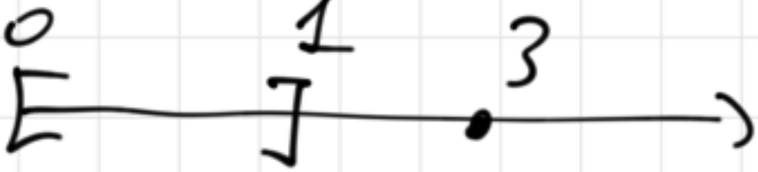
(2) $M \in A$

Es 1: $A = [0,1]$



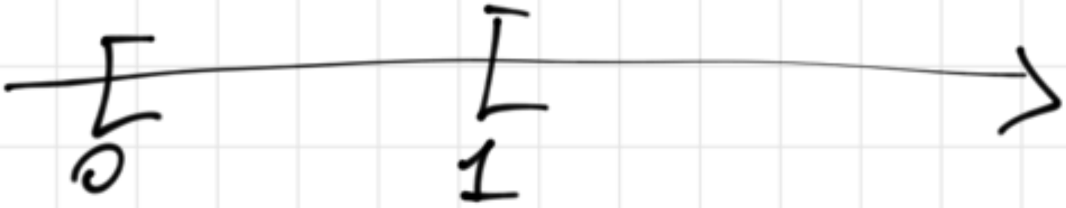
massimo 1.

Es 2: $A = [0,1] \cup \{3\}$



massimo 3.

Es 3: $A = [0, 1[$



1 non è il massimo

$A \subseteq \mathbb{R}$ minimo per A

$m \in \mathbb{R}$ (1) m minorante per A

(2) $m \in A$.

Estremo superiore di

$A \subseteq \mathbb{R}$ $(A \neq \emptyset)$

(1) A non ha maggioranti

l'estremo superiore di A

è $+\infty$.

(2) A ha maggioranti

l'estremo superiore è il

minimo dei maggioranti.

Esempio:

$A = [0, 1[$

1 è un maggiorante

($0 \leq \underbrace{a < 1}$

$\forall \varepsilon > 0 \quad 1 - \varepsilon < q < 1 \quad q \in \mathbb{Q}$

$\Rightarrow$ $1-\varepsilon$ non è un maggiorante per A)

$L = \sup A$ (estremo superiore)

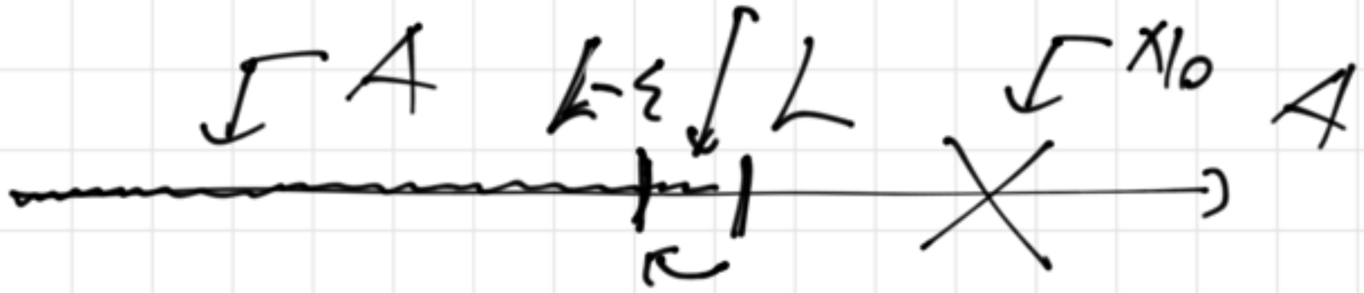
(1) $\forall a \in A : \quad a \leq L$

(L maggiorante per A)

(2) $\forall \varepsilon > 0 \ \exists a \in A :$

$L - \varepsilon < a$

(L minimo dei maggioranti)



Estremo inferiore

$A \subseteq \mathbb{R}$

$\inf A$

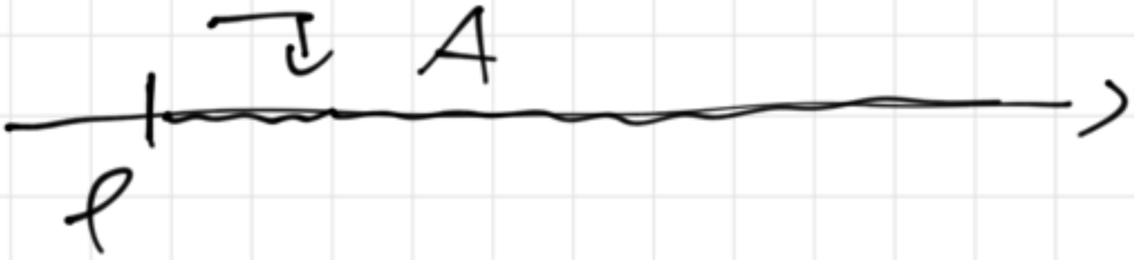
(1) A non inferiormente limitato $\inf A = -\infty$

(2) A inferiormente limitato

$\inf A$ "=" massimo dei minoranti di A

$\inf A = l$

(1) $\forall a \in A \quad a \geq l$



(2) $\forall \varepsilon > 0 \ \exists a \in A :$

$$l + \varepsilon > a$$

$l+\varepsilon$ non è un minorante.

Assioma di completezza:

$\forall A \subseteq \mathbb{R}$ A superiormente limitato $\Longrightarrow \exists \sup A \ (\in \mathbb{R})$.

Oss.: Se A ha massimo $(\max A)$

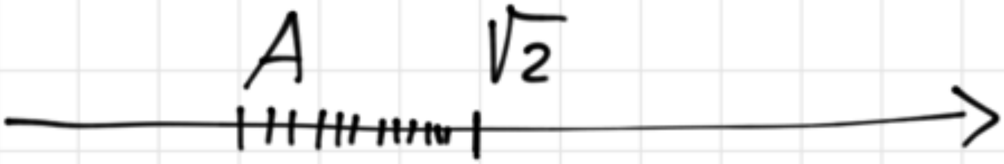
$\Rightarrow \max A = \sup A$

Se A ha minimo $(\min A)$ $\Rightarrow \min A = \inf A$

Problema: mostrare che l'equazione $x^2-2=0$ ammette soluzione in $\mathbb{R}$.

A.C.: $A \subseteq \mathbb{R}$ superiormente limitato $\Rightarrow \sup A \in \mathbb{R}$.

$$A=\{q\in\mathbb{Q} : q>0 \quad q^2<2\}$$



$A = \{ q \in \mathbb{Q} : \quad q > 0 \quad q^2 < 2 \}$

- A è superiormente limitato

2 è un maggiorante per A?

$$\boxed{\forall q \in A \qquad q \leq 2}$$

Per assurdo: $\exists q \in A \quad q > 2$

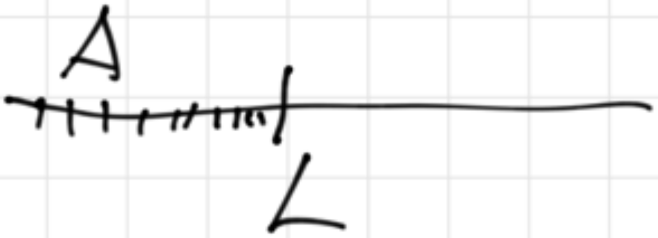
$\Rightarrow \quad q^2 > 4$ per definiz. di A $\quad q^2 < 2$ assurdo

$\forall q \in A \qquad q \leq 2$

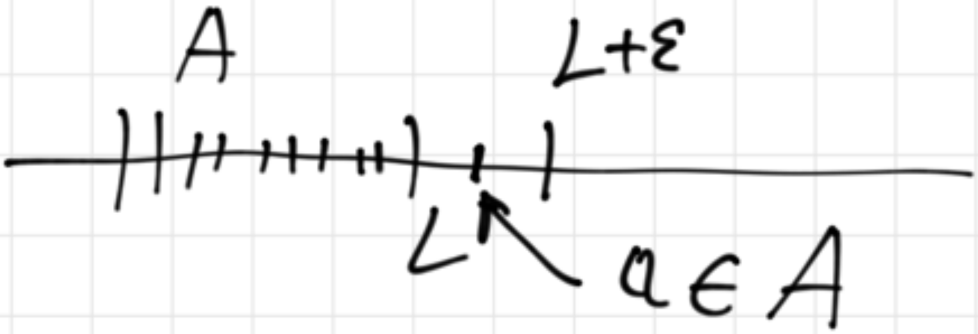
$\Longleftrightarrow$ 2 è un maggiorante per A.

A.C. $\Longrightarrow \exists \sup A = L$

$\boxed{L^2 \overset{?}{=} 2}$



(1) $L^2 < 2$ ($\Longrightarrow$ assurdo)



$$\exists\, \varepsilon > 0 \qquad (L+\varepsilon)^2 < 2$$

$$\varepsilon^2 + 2L\varepsilon + L^2 < 2$$

$$\underbrace{\varepsilon^2 + 2L\varepsilon < \underbrace{2 - L^2}_{\substack{\vee \\ 0}}}_{}$$

$$\varepsilon \in \left]0, 1\right[$$

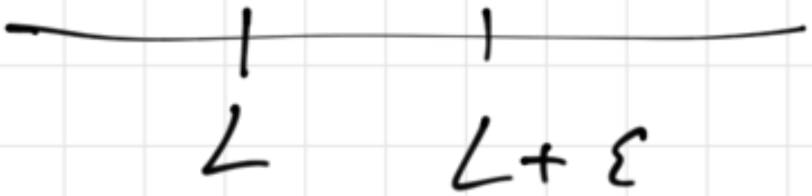
$$\varepsilon^2 + 2L\varepsilon < \underbrace{\varepsilon + 2L\varepsilon}_{}$$

$$\left.\begin{aligned} &\varepsilon + 2L\varepsilon = \varepsilon(1 + 2L) \\ &\overset{?}{<} 2 - L^2 \end{aligned}\right]$$

$$\varepsilon(1+2L) \overset{?}{<} 2-L^2$$

$$\varepsilon < \frac{2-L^2}{1+2L}$$

$$(L+\varepsilon)^2 < 2 \qquad L^2 < 2$$



densità di $\mathbb{Q}$ $\exists\, q \in \mathbb{Q}$:

$$\sup A = L < q < L+\varepsilon \qquad q > 0$$

$$q^2 < (L+\varepsilon)^2 < 2 \Rightarrow \boxed{q \in A}$$

(2) $L^2 > 2$ ($\Longrightarrow$ assurdo).

Esercizi:

$$x^2 - 4x + 7 \geq 0$$

$$x^2 - 4x + 7 = 0$$

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\underbrace{x^2+2\,\frac{b}{2a}\,x}+\frac{c}{a}=0$$

$$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$$

(completamento del quadrato)

$$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}$$

#

$$x = \frac{4 \pm \sqrt{16-28}}{2} = \frac{4 \pm \sqrt{-12}}{2}$$

$$x^2 - 4x + 7 \neq 0 \quad \forall x \in \mathbb{R}$$

$$\{(x,y) : \quad y = x^2 - 4x + 7\}$$

$$\begin{cases} y = x^2 - 4x + 7 \\ y = 0 \end{cases}$$

$\mathbb{R}$ $\quad]-\infty, +\infty[$

Es 2:

$$\frac{x^2-5x+6}{x-7}<0$$

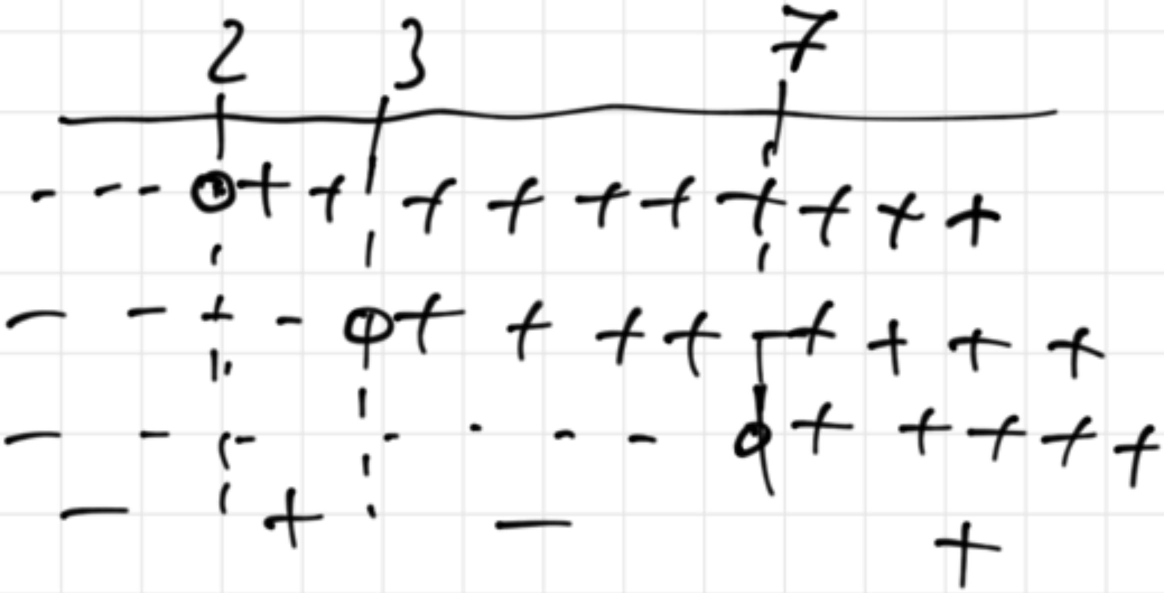
$$x^2-5x+6$$

$x=2$

$x=3$

$$x^2-5x+6=(x-2)(x-3)$$

$$\frac{(x-2)(x-3)}{x-7}<0$$



$x < 2$ oppure $3 < x < 7$.

Eg :

$$\frac{x^4+3x^2+2x+1}{2x^2+3}$$

$$\begin{array}{l|l}
\underbrace{x^4}_{}+3x^2+2x+1 & \underbrace{2x^2}_{}+3 \\ \hline
-x^4-\frac{3}{2}x^2 & \frac{1}{2}x^2+\frac{3}{4} \\ \hline
\frac{3}{2}x^2+2x+1 & \\
-\frac{3}{2}x^2-\frac{9}{4} & \\ \hline
2x-\frac{5}{4} &
\end{array}$$

$$x^4+3x^2+2x+1=(2x^2+3)\cdot\left(\frac{x^2}{2}+\frac{3}{4}\right)$$

$$+\ 2x-\frac{5}{4}$$

$$\frac{x^4+3x^2+2x+1}{2x^2+3}=\frac{x^2}{2}+\frac{3}{4}$$

$$+\ \frac{2x-\frac{5}{4}}{2x^2+3}$$

E1 :

$$\frac{x^5+7x^3+2x+1}{x^2+1}$$

$$\begin{array}{l|l}
x^5+7x^3+2x+1 & x^2+1 \\ \cline{2-2}
-x^5-x^3 & x^3+6x \\ \cline{1-1}
\quad 6x^3+2x+1 & \\
-6x^3-6x & \\ \cline{1-1}
\qquad -4x+1 &
\end{array}$$

$$x^5+7x^3+2x+1=(x^2+1)(x^3+6x)$$

$$-4x+1$$

$$\frac{x^5+7x^3+2x+1}{x^2+1} = x^3+6x$$

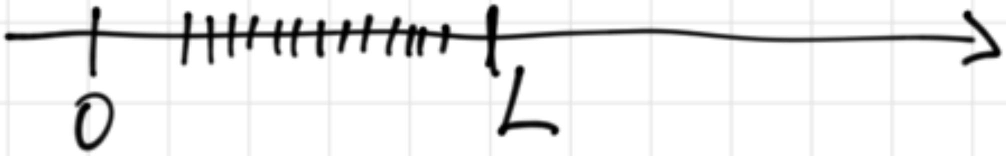
$$+\frac{1-4x}{x^2+1}$$

Lezione 25/9 :

Conclusione della dim. dell'esistenza di $\sqrt{2}$:

$$A = \{ q \in \mathbb{Q} : q > 0 \text{ e } q^2 < 2 \}$$

$$\exists\, L = \sup A \in \mathbb{R}$$



(1) $L^2 < 2$ (non può essere)

(2) $L^2 > 2$ (è falso)

$$\boxed{\exists \varepsilon > 0 \quad (L-\varepsilon)^2 > 2}$$

Problema $\varepsilon = ?$

$$L^2 - 2\varepsilon L + \varepsilon^2 \overset{?}{>} 2$$

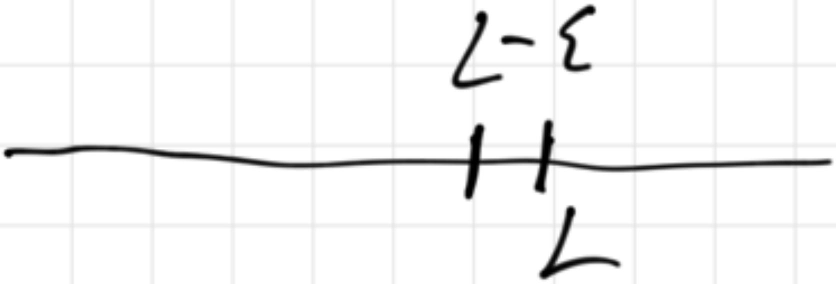
$$L^2 - 2\varepsilon L + \varepsilon^2 > L^2 - 2\varepsilon L$$

$$\exists \varepsilon > 0 : \quad L^2 - 2\varepsilon L > 2$$

$$\Longleftrightarrow \quad L^2 - 2 > 2\varepsilon L$$

$$(0<)\, \varepsilon < \frac{L^2-2}{2L}$$

$L-\varepsilon$ è un maggiorante per A



(l'assurdo deriva dal fatto che L è il minimo dei maggioranti).

Se $L-\varepsilon$ non fosse un maggiorante $\left((L-\varepsilon)^2 > 2 \right)$

$\Longrightarrow$ esiste $q \in A \quad q > L-\varepsilon$

$$q^2 > (L-\varepsilon)^2 > 2$$

$q \in A$ $\quad q^2 < 2$

$2 > 2$ assurdo.

$\Rightarrow$ $L - \varepsilon$ è un maggiorante

(in contrasto $L = \sup A$).

$\Rightarrow$ $L^2 = 2$. #

Esercizio: $A \subseteq \mathbb{R}$ $(A \neq \emptyset)$

dimostrare che $\sup A$ è unico.

Oss.: $a > 0$

$\exists x \in \mathbb{R} : x^n = a$

$(\iff \exists \sqrt[n]{a})$

Oss.: per definizione

$a > 0$ $\quad \sqrt[n]{a} > 0$.

Es 1: dim. per induzione

$$2^n \geq 1+n \quad \forall n \in \mathbb{N}.$$

Svolgimento:

I) $n=1$ $\qquad 2^1=2 \geq 1+1=2$

vero.

