METRIC NORMAL AND DISTANCE FUNCTION IN THE HEISENBERG GROUP

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ABSTRACT. We introduce a notion which is equivalent in the Heisenberg group $\mathbb{H}$ to that of segment normal to a surface. Then, we study some regularity properties of the function measuring the Carnot-Carathéodory distance from an Euclidean surface S in $\mathbb{H}$ in terms of the regularity of S.

CONTENTS

1. INTRODUCTION

Let $\mathbb{H} = \mathbb{H}^1$ be $\mathbb{R}^3$ with the Heisenberg group structure and let d be the associated Carnot-Charathéodory distance. If f is a real valued function defined on an open subset of $\mathbb{H}$, its *horizontal gradient* will be denoted by $\nabla_{\mathbb{H}} f$, moreover if S is a surface in $\mathbb{H}$ which is C^1 in the Euclidean sense, and Q is a point on S, we say that Q is *characteristic* for S if the space tangent to S at Q, $T_Q S$, contains all horizontal vectors at Q, and donote by $Char(S)$ the set of all characteristic points of S, (see below and Section 2 for the definition of this and other intrinsic objects of the analysis in $\mathbb{H}$). If S is a closed subset of $\mathbb{H}$, the distance from a point P to S is

$$d_E(P) = \inf_{Q \in E} d(P, Q).$$

If $S = \partial\Omega$ is the boundary of an open set Ω in $\mathbb{H}$, we define the **signed distance** from S as follows:

$$\delta_S(P) = \begin{cases} -d_S(P) & \text{if } P \in \Omega \\ d_S(P) & \text{if } P \notin \Omega. \end{cases} \tag{1}$$

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In this paper we prove the following result.

Theorem 1.1. *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H}-\overline{\Omega}$.*

(i) If S is $C^{1,1}$ in the Euclidean sense, then $\nabla_{\mathbb{H}}\delta_S$ is a continuous function in an open neighborhood of $S-Char(S)$ in $\mathbb{H}$.

(ii) If S is C^k in the Euclidean sense, $k\geq 2$, then $\nabla_{\mathbb{H}}\delta_S$ and δ_S are of class C^{k-1}, in the Euclidean sense, in an open neighborhood of $S-Char(S)$ in $\mathbb{H}$.

In Section 6 we show in fact a more general result, see Theorem 6.1, involving the notion of *set of positive reach*, see Section 5 for the definition.

In Euclidean space this kind of result was proved by Federer, [8], for the nonsigned distance, and by Krantz and Parks, [12], for the signed distance. The advantage of the signed distance over the nonsigned one is that it can be regular even on the surface S. For an updated survey in the Euclidean space, also containing new results, see [7].

Concerning the Heisenberg group and properties of the distance function from a set, we refer to the paper by Monti and Serra-Cassano [15] where, in particular, they proved that if S is a closed subset of $\mathbb{H}$, then δ_S satifies the Eikonal equation *a.e.*,

$$|\nabla_{\mathbb{H}}\delta_S|=1.$$

The proofs of the regularity results in Euclidean space rely on the following fact. If the hypersurface S is regular enough, then for each point Q in S there exists a segment n passing through Q which is normal to S and, moreover, the distance from any point P on n to S is realized by the distance between P and Q. In order to determine n, it suffices to know the vector normal to S at Q. These considerations extend to the case of a hypersurface in a Riemannian manifold M, since the correspondence $\gamma\mapsto\dot\gamma(0)$ establishes a 1-to-1 correspondence between geodesics γ such that $\gamma(0)=Q$ is fixed in M and vectors tangent to M at Q. This fact has no clear analogue in sub-Riemannian geometry, since, given a *horizontal vector* V at Q, a point in $\mathbb{H}$, there are infinitely many geodesics leaving Q and having speed V at Q. To overcome this difficulty, we define the notion of *metric normal* to a surface in $\mathbb{H}$.

Before proceeding, we fix some notation. In the realization we will work with, the Heisenberg group is the set of the triples (x,y,t) in $\mathbb{R}^3$ with the product

$$(x,y,t)\cdot(x',y',t')=(x+x',y+y',t+t'-2(xy'-x'y)).$$

A left invariant vector field V on $\mathbb{H}$ is *horizontal* if it is the linear combination of X and Y,

$$X=\partial_x+2y\partial_t,\ Y=\partial_y-2x\partial_t.$$

As usual, we identify a vector V at P with the unique left invariant vector field having value V at P. Hence, we can talk about *horizontal vectors.* The linear space of the horizontal vector fields is denoted by $\mathcal{H}$, while $\mathcal{H}_P$ is the linear space of the horizontal vectors in $T_P\mathbb{H}$, the space tangent to $\mathbb{H}$ at P. If S is a surface in $\mathbb{H}$ and it is smooth in the Euclidean sense, a point C in S is *characteristic* if $T_CS=\mathcal{H}_C$.

Let Q be a point in S. The *metric normal* to S at Q, $\mathcal{N}_QS$, is the set of those points P such that

$$d_S(P)=d(P,Q).$$

When S is a surface which is smooth enough, then $\mathcal{N}_QS$ reduces to the point Q, if Q is characteristic, and is a nontrivial geodesic arc through Q. In Theorem 4.1 we give a geometric description of $\mathcal{N}_QS$.

