Mirror Symmetry for Fano varieties

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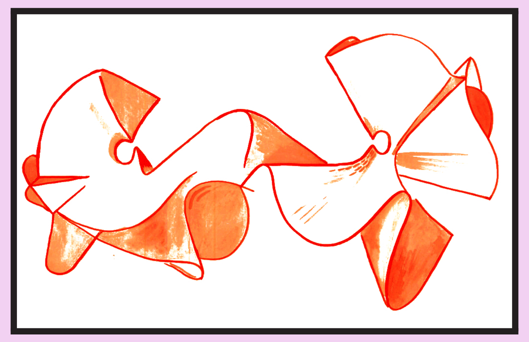
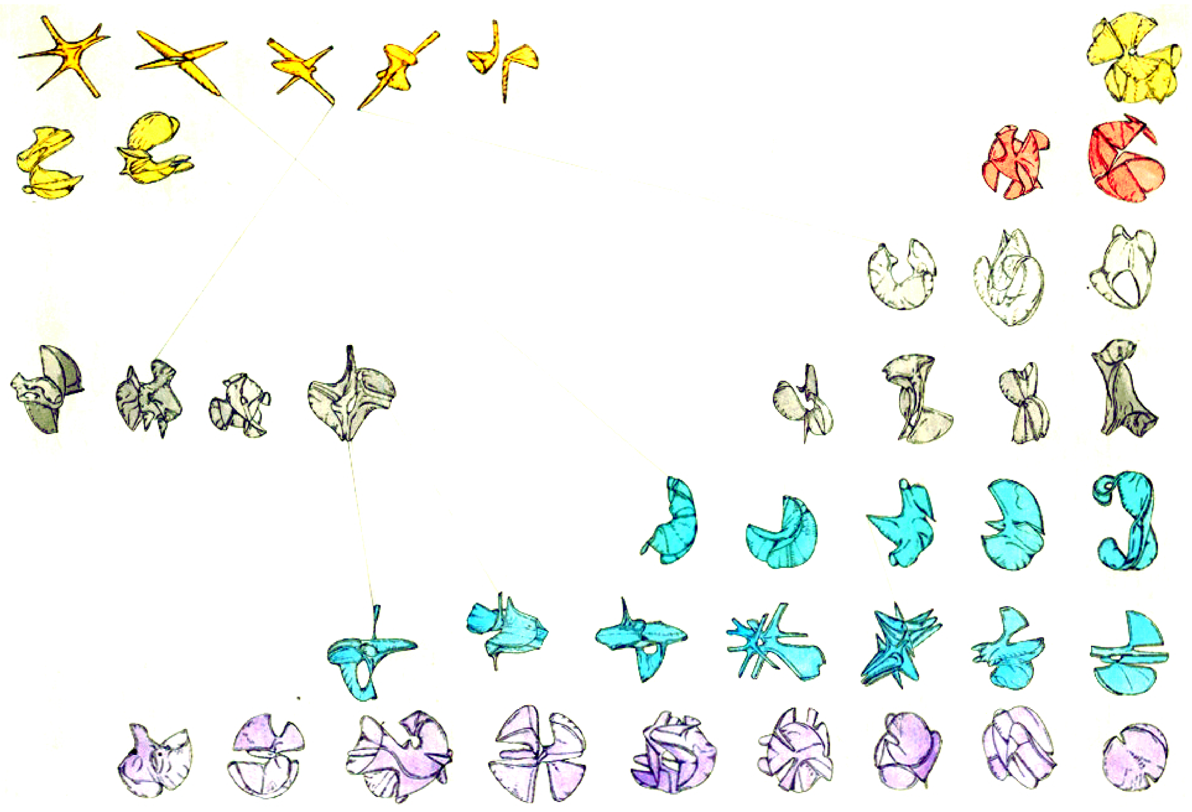


Fano varieties

Fano varieties are a special class of complex varieties, which are among the basic **building blocks** of algebraic geometry. In every dimension the number of deformation families of Fano varieties is finite, but a complete classification has been obtained in the following examples only.