II) $$\underbrace{2^n \geq 1+n}_{\text{ipotesi induttiva}} \Longrightarrow \underbrace{2}^{n+1} \geq 2+n$$

$$\overbrace{2\cdot 2^n}^{2^{n+1}} \geq 2\cdot(1+n) = \underbrace{2+n+n}$$

$$2^{n+1} \geq 2+n+n \underset{\substack{\uparrow \\ (n \geq 1)}}{>} 2+n$$

E12: $h > -1$

$$(1+h)^n \geq 1+nh$$

Svolgimento:

I) $n=1$ $\quad (1+h)^1 \geq 1+1\cdot h$

vero

II) $(1+h)^n \geq 1+nh$

$$\Rightarrow (1+h)^{n+1} \geq 1+(n+1)h$$

$$(1+h)(1+h)^n \geq (1+nh)(1+h)$$

$$\uparrow$$

$$1+h>0 \Leftrightarrow h>-1$$

$$(1+h)^{n+1} \geq \underbrace{1+(n+1)h}_{} + nh^2$$

$$\geq 1+(n+1)h$$

$$\Rightarrow (1+h)^{n+1} \geq 1+(n+1)h. \quad \#$$

E13: $n! = 1 \cdot 2 \cdot 3 \cdots n$

$0! = 1$

$n \in \mathbb{N} \quad k \in \mathbb{N} \quad k \leq n$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

coeff. binomiale

$a, b \in \mathbb{R}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \, b^k$$

Oss: $n=2 \quad (a+b)^2 = \binom{2}{0} a^{2-0} b^0 + \binom{2}{1} a^{2-1} b^1$

$$+\binom{2}{2} a^{2-2} b^2 = \binom{2}{0} a^2 + \binom{2}{1} ab +$$

$$+\binom{2}{2} b^2$$

$$\binom{2}{0} = \frac{2!}{(2-0)!\ 0!} = \frac{2}{2 \cdot 1} = 1$$

$$\binom{2}{1} = \frac{2!}{(2-1)!\ 1!} = \frac{2}{1 \cdot 1} = 2$$

$$\binom{2}{2} = \frac{2!}{(2-2)!\ 2!} = \frac{2!}{0!\ 2!} = \frac{2}{1 \cdot 2}$$

$$= 1 \qquad (a+b)^2 = a^2 + 2ab + b^2$$

Verifica per induzione

I) $(a+b)^1 \overset{?}{=} \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k$

$$= \binom{1}{0} a \underbrace{b^0}_{\substack{\shortparallel \\ 1}} + \binom{1}{1} \underbrace{a^0}_{\substack{\shortparallel \\ 1}} b$$

$$= a+b$$

II) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$$\Rightarrow (a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

$$(a+b)^{n+1}=(a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}$$

$$=a\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}+$$

$$b\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}=$$

$$=\sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^{k}+\underbrace{\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{\overset{k+1}{\overbrace{k}}}}$$

$$\boxed{l=k+1} \qquad l=0+1=1 \quad l=n+1$$

$$=\sum_{l=1}^{n+1}\binom{n}{l-1}a^{n+1-l}b^{l}+\sum_{l=0}^{n}\binom{n}{l}a^{n+1-l}b^{l}$$

$$a_1 + \dots + a_n + b_1 + \dots + b_m =$$

$$\sum_{\ell=1}^{n} a_\ell + \sum_{\ell=1}^{m} b_\ell$$

$$(a+b)^{n+1} = \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} a^{n+1-\ell} b^{\ell}$$

$$+ \sum_{\ell=0}^{n} \binom{n}{\ell} a^{n+1-\ell} b^{\ell} =$$

$$= \sum_{\ell=1}^{n} \binom{n}{\ell-1} a^{n+1-\ell} b^{\ell} + \binom{n}{n} b^{n+1} +$$

$$a_1 + \dots + a_{n+1} = \sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}$$

$$+\binom{n}{0} a^{n+1} + \sum_{l=1}^{n} \binom{n}{l} a^{n+1-l} b^{l} =$$

$$\left[\binom{n}{n} = \binom{n}{0} = 1 \right]$$

$$= a^{n+1} + b^{n+1} + \sum_{l=1}^{n} \left[\binom{n}{l-1} + \binom{n}{l} \right] \cdot$$

$$\cdot\, a^{n+1-l} b^{l}$$

$$\binom{n}{l-1} + \binom{n}{l} \overset{?}{=} \binom{n+1}{l}$$

$$(a+b)^{n+1} = a^{n+1} + b^{n+1} +$$

$$+ \sum_{l=1}^{n} \binom{n+1}{l} a^{n+1-l} b^{l} =$$

$$= \binom{n+1}{0} a^{n+1} + \binom{n+1}{n+1} b^{n+1}$$

$$+ \sum_{l=1}^{n} \binom{n+1}{l} a^{n+1-l} b^{l} =$$

$$= \sum_{l=0}^{n+1} \binom{n+1}{l} a^{n+1-l} b^{l} .$$

Verifica: $\binom{n}{l} + \binom{n}{l-1} = ?$

$$\frac{n!}{(n-l)!\,l!} + \frac{n!}{(n+1-l)!\,(l-1)!} =$$

$$= \frac{n!}{(l-1)!\,(n-l)!} \left[\frac{1}{l} + \frac{1}{n+1-l} \right]$$

$$\Big[\; l\,(l-1)! = l!\,,\; (n+1-l)! =$$

$$= (n+1-l)\,(n-l)! \;\Big]$$

$$= \frac{n!}{(l-1)!\,(n-l)!} \; \frac{n+1}{l\,(n+1-l)} = \frac{(n+1)!}{l!\,(n+1-l)!}$$

$$\binom{n+1}{l} \#$$

Elementi di analisi combinatoria

PERMUTAZIONI:

n oggetti $(1, 2, \dots, n)$

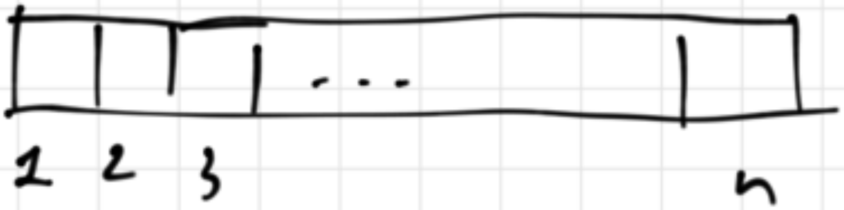
<u>E1</u>: $n = 3$

123 132 213
231 312 321

P_n = numero di permutazioni di n oggetti

$P_3 = 6$

$P_n = ?$ n oggetti



scelte per la casella $\boxed{1}$

n $\longrightarrow$ restano (a scelta fatta) n−1 oggetti

per casella $\boxed{2}$ (n−2)

scelte $\longrightarrow$ … $\longrightarrow$ $\boxed{n}$ 1 scelta

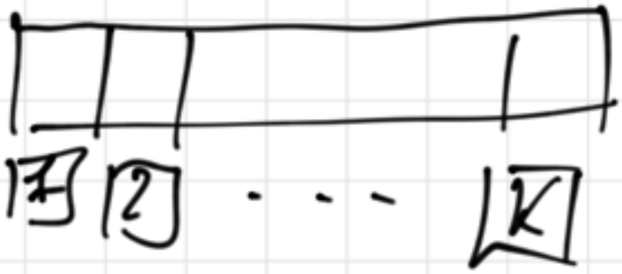
$$P_n = n(n-1)\dots 3\cdot 2\cdot 1$$

$$= n!$$

Disposizioni di n oggetti presi k a k

$(\ k \in \{1, 2, \dots, n\}\ )$

$1, 2, \dots, n$



$D_{n,k}$ = numero disposizioni di n oggetti presi k a k.

$\boxed{1} \longrightarrow n$ scelte

$\boxed{2} \longrightarrow n-1$ scelte

$\vdots$

$\boxed{k} \longrightarrow n-k+1$ scelte

$$D_{n,k} = n(n-1)\cdots(n-k+1)$$

$$= n(n-1)\cdots(n-k+1)\frac{(n-k)!}{(n-k)!}$$

$$= \frac{n!}{(n-k)!}$$

Combinazioni: n oggetti k a k (non importa l'ordine di scelta)

$$C_{n,k} = \frac{D_{n,k}}{k!} = \frac{n!}{(n-k)!\,k!}$$

$$= \binom{n}{k}$$

Binomio di Newton

$$(a+b)^3 = (a+b)(a+b)(a+b)$$

*	*	
*		*
	*	*

⊙ a^2b

$$C_{3,2} = \frac{3!}{(3-2)!\,2!} = 3$$

$$a^3 + 3a^2b + 3ab^2 + 1$$

↑

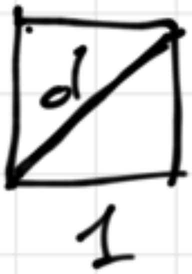
Lezione 30/9:

1. Dimostrazioni per induzione

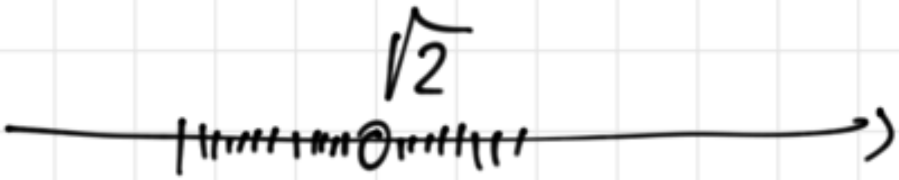
2. Definizione (ed uso) di $\inf A$ e $\sup A$.

Riepilogo:

- Assioma di completezza:



$1 \in \mathbb{Q}$ $\quad d \notin \mathbb{Q}$



$A \subseteq \mathbb{R}$ A superiormente limitato

$$\left(\exists M : \forall a \in A \quad a \le M \right)$$

$$\overset{A.C.}{\Longrightarrow} \quad \exists \sup A \in \mathbb{R}.$$

(oo) $x^2 = 2$ $\quad (x = \sqrt{2})$

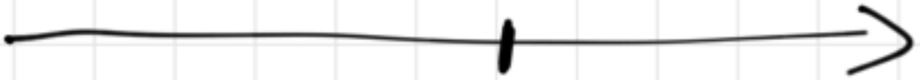
ha soluzione in $\mathbb{R}$.

(...) $a > 0$ $\quad n \in \mathbb{N}$

$n = 3, 4, 5, \dots$

$x^n = a$ ha soluzione?

($x = \sqrt[n]{a}$ radice n-esima).



$A = \{ q \in \mathbb{Q} : q > 0 \quad q^n < a \}$.

(i) A è superiormente limitato

(ii) Assioma di completezza $\exists \sup A$

(iii) $(\sup A)^n = a$

(•) $(\sup A)^n < a$

$(\sup A + \varepsilon)^n < a \; \cdots\cdots > \varepsilon > 0$

$\Rightarrow \exists\, b \in A : \quad b > \sup A$ (assurd

(••) $(\sup A)^n > a$

.... assurdo con la def. di estremo superiore di A.

Valore assoluto di un numero reale:

$a \in \mathbb{R}$

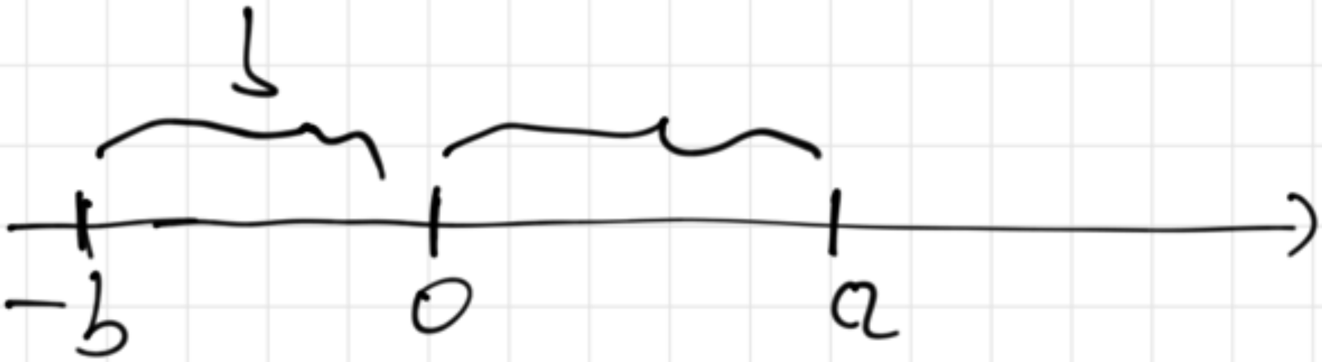
$$|a| = \begin{cases} a, & \text{se } a \geq 0 \\ -a, & \text{se } a < 0 \end{cases}$$

Esempi:

$|0| = 0$, $\left|-\frac{5}{7}\right| = \frac{5}{7}$

$|43| = 43$.

Oss. (significato "geometrico")



$b>0$ $\qquad |a| = a$

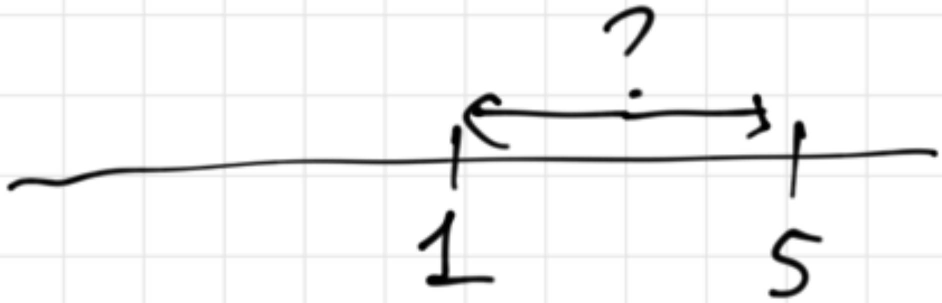
$$|-b| = b$$

$|a|$ "=" distanza di a da 0.

distanza tra a e b $= |a-b|$

Esempi:

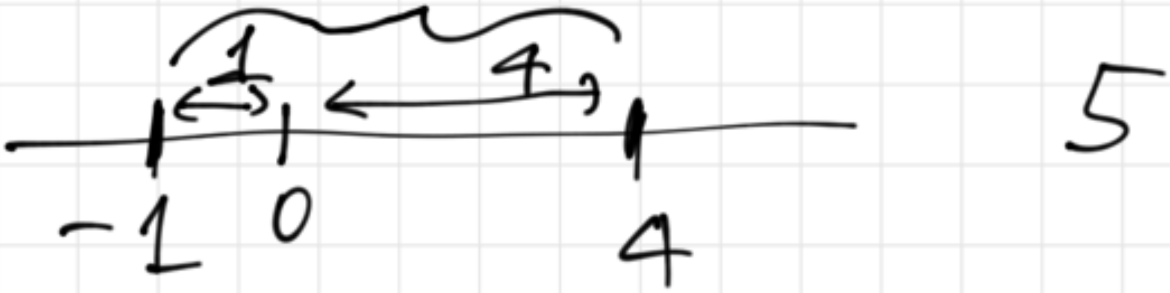
①



$$? = 5 - 1 = 4$$

$$\overset{?}{=} |1-5| = |-4| = 4$$

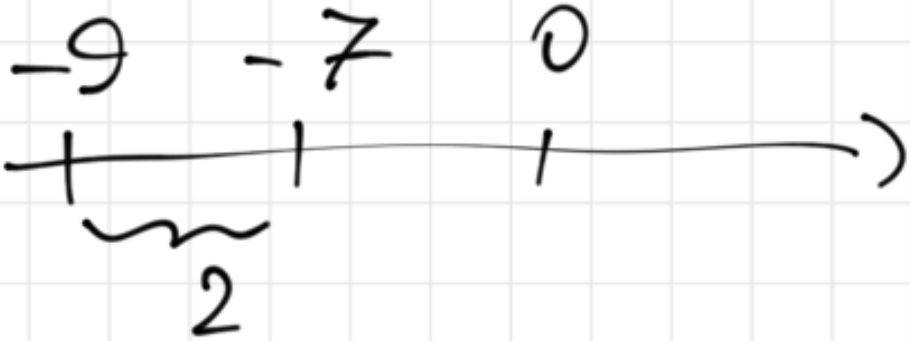
②



$$|-1-4| = |-5| = 5$$

$$|4-(-1)| = |5| = 5$$

③



$$|-9-(-7)|=|-9+7|$$

$$=|-2|=2$$

Proprietà del valore assoluto:

(1) $\forall x \in \mathbb{R} \qquad |x| \geq 0$,

$$|x|=0 \iff x=0 .$$

(2) (simmetrica $\forall x,y \in \mathbb{R}$) $|x-y|=|y-x|$

(3) (proprietà triangolare):

$\forall x, y, z \in \mathbb{R}$

$$|x-y| \le |x-z| + |z-y|$$

$$(\, |\underbrace{x-z}_{a} + \underbrace{z-y}_{b}| \le |x-z| + |z-y|$$

$$\boxed{\forall a, b \in \mathbb{R} : \; |a+b| \le |a| + |b|}.$$

se a oppure b sono 0 (banale).

a, b segni concordi

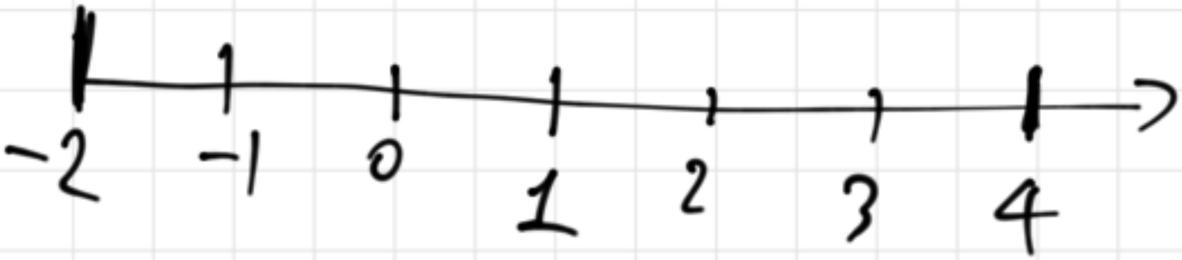
a, b segni discordi

(4) $\forall a, b \in \mathbb{R}$

$$|ab| = |a| \, |b|$$

Esempi:

(1) $|x-1| \le 3$ $\qquad$ $x = ?$



$$[-2, 4]$$

$$\underset{-2}{\underset{\|}{1-3}} \le x \le 1+3 = 4$$

(1 bis) $b > 0$ $\underline{a \in \mathbb{R}}$

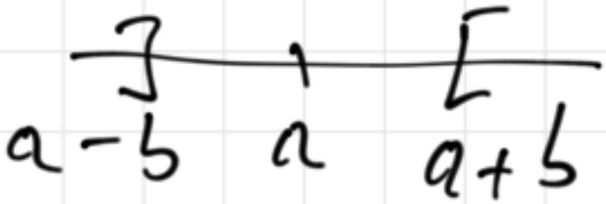
$$|x - a| \leq b$$

$$a - b \leq x \leq a + b$$

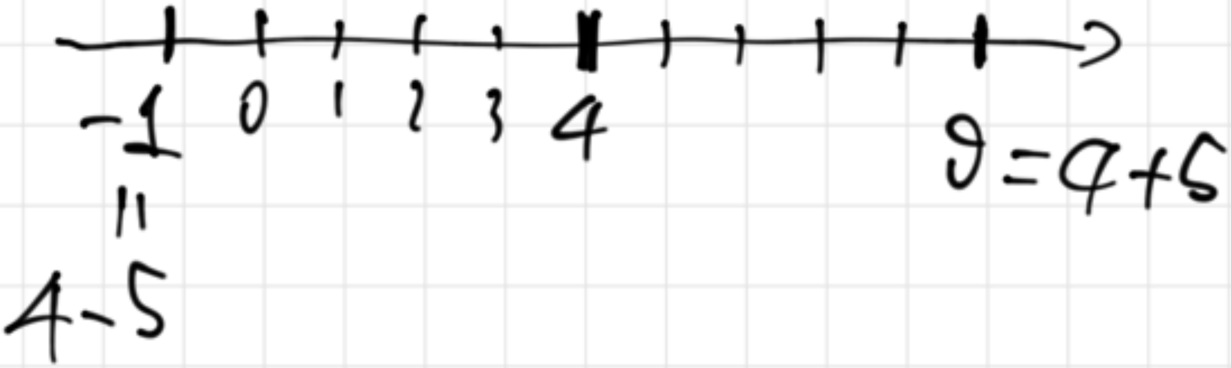
$$[a - b, a + b]$$

<u>Def.:</u> $]a - b, a + b[$ $b > 0$

$a \in \mathbb{R}$ intervallo centrato in a



(2) $|x-4|>5$



$x<-1$ oppure $x>9$.