Theorem 1.2. *Let S be a smooth enough surface (e.g., by the results of Section 5, $C^{1,1}$), let Q be a non-characteristic point on S and let Π_QS be the Euclidean plane in $\mathbb{H}$ which is tangent to S at Q, in the Euclidean sense. Let C be the characteristic point on Π_QS (if Π_QS has none, below we set $d(Q,C)=\infty$). Then, $\mathcal{N}_QS$ is a nontrivial arc on the geodesic γ passing through*

Q such that (i) $\dot\gamma(0) = N_P S$ is the horizontal vector normal to S at Q; (ii) the maximal length over which γ is length minimizing in $\mathbb{H}$ is $\pi d(P,C)$.

The plane $\Pi_Q S$ is that tangent to S in our realization of $\mathbb{H}$. Geodesics in $\mathbb{H}$ can be grouped in equivalence classes of mutually isometric geodesics. Two geodesics γ_1 and γ_2 belong to the same equivalence class if and only if the maximal extention over which each geodesic is lenght minimizing is the same for γ_1 and γ_2, see Section 2.

This is the way in which the paper is structured. In Section 2, we give some preliminaries on the Heisenberg group. Section 3 and 4 are devoted to the determination of the metric normal for a Euclidean plane in $\mathbb{H}$, hence for a smooth surface. In Section 5 we prove some properties of the sets of *positive reach* in $\mathbb{H}$. Section 6 contains statements and proofs of various regularity results for the distance function. Finally, some properties of the cutlocus of a surface in $\mathbb{H}$ are discussed in Section 7.

In this paper we considered the Heisenberg group with the lowest dimension. This object is interesting in itself. For instance, it is related with the study of the isoperimetric inequality in the Euclidean plane [13]. More recently, it was realized that sub-Riemannian structures modelled on the lowest dimensional Heisenberg group can be used to model the human visual system, see [5], [6], [19] and the references quoted therein.

In [2] we use the results of this paper to study the *horizontal Hessian* of the distance function from a surfaces and some properties of the *mean curvature.*

2. Notation and preliminaries

In this section, we collect some basic definitions and known facts about the structure and the geometry of $\mathbb{H}$. There is a vast literature on sub-Riemannian geometry and Carnot groups. Justs to quote a few titles, we refer the reader to [4], [9], [10], [13], [16], [17], [18], [21].

The Heisenberg group $\mathbb{H} = \mathbb{H}^1$ is the Euclidean space $\mathbb{R}^3$ endowed with the noncommutative product

$$(x,y,t)\cdot(x',y',t') = (x+x', y+y', t+t'+2(x'y-xy')).$$

Sometimes it is convenient to think of the elements of $\mathbb{H}$ as $(z,t) \in \mathbb{C}\times\mathbb{R}$. The *Carnot-Charathéodory distance* in $\mathbb{H}$ is the sub-Riemannian metric that makes pointwise orthonormal the left invariant vector fields X and Y,

$$X = \partial_x + 2y\partial_t, \;\; Y = \partial_y - 2x\partial_t.$$

The vector fields X, Y do not commute, $[X,Y] = -4\partial_t$. The distance between two points P and Q in $\mathbb{H}$ is denoted by $d(P,Q)$. The span of the vector fields X and Y is called *horizontal distribution*, and it is denoted by $\mathcal{H}$. The *fiber* of $\mathcal{H}$ at a point P of $\mathbb{H}$ is $\mathcal{H}_P = \mathrm{span}\{X_P, Y_P\}$. The inner product in $\mathcal{H}_P$ is denoted by $\langle\cdot,\cdot\rangle$ and the associated norm by $|\cdot|$.

An important element of $\mathbb{H}$'s structure is the *dilation group at the origin* $\{\delta_\lambda : \lambda \neq 0\}$,

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \;\; z = x+iy$$

By left translation, a dilation group is defined at each point P of $\mathbb{H}$.

The Heisenberg group is also endowed with a *rotation group*, which is useful in simplifying some calculations. For $\theta\in\mathbb{R}$, let

$$R_\theta(z,t) = (e^{i\theta}, t)$$

be the rotation by θ around the t-axis. Composing with left translation, one could define rotations around any vertical line $(x,y) = (a,b)$. R_θ is an isometry of $\mathbb{H}$ and its differential acts on the fiber $\mathcal{H}_O$ as a rotation by θ. Under the usual identification between the Riemannian tangent space of $\mathbb{H}$ at O, $T_O\mathbb{H}$, and the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$, the differential of R_θ can be thought of as a rotation on $\mathrm{span}\{X,Y\}$, the first stratum of $\mathfrak{h}$. With respect to the basis $\{X,Y\}$,

$$dR_\theta V = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix} V$$

whenever $V \in \text{span}\{X, Y\}$. With some abuse of notation, we denote dR_θ by R_θ.

The distance between two points P and Q in $\mathbb{H}$ is defined as follows. Consider an absolutely continuous curve γ in $\mathbb{R}^3$, joining P and Q, which is *horizontal*. That is, $\dot\gamma(t) = a(t)X_{\gamma(t)} + b(t)Y_{\gamma(t)}$ lies in $\mathcal{H}_{\gamma(t)}$. Its Carnot-Charathéodory length is $l_{\mathbb{H}}(\gamma)$,

$$l_{\mathbb{H}}(\gamma) = \int (a(t)^2 + b(t)^2)^{1/2} dt$$

The Carnot-Charathéodory distance between P and Q, $d(P, Q)$, is the infimum of the Carnot-Charathéodory lengths of such curves. The infimum is actually a minimum, the distance between P and Q is realized by the length of a geodesic. By translation invariance, all geodesics are left translations of geodesics passing through the origin. The unit-speed geodesics at the origin [14], [13] are

$$\gamma_{O,\phi,W}(\sigma) = \begin{cases} x(\sigma) = \sin(\alpha(W))\frac{1-\cos(\phi\sigma)}{\phi} + \cos(\alpha(W))\frac{\sin(\phi\sigma)}{\phi} \\ y(\sigma) = \sin(\alpha(W))\frac{\sin(\phi\sigma)}{\phi} - \cos(\alpha(W))\frac{1-\cos(\phi\sigma)}{\phi} \\ t(\sigma) = 2\frac{\phi\sigma-\sin(\phi\sigma)}{\phi^2}. \end{cases} \tag{2}$$