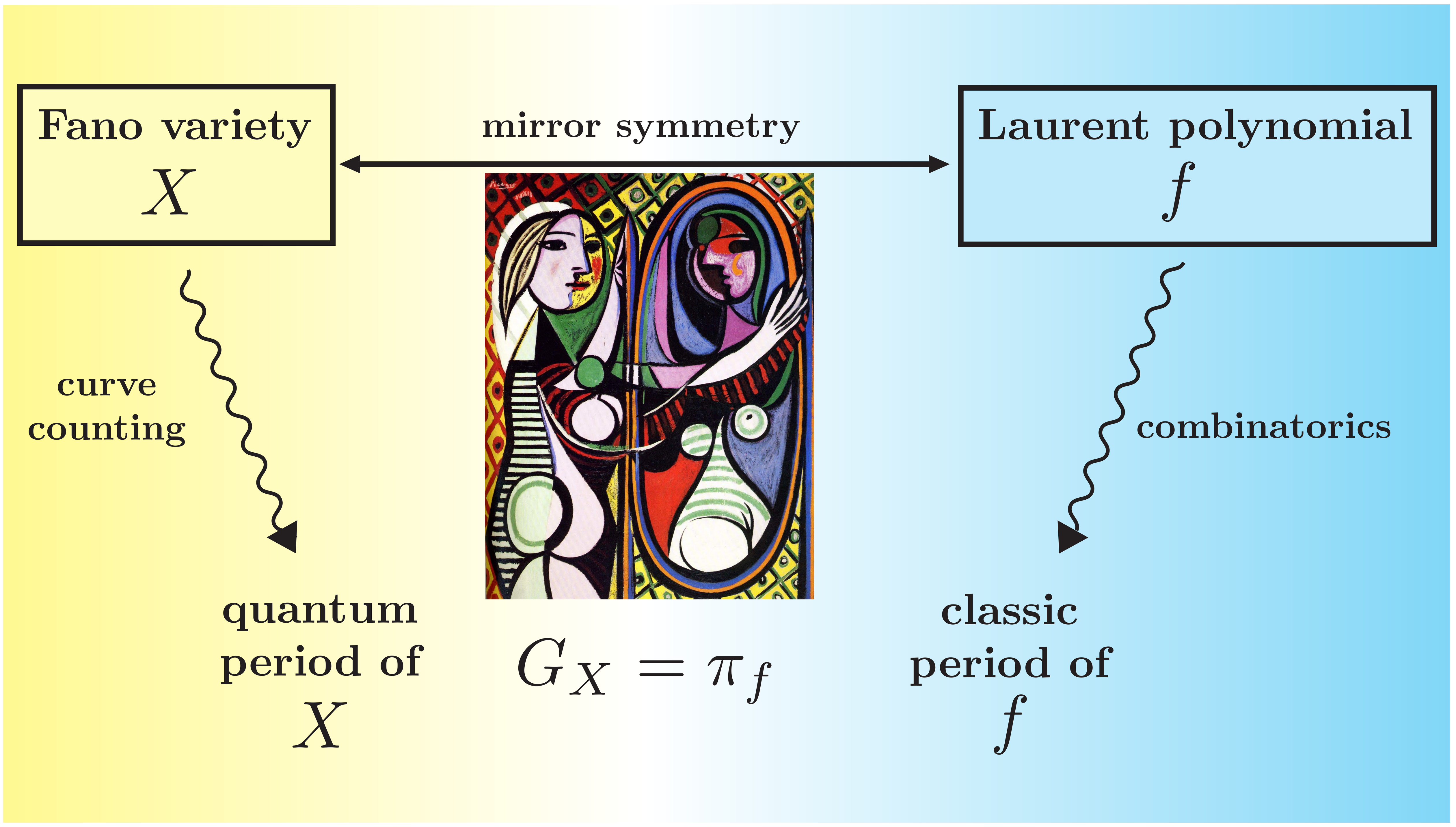
- The Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ is the unique Fano variety of dimension 1.
- The smooth Fano varieties of dimension 2 are 10 (e.g. $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$).
- The smooth Fano varieties of dimension 3 are 105 (Iskovskikh 1977, Mori–Mukai 1981).