(2bis) $a\in\mathbb{R}$ $b>0$

$|x-a|>b \iff x<a-b$

oppure $x>a+b$.

#

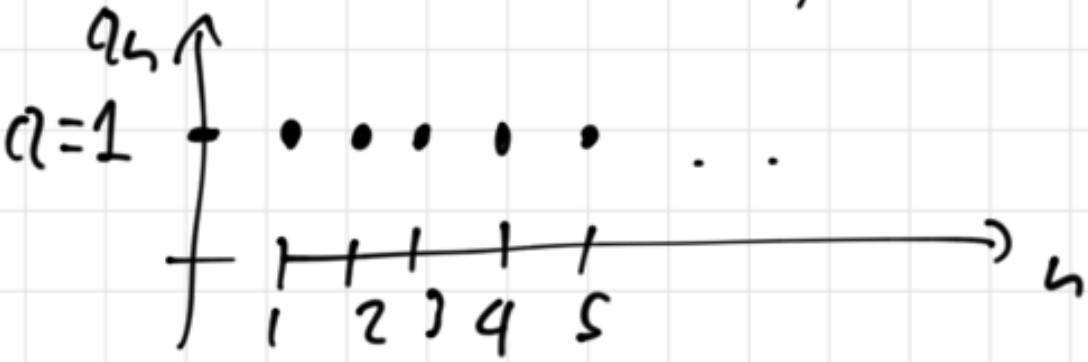
Successioni in $\mathbb{R}$:

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longmapsto f(n)$$

$$\underline{a_n} \qquad \left(\quad \{a_n\}_{n \in \mathbb{N}} \quad \right)$$

Esempi:

1. $a \in \mathbb{R}$ $\qquad a_n = a$



2. $a_n = \frac{1}{n}$

3. $a_n = n$

Def.: una certa proprietà vale definitivamente per una successione a_n se $\exists N \in \mathbb{R}$: la proprietà vale $\forall n > N$.

Esempio: $a_n = \frac{1}{n}$

definitivamente $a_n < \frac{1}{10}$.

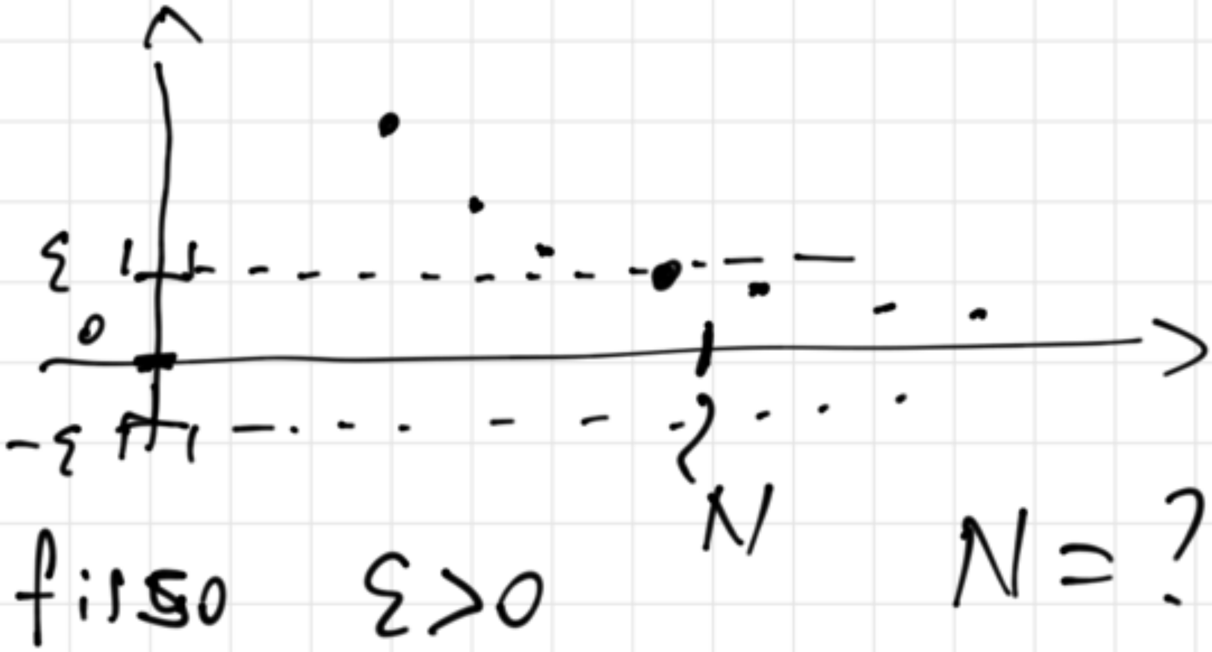
$(N = 10 \quad \forall n > 10 \quad a_n < \frac{1}{10})$.

Limiti di successioni :

$$\lim_{n\to\infty} \frac{1}{n} \overset{?}{=} 0$$

$$\overbrace{\forall \varepsilon>0 \ \exists N\in\mathbb{R}} :$$

$$\underbrace{|a_n-0|<\varepsilon} \quad \forall n>N .$$



fisso $\varepsilon>0$ $\quad N=?$ $\quad \boxed{\frac{1}{n}\overset{?}{<}\varepsilon}$

$$\frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$$

$$\forall n > N : \quad \frac{1}{\varepsilon} < n$$

$$\boxed{N = \frac{1}{\varepsilon}}$$

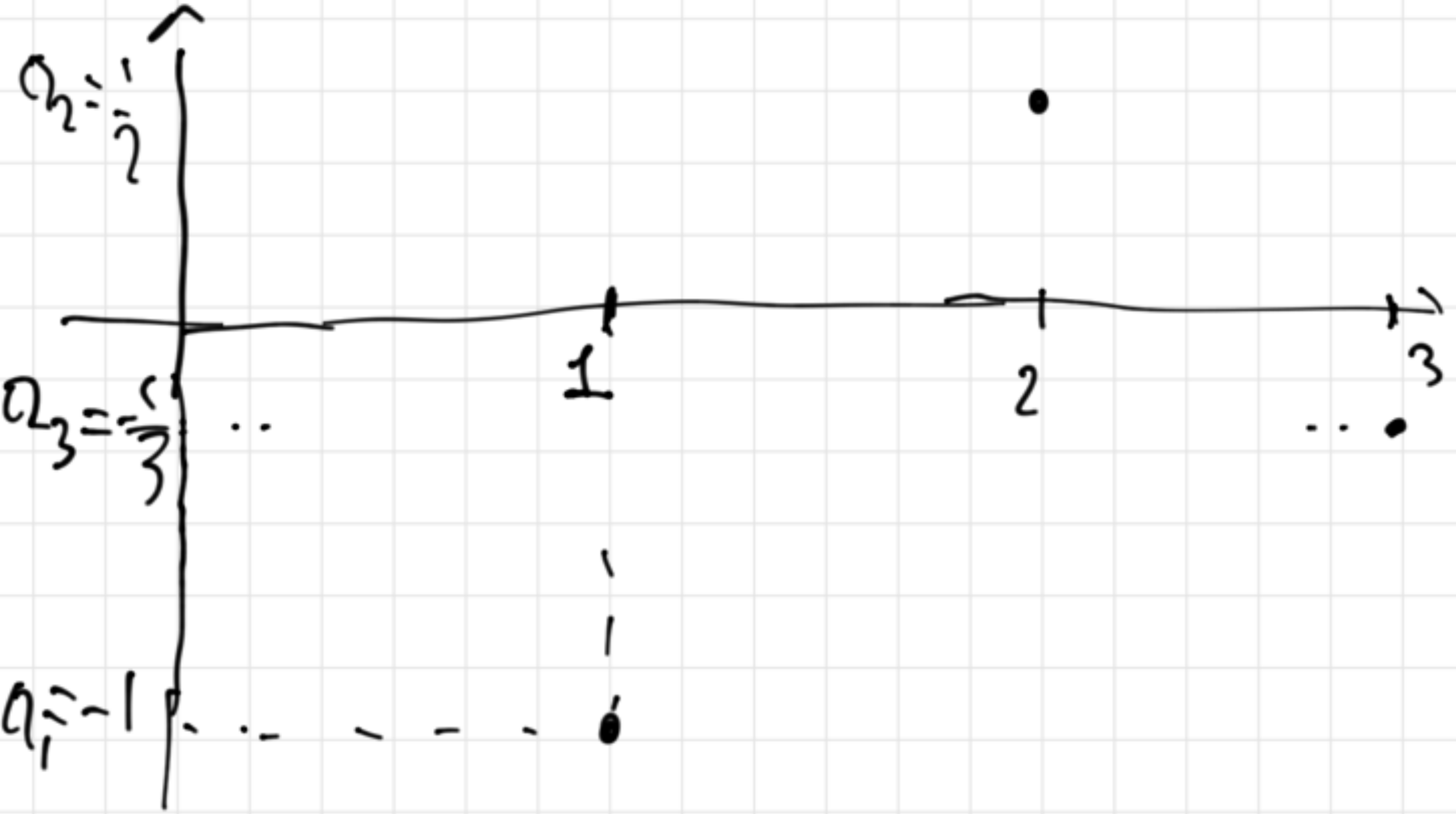
Def.: successione convergente

a_n converge $\ell \in \mathbb{R}$ se

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{R} : |a_n - \ell| < \varepsilon$$

$$\forall n > N.$$

E1: $a_n = \frac{(-1)^n}{n}$



$$\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$$

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{R} : \left| \frac{(-1)^n}{n} - 0 \right| < \varepsilon \ \forall n > N$$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{R}$ $\quad \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} < \varepsilon$

$\forall n > N$.

Verifica: fisso $\varepsilon > 0$

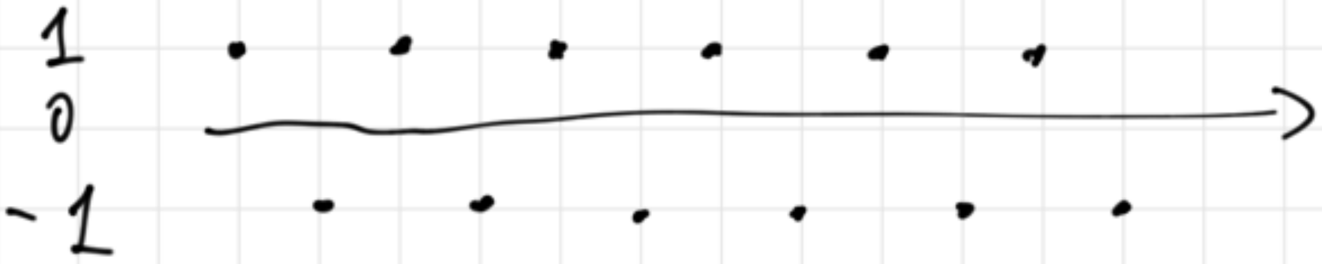
$N = ?$ $\quad \boxed{\frac{1}{n} < \varepsilon} \ \forall n > N$

$$\frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$$

$$\boxed{N = \frac{1}{\varepsilon}}$$

Esempio (di non convergenza):

$a_n = (-1)^n$



$$\lim_{n \to \infty} (-1)^n \overset{??}{=} 0$$

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{R} : \overbrace{|(-1)^n - 0|}^{1} < \varepsilon$$

$$\forall n > N$$

$\varepsilon = \frac{1}{2}$ $\quad N = ???$ $\quad 1 < \frac{1}{2} \ \forall n > N$

$$\Rightarrow \quad (-1)^n \not\to 0$$

($(-1)^n$ non converge a 0).

$$\lim_{n\to\infty} (-1)^n \overset{??}{=} 1$$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{R} : |(-1)^n - 1| < \varepsilon$

$\forall n > N$

$$|(-1)^n - 1| = \begin{cases} 0, & \text{se } n \text{ pari} \\ |-2| = 2, & n \text{ dispari} \end{cases}$$

Fisso $\varepsilon = \frac{1}{2}$

n pari $\quad 0 < \frac{1}{2}$ (vero)

n dispari $\quad 2 < \frac{1}{2}$ (falso)

$\forall N > 0 \quad \exists n$ dispari

$n > N$: $\quad 2 < \frac{1}{2}$ (falso).

$\Rightarrow \quad a_n \not\to 1$.

Lezione 1/10 :

Incontro 2/10 ore 15:30
aula PINCHERLE (dip. Matematica)

Data a_n $\quad \ell \in \mathbb{R}$

$$\lim_{n\to\infty} a_n = \ell \quad (a_n \to \ell)$$

$$\Longleftrightarrow \forall \varepsilon > 0 \; \exists N > 0 :$$

$$|a_n - \ell| < \varepsilon \qquad \forall n > N$$

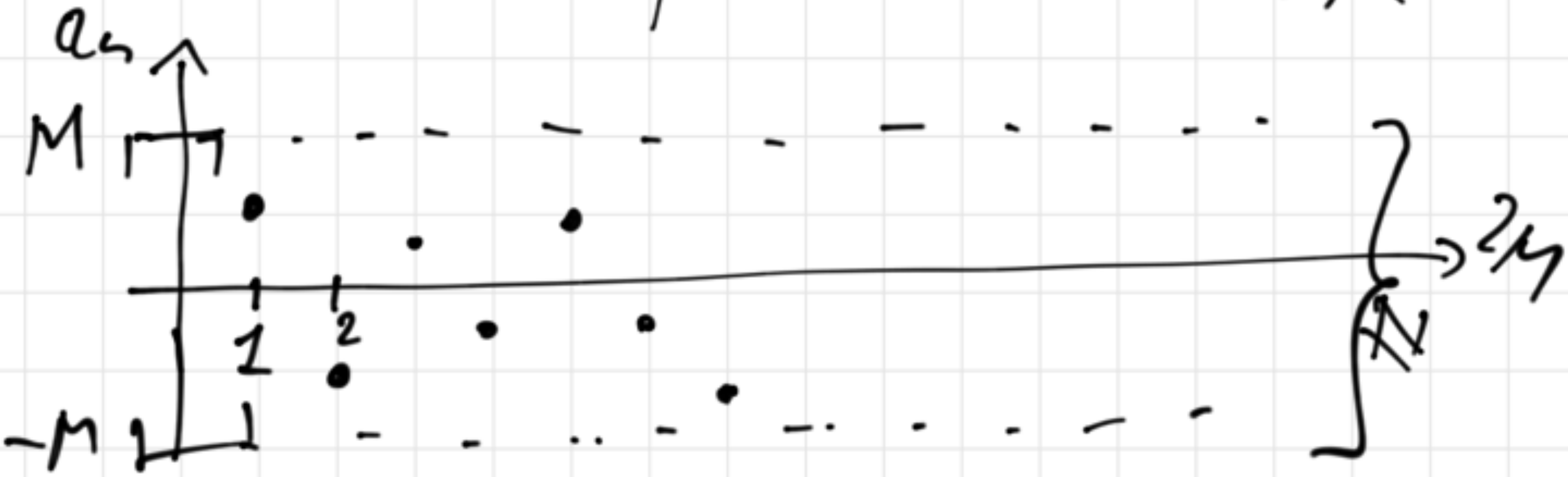
$$(\ell - \varepsilon < a_n < \ell + \varepsilon)$$

Def: a_n è limitata

se $\exists M > 0 : |a_n| \leq M$

$(\forall n \in \mathbb{N})$

In altre parole: $-M \leq a_n \leq M$



Lemma:

$a_n \to \ell \implies a_n$ è limitata

dimostrazione:

$\forall \varepsilon > 0 \ \exists N > 0 : |a_n - \ell| < \varepsilon$

$\forall n > N$

Fisso $\varepsilon = 1 \to \exists N_1 > 0$:

- $|a_n - \ell| < 1 \qquad \forall n > N_1$

$\cdot\cdot$ $|a_n| = |(a_n - \ell) + \ell| \le |a_n - \ell| + |\ell|$

dis. triangolare

(dis. triangolare: $|a+b| \le |a| + |b|$).

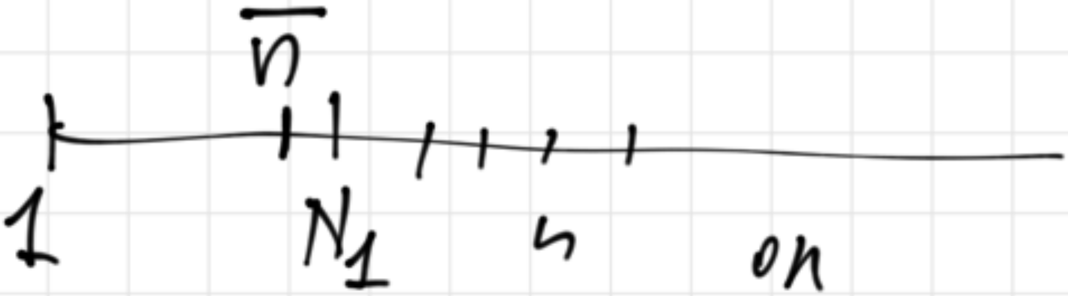
$\forall n > N_1$

$$|a_n| \le |a_n - \ell| + |\ell|$$

$$\underset{\uparrow}{\le} 1 + |\ell|$$

$\bullet$

$\exists M > 0 : \quad |a_n| \le M \quad \forall n \in \mathbb{N}$



$$M = 1 + |\ell| + |a_1| + \dots + |a_{\bar{n}}|$$

$$\mathbb{N} = \{1, 2, \dots, \bar{n}\} \cup \{n \in \mathbb{N} : n > N_1\}$$

$$M > 1 + |\ell|$$

$$|a_n| < M \qquad \forall n > N_1$$

$$|a_1| < \underbrace{1 + |\ell| + \underset{\uparrow}{|a_1|} + \dots + |a_{\bar{n}}|} = M .$$

$$\forall n \in \{1, \dots, \bar{n}\} :$$

$$|a_n| \leq M . \Rightarrow$$

$$\forall n \in \mathbb{N} : \quad |a_n| \leq M . \qquad \#$$

Proposizione (permanenza del segno):

$a_n \longrightarrow \ell > 0$ (analogamente $\ell < 0$) $\Rightarrow \exists N > 0 : a_n > 0$ $\forall n > N$.