Here, W is a unitary vector in $\mathcal{H}_O$ and $\alpha(W) \in [0, 2\pi)$ is unique with the property $\dot\gamma_{O,\phi,W}(0) = W$. $\phi \in \mathbb{R}$, and the geodesic is length minimizing over any interval of length $2\pi/|\phi|$. In the case $\phi = 0$, the geodesic is a straight line in the plane $\{t = 0\}$,

$$x(\sigma) = \cos(\alpha(W))\sigma, \ y(\sigma) = \sin(\alpha(W))\sigma,$$

and we say that the geodesic is *straight*.

¿From these equations we deduce the parametric equations of the boundary of the ball $B(0, r)$ and in particular $(z, t) \in \partial B(0, r)$ if and only if there is $\phi \in [-2\pi/r, 2\pi/r]$ so that

$$\begin{cases} |z| = 2\frac{\sin(\phi r/2)}{\phi} \\ t = 2\frac{\phi r-\sin(\phi r)}{\phi^2}. \end{cases} \tag{3}$$

If $P = (z, t)$ and $z \neq 0$, then there exists a unique length minimizing geodesic connecting P and O. If $P = (0, t)$, $t \neq 0$, (i.e., if P belongs to the center of $\mathbb{H}$) then there is a one parameter family of length minimizing geodesics joining P and O, obtained by rotation of a single geodesic around the t-axis.

Given points $P = (z, t)$ and $P' = (z', t')$, they are joined by a unique length minimizing geodesic, unless $z = z'$.

Let

$$\gamma_{P,\phi,\alpha} = L_P\gamma_{O,\phi,\alpha}$$

The parameter ϕ is geometric in the following sense: $2\pi/|\phi|$ is the length of $\gamma_{P,\phi,W}$ and $sgn(\phi)$ is positive if and only if the t-coordinate increases with σ. Recall that in $\mathbb{H}$ the orientation of the t-axis is an intrinsic notion, unlike the Euclidean space. If $\gamma_{P,\phi,W}$ and $\gamma_{P',\phi',W'}$ have an arc in common, then $\phi = \pm\phi'$, while no such easy relation exists for the parameter W. To change the orientation of a geodesic, observe that

$$\gamma_{P,\phi,W}(\sigma) = \gamma_{P,-\phi,-W}(-\sigma).$$

Unlike the Euclidean case, a geodesic γ leaving O is not determined by its tangent vector at the origin, $\dot\gamma(0) = W = \cos(\alpha(W))X_O + \sin(\alpha(W))Y_O$. The extra parameter we need is ϕ. Notice that ϕ is related to the dilation group as follows: $\delta_\lambda(\gamma_{O,\phi,W})$ is a reparametrization of the geodesic $\gamma_{O,\phi/\lambda,W}$. That is, all geodesics γ leaving O and having fixed initial velocity $\dot\gamma(0) = v$ in $\mathcal{H}_O$ are dilated of each other (with the exception corresponding to $\phi = 0$) but, contrary to the Euclidean case, a geodesic's dilated is a different geodesic. The case of the straight geodesics is the limiting one, corresponding to $\lambda \to 0$. In a precise sense, then, the set of non-straight geodesics at O is parametrized by the unit circle in $\mathcal{H}_O$ and by the dilation group, a feature of $\mathbb{H}$ with no Euclidean counterpart.

Let S be a surface in $\mathbb{H}$ which is C^1 in the Euclidean sense and such that, for some open Ω in $\mathbb{H}$, $S = \partial\Omega = \partial(\mathbb{H} - \overline{\Omega})$. We need some differential geometric notions about S.

Definition 2.1. *Let S be a surface in $\mathbb{H}$ as above and let $P \in S$ be a non-characteristic point. The Euclidean tangent space to S at P is denoted by $T_P S$. The* **direction tangent to S at P** *is the 1-dimensional space $V_P S = T_P S \cap \mathcal{H}_P$. The* **plane tangent to S at P**, *$\Pi_P S$, is the Euclidean plane in $\mathbb{H}$, tangent to S at P in the Euclidean sense. The* **direction normal to S** *at P is $N_P S = \mathcal{H}_P \ominus V_P S$.*

The **Pansu exterior normal to** *S at P, denoted by $N_P^+ S$, is the unique horizontal vector V in the direction $N_P S$ such that, denoted by ν the Euclidean exterior normal to S at P, $\langle V, \nu\rangle > 0$. Here, $\langle\cdot,\cdot\rangle$ is the Euclidean inner product.*

The **group tangent to S at P** [17] is the 2-dimensional vector space $G_P S = \mathcal{V}_P S \oplus \mathcal{T}$, where $\mathcal{T} = \{(0,0,t) : \ t \in \mathbb{R}\}$ is the center of $\mathbb{H}$ and $\mathcal{V}_P S$ is the one parameter subgroup of $\mathbb{H}$ generated by $V_P S$. One point we want to make in the present paper is that $G_P S$ does not seem to capture the complexity of the geodesics' set, while $T_P S$ does, in a precise sense.