Question. What happens **in higher dimension**? What about **singular Fano varieties**?

Fano varieties and Laurent polynomials

Idea ([2]). Use mirror symmetry to classify Fano varieties.



A Fano variety X of dimension n is **mirror** to a Laurent polynomial f in n variables if the quantum period of X equals the classic period of f, i.e. $G_X(t) = \pi_f(t)$.

Mirrors to smooth Fano varieties of dimension 2 and 3 have been found in [3]. Conjectures about some singular Fano varieties of dimension 2 appear in [1]. The classification of smooth Fano varieties of dimension 4 is still an open problem.

Curve counting

The **quantum period** of a Fano variety X is a power series

$$G_X(t) := \sum_{d \geq 0} c_d t^d$$

where c_d is the number of rational curves in X of degree d satisfying certain conditions. It is a **generating function that counts curves** in X. The numbers c_d are thus some Gromov–Witten invariants of X of genus 0.

Laurent polynomials

A Laurent polynomial is a linear combination of products of positive and negative powers of some variables. If f is a Laurent polynomials in the variables $x_1, \dots, x_n$, then its **classic period** is defined to be

$$\pi_f(t) := \sum_{k \geq 0} \mathrm{coeff}_1(f^k) t^k$$

where $\mathrm{coeff}_1(f^k)$ is the coefficient of the monomial $1 = x_1^0 \cdots x_n^0$ in the kth power of f.

Mirror Symmetry

Mirror Symmetry consists of **predictions**, motivated by the work of physicists, about relations between mathematical objects which are not apparently related. There are many mirror symmetry type conjectures, but the structure of all of them is the following.

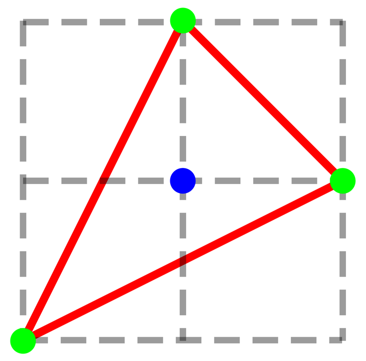
The starting point is two distinct classes of objects: $\mathcal{A}$ and $\mathcal{B}$. For every object $A \in \mathcal{A}$ in the first class one constructs a certain invariant ϕ_A, which may be a number, a vector space, a group, a power series, a category, etc. Then, one constructs a certain invariant ψ_B, for every object $B \in \mathcal{B}$ in the second class. Then the pair (A, B) is called a **mirror pair** if $\phi_A = \psi_B$.

A mirror symmetry theory is the study of mirror pairs. In other words, $\mathcal{A}$ and $\mathcal{B}$ are separate worlds that communicate only through a mirror.

An example

A mirror of the **projective plane** $\mathbb{P}^2_{\mathbb{C}}$ is $f = x_1 + x_2 + x_1^{-1} x_2^{-1}$. Since $\mathbb{P}^2_{\mathbb{C}}$ is the toric variety associated to the Newton polytope of f, the work of Givental implies the following equality.

$$G_{\mathbb{P}^2_{\mathbb{C}}}(t) = \sum_{d \geq 0} \frac{(3d)!}{d!^3} t^{3d} = \pi_f(t)$$



References

[1] M. Akhtar, T. Coates, A. Corti, L. Heuberger, A. Kasprzyk, A. Oneto, A. Petracci, T. Prince, K. Tveiten, *Mirror Symmetry and the Classification of Orbifold del Pezzo Surfaces*, arXiv:1501.05334.

[2] T. Coates, A. Corti, S. Galkin, V. Golyshev, A. Kasprzyk, *Mirror symmetry and Fano manifolds*, in European Congress of Mathematics Kraków, 2–7 July 2012, 2014 pp. 285–300.

[3] T. Coates, A. Corti, S. Galkin, A. Kasprzyk, *Quantum periods for 3-dimensional Fano manifolds*, arXiv:1303.3288.

[4] `http://coates.ma.ic.ac.uk/fanosearch/`

[5] `http://www.gemma-anderson.co.uk/maths.html`