<u>Def</u>: a_n sottosuccessione

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \dots$$

$$a_{k_1 = \textcircled{1}} \qquad a_{k_2 = 4} \qquad a_{7 = k_3} \dots$$

$k_n : \mathbb{N} \longrightarrow \mathbb{N}$ crescente

$k_{n+1} > k_n \quad (\forall n \in \mathbb{N})$

a_{k_n} sottosuccessione di a_n.

<u>Def</u>: a_n Successione di Cauchy

se

$\forall \varepsilon > 0 \ \exists N > 0 : |a_n - a_m| < \varepsilon$

$\forall n, m > N$.

Teorema:

Una successione è convergente se e solo se è di Cauchy.

Dimostrazione:

$a_n \to l \overset{?}{\Longrightarrow} a_n$ di Cauchy.

$\forall \varepsilon > 0 \ \exists N > 0 : |a_n - l| < \varepsilon$

$\forall n > N$.

Fisso $\varepsilon > 0 \longrightarrow N > 0 \quad n, m > N$

$$|a_n - a_m| = |\underbrace{a_n - l} - \underbrace{(a_m - l)}|$$

$$|a_n - a_m| \le |a_n - \ell| + |a_m - \ell|$$

↑ dis. triangolare

visto che $n > N$ $m > N$

$|a_n - \ell|, |a_m - \ell| < \varepsilon$

Fissato $\varepsilon > 0$ $\exists N > 0$

$\forall n, m > N$ $|a_n - a_m| < 2\varepsilon$

↑

$\Leftarrow$ a_n è una succ. di Cauchy.

Succ. di Cauchy ⇒ succ. convergente.

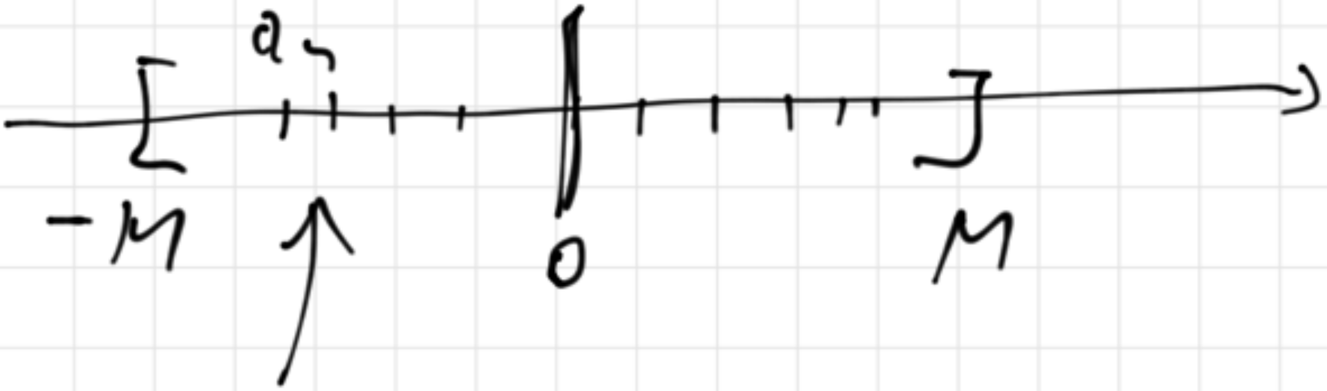
1° passo: le successioni di Cauchy sono limitate.

2° passo: ogni successione limitata ammette una sottosuccessione convergente.

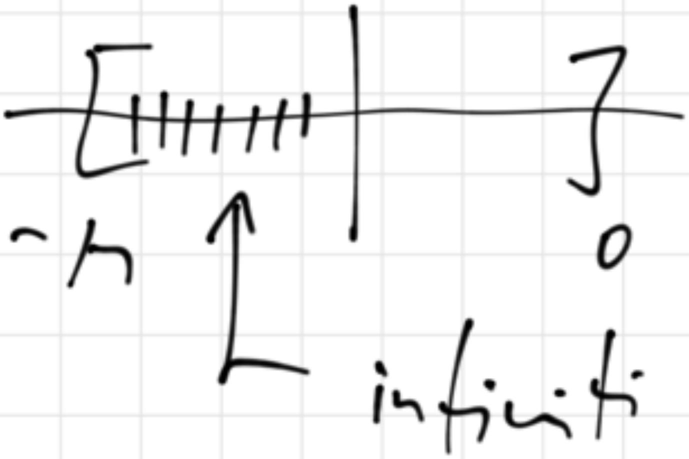
3° passo: una succ. di Cauchy che ammette una sottosucc.

convergente è convergente.

Idea della dim. 2° passo:



ci sono infiniti a_n



infiniti a_n ... #

Def: a_n successione crescente (decrescente) $\forall n \in \mathbb{N}$ $a_n \leq a_{n+1}$ ($\forall n \in \mathbb{N}$ $a_{n+1} \leq a_n$).

Teorema:

Sia a_n una successione crescente e limitata $\Rightarrow$ a_n è convergente.

dim.:

$l = \sup \{ a_n \ : \ n \in \mathbb{N} \}$

(l esiste per le ipotesi $|a_n| \leq M \ \forall n \in \mathbb{N}$ e dell'assioma di completezza).

$\forall \varepsilon > 0 \ \exists N > 0 : \underbrace{|a_n - l| < \varepsilon}_{}$

$\forall n > N$

$l - \varepsilon < \underbrace{a_n < l + \varepsilon}_{}$ per definizione

$a_n \le \ell < \ell + \varepsilon \quad \forall \varepsilon > 0$

$\forall n \in \mathbb{N}$

Fisso $\varepsilon > 0$ $\quad \exists N \in \mathbb{N}$

$\ell - \varepsilon \underset{\uparrow}{<} a_N$

(dalla definizione di sup)

a_n crescente

- $a_{N+1} \ge a_N$

·· $a_{N+2} \ge a_{N+1} \Rightarrow a_{N+2} \ge a_N$

$a_n \geq a_N \quad \forall n \geq N$

$l - \varepsilon < a_N \leq a_n \quad \forall n \geq N.$

$\nexists$

Il numero e:

$a_n = \left(1 + \frac{1}{n}\right)^n$

- a_n è convergente

$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

a_n è crescente :

$$\frac{a_{n+1}}{a_n} \overset{?}{>} 1 \quad \left(a_{n+1} > a_n \right)$$

$$a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = \left(\frac{n+2}{n+1}\right)^{n+1}$$

$$a_n = \left(1+\frac{1}{n}\right)^{n} = \left(\frac{n+1}{n}\right)^{n}$$

$$\frac{1}{a_n} = \left(\frac{n}{n+1}\right)^{n}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+2}{n+1}\right)^{\uparrow\, n\boxed{+1}} \left(\frac{n}{n+1}\right)^{n}$$

$$= \underset{\downarrow}{\frac{n+2}{n+1}} \left(\frac{n+2}{n+1} \frac{n}{n+1} \right)^n =$$

$$= \left(1+\frac{1}{n+1}\right) \left(\frac{n^2+2n+1-1}{n^2+2n+1} \right)^n =$$

$$= \left(1+\frac{1}{n+1}\right) \left(1 - \frac{1}{\underbrace{n^2+2n+1}_{(n+1)^2}} \right)^n$$

$$\left[\left(1 \underbrace{- \frac{1}{(n+1)^2}}_{h > -1} \right)^n \geq 1 + nh \qquad h = -\frac{1}{(n+1)^2} \right]$$

Bernoulli

$$\boxed{\frac{a_{n+1}}{a_n}} \geq \left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{(n+1)^2}\right)$$

$$= 1 + \frac{1}{n+1} - \frac{n}{(n+1)^2} - \frac{n}{(n+1)^3}$$

$$= \frac{\cdots\cdots}{(n+1)^3} = 1 + \frac{1}{(n+1)^3} \boxed{> 1}$$

(••) $\left(1+\frac{1}{n}\right)^n$ è limitata

$$(a+b)^n = \sum_{j=0}^{n} \frac{n!}{(n-j)!\, j!} a^{n-j} b^j$$

$a=1 \quad b=\frac{1}{n}$

$$\left(1+\frac{1}{n}\right)^n = \sum_{j=0}^{n} \frac{n!}{(n-j)!\,\boxed{j!}}\, \underset{1}{\boxed{1^{n-j}}}\, \frac{1}{n^j}$$

$$\frac{n!}{(n-j)!} = \frac{n\,(n-1)\ldots(n-j+1)(n-j)(n-j-1)\ldots}{(n-j)(n-j-1)\ldots}$$

$$= n\,(n-1)\;\ldots\;(n-j+1)$$

$$\frac{n!}{(n-j)!\,n^j} = \boxed{\frac{n}{n}}\,\boxed{\frac{(n-1)}{n}}\cdots\frac{(n-(j-1))}{n}$$

$$= \underbrace{1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)}_{j \text{ fattori}} \le 1$$

$$\left(1+\frac{1}{n}\right)^n \leq \sum_{j=0}^{n} \frac{1}{j!} \overset{?}{\leq} 3$$

$$\sum_{j=0}^{n} \frac{1}{j!} = \frac{1}{0!} + \frac{1}{1!} + \sum_{j=2}^{n} \frac{1}{j!}$$

$$\left[\, 0! = 1 \qquad 1! = 1 \,\right]$$

$$= 2 + \sum_{j=2}^{n} \boxed{\frac{1}{j!}}$$

$$j \geq 2 \qquad \frac{1}{j!} = \frac{1}{j(j-1)(j-2)!} \leq \boxed{\frac{1}{j(j-1)}}$$

$$\left(1+\frac{1}{n}\right)^n \le 2 + \sum_{j=2}^{n} \frac{1}{j(j-1)} =$$

$$\uparrow$$

somma telescopica.

$$\left[\frac{1}{j(j-1)} = -\frac{1}{j} + \frac{1}{j-1}\right]$$

$$= 2 + \sum_{j=2}^{n} \left(\overset{[}{\frac{1}{j-1}} - \frac{1}{j} \right) =$$

$$= 2 + 1 + \cancel{\frac{1}{2}} + \dots + \frac{1}{n-1} -$$

$$- \cancel{\frac{1}{2}} - \frac{1}{3} - \dots - \frac{1}{n} =$$

$$= 3 - \frac{1}{n} < 3$$

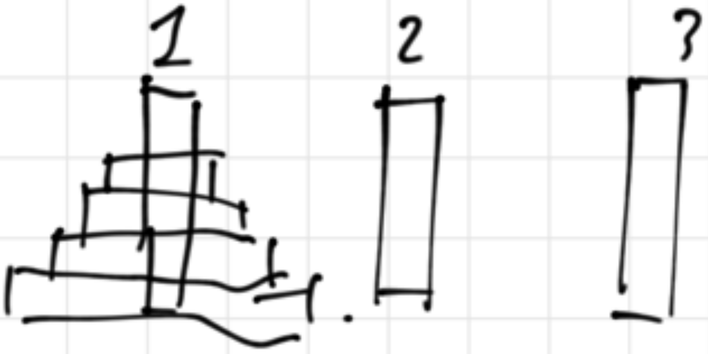
$$\sum_{j=2}^{n} \left(- \frac{1}{\underset{\substack{\uparrow \\ n}}{j}} + \frac{1}{\underset{\substack{\uparrow \\ 2}}{j-1}} \right) = 1 - \frac{1}{n}$$

$\xrightarrow{\text{teorema precedente}}$ $\left(1+\frac{1}{n}\right)^n$ è converg.

$2 < e < 3$.

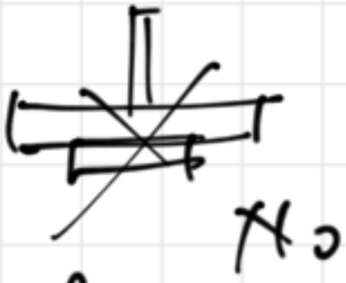
#

Torre di Hanoi:



1 disco alla volta

2 disco → piolo libero
→ sopra disco più grande

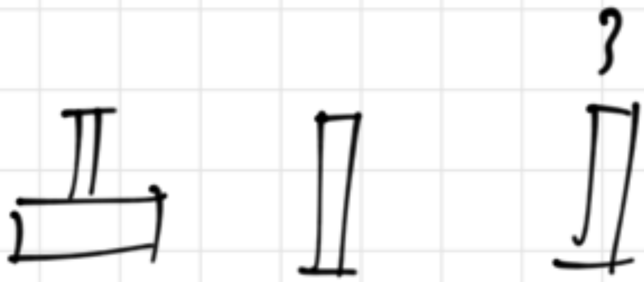


n dischi numero minimo di mosse

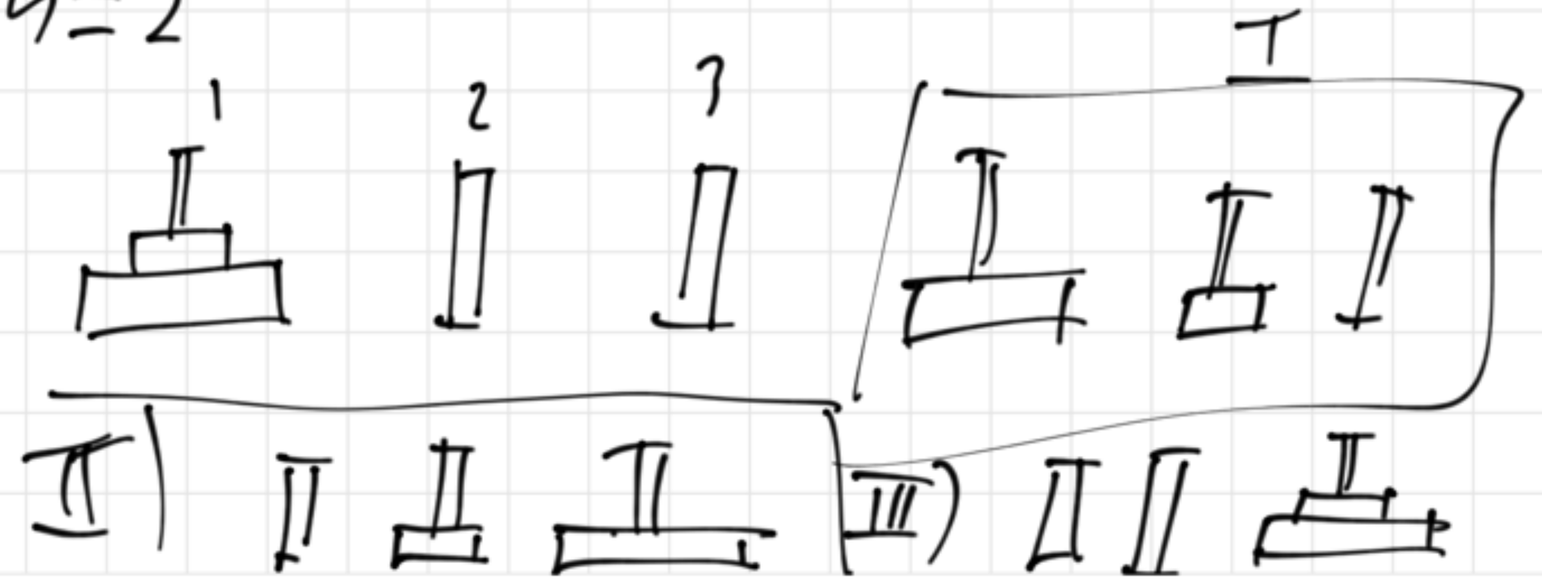
per portare tutti i dischi a 3 a_n.

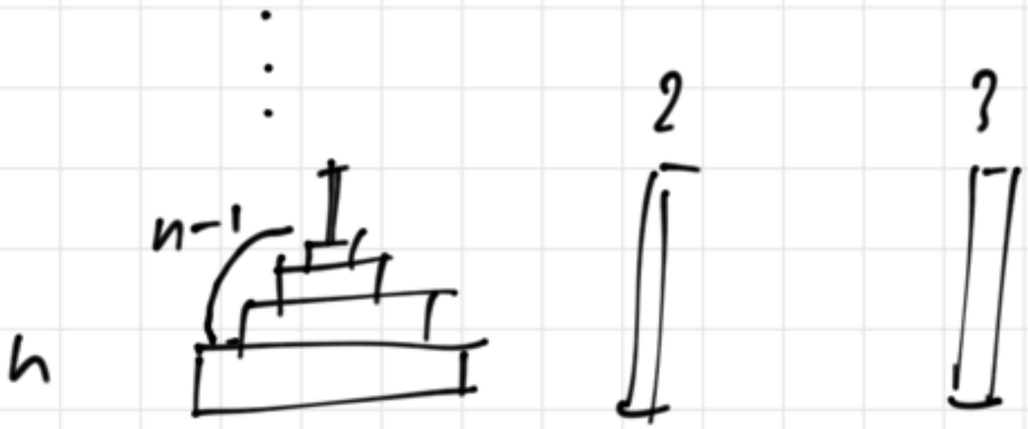
(Knuth "Concrete mathematics")

$n=1$ $a_1 = 1$



$n=2$





$$a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1$$

$$\left\{\begin{array}{l} a_1 = 1 \\ a_{n+1} = 2a_n + 1 \end{array}\right]$$

$$a_1 = 1 \quad a_2 = 3 \quad a_3 = 7$$

$$a_4 = 15 \quad \dots \quad \boxed{a_n = 2^n - 1}$$

Esercizio:

Sia a_n definita

(per ricorrenza)

$a_1 = 1 \qquad a_{n+1} = 2a_n + 1$

dimostrare (per induzione)

$a_n = 2^n - 1$.