The following facts are easily established by direct calculation.

Proposition 2.1. *Let S be a smooth surface in $\mathbb{H}$, implicitely defined by $g(x,y,t) = 0$. Let $P \in S$ be non-characteristic. Then,*

$$V_P S = span\{Yg \cdot X - Xg \cdot Y\}, \ N_P S = span\{Xg \cdot X + Yg \cdot Y\} = span\{\nabla_{\mathbb{H}} g\} \tag{4}$$

3. The metric normal to a plane and a sphere

In this section, we compute the metric normal for a plane and a sphere.

Definition 3.1. *Let E be a closed subset of $\mathbb{H}$, $P \in E$. The* **metric normal** *to E at P is the set $\mathcal{N}_P S$ of the points $Q \in \mathbb{H}$ such that $d(Q,E) = d(Q,P)$.*

See also [3], where a pathological occurrence of the metric normal to the unit sphere was used to study bi-Lipschitz functions in $\mathbb{H}$.

Lemma 3.1. *Let E be a closed subset of $\mathbb{H}$, $P \in E$. Let Q in $\mathcal{N}_P S$ and $\gamma : I \to \mathbb{H}$ be a length minimizing geodesic from Q to P. Then*

$$\gamma(I) \subseteq \mathcal{N}_P S.$$

Proof. Let A be any point in γ, then $d_S(A) \le d(A,P)$. If there were P' in S such that $d(A,P') < d(A,P)$, then by the triangle inequality

$$d(Q,P') \le d(Q,A) + d(P',A) < d(Q,A) + d(P,A) = d(Q,P),$$

contradicting $Q \in \mathcal{N}_P S$.

□

Theorem 3.1. *Let $\mathcal{P}$ be a plane in $\mathbb{H}$ and let $P \in \mathcal{P}$ be non-characteristic. Suppose that $\mathcal{P}$ has a characteristic point C and consider $\mathcal{P} = \partial\Omega$, where Ω is one of the half-spaces having $\mathcal{P}i$ as boundary. Then,*

$$\mathcal{N}_P \mathcal{P} = \gamma_{P, 2/d(P,C), N_P^+ \mathcal{P}}([-\frac{\pi}{2} d(P,C), \frac{\pi}{2} d(P,C)])$$

If $\mathcal{P}$ is a vertical plane, then $\mathcal{N}_P \mathcal{P}$ is the straight geodesic through P, in the direction $N_P \mathcal{P}$.

Another way to state this result is the following. The projection onto the $t = 0$ plane of γ is the circle c having as diameter the line joining C_1, the projection of C, and P_1, the projection of P.

Proof of the Proposition 3.1. The case when $\mathcal{P}$ is a vertical plane is elementary. Consider the case when $\mathcal{P}$ has a characteristic point.

In the statement of the proposition, everything is invariant under isometries, we can then left-translate everything so that $\mathcal{P}$ is the plane $t = 0$ and $C = O$. Eventually after a mirror symmetry $(z, t) \mapsto (z, -t)$, we can assume that Ω is the unique half-space having bounday $\mathcal{P}$ and containing a half line on the negative t-axis. Let $P \neq O$ be a point in $\mathcal{P}$. There exists a point $(0, 0, T)$ on the vertical axis, $T > 0$, so that a length-minimizing geodesic from $(0, 0, -T)$ to $(0, 0, T)$, say γ, intersects $\mathcal{P}$ at P. The existence of T with the stated properties is guaranteed by a simple continuity argument. By the equation of geodesics we deduce $T = \frac{\pi}{2}d(O, P)^2$. $\gamma = \gamma_{P,\phi,W}$ is uniquely determined. We need to show that the parameters of γ are those in the theorem's statement, and that γ is a parametrization of the metric normal.

Step 1: γ has the right parameter ϕ. Note that γ points upward, hence, by the geodesic equation we have $\phi \geq 0$. By symmetry with respect to $t = 0$, $d(P, (0, 0, T)) = \pi/\phi$, one half of the total length of γ. Inserting this in the equation for γ, and assuming after a rotation that P lies on the positive x-axis, we obtain that $P = (2/\phi, 0, 0)$. This shows that $d(P, 0) = 2/\phi$.

Step 2. γ lies in the metric normal. We show that $(0, 0, -T)$ belongs to $\mathcal{N}_P S$. Consider the ball $B = B((0, 0, -T), d(P, (0, 0, -T)))$. By maximizing the t coordinate in (3), we see that $B \cap \{(z, t) :\ t > 0\}$ is empty and that $\overline{B} \cap \{(z, t) :\ t > 0\}$ is the circle $c = \mathcal{P} \cap B(O, d(O, P))$ on $\mathcal{P}$, having radius $d(P, O)$. If $Q \in c$, then $d(Q, (0, 0, -T))$, by rotational symmetry, while $Q \in \mathcal{P} - c$ implies that $d(Q, (0, 0, -T)) > d(P, (0, 0, -T))$.

Since $(0, 0, -T)$ belongs to $\mathcal{N}_P S$, by Lemma 3.1 the lower half of γ's trace belongs to $\mathcal{N}_P S$. By symmetry, the upper half does, too.

Step 3. γ has the right parameter W. Let $c = c(s)$ be the projection of γ on $\mathcal{P}$. Then, $c(0) = c(2\pi/\phi) = 0$, $c(\pi/\phi) = P$ and c is a circle having Euclidean diameter OP. As s increases, $c(s)$ runs the circle clockwise. As to the tangent vectors, $\dot{c}(s)$ is the projection on the plane $\mathcal{P} = \{t = 0\}$ of the horizontal vector $\dot{\gamma}(s)$. After a rotaion, we can suppose that $P = (2/\phi, 0, 0)$ lies on the positive x-axis. Then, $\dot{c}(\pi/\phi) = (0, -1)$, hence $\dot{\gamma}(\pi/\phi) = -Y(P)$. We only have to show, then, that $-Y(P) = N_P^+\mathcal{P}$. Now, the space tangent to $\mathcal{P}$ at P contains just one horizontal direction (P is non-characteristic), and this direction is that of the x-axis, X. Then, $N_P\mathcal{P} = span\{Y(P)\}$, hence, since $N_P^+\mathcal{P}$ has to point upward, $N_P^+\mathcal{P} = -Y(P) = -\partial_y + \frac{4}{\phi}\partial_t$, as wished.

Step 4. γ contains the metric normal. Suppose now that Q belongs to both $\mathcal{N}_P\mathcal{P}$ and $\mathcal{N}_{P'}\mathcal{P}$. We show that, then, Q lies on the axis $x = y = 0$ and that P and P' lie on a Euclidean circle centered at O in $\mathcal{P}$.

After a group translation, the plane $L_{Q^{-1}}\mathcal{P}$ touches a closed Carnot Charathèodory ball B centered at the origin in $Q^{-1}P$ and $Q^{-1}P'$. This may happen only if $Q^{-1}P$ and $Q^{-1}P'$ both lie on the circle c of the points in B having highest (or lowest) t-coordinate, and $c = B \cap L_{Q^{-1}}\mathcal{P}$. Hence, the characteristic point of $\mathcal{P}$ lies on the same vertical line as the center of B, and this property is invariant under left translations. As a consequence, Q lies on $x = y = 0$ and P, P' are taken one into the other by a rotation around $x = y = 0$.

For each point P in $\mathcal{P} - \{O\}$, consider the geodesic arc

$$\beta_P = \gamma_{P,2/d(P,C),N_P^+\mathcal{P}}([-\frac{\pi}{2}d(P, C), \frac{\pi}{2}d(P, C)])$$

which, by steps 1-3 lies in $\mathcal{N}_P\mathcal{P}$. As P varies, the union of the sets β_P fills up $\mathbb{H} - \{O\}$.

Suppose now that Q belongs to $\mathcal{N}_P\mathcal{P} - \beta_P$. Clearly, $Q \neq O$, and Q belongs to some $\beta^{P'} \subset \mathcal{N}_{P'}\mathcal{P}$. By the considerations above, then, Q lies on $x = y = 0$ and P' is taken to P by a rotation around $x = y = 0$. The same rotation brings $\beta_{P'}$ onto β_P and fixes Q, hence $Q \in \beta_P$, a contradiction. $\square$

By Theorem 3.1 and (2) we have an explicit expression for the metric normal to a plane.

Proposition 3.1. *The metric normal to $\Pi = \{t = 0\}$ at $P = (z, t)$, $z = x + iy$, is the support of the geodesic arc*

$$\gamma_P(\sigma) = (u(\sigma),\ v(\sigma),\ s(\sigma)) = \begin{cases} \frac{x}{2}\left(1+\cos\left(\frac{2\sigma}{|z|}\right)\right) + \frac{y}{2}\sin\left(\frac{2\sigma}{|z|}\right) \\ \frac{y}{2}\left(1+\cos\left(\frac{2\sigma}{|z|}\right)\right) - \frac{x}{2}\sin\left(\frac{2\sigma}{|z|}\right) \\ \frac{|z|^2}{2}\left(\frac{2\sigma}{|z|} + \sin\left(\frac{2\sigma}{|z|}\right)\right) \end{cases},\ |\sigma| \le \frac{\pi}{2}|z| \tag{5}$$

Let $w(\sigma) = u(\sigma) + iv(\sigma)$, then, by (5),

$$\begin{cases} |w| = |z|\cos\left(\frac{\sigma}{|z|}\right) \\ s = |z|^2\left(\frac{\sigma}{|z|} + \frac{1}{2}\sin\left(\frac{2\sigma}{|z|}\right)\right). \end{cases} \tag{6}$$

Notice that for σ fixed, (6) gives a parametrizazion of the set of points having distance σ from Π.

The next lemma helps with calculations.

Remark 3.1. *If p is the plane $t = ax + by + c$, its characteristic point is*

$$C = (-b/2, a/2, c)$$

Proof. The tangent space of p at (x, y, t) is spanned by $(1, 0, a)$ and $(0, 1, b)$. They are both horizontal iff $a = 2y$ and $b = -2x$. □

We also need this fact.

Lemma 3.2. *Let $B = B(A, r)$ be a ball in $\mathbb{H}$, $Q \in \partial B(A, r)$ and γ be a geodesic from A to Q. Then, if A' is a point of γ, $B(A', r - d(A, A'))$ is contained in $B(A, r)$ and Q belongs to $\partial B(A', r - d(A, A'))$.*

Moreover, if $A' \neq A$, then $\partial B(A, r) \cap \partial B(A', r - d(A, A')) = \{Q\}$.