$n=41 \quad 2^{41} - 1 \sim 2 \cdot 10^{12} \Rightarrow 1000$ anni con 1 mossa al secondo

"Algebra" dei limiti:

$a_n \longrightarrow l \qquad b_n \longrightarrow m$

$a_n + b_n \longrightarrow l+m$

$a_n - b_n \longrightarrow l-m$

⊙ $a_n b_n \longrightarrow lm$

$\dfrac{a_n}{b_n} \longrightarrow \dfrac{l}{m} \qquad \left(\begin{array}{l} b_n \neq 0 \\ m \neq 0 \end{array}\right)$

$$\forall \varepsilon > 0 \ \exists N > 0 \quad |a_n - \ell| < \varepsilon \quad \forall n > N$$

$$\forall \varepsilon > 0 \ \exists N > 0 \quad |b_n - m| < \varepsilon \quad \forall n > N$$

$$\Downarrow ?$$

$$\forall \varepsilon > 0 \ \exists N > 0 : \underbrace{|a_n b_n - \ell m| < \varepsilon}_{} \quad \forall n > N .$$

$$|a_n b_n - \ell m| = |(a_n - \ell) b_n + \underbrace{\ell b_n - \ell m}_{}| = |(a_n - \ell) b_n + \ell (b_n - m)|$$

$$|a_n b_n - lm| = |b_n(a_n - l) + l(b_n - m)|$$

$$\le |b_n(a_n - l)| + |l(b_n - m)|$$

$$= \underbrace{|b_n|}_{\le M} \underbrace{|a_n - l|}_{\downarrow \atop 0} + |l| \underbrace{|b_n - m|}_{\downarrow \atop 0}$$

succ. conv. è limitata

$$\Rightarrow \quad a_n b_n \longrightarrow lm .$$

$$\left(\begin{array}{l} a_n \longrightarrow l \\ \qquad |a_n - l| \longrightarrow 0 \end{array} \right)$$

Lezione 7/10

1) $a_n \to \ell \qquad b_n \to m$

$$\Rightarrow \quad a_n b_n \to \ell m$$

$$|a_n b_n - \ell m| \le |b_n|\,|a_n - \ell|$$
$$+ |\ell|\,|b_n - m|$$

$$\boxed{\begin{array}{l}\forall \varepsilon>0 \ \exists N>0 \quad |a_n b_n - \ell m| < \varepsilon \\ \quad \forall n > N.\end{array}}\ ?$$

Fisso $\varepsilon > 0$ $\exists N_1 > 0$:

$$\begin{cases} |b_n| \le M & (\text{per un certo } M>0) \\ |a_n - \ell| < \varepsilon & \forall n > N_1 \\ |b_n - m| < \varepsilon \end{cases}$$

$$\to |a_n b_n - \ell m| \le |b_n| \, |a_n - \ell| + |\ell| \, |b_n - m|$$

$$\le M \varepsilon + |\ell| \varepsilon = \underbrace{(M + |\ell|)}_{\text{numero dato}} \varepsilon$$

$$\forall n > N_1$$

$$\forall \varepsilon > 0 \ \exists N_1 > 0 : |a_n b_n - \ell m| \le \overbrace{(M + |\ell|)} \varepsilon$$

$$\forall n > N_1$$

$\#$

2) $a_n \longrightarrow \ell \qquad b_n \longrightarrow m \qquad m \neq 0$

$$\Rightarrow \quad \frac{a_n}{b_n} \longrightarrow \frac{\ell}{m}$$

dimostrazione:

$$\frac{a_n}{b_n} = \underset{\substack{\downarrow \\ \ell}}{a_n} \underset{\substack{?\downarrow \\ \frac{1}{m}}}{\left(\frac{1}{b_n}\right)}$$

$$b_n \longrightarrow m \neq 0 \overset{?}{\Rightarrow} \frac{1}{b_n} \longrightarrow \frac{1}{m}$$

$$\forall \varepsilon > 0 \ \exists N > 0 : \left| \frac{1}{b_n} - \frac{1}{m} \right| < \varepsilon$$

$$\forall n > N \qquad [\, b_n \to m \neq 0 \,]$$

Fisso $\varepsilon > 0$

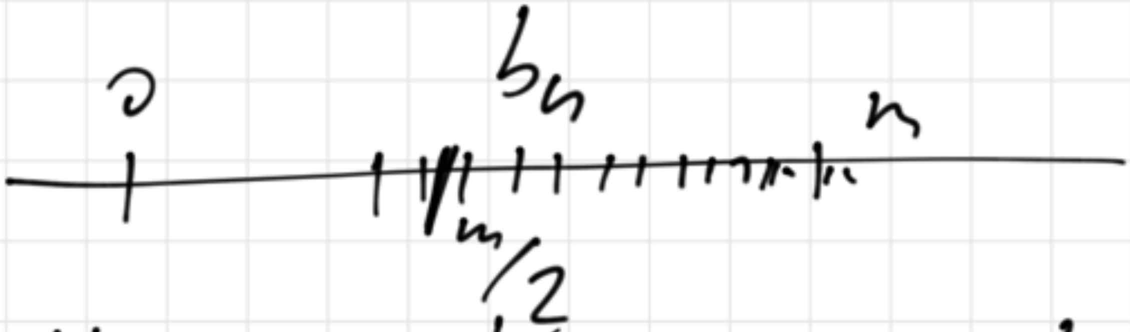
$$\left| \frac{1}{b_n} - \frac{1}{m} \right| \overset{?}{<} \varepsilon$$

$$\left| \frac{m - b_n}{b_n m} \right| = \frac{|b_n - m|}{\underbrace{|b_n|}\, |m|} \overset{?}{<} \varepsilon$$

$$\left(\left| \frac{a}{b} \right| = \frac{|a|}{|b|} \qquad |ab| = |a|\,|b| \right)$$

devo minorare $|b_n|$?

Per fissare le idee $m > 0$



$\exists N_1 > 0 : \forall n > N_1 \quad b_n > \frac{m}{2}$

$$\frac{|b_n - m|}{|b_n||m|} = \frac{|b_n - m|}{b_n \quad m} < \quad \forall n > N_1$$

$$< \frac{|b_n - m|}{m/2 \quad m} = \frac{2|b_n - m|}{m^2}$$

$\exists N \geq N_1 : \quad \forall n > N$

$$\left| \frac{1}{b_n} - \frac{1}{m} \right| = \frac{|b_n - m|}{|b_n||m|} \leq$$

$$\leq \frac{2|b_n - m|}{m^2} < \frac{2}{m^2} \varepsilon$$

$b_n \to m$.

#

Teorema (dei due carabinieri):

$a_n \leq b_n \leq c_n \ , \quad \forall n \in \mathbb{N}$

$a_n \longrightarrow l \qquad c_n \longrightarrow l \quad (l \in \mathbb{R}).$

Conclusione: $b_n \longrightarrow l$.

Esercizio: dimostrare il teorema.

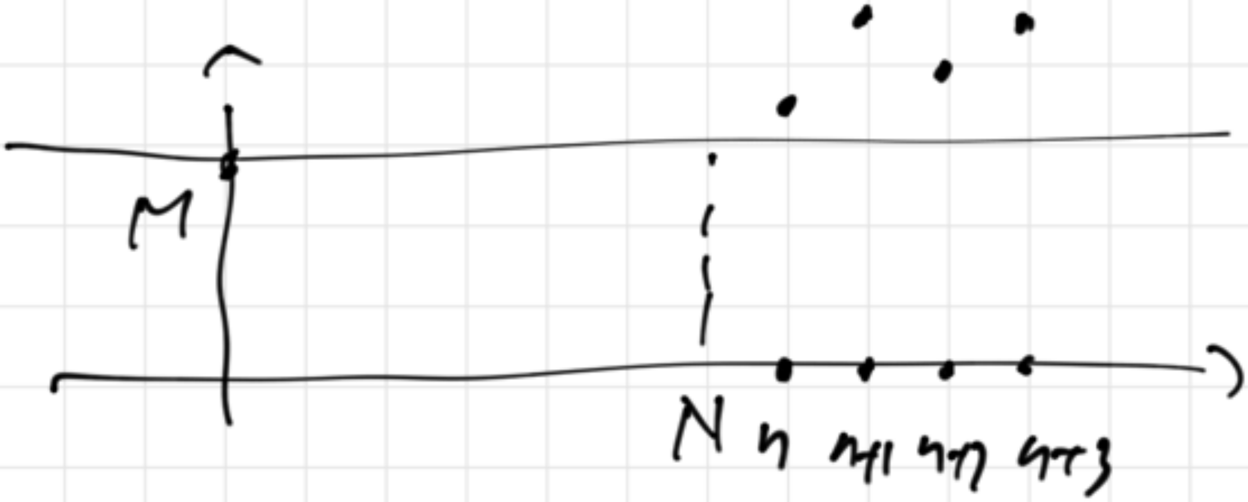
Successioni divergenti:

$$\lim_{n\to\infty} a_n = +\infty \qquad \lim_{n\to\infty} a_n = -\infty$$

a_n diverge a $+\infty$ se

$\forall M>0\ \exists N>0 : \quad a_n > M$

$\forall n > N$.



Esempi:

1) $a_n = n \qquad n \longrightarrow +\infty$

$\forall M > 0 \ \exists N > 0 : \quad n > M$
$\forall n > N$

Fisso $M > 0 \qquad N = ?$

$n > N \Rightarrow n > M$

scelgo $N = M$.

2) $a_n = n^2$

$$\forall M > 0 \ \exists N > 0 : \boxed{n^2 > M} \quad \forall n > N.$$

Fisso $M > 0$ $\qquad N = ?$

$$n^2 > M \iff \begin{cases} \boxed{n < -\sqrt{M}} \\ n > \sqrt{M} \end{cases}$$

MAI
$n \in \mathbb{N}$

$$N = \sqrt{M}.$$

a_n diverge a $-\infty$

$\left(\lim_{n\to\infty} a_n = -\infty \, , \quad a_n \to -\infty \right).$

$\forall \underline{M>0} \ \exists N>0 : \quad a_n < -M$

$\forall n > N.$

Esempio: $\quad a_n = -n^3 \xrightarrow{?} -\infty$

$\forall M>0 \ \exists N>0 : \underbrace{-n^3 < -M}$

$\forall n > N.$ Fisso $M>0$ $N=?$

$-n^3 < -M \iff M < n^3 \iff$

$\sqrt[3]{M} < n$

$N = \sqrt[3]{M}$.

Algebra dei limiti "estesa":

$a_n \to \underline{l \in \mathbb{R}}$ $\qquad b_n \to m = \pm\infty$

$a_n + b_n \longrightarrow m$

$a_n - b_n \longrightarrow -m$

$a_n \to l = \pm\infty$ $\qquad b_n \to m = \pm\infty$

$$\Rightarrow \quad a_n + b_n \longrightarrow l$$

(se il segno di l = segno di m).

Esempio:

(i) $a_n = n \qquad b_n = -n^2$

$$a_n + b_n = n - n^2 = \underset{\substack{\downarrow \\ +\infty}}{n^2} \underset{\substack{\downarrow \\ -1}}{\underbrace{\left(\overset{\nearrow 0}{\frac{1}{n}} - 1\right)}}$$

$$a_n + b_n \longrightarrow -\infty$$

(ii) $a_n = -n \qquad b_n = n^2 \qquad a_n + b_n \to +\infty.$

$(\cdot\cdot)$ $a_n \cdot b_n$ $\qquad$ $a_n \longrightarrow l$

$b_n \longrightarrow m$

indeterminato : $l = 0 \quad m = \pm\infty$

$(l = \pm\infty \quad m = 0)$

$a > 0 \qquad a(+\infty) = +\infty$

$a < 0 \qquad a(+\infty) = -\infty$

$a > 0 \qquad a(-\infty) = -\infty$

$a < 0 \qquad a(-\infty) = +\infty .$

$(\cdot\cdot\cdot)$ $\frac{a_n}{b_n}$

NON INDET.: $a_n \to \ell \in \mathbb{R}$

$b_n \to \pm\infty \Rightarrow \frac{a_n}{b_n} \to 0$

- $a_n, b_n \to 0$ $\frac{a_n}{b_n} \to ?$

- $a_n, b_n \to \pm\infty$ $\frac{a_n}{b_n} \to ?$

Osservazione:

$a_n \longrightarrow +\infty \qquad b_n \longrightarrow 0$

b_n di segno costante

$(b_n > 0)$

$$\frac{a_n}{b_n} \longrightarrow \underset{\substack{\downarrow \\ +\infty}}{a_n} \overset{\substack{+\infty \\ \uparrow}}{\left(\frac{1}{b_n}\right)} = +\infty$$

$$(+\infty)(+\infty) = +\infty$$

$$(+\infty)(-\infty) = -\infty$$

$$(-\infty)(-\infty) = +\infty$$

#

Teorema (del confronto):

1) $a_n \leq b_n$, $a_n \longrightarrow +\infty$

$\Longrightarrow b_n \longrightarrow +\infty$

2) $a_n \leq b_n$, $b_n \longrightarrow -\infty$

$\Longrightarrow a_n \longrightarrow -\infty$.

#

Esempi:

1) $a \in]0,1[$

$$a_n = a^n \longrightarrow ?$$

$$a = \frac{1}{10} \qquad a_n = \left(\frac{1}{10}\right)^n = \frac{1}{10^n}$$

$$a_1 = \frac{1}{10} \qquad a_2 = \frac{1}{100} \qquad a_3 = \frac{1}{1000}$$

$$\dots \; a_6 = \frac{1}{1.000.000} \qquad a_n > 0$$

$$a_n = \frac{1}{10^n} \xrightarrow{?} 0$$

$$0 < \frac{1}{10^n} <$$

$$10^n = (1+9)^n \geq 1 + 9n$$

Bernoulli

$$\boxed{0} < \frac{1}{10^n} < \frac{1}{1+9n} < \frac{1}{9n} < \boxed{\frac{1}{n}}$$

$$0 < \frac{1}{10^n} < \frac{1}{n}$$

2 carabinieri

$$0 \leq \lim_{n\to\infty} \frac{1}{10^n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

$a \in]0,1[$ fissato

$$0 < a^n = \frac{1}{\left(\frac{1}{a}\right)^n}$$

$\frac{1}{a} > 1$ $\quad \frac{1}{a} = 1 + h \quad h > 0$

$$0 < \frac{1}{(1+h)^n} \leq \frac{1}{1+nh} \leq \frac{1}{h}\frac{1}{n}$$

$$0 \leq \lim_{n\to\infty} a^n \leq \lim_{n\to\infty} \frac{1}{h}\frac{1}{n} = 0$$

$\Rightarrow \quad a^n \longrightarrow 0.$

2) $a > 0$ $\qquad a_n = \sqrt[n]{a} = a^{\frac{1}{n}}$

(1) $a \geq 1$ $\qquad \sqrt[n]{a} \geq 1$

$$\boxed{\sqrt[n]{a} = 1 + h_n} \text{ --- } \boxed{h_n \geq 0}$$

$$\sqrt[n]{a} \to 1 \iff \boxed{h_n \to 0}$$

?

$$(\sqrt[n]{a}) = \underbrace{a} = (1+h_n)^n \geq \underbrace{1 + n h_n}$$

$$0 \leq h_n \leq \frac{\overbrace{a-1} \geq 0}{n} \xrightarrow[n\to\infty]{} 0$$

$b_n \longrightarrow 0$ (per confronto)

$\Rightarrow \sqrt[n]{a} \longrightarrow 1 \qquad (a \geq 1)$

(ii) $\quad a \in]0,1[$

$$a_n = \sqrt[n]{a} = \sqrt[n]{\frac{1}{\left(\frac{1}{a}\right)}} = \frac{1}{\sqrt[n]{\frac{1}{a}}}$$

$\frac{1}{a} > 1$

$$\sqrt[n]{\frac{1}{a}} \longrightarrow 1$$

x algebra dei limiti $a_n \longrightarrow \frac{1}{1} = 1.$

3) $a_n = \sqrt[n]{n}$

$n^{\frac{1}{n}}$

$n \to +\infty \qquad \frac{1}{n} \to 0$

$a_n \longrightarrow 1$

$a_n = \sqrt[n]{n} \qquad n \geq 1$

$1 \leq \sqrt[n]{n} = 1 + h_n$

$h_n \to 0$?

$$n = (1+h_n)^n \geq 1 + h_n n$$

$$h_n \leq \frac{n-1}{n} = 1 - \frac{1}{n}$$

$$\sqrt[n]{a_n} = (\sqrt{n})^{1/n} = 1 + h_n \,, \quad h_n \geq 0$$

$$\sqrt{n} = (1+h_n)^n \geq 1 + n h_n$$

$$0 \leq h_n \leq \frac{\sqrt{n}-1}{n} < \frac{1}{\sqrt{n}}$$

<u>Esercizio:</u> $\frac{1}{\sqrt{n}} \longrightarrow 0$ (usare def.)

contrasta $\quad b_n \to 0$

$\Rightarrow \quad \sqrt{a_n} \longrightarrow 1$

$$a_n = \sqrt{a_n}\,\sqrt{a_n} \longrightarrow 1 \cdot 1 = 1$$

algebra dei limiti.

$$\sqrt[n]{n} \longrightarrow 1 .$$

Lezione 8/10:

Alcune successioni.

$K \in \mathbb{N}$ $\qquad a_n = n^K$

Es.: $a_n \to +\infty$ $\quad (N = \sqrt[K]{M})$.

$a > 1$ $\qquad b_n = a^n$

Es.: $b_n \to +\infty$ $\quad \left(N = \frac{M}{a-1}\right)$

$c_n = n!$

Es.: $c_n \to +\infty$ $\quad (N = M)$

$d_n = n^n$

Es.: $d_n \to +\infty$ $\quad (N = M)$.

I confronto (potenza con esponenziale).

$$\frac{a_n}{b_n} = \frac{n^k}{a^n} \qquad (k \text{ fissato}) \quad a > 1$$

$$\sqrt[2k]{\frac{n^k}{a^n}} = \frac{\sqrt{n}}{\left(\sqrt[2k]{a}\right)^n}$$

$$\begin{gathered} \sqrt[2k]{a} > 1 \\ \shortparallel \\ 1+h \ (h>0) \end{gathered}$$

$$\left[(1+h)^n \geq 1+nh \right]$$

$$0 \leq \sqrt[2k]{\frac{n^k}{a^n}} \leq \frac{\sqrt{n}}{1+nh} \leq \frac{1}{h\sqrt{n}}$$

$$\sqrt[2k]{\frac{n^k}{a^n}} \longrightarrow 0 \quad \left(\text{per confronto} \quad \frac{1}{a\sqrt{n}} \longrightarrow 0 \right)$$

$$\frac{n^k}{a^n} = \underbrace{\underset{\downarrow\, 0}{\sqrt[2k]{\frac{n^k}{a^n}}} \cdots \underset{\downarrow\, 0}{\sqrt[2k]{\frac{n^k}{a^n}}}}_{2k\text{-fattori}}$$

$$\longrightarrow 0$$

[limite di un prodotto "=" prodotto dei limiti

I) l'esponenziale diverge più rapidamente delle potenze

II) esponenziale "contro" fattoriale

$$\frac{a^n}{n!} \qquad a > 1$$

<u>1° modo:</u> $\underline{a \geq 2}$

$k \leq a < k+1 \qquad k \in \mathbb{N}$

Es.: $a = 3,25 \qquad k = 3$

$$0 \leq \frac{a^n}{n!} < \frac{(k+1)^n}{n!} =$$

$$= \frac{\overbrace{(k+1)\,(k+1)\cdots(k+1)\,(k+1)}^{n\text{-volte}}}{\underbrace{n\,(n-1)\cdots(k+2)\,(k+1)\cdots 1}_{n}} =$$

$$\frac{(k+1)\,(k+1)}{(k+1)\cdots 1} \longleftarrow C_n$$

$$= \frac{k+1}{n}\,\underbrace{\frac{k+1}{n-1}}_{\leq 1}\cdots\underbrace{\frac{k+1}{k+2}}_{\leq 1}\underbrace{\left(\frac{k+1}{k+1}\,\frac{k+1}{k}\cdots\frac{k+1}{1}\right)}_{C_n}$$

$$\leq \frac{k+1}{n}\,1\cdots 1\cdot C_n \xrightarrow[n\to+\infty]{} 0$$

Il fattoriale diverge più rapidamente dell'esponenziale

$$\left(\frac{a^n}{n!} \longrightarrow 0 \right)$$

2° metodo: $f_n = a^n / n!$

$$\frac{f_{n+1}}{f_n} = \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} = \frac{a}{n+1}$$

$$\exists N \in \mathbb{N} \quad \forall n \geq N \quad \frac{a}{n+1} < \frac{1}{2}$$

$\exists N \in \mathbb{N} \quad \forall n \geq N$

$$\frac{f_{n+1}}{f_n} \leq \frac{1}{2} \iff f_{n+1} \leq \frac{f_n}{2}$$

$n = N$ $$\frac{f_{N+1}}{f_N} \leq \frac{1}{2} \iff \underbrace{f_{N+1} \leq \frac{f_N}{2}}$$