Proof. Let R be a point in $B(A', r - d(A, A'))$, then

$$d(R, A) \le d(R, A') + d(A', A) = r.$$

Suppose, now that $A \neq A'$ and that $Q \neq Q' \in \partial B(A', r - d(A, A'))$. Since $Q' \notin \gamma$, $d(A, Q') < d(A, A') + d(A', Q') = d(A, Q)$, and this contradicts $Q' \in \partial B(A, r)$, proving the last statement of the lemma. □

Lemma 3.3. *Let $r > 0$, $\phi \in \mathbb{R}$, $\mid \phi r \mid < \pi$ and $Q \in \mathbb{H}$. If γ is a geodesic of parameter ϕ and endpoints Q and P, $P \in \partial B(Q, r)$, and $B = B(Q, r)$, then*

$$\Pi_P(\partial B) \cap \bar{B} = \{P\}.$$

Proof. Consider the prolongement of γ from Q to Q', which we still call γ, where $d(P, Q') = \frac{\pi}{\phi} \ge r$. We can assume that $Q' = O$ is the origin and that the geodesic γ points upward from $Q' = O$ to P. Consider the ball $B' = B(Q', P)$. Then, P is a point of maximal t-coordinate on B', hence $\Pi_P B'$ is a plane parallel to $t = 0$ (more precisely, $\Pi_P B'$ is the plane having equation $t = \frac{2R^2}{\pi}$), intersecting $\partial B'$ in a Euclideean circle. By Lemma 3.2, $B \subset B'$ and P lies on the boundary of both B and B'. Hence, $\Pi_P B = \Pi_P B'$ and we obtain the thesis, again by Lemma 3.2. □

Proposition 3.2. *Let $B = B(Q, r)$ be a ball and P a point where its boundary is smooth. Let $\gamma_P : I \to \mathbb{H}$ be the maximal lenght minimizing geodesic starting at Q and passing through P. Then*

i) $\mathcal{N}_P(\partial B) = \gamma_P(I)$.

ii) *An open arc of γ_P containing P is contained in $\mathcal{N}_P(\partial B) \cap \mathcal{N}_P(\Pi_P(\partial B))$.*

Proof. We prove (i) first. Let Q' be the other endpoint of γ_P. As a consequence of the triangle inequality, $d(Q', P) = d_{\partial B}(Q')$. Hence by Lemma 3.1 the arc of γ_P between P and Q' is contained in $\mathcal{N}_P(\partial B)$. Clearly the arc of γ_P between P and Q is contained in $\mathcal{N}_P(\partial B)$, too. Hence $\gamma_P \subseteq \mathcal{N}_P(\partial B)$.

Suppose now that $A \in \mathcal{N}_P \partial B$ and $A \notin \gamma_P$. We assume, first, that $A \notin B$. Let η be a length minimizing geodesic between A and Q and let P' be its intersection with ∂B. Thus $d(A, P') \geq d_{\partial B}(A) = d(A, P)$ and $d(Q, P') = r = d(Q, P')$. Hence, $d(A, Q) \leq d(Q, P) + d(P, A) \leq d(Q, P') + d(P', A) = d(A, Q)$, and this is possible only if P belongs to a length minimizing geodesic between A and Q, which prolongs the arc of γ_P between Q and P. i.e., A belongs to a length minimizing prolongment of the geodesic between Q and P, but the maximal such prolongment is γ_P.

If $A \in B$, a similar argument shows that $A \in \gamma_P$. Details are left to the reader.

We now show (ii). By lemma 3.3, there exists a point Q' on γ_P such that the closure of $B' = B(Q', d(Q', P))$ meets $\Pi_P(\partial B') = \Pi_P(\partial B)$ in P only. Thus, by definition of metric normal, the arc of γ_P between P and Q' lies in $\mathcal{N}_P(\Pi_P(\partial B))$. On the other hand, by (i), γ_P is contained in $\mathcal{N}_P(\partial B)$.

A similar argument shows that there exists an arc of γ_P external to B which lies in $\mathcal{N}_P(\partial B) \cap \mathcal{N}_P(\Pi_P(\partial B))$.

Remark 3.2. *Lemma 3.3 implies that part of ∂B is* convex *in the Euclidean sense. In other words that both principal curvatures are non-negative.*

□

4. The metric normal for a smooth surface

In this section, we discuss the metric normal to smooth surfaces. We consider surfaces satisfying an inner-and-outer ball condition. Many of the results extend, with obvious modifications, to surfaces just satisfying an inner ball condition. When no ambiguity is possible, we shall identify a curve $\gamma : I \to \mathbb{H}$ with its trace $\gamma(I)$.

Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$. We say that S satisfies condition (TB) at P if there are open balls $B_1 \subseteq \Omega$ and $B_2 \subseteq \mathbb{H} - \overline{\Omega}$ containing P on their boundary.

Theorem 4.1. *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$.*

(i) *If P belongs to S and S has tangent plane $\Pi_P S$ at P, then $\mathcal{N}_P S$ is an arc (eventually degenerate) of a geodesic. Moreover, if $\mathcal{N}_P S$ is nontrivial, then its intersection with $\mathcal{N}_P \Pi_P S$ is a nontrivial geodesic arc containing P.*

(ii) *Suppose, more, that S satisfies (TB) at P. Then S has tangent plane P and $\mathcal{N}_P S$ is a nontrivial geodesic arc having endpoints in Ω and $\mathbb{H} - \overline{\Omega}$, respectively.*

An arc on a curve is degenerate if it reduces to a point. We say that S has tangent plane $\Pi_P S$ at P if $d_{Euc}(Q, \Pi_P S) = o(|Q - P|)$ as $Q \to P$ in S. Observe that, by (i), if $\mathcal{N}_P S$ is nontrivial, then it can be parametrized by any equation parametrizing $\mathcal{N}_P \Pi_P S$.

Proof. (i) If $\mathcal{N}_P S$ reduces to P alone, there is nothing to prove. Otherwise, let $P \neq Q \in \mathcal{N}_P S$, $Q \in \Omega$ and let γ be any geodesic from P to Q (actually, there is only one such geodesic).

Then, $B(Q, d(P, Q)) \subseteq \Omega$ and $P \in \partial B(Q, d(P, Q))$. Now, either $\partial B(Q, d(P, Q))$ is smooth at P or P is one of the "poles" of $\partial B(Q, d(P, Q))$. In the first case, γ lies in $\mathcal{N}_P \partial B(Q, d(P, Q))$, hence, by Proposition 3.2, a non trivial arc of γ lies in $\mathcal{N}_P \Pi_P(\partial B(Q, d(P, Q))) = \mathcal{N}_P \Pi_P S$, since $\Pi_P(\partial B(Q, d(P, Q))) = \Pi_P S$. The equation of γ is then completely determined by that of $\mathcal{N}_P \Pi_P$.

If P is, say, the North Pole of $B(Q,d(P,Q))$, S, being external to $B(Q,d(P,Q))$, can not have tangent plane at P, since there is a proper Euclidean cone external to $B(Q,d(P,Q))$, which tangent to the boundary of $B(Q,d(P,Q))$ at P.

So far, we have proved that $\gamma_1 = \mathcal{N}_P S \cap \overline{\Omega}$ and $\gamma_2 = \mathcal{N}_P S \cap (\mathbb{H} - \Omega)$ are geodesic arcs and that any of them, if nontrivial, shares a nontrivial arc with $\mathcal{N}_P \Pi_P S$. We show that also $\mathcal{N}_P S = \gamma_1 \cup \gamma_2$ is a length minimizing geodesic. Let Q_j the endpoint of γ_j other than P and let l_j be the (intrinsic) length of γ_j. Then, the balls $B_j = B(Q_j, l_j)$ do not intersect and P belongs to the boundary of both. Let η be any horizontal curve between Q_1 and Q_2. Since η has to meet the boundaries of B_1 and B_2, $l_{\mathbb{H}}(\eta) \geq l_{\mathbb{H}}(\gamma_1) + l_{\mathbb{H}}(\gamma_2)$, hence $d(Q_1,Q_2) = l_{\mathbb{H}}(\gamma_1) + l_{\mathbb{H}}(\gamma_2)$, i.e., $\gamma_1 \cup \gamma_2$ is a (length minimizing) geodesic arc.

(ii) Since B_1 and B_2 have empty intersection, the point P can not be a pole of either B_1 or B_2. Being ∂B_1 and ∂B_2 smooth at P in the Euclidean sense, ∂B_1, ∂B_2 and S have the same tangent plane $\Pi = \Pi_P S$ at P. The centers Q_1 and Q_2 of B_1 and B_2 belong to $\mathcal{N}_P S$, by definition, hence, by Proposition 3.1, the geodesic γ_j between Q_j and P is contained in $\mathcal{N}_P S$, $j = 1,2$. Hence, $\mathcal{N}_P S$, which is a geodesic arc by (i), is non degenerate and does not have P as endpoint. □

Corollary 4.1. *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$ and let C be a characteristic point of S. If S has tangent plane at C, then S can not satisfy (TB) at C.*

Proof. If (TB) held at C, $\mathcal{N}_C S$ would be nontrivial, hence $\mathcal{N}_C \Pi_C S$ would nontrivial, but C is characteristic for $\Pi_C S$, hence $\mathcal{N}_C \Pi_C S$ is degenerate, by Theorem 3.1. □

Definition 4.1. *Let S be a C^1 surface in the Euclidean sense in $\mathbb{H}$, which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$ and let $P \in S$. The* **oriented metric normal to S at P, $\mathcal{N}_P^+ S$,** *is the unique parametrization of $\mathcal{N}_P S$ such that $\delta_S(\mathcal{N}_P^+ S(\sigma), P) = \sigma$.*

This means that $\mathcal{N}_P^+ S(\sigma) \in \Omega$ for $\sigma < 0$ and $\mathcal{N}_P^+ S(\sigma) \in \mathbb{H} - \overline{\Omega}$ for $\sigma > 0$. In particular, if $\mathcal{N}_P^+ S$ is nontrivial, then

$$\dot{\mathcal{N}}_P^+ S(0) = N_P^+ S.$$

Let S be a differentiable surface (in the Euclidean sense) in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$ and suppose that, locally, S has equation $g = 0$, where $g : \mathbb{H} \to \mathbb{R}$ is differentiable in the Euclidean sense and $\nabla g \neq 0$ pointwise on S. Let P be a point in S. P is characteristic for S if and only if $\nabla_{\mathbb{H}} g(P) = 0$. Let P be noncharaceristic for S and let C be the characteristic point of $\Pi_P S$. (If $\Pi_P S$ has no characteristic point, i.e., if it has equation $ax + by + c = 0$, then we say that $\Pi_P S$ *has characteristic point at infinity*). Then,

$$\Pi_P S = \{(x,y,t) : \ \partial_x g(P)(x - x(P)) + \partial_y g(P)(y - y(P)) + \partial_t g(P)(t - t(P)) = 0\}.$$

and

$$C = \left(\frac{g_y(P)}{2g_t(P)}, -\frac{g_x(P)}{2g_t(P)}, t_0 + x_0 \frac{g_y(P)}{g_t(P)} + y_0 \frac{g_x(P)}{g_t(P)} \right), \tag{7}$$

$$d(P,C) = 2\frac{|\nabla_{\mathbb{H}} g|}{|[X,Y]g|} = \frac{|\nabla_{\mathbb{H}} g|}{2|\partial_t g|} \tag{8}$$