$n = N+1$ $$\underline{f_{N+2} \leq \frac{f_{N+1}}{2}} \leq \frac{f_N}{2^2}$$

$n = N+2$ $$f_{N+3} \leq \frac{f_{N+2}}{2} \leq \frac{f_N}{2^3}$$

. . . .

$$f_{N+h} \le \frac{f_N}{2^h} \quad (h \in \mathbb{N}).$$

confronto

$$0 \le \lim_{h\to\infty} f_{N+h} \le \underbrace{f_N \lim_{h\to\infty} \frac{1}{2^h}}_{\overset{\shortparallel}{0}}$$

$$\Longrightarrow \quad \frac{a^n}{n!} \xrightarrow[n\to\infty]{} 0 \qquad \#$$

III $$\frac{n!}{n^n} = g_n$$

$$\frac{g_{n+1}}{g_n} = \frac{(n+1)!}{(n+1)^{n+1}} \quad \frac{n^n}{n!}$$

$$= \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\left(1+\frac{1}{n}\right)^n \longrightarrow e > 2$$

$\Longrightarrow \exists\, \alpha \in]0,1[\quad \exists\, N \in \mathbb{N}$
$\forall n \geq N \qquad g_{n+1}/g_n \leq \alpha$

$\exists \alpha \in]0,1[\quad \exists N \in \mathbb{N} \quad \forall n \geq N$

$$\frac{\rho_{n+1}}{\rho_n} \leq \alpha \left(\Leftrightarrow \rho_{n+1} \leq \alpha \rho_n \right)$$

$n = N \qquad \underline{\rho_{N+1} \leq \alpha \rho_N}$

$n = N+1 \qquad \rho_{N+2} \leq \underline{\alpha \rho_{N+1} \leq} \alpha \cdot \alpha \rho_N =$

$= \alpha^2 \rho_N$

$n = N+2 \qquad \rho_{N+3} \leq \dots \leq \alpha^3 \rho_N$

$\cdot \cdot \cdot$

$\forall h \in \mathbb{N}$

$$g_{N+h} \le g_N \, \alpha^h$$

$$0 \le \lim_{h \to +\infty} g_{N+h} \le g_N \underbrace{\lim_{h \to +\infty} \alpha^h}_{0}$$

lezione del 7/10.

$$\boxed{\frac{n!}{n^n} \xrightarrow[n \to +\infty]{} 0}$$

Osservazione:

$$\left.\begin{array}{l} a_n \longrightarrow \; ? \\[1em] a_{\underline{100}+n} \longrightarrow \ell \end{array}\right\} \Longrightarrow a_n \longrightarrow \ell$$

Serie numeriche:

$a_1, a_2, \dots, a_n, \dots$

$a_1 + a_2 + a_3 + \dots + a_n + \dots$

$= \sum_{n=1}^{\infty} a_n$

Successione delle somme parziali: $n \in \mathbb{N}$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

Si studia il $\lim_{n \to +\infty} S_n$

Situazioni possibili

$\lim_{n\to+\infty} S_n$

- → convergenza
- → divergenza a $+\infty$
- → divergenza a $-\infty$
- → il limite non esiste.

Esempio: la serie geometrica

$$a\in\mathbb{R} \qquad \sum_{n=1}^{\infty} a^{n-1}$$

(1) $a=1$ $\quad S_n=\sum_{h=1}^{n} 1^{h-1}=n$

$$S_n \longrightarrow +\infty$$

(2) $a \neq 1$

$$S_n = \sum_{k=1}^{n} a^{k-1} = 1 + a + \dots + a^{n-1}$$

$$S_n \frac{\overset{\downarrow}{1}-a}{1-a} = \frac{1}{1-a}\Big[\underset{\uparrow}{1} + a + \dots + a^{n-1}$$

$$- a - a^2 - \dots - \underset{\uparrow}{a^n}\Big] = \frac{1-a^n}{1-a}$$

somma telescopica

(2)(i) $a > 1$

$$S_n = \frac{1-a^n}{1-a} = \frac{a^n-1}{a-1}$$

$$S_n \longrightarrow +\infty$$

(ii) $-1 < a < 1$ $\left(\Leftrightarrow |a| < 1\right.$

$$S_n = \frac{1-a^n}{1-a} \qquad \left. a^n \longrightarrow 0\right)^{(*)}$$

$$S_n \longrightarrow \frac{1}{1-a}$$

(*) $|a|<1$

$$a^n \xrightarrow{?} 0$$

$0<a<1 \Longrightarrow \quad a^n \longrightarrow 0$

$-1<a<0 \qquad |a^n| \xrightarrow{?} 0$

$$|a^n| = |a|^n \qquad -1<a<0$$

$0<|a|<1 \qquad \lim_{n\to\infty} |a|^n = 0)$.

(iii) $a \leq -1$ NON ESISTE $\downarrow$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1-a^n}{1-a}$$

esempio:

- $a=-1$ $$\frac{1-(-1)^n}{1-(-1)} = \frac{1-(-1)^n}{2} =$$

$$= \begin{cases} \to 1 & n \text{ dispari.} \\ \to 0 & n \text{ pari} \end{cases}$$

- $a=-2$ $$\frac{1-(-2)^n}{1-(-2)} = \frac{1-(-2)^n}{3}$$

$$a_{2n} = \frac{1-2^{2n}}{3} = \frac{1-4^n}{3} \to -\infty$$

$$a_{2n+1} \to +\infty .$$

Condizioni NECESSARIE per la convergenza:

Teorema:

Sia $\sum_{n=1}^{\infty} a_n$ una serie convergente $\Longrightarrow$ $a_n \longrightarrow 0$

dimostrazione:

$\sum_{n=1}^{\infty} a_n$ convergente $\Longrightarrow$

$s_n = \sum_{h=1}^{n} a_h$ è convergente

$\Longleftrightarrow$ s_n è di Cauchy.

$\forall \varepsilon > 0 \; \exists N > 0 : \; |S_n - S_m| < \varepsilon$

$\forall n, m > N.$

$m = n+1$

$\forall \varepsilon > 0 \; \exists N > 0 : \; |S_n - S_{n+1}| < \varepsilon$

$\forall \; n > N$

$S_n - S_{n+1} = a_1 + \dots + a_n - (a_1 + \dots + a_{n+1})$

$= -a_{n+1}$

$\forall \varepsilon > 0 \; \exists N > 0 : \; |a_{n+1}| < \varepsilon$

$\forall n > N \iff \; a_n \longrightarrow 0 \;. \;\#$

Oss.

$$\frac{n!}{n^n} = \underset{\le 1}{\frac{n}{n}} \cdot \underset{\le 1}{\frac{(n-1)}{n}} \cdot \underset{\le 1}{\frac{\cdots}{\cdots}} \cdots \left(\frac{1}{n}\right) \le \frac{1}{n}$$

Serie a termini di segno costante ($a_n \geq 0$)

convergenza $\sum_{n=1}^{\infty} a_n$ $(a_n \to 0)$?

Teorema (criterio del rapporto):

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \ell \qquad (a_n \geq 0)$$

$$(1) \quad \ell < 1 \implies \underbrace{\sum_{n=1}^{\infty} a_n < +\infty}_{\text{la serie converge}}$$

$$(2) \quad \ell > 1 \implies \underbrace{\sum_{n=1}^{\infty} a_n = +\infty}_{\text{diverge a } +\infty}$$

$$(3) \quad \ell = 1 \qquad ?$$

Oss.: Se $a_n \geq 0$

$$S_n = \sum_{h=1}^{n} a_h \quad \text{è crescente:}$$

$$S_{n+1} = S_n + a_{n+1} \geq S_n \quad \forall n \in \mathbb{N}$$

S_n crescente $\nearrow$ se è limitata è convergente

$\searrow$ diverge a $+\infty$.

Quindi se $\exists M>0$: $\left.\begin{array}{l} \\ S_n \leq M \end{array}\right\} \Rightarrow$ la serie converge.

Dimostrazione:

(1) $$\frac{a_{n+1}}{a_n} \longrightarrow \ell < 1$$

$$\forall \varepsilon > 0 \ \exists N > 0 : \ell - \varepsilon < \frac{a_{n+1}}{a_n} < \ell + \varepsilon$$

$\forall n > N$

Sia $\varepsilon > 0$ $\qquad \ell + \varepsilon < 1$

$$\exists N \in \mathbb{N} : \forall n \geq N \qquad \frac{a_{n+1}}{a_n} < \underbrace{\ell + \varepsilon}_{\overset{\wedge}{1}}$$

$\exists N \in \mathbb{N} \ \forall n \geq N$ $\boxed{l+\varepsilon < 1}$

$$a_{n+1} \leq (l+\varepsilon)\, a_n$$

$n = N$ $a_{N+1} \leq (l+\varepsilon)\, a_N$

$n = N+1$ $a_{N+2} \leq (l+\varepsilon)\, a_{N+1} \leq$

$$\leq (l+\varepsilon)^2 a_N$$

. . .

$h \in \mathbb{N}$ $a_{N+h} \leq (l+\varepsilon)^h a_N$

$$\sum_{h=1}^{\infty} a_{N+h} \leq a_N \underbrace{\sum_{h=1}^{\infty} (l+\varepsilon)^h}$$

ho maggiorato la serie con una serie geometrica convergente

($a = \ell + \varepsilon \in]0,1[\dots$).

(2) $\frac{a_{n+1}}{a_n} \longrightarrow \ell > 1$

"come" nel punto (1):

scelgo $\varepsilon > 0$: $\ell - \varepsilon > 1$

dalla def. di limite

$\exists N \in \mathbb{N} \ \forall n \geq N \quad \frac{a_{n+1}}{a_n} \geq \ell - \varepsilon$

$\forall h \in \mathbb{N} \quad a_{N+h} \geq a_N (l-\varepsilon)^h$

$$\sum_{h=1}^{\infty} a_{N+h} \geq a_N \underbrace{\sum_{h=1}^{\infty} (l-\varepsilon)^h}_{l-\varepsilon > 1}$$

serie geometrica divergente

a $+\infty$ $\quad (a = l-\varepsilon > 1)$.

Lezione 14/10:

$a_n \quad n \in \mathbb{N}$

$$a_1 + a_2 + .. + a_n + \ldots = \sum_{n=1}^{\infty} a_n$$

somme parziali: $S_n = \sum_{h=1}^{n} a_h =$

$= a_1 + a_2 + .. + a_n$

$$\begin{tikzcd} & \text{converge} \\ S_n \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \text{diverge a } +\infty \\ & \text{diverge a } -\infty \\ & \text{non esiste il limite} \end{tikzcd}$$

Teorema: $\sum_{n=1}^{+\infty} a_n$ converge $\Rightarrow a_n \xrightarrow[n\to\infty]{} 0$

Serie a termini positivi

$a_n \geq 0 \quad \forall n \in \mathbb{N}$

$$s_n = \sum_{k=1}^{n} a_k \quad \text{è crescente}$$

$$s_{n+1} = \underbrace{a_{n+1}}_{\geq 0} + s_n \geq s_n \quad \forall n \in \mathbb{N}$$

$\nearrow$ la serie converge ($\iff s_n$ è limitata)

$\searrow$ la serie diverge a $+\infty$.

Criterio del rapporto: $a_n \geq 0$

$\frac{a_{n+1}}{a_n} \longrightarrow \ell \qquad \Longrightarrow$

$\ell > 1 \Rightarrow$ la serie $\sum_{n=1}^{\infty} a_n$ diverge

$$\left(\sum_{n=1}^{+\infty} a_n = +\infty \right)$$

$\ell < 1 \Rightarrow$ la serie converge

$$\left(\sum_{n=1}^{\infty} a_n < +\infty \right)$$

$\ell = 1$??

Esempio: ① $a > 0$ $\qquad \sum_{n=1}^{\infty} \underbrace{a^n}_{a_n}$

(serie geometrica)

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{a^n} = a$$

$\nearrow$ $a \in]0,1[$ converg.

$\rightarrow$ $a > 1$ div.

$\searrow$ $a = 1$ div.

NON DA TEO. $\longrightarrow$ $a = 1$

② $\sum_{n=0}^{\infty} \frac{1}{n!}$ $a_n = \frac{1}{n!}$ $\boxed{0! = 1}$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!} \; n! = \frac{1}{n+1}$$

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \to +\infty} \frac{1}{n+1} = 0$$

$$0 < 1 \xRightarrow{\text{T. del rapporto}} \sum_{n=0}^{\infty} \frac{1}{n!} < +\infty$$

Affermazione: $\sum_{n=0}^{\infty} \frac{1}{n!} = e$

$$\left(1 + \frac{1}{n}\right)^n \leq \sum_{h=0}^{n} \frac{1}{h!} < 3$$

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}\,\frac{1}{n^k} =$$

$$= \sum_{k=1}^{n} \frac{1}{k!}\left(\frac{n}{n}\,\frac{(n-1)}{n}\cdots\frac{(n-k+1)}{n}\right)$$

$$1\,\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}_{\geq\ ?}$$

$$(1-b_1)(1-b_2)\cdots(1-b_k) \geq$$

$$\left[\,b_1, b_2, \dots, b_k \in \left]0,1\right[\,\right]$$

$$\geq 1 - b_1 - b_2 - \dots - b_k$$

$b_1, \dots, b_h \in]0,1[$

$(1-b_1)(1-b_2)\cdots(1-b_h) \geq 1-(b_1+b_2+\dots$

$\dots + b_h)$.

verifica per induzione

I) $h=1$ $\quad 1-b_1 \geq 1-b_1$ vera

II) $(1-b_1)\cdots(1-b_h) \geq 1-(b_1+\dots+b_h)$

$\overset{?}{\Longrightarrow} (1-b_1)\cdots(1-b_h)(1-b_{h+1}) \geq$

$1-(b_1+\dots+b_h+b_{h+1})$.

Moltiplico la disug. nell'ip. ind. per $(1-b_{h+1})$

$$(1-b_1)\cdots(1-b_h)(1-b_{h+1}) \geq$$

$$(1-(b_1+\dots+b_h))(1-b_{h+1}) =$$

$$= 1-(b_1+\dots+b_h+b_{h+1}) + \underbrace{b_{h+1}(b_1+\dots+b_h)}_{\substack{\vee \\ 0}}$$

$$> 1-(b_1+\dots+b_h+b_{h+1})$$

$$\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{h-1}{n}\right) \geq$$

$$\geq 1-\left(\frac{1}{n}+\frac{2}{n}+\dots+\frac{h-1}{n}\right)$$

$$= 1-\frac{1}{n}(\underbrace{1+2+\dots+h-1}_{\frac{(h-1)h}{2}})$$

$$\boxed{h \geq 2}$$

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!} \underbrace{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}$$

$$= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\geq 2 + \sum_{k=2}^{n} \frac{1}{k!} \left(\underset{\uparrow}{1} - \frac{1}{2n} k(k-1)\right)$$

$$= \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{2n} \sum_{k=2}^{n} \frac{1}{k!} k(k-1)$$

$$= \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{2n} \sum_{k=2}^{n} \frac{1}{(k-2)!}$$

$$\underbrace{\sum_{k=0}^{n}\frac{1}{k!}}_{\to m} - \overset{0}{\overset{\uparrow}{\boxed{\frac{1}{2n}}}}\overbrace{\sum_{k=2}^{n}\frac{1}{(k-2)!}}^{\to m} \le$$

$$\le \underbrace{\left(1+\frac{1}{n}\right)^{n}}_{\substack{\downarrow \; n\to+\infty \\ e}} \le \underbrace{\sum_{k=0}^{n}\frac{1}{k!}}_{\substack{\downarrow \; n\to+\infty \\ m}} \le 3$$

Confronto

$$\Longrightarrow \quad e = m.$$

$$\sum_{n=0}^{+\infty}\frac{1}{n!} = e$$

\#

Teorema (criterio della radice):

$$a_n \geq 0 \qquad \sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = l \begin{cases} \to l<1 & \sum_{n=1}^{+\infty} a_n < +\infty \\ \to l>1 & \sum_{n=1}^{\infty} a_n = +\infty \\ \to l=1 & ?? \end{cases}$$

dimostrazione:

(i) $l<1 \qquad \underbrace{\sqrt[n]{a_n} \to l}$

fisso $\varepsilon > 0$: $l+\varepsilon < 1$

$\exists N \in \mathbb{N} : \forall n \geq N \quad \underbrace{\sqrt[n]{a_n} < l+\varepsilon}$

$a_n < (l+\varepsilon)^n \iff$

$\exists N \in \mathbb{N} \quad \forall n \geq N : \quad a_n \leq (\ell+\varepsilon)^n$

$\ell + \varepsilon \in \,]0, 1[$

$$\sum_{n=N}^{+\infty} a_n \leq \sum_{n=N}^{+\infty} (\ell+\varepsilon)^n < +\infty$$

$\uparrow$

serie geometrica convergente

$$\xRightarrow{\text{confronto}} \sum_{n=N}^{+\infty} a_n < +\infty$$

$$\left(\sum_{n=1}^{+\infty} a_n < +\infty \right)$$

II : $\sqrt[n]{a_n} \longrightarrow l > 1$

Sia $\varepsilon > 0$ $\quad l - \varepsilon > 1 \implies$

$\exists N \in \mathbb{N} \quad \forall n \geq N \quad \sqrt[n]{a_n} > l - \varepsilon$

$\left(\iff \quad a_n > (l-\varepsilon)^n \right)$

$$\sum_{n=N}^{+\infty} a_n \geq \underbrace{\sum_{n=N}^{+\infty} (l-\varepsilon)^n}$$

$l - \varepsilon > 1 \implies$ serie geometrica divergente $\overset{\text{confronto}}{\implies} \sum_{n=N}^{+\infty} a_n = +\infty$

$\left(\sum_{n=1}^{\infty} a_n = +\infty \right)$. #

Esempio: $\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$

$a_n = \left(\frac{2}{n}\right)^n$ criterio delle radici

$$\sqrt[n]{a_n} = \frac{2}{n} \xrightarrow[n\to+\infty]{} 0 \ (<1)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n < +\infty . \ \#$$

Divagazione:

$a_1 = 1 \quad a_{n+1} = 2a_n + 1$

per induzione $a_n = 2^n - 1$

$a_1 = 1 \qquad \underbrace{a_{n+1} = 2a_n + 1}_{} \qquad a_n \overset{?}{=} 2^n - 1$

I passo: $n = 1$

$$a_1 = 1 \quad = \quad 2^1 - 1 = 1 \quad \text{ok}$$

II passo: $a_n = 2^n - 1 \Longrightarrow$

$$a_{n+1} = 2^{n+1} - 1$$

$a_n = 2^n - 1 \qquad a_{n+1} = 2a_n + 1$

$= 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 =$

$= 2^{n+1} - 1 \iff a_{n+1} = 2^{n+1} - 1$ #

La serie armonica:

$$\sum_{n=1}^{\infty} \frac{1}{n} < +\infty \iff S_n = \sum_{k=1}^{n} \frac{1}{k}$$

è convergente $\iff S_n$ di Cauchy.

$$\iff \forall \varepsilon > 0 \ \exists N > 0 \ \forall n, m > N$$

$$(n<m) \qquad |S_n - S_m| < \varepsilon$$

$$|S_n - S_m| = \Big| \overbrace{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}^{S_n} - \Big(\overbrace{1+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{m}}^{S_m}\Big) \Big| = \frac{1}{n+1}+\dots+\frac{1}{m}$$

$$|S_n - S_m| = \frac{1}{n+1} + \dots + \frac{1}{m}$$

$$m = 2(n+1) = 2n+2$$

$$|S_n - S_{2n+2}| = \frac{1}{n+1} + \dots + \frac{1}{2(n+1)}$$

$$\geq (n+1) \frac{1}{2(n+1)} = \frac{1}{2}$$

fisso $\varepsilon = \frac{1}{10} \to N$

$n > N \quad (\Rightarrow \; 2n+2 > N)$

$\frac{1}{10} > \frac{1}{2}$ falso

$\Rightarrow$ S_n non è di Cauchy

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} = +\infty .$$

Serie armonica generalizzata

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \qquad \alpha > 0$$

$$\left(\Leftarrow \frac{1}{n^\alpha} \longrightarrow 0 \right)$$

$0 < \alpha < 1$ $\qquad \dfrac{1}{n^\alpha} \geq \dfrac{1}{n}$ (confronto)

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$\alpha > 1$?

Teorema (criterio del 2^n):

$a_n \geq 0$ a_n decrescente

$a_n \to 0$ $\sum_{n=1}^{\infty} 2^n a_{2^n} < +\infty$

$$\Rightarrow \sum_{n=1}^{\infty} a_n < +\infty .$$

Applicazione: $\alpha > 1$ $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$

$$\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^\alpha} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{\alpha-1}} \right)^n$$

$$\frac{1}{2^{\alpha-1}} < 1 \Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{2^{\alpha-1}}\right)^n$$

serie geometrica convergente

$$\xRightarrow{\text{criterio } 2^n} \quad \sum_{n=1}^{\infty}\frac{1}{n^{\alpha}} \qquad (\alpha > 1)$$

è convergente.

$\neq 1$

Dimostrazione del criterio del 2^n:

$$a_1 + \underbrace{a_2 + a_3}_{} + \underbrace{a_4 + a_5 + a_6 + a_7}_{} + \underbrace{a_8 + \dots + a_{15}}_{} + \dots$$

$$\le a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$

$$\underbrace{a_{2^n} + \dots + a_{2^{n+1}-1}}_{} \le a_{2^n} 2^n$$

$2^n = 2^{n+1} - 1 - 2^n + 1$ termini

Lezione 15/10:

Giovedì dalle 13:15 aula CREMONA (presso Dip. di Matematica): ESERCIZI.

Serie (senza condizioni di segno):

- la condizione necessaria per la convergenza ($a_n \longrightarrow 0$) deve valere.

Def.: la serie $\sum_{n=1}^{\infty} a_n$ si dice ASSOLUTAMENTE convergente se è convergente $\sum_{n=1}^{\infty} |a_n|$.

Teorema: Se $\sum_{n=1}^{\infty} a_n$ è assolutamente convergente allora è convergente.

dimostrazione:

$\sum_{n=1}^{\infty} a_n$ è convergente $\Longleftrightarrow$

$S_n = \sum_{h=1}^{n} a_h$ succ. di Cauchy

Ipotesi $\sum_{n=1}^{\infty} |a_n|$ è conv.

$\Longleftrightarrow$ $S_n = \sum_{h=1}^{n} |a_h|$ è di Cauchy $\overset{?}{\Longrightarrow}$ $\sum_{h=1}^{n} a_h$ è una succ. di Cauchy.

S_n di Cauchy $\Longleftrightarrow$

$\forall \varepsilon > 0 \; \exists N > 0 : \forall n, m > N$

$$|S_n - S_m| < \varepsilon$$

$$n < m \quad \underbrace{|S_n - S_m|} =$$

$$= \big||a_1| + .. + |a_n| - |a_1| - |a_2| - \dots - |a_n| - |a_{n+1}| - .. - |a_m|\big|$$

$$= |a_{n+1}| + .. + |a_m| \geq$$

↑ disug. triangolare

$$\underbrace{\geq |a_{n+1} + \dots + a_m|}$$

$\forall \varepsilon > 0 \ \exists N > 0 : \forall n, m > N \quad (n<m)$

$$\left| a_{n+1} + \dots + a_m \right| < \varepsilon$$

$$\shortparallel$$

$$\left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right|$$

$\iff \displaystyle\sum_{k=1}^{n} a_k$ è d.

Cauchy

#

Serie convergente $\not\Rightarrow$ Serie assolutamente conv. Falso.

Esempio:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

conv. assoluta:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverge a $+\infty$

Teorema (criterio di Leibniz):

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{è conv.}$$

dim:

$$S_{2n} = \sum_{h=1}^{2n} \frac{(-1)^h}{h} = S_{2n-1} + \underbrace{\frac{(-1)^{2n}}{2n}}_{\overset{\vee}{0}}$$

$$S_{2n} > S_{2n-1}$$

- S_{2n} è una succ. decresc.

•• S_{2n-1} è una succ. cresc.

- $S_{2(n+1)} - S_{2n} =$

$$= \frac{(-1)^{2(n+1)}}{2(n+1)} + \frac{(-1)^{2n+1}}{2n+1} = \frac{1}{2(n+1)} - \frac{1}{2n+1}$$

$< 0 \iff S_{2n}$ è decrescente

•• S_{2n-1} è crescente.

cresc. $S_{2n-1} < S_{2n}$

decresc.

$$S_{2n-1} \longrightarrow l \leq m \longleftarrow S_{2n}$$

$$S_{2n-1} \leq l \leq m \leq S_{2n}$$

$$S_{2n-1} \leq l \leq m \leq S_{2n}$$

$$0 \leq m - l \leq S_{2n} - S_{2n-1} =$$

$$= a_{2n} = \frac{(-1)^{2n}}{2n} = \frac{1}{2n}$$

$$0 \leq m - l \leq \frac{1}{2n} \xrightarrow[n \to +\infty]{} 0$$

$$\Rightarrow \quad m - l = 0 \; (\Leftrightarrow m = l)$$

#

NUMERI COMPLESSI:

$$x^2+1=0 \iff x^2=-1$$

i unità immaginaria

$$i^2=-1$$

Considerazione: $t>0$

$$\frac{1}{1+t}=a_0+a_1t+a_2t^2+\dots$$

$$a_0=?\quad a_1=?\quad \dots$$

$$1=\underbrace{(1+t)(a_0+a_1t+a_2t^2+\dots)}_{}$$

$$1 = a_0 + t\,(a_0 + a_1) + t^2(a_1 + a_2)$$

$$+ \dots + t^n(a_{n-1} + a_n) + \dots$$

$$a_0 = 1 \qquad a_0 + a_1 = 0 \quad \dots \quad a_{n-1} + a_n = 0$$

$$\Big[\quad \underbrace{2}_{} = \underbrace{a}_{} + \underbrace{bt}_{} \qquad a = ? \quad b = ?$$

$$a = 2 \qquad b = 0$$

$$a_0 = 1 \qquad a_1 = -1 \qquad a_2 = 1 \quad \dots$$

$$a_n = (-1)^n$$

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots = \sum_{n=0}^{+\infty} (-1)^n t^n$$

$$t = x^2$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$$ per $x=1$ non esiste

↑ ↑ geometrica

serie converge se $|x|<1$

$\frac{1}{2}$ per $x=1$

(riprenderemo ...)

$$\mathbb{C} = \{ z = a + i b \ : \ a, b \in \mathbb{R} \} \qquad i^2 = -1$$

a = parte reale di z
b = parte immaginaria

$\mathbb{C}$ è un campo (come $\mathbb{R}$)

$z = a + ib \qquad z' = a' + ib'$

$z + z' = (a + a') + i(b + b')$

$zz' = (a + ib)(a' + ib') =$

$aa' + iab' + iba' + \overset{=-1}{\boxed{i^2}}\, bb'$

$= (aa' - bb') + i\,(ab' + ba')$

$\mathbb{C}$ un campo $\Rightarrow$ $z \neq 0$

$\exists\ z^{-1}$

Def: (i) $z = a + ib$

il coniugato di z

$$\overline{z} = a - ib$$

(ii) modulo di un numero complesso $z = a + ib$

$$|z| = \sqrt{a^2 + b^2} \quad (\geq 0)$$

Conto:

$$z\bar{z} = (a+ib)(a-ib) =$$

$$= a^2 - i\cancel{ab} + i\cancel{ab} - i^2 b^2$$

$$= a^2 + b^2 = |z|^2$$

Se $z \neq 0 \iff |z| \neq 0$

$$z\left(\frac{\bar{z}}{|z|^2}\right) = 1$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Esercizio: $z = \underset{\text{dati}}{\underset{\uparrow}{a}} + i\underset{\uparrow}{b} \neq 0$

$z^{-1} = c + id$ $\qquad c = ? \quad d = ?$

$z\, z^{-1} = 1$

$$(a+ib)(c+id) = \underset{\uparrow}{1} + i \cdot \underset{\uparrow}{0}$$

$$\begin{cases} ac - bd = 1 \\ ad + bc = 0 \end{cases}$$

È sistema 2×2 $\quad c, d$

$\ldots \qquad c = \dfrac{a}{a^2+b^2} \qquad d = \dfrac{-b}{a^2+b^2}$

Esercizio: trovare la radici

di $4x^2+x+1=0$

$$x=\frac{-1\pm\sqrt{1-16}}{8}=$$

$$=\frac{-1\pm\sqrt{-15}}{8}$$

formale: $\sqrt{(-1)(15)}=\underbrace{\sqrt{-1}}_{i}\sqrt{15}$

$$x=\frac{-1\pm i\sqrt{15}}{8}$$

Interpretazione geometrica:

$$\mathbb{C} = \{ z = a + ib \ : \ a, b \in \mathbb{R} \}$$

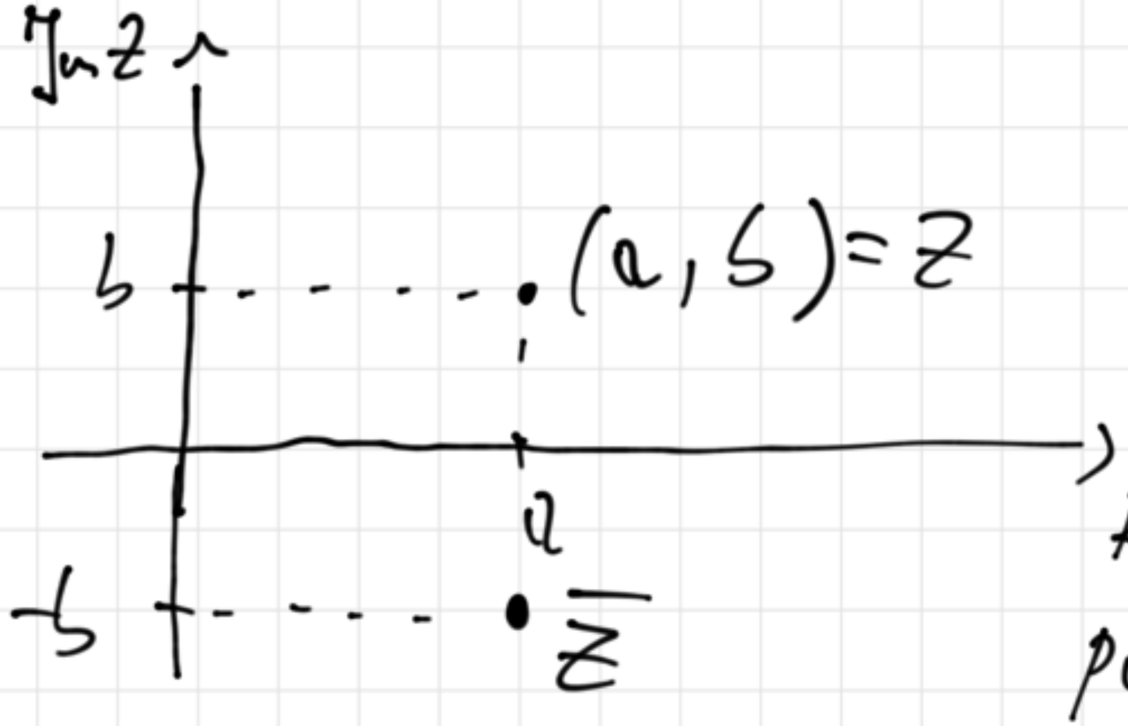
$$\mathbb{R} = \{ z = a + i0 : \ a \in \mathbb{R} \}$$

$$\mathbb{R} \subset \mathbb{C} .$$

$$z \simeq (a, b) \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

[A e B insiemi:

$$A \times B := \{ (a, b) \ : \ a \in A \ \ b \in B \}]$$



$z = a + ib$

$\simeq (a, b)$

Re z

parte reale di z

$\bar{z} = a - ib$

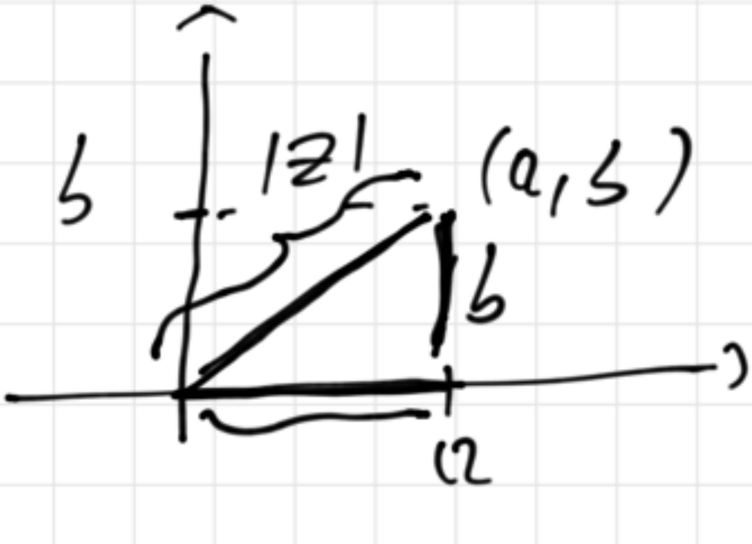
(a, b)

$(a, -b)$

$|z| = \sqrt{a^2 + b^2}$

$\parallel$

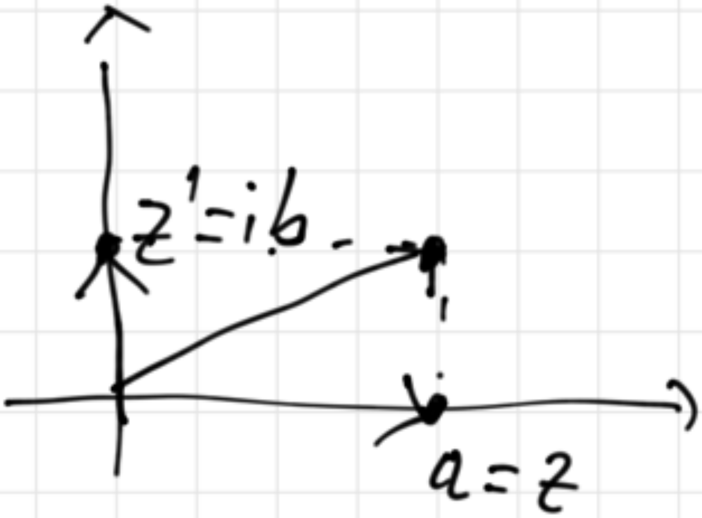
distanza di (a, b) da $(0, 0)$

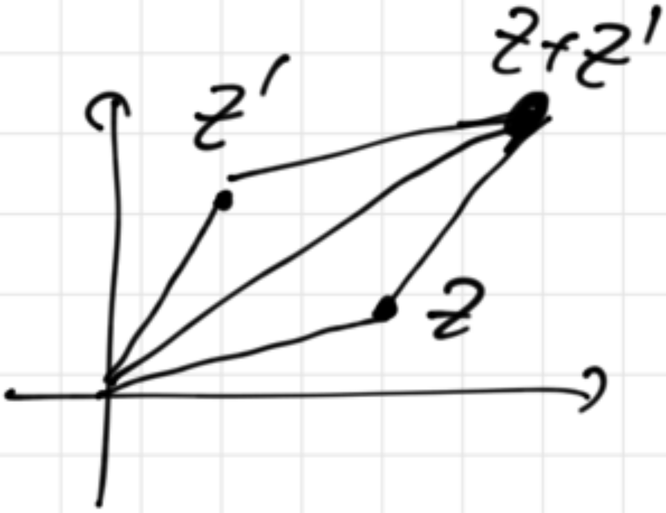


$z + z' =$ $\qquad$ $z = a$ $\qquad$ $\underbrace{z' = ib}_{b \neq 0}$

puramente immaginario

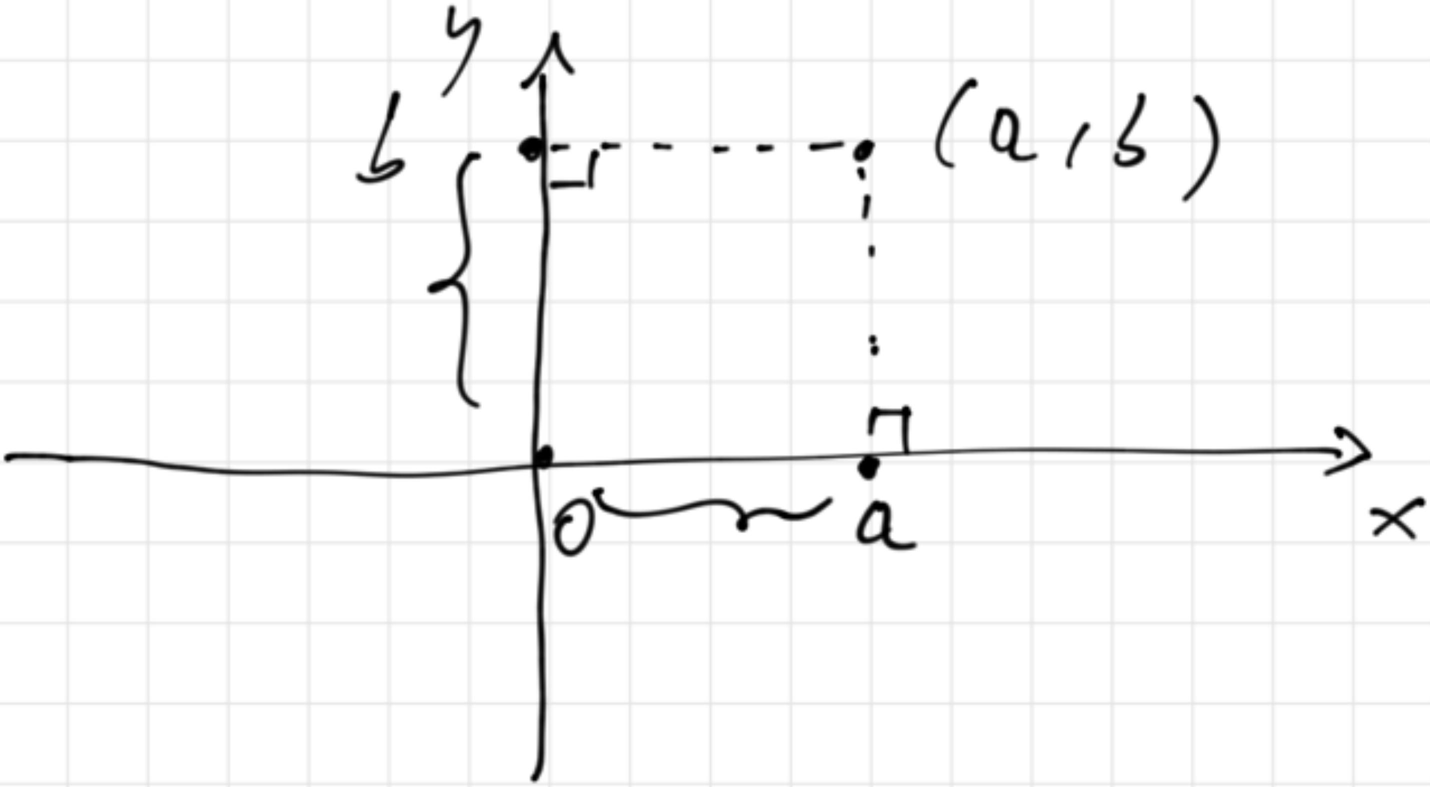
$= a + ib$

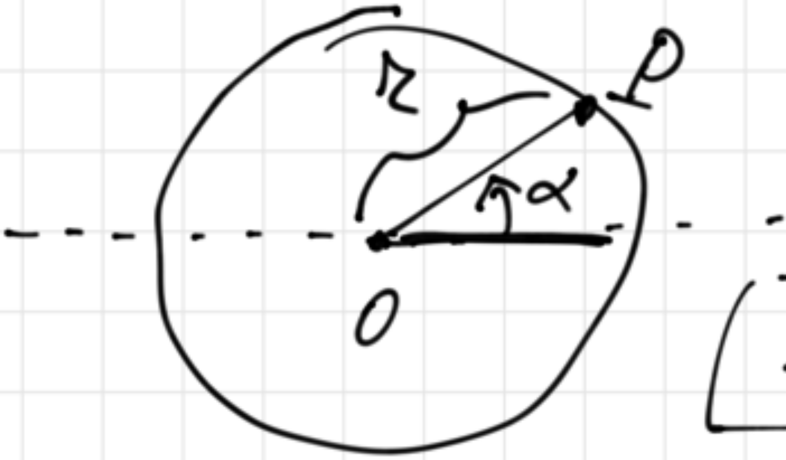




#

Coordinate polari:

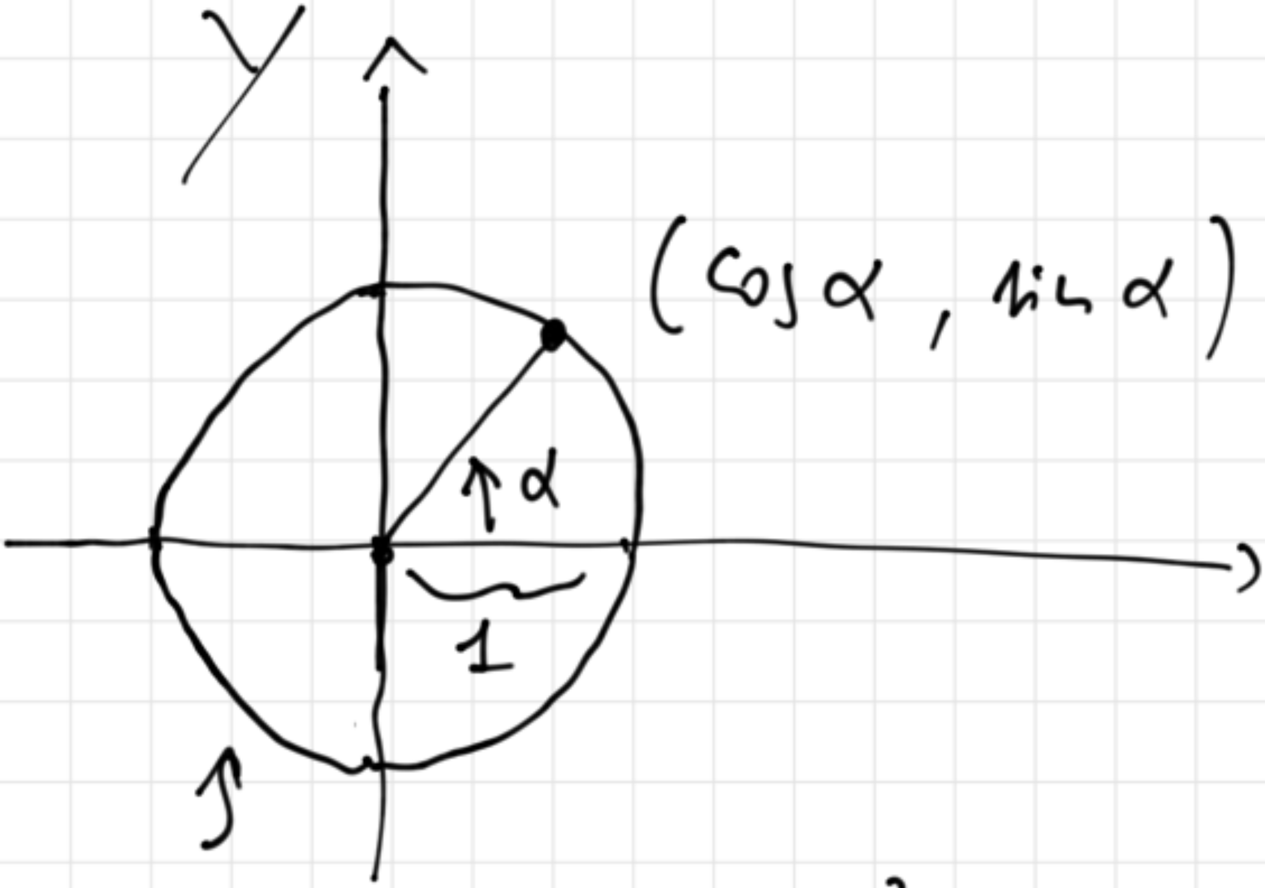




$$\boxed{360^\circ = 2\pi}$$

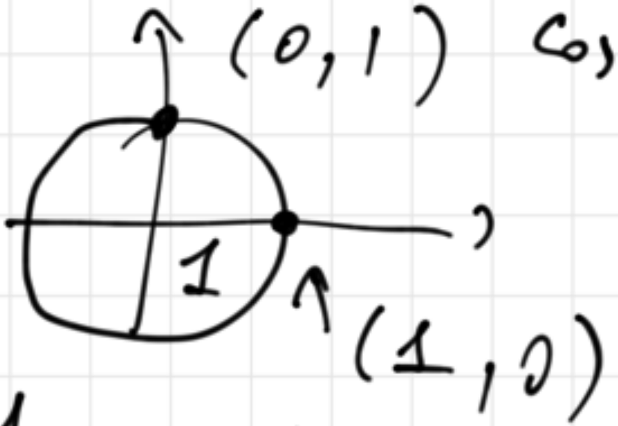
$r \geq 0$ $\alpha \in [0, 2\pi]$

Seno e coseno:



$$C = \{(X,Y) \in \mathbb{R}^2 : X^2 + Y^2 = 1\}$$

$\alpha = 0$

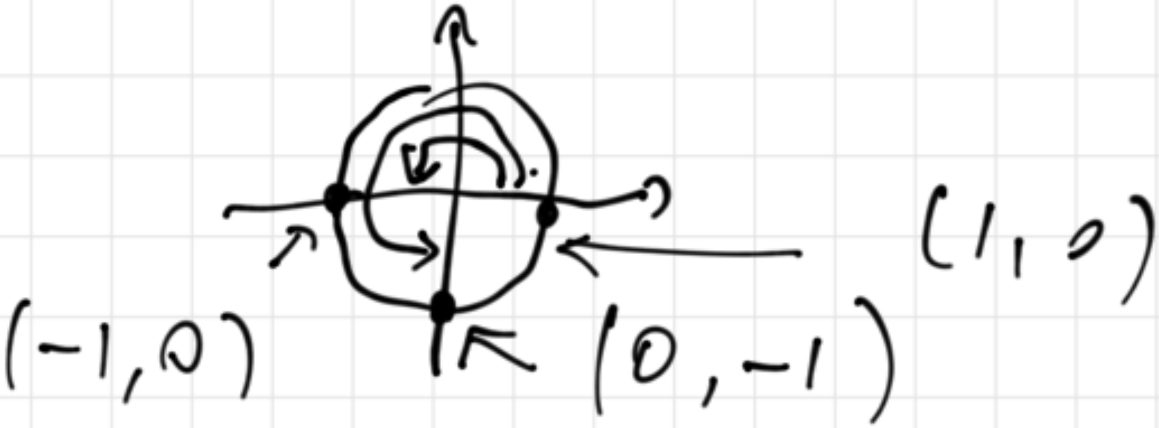


$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$

$\cos 0 = 1$

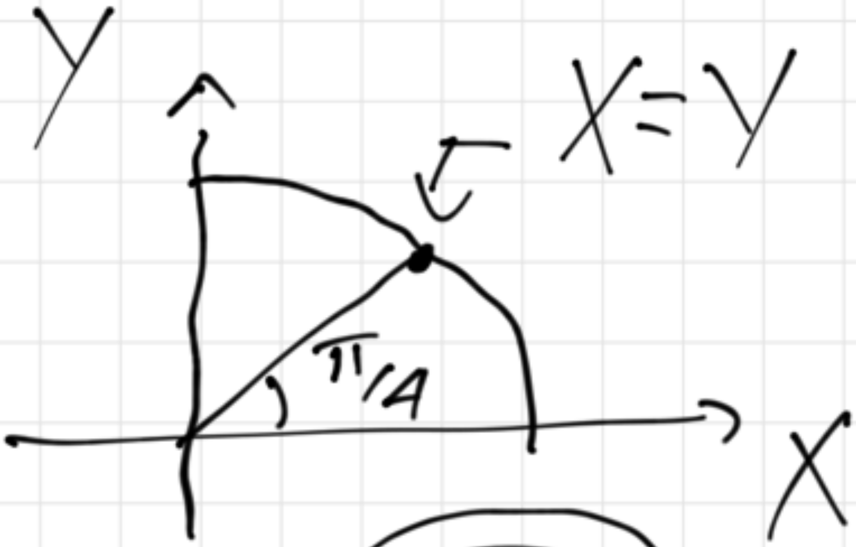
$\sin 0 = 0$



$\cos \pi = -1 \quad \sin \pi = 0$

$\cos\left(\frac{3}{2}\pi\right) = 0 \quad \sin\left(3/2\,\pi\right) = -1$

$\cos(2\pi) = 1 \quad \sin(2\pi) = 0 \quad \ldots$



$\frac{\pi}{4}\ (45°)$

$$\left(\cos\frac{\pi}{4}\right)^2 + \left(\sin\frac{\pi}{4}\right)^2 = 1$$

$$2\left(\cos\frac{\pi}{4}\right)^2 = 1$$

$$\cos\frac{\pi}{4} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

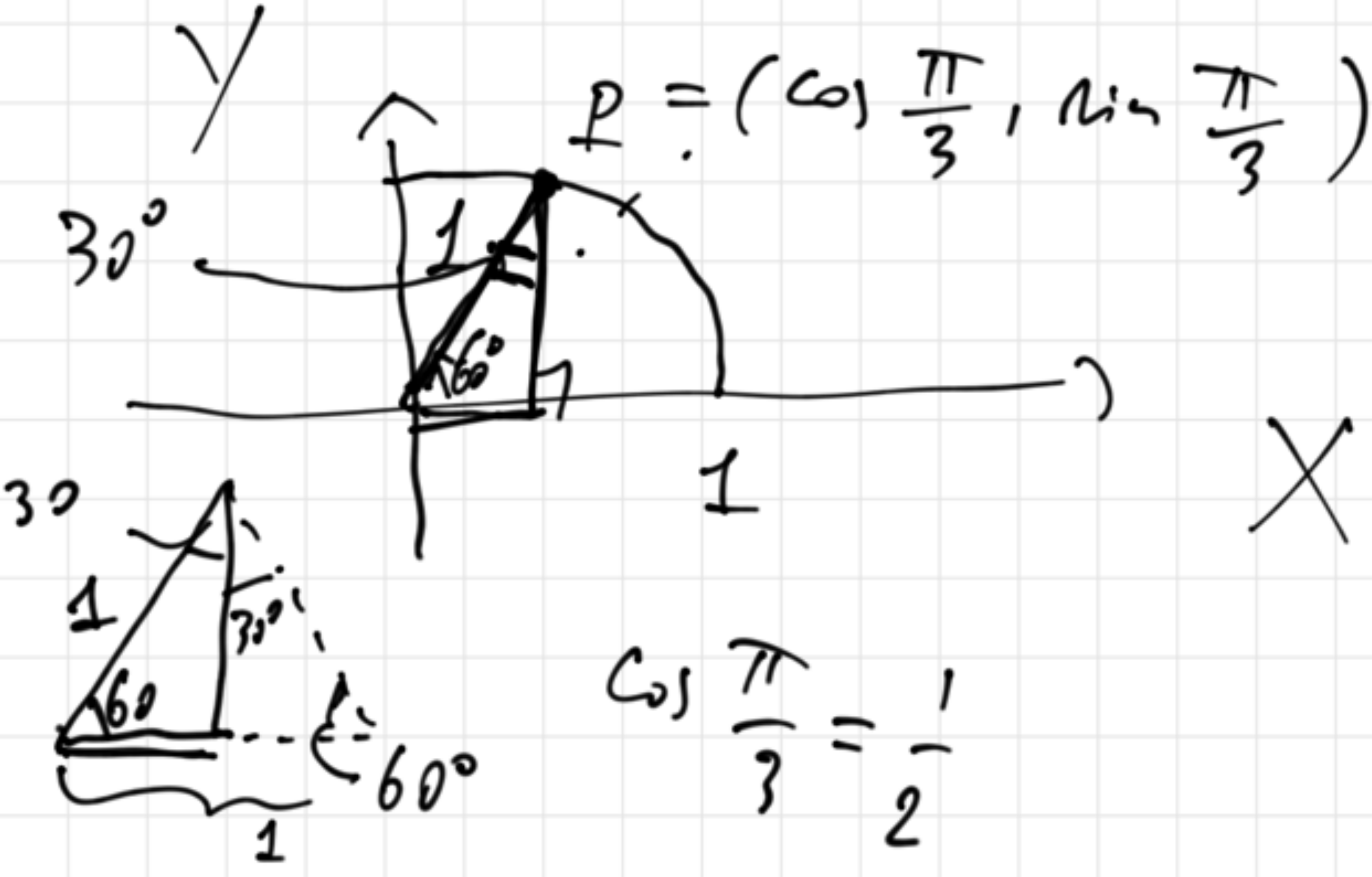
dalla figura $\cos\frac{\pi}{4}, \sin\frac{\pi}{4} > 0$

(I quadrante)

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$30^\circ \qquad \frac{\pi}{6} < \frac{\pi}{4}$

$60^\circ \qquad \frac{\pi}{3} > \frac{\pi}{4}$



$P = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$$\left(\sin\frac{\pi}{3}\right)^2 + \left(\cos\frac{\pi}{3}\right)^2 = 1$$

$$\left(\sin\frac{\pi}{3}\right)^2 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

$$\sin\frac{\pi}{3} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

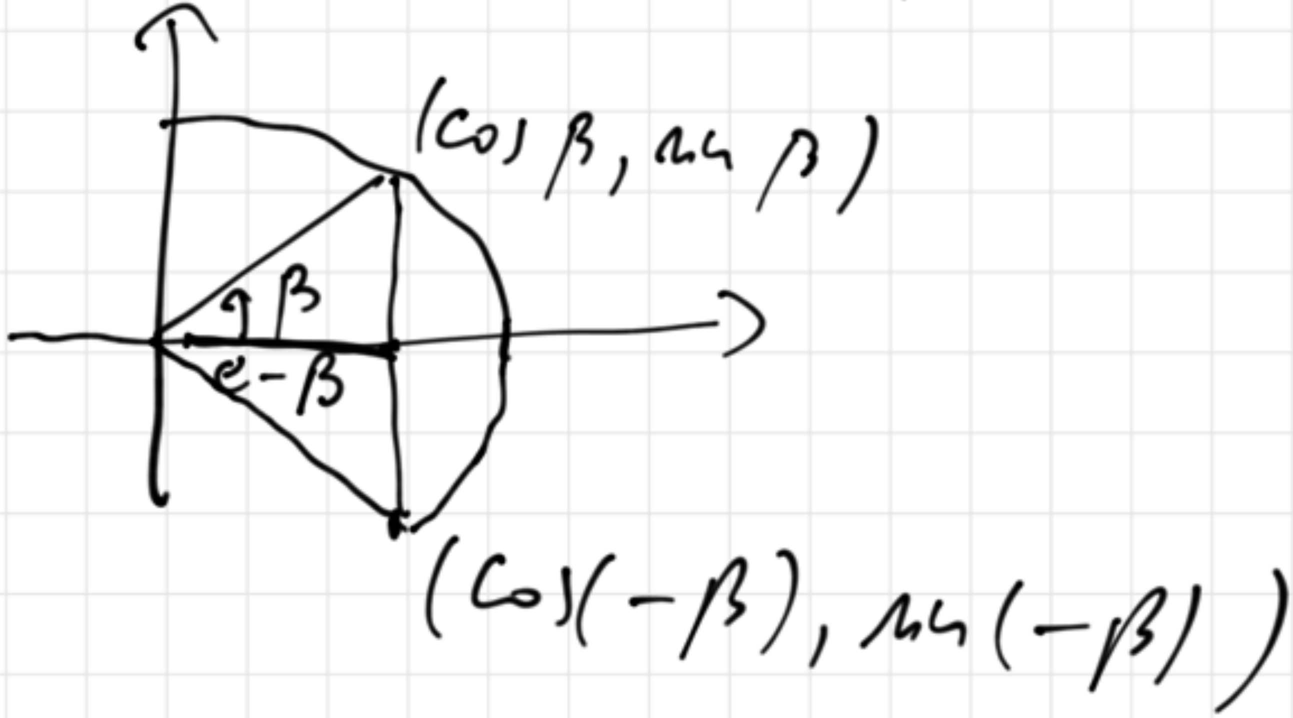
I quadrante

+ $\Downarrow$

$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\,\cos(-\beta) - \sin\alpha\,\sin(-\beta)$$



$$\cos(-\beta) = \cos\beta \qquad \sin(-\beta) = -\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\,\cos\beta + \sin\beta\,\cos\alpha$$

$$\sin(\alpha-\beta) = \sin\alpha\,\cos\beta - \sin\beta\,\cos\alpha$$

$\sin x - \sin y$ $\qquad$ x, y dati

$$\begin{cases} x = \alpha + \beta \\ y = \alpha - \beta \end{cases} \qquad \alpha = \frac{x+y}{2} \qquad \beta = \frac{x-y}{2}$$

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) =$$

$$\cancel{\sin\alpha\cos\beta} + \sin\beta\cos\alpha -$$

$$-\left(\cancel{\sin\alpha\cos\beta} - \sin\beta\cos\alpha\right)$$

$$= 2\sin\beta\cos\alpha$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$