If $\Omega = \{g < 0\}$ in a neighborhood of P , the Pansu unit normal vector pointing outside S has equation

$$N_P^+ S = \frac{\nabla_{\mathbb{H}} g(P)}{|\nabla_{\mathbb{H}} g(P)|}.$$

By Theorems 4.1 and 3.1, the equation of $\mathcal{N}_P^+S$ can be written in terms of g's partial derivatives. $\mathcal{N}_P^+S = P\cdot\eta$ (left translation by P), where $\eta=(u,v,s)$ and

$$\eta(\sigma)=\begin{cases} u(\sigma)=\frac{1}{4\partial_t g}\left\{Yg(P)\left(1-\cos\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}}g(P)|}\right)\right)+Xg(P)\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}}g(P)|}\right)\right\} \\ v(\sigma)=\frac{1}{4\partial_t g}\left\{-Xg(P)\left(1-\cos\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}}g(P)|}\right)\right)+Yg(P)\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}}g(P)|}\right)\right\} \\ s(\sigma)=\frac{|\nabla_{\mathbb{H}}g(P)|^2}{8(\partial_t g(P))^2}\left\{\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}}g(P)|}-\sin\left(\frac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}}g(P)|}\right)\right\}\end{cases}\tag{9}$$

Observe that $\mathcal{N}_P^+S$ points upwards if $\partial_t g(P)>0$ and downwards if $\partial_t g(P)<0$. If $\partial_t g(P)=0$, then $\Pi_P S$ has characteristic point C at infinity, $d(P,C)=\infty$ and (9) becomes

$$\eta(\sigma)=\left(\frac{Xg(P)}{|\nabla_{\mathbb{H}}g(P)|}\sigma,\frac{Yg(P)}{|\nabla_{\mathbb{H}}g(P)|}\sigma,0\right)\tag{10}$$

These equations show that, for some smooth function Φ,

$$\mathcal{N}_P^+S(\sigma)=\Phi(\sigma,P,\nabla_{\mathbb{H}}g(P),[X,Y]g(P)),$$

i.e., in order to write $\mathcal{N}_P^+S$, the knowledge of $\nabla_{\mathbb{H}}g$ alone is not sufficient. On the other hand, we do not need *all* of the horizontal second order derivatives, but just $[X,Y]$. The continuity of $\nabla_{\mathbb{H}}g$ and $[X,Y]g$ is equivalent to the requirement that g is C^1 in the Euclidean sense.

We think that this is a sufficient justification for our choice of considering surfaces which are C^1 in the Euclidean sense, at least outside their characteristic set. C^1 regularity should be henceforth considered as an intrinsic requirement which is intermediate between $C^1_{\mathbb{H}}$ and $C^2_{\mathbb{H}}$.

5. Sets of positive reach

Definition 5.1. *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H}-\overline{\Omega}$ and let S_+ an open subset of S. We say that S satisfies condition (UTB) (uniform tangent ball) on S_+ if for each point $P_0\in S_+$ we can find $r>0$ and $h>0$ such that, for all P in $B(P_0,r)$, condition (TB) holds at P with balls B_1 and B_2 of radius h.*

Theorem 5.1. *Let S be a surface in $\mathbb{H}$, $S=\partial\Omega$ and $S=\partial(\mathbb{H}-\overline{\Omega})$, where Ω is open in $\mathbb{H}$. If S is $C^{1,1}$ in the Euclidean sense and if $Char(S)$ is the set of S's characteristic points, then S satisfies (UTB) on $S-Char(S)$.*

Proof. Federer showed that a $C^{1,1}$ Euclidean surface S satisfies (UTB) on the whole of S with Euclidean balls instead of Heisenberg balls.

Let P be a non characteristic point on S, let $\Pi_P S$ be the Euclidean plane tangent to S at P and let γ be the metric normal to $\Pi_P S$ at P pointing inside Ω. Let $s\geq 0$ be a number smaller than the length of γ and consider the point C_s on γ having distance s from P. Let τ_s be the left translation carrying C_s in O and let S', P' γ', Ω' be the images of S, P and γ, Ω, respectively. By continuity and by the assumption that S is C^2, there is $r>0$ such that, for each $s<\delta(r)$, there exists a Euclidean ball $B^s_{Euc}(D_s,r)$ of radius r contained in Ω' and having P' on its boundary.

We will show that there exists a Heisenberg ball $B(R,\eta)$ centered in point R of γ', contained in $B^s_{Euc}(D_s,r)$ and having P' on its boundary. The ball $\tau_s^{-1}B(R,\eta)=B(\tau_s^{-1}R,\eta)$ is contained in Ω and has P on its boundary. Exchanging Ω with $\mathbb{H}-\overline{\Omega}$, we see that (TB) holds at P. Since our procedure is stable as P varies in $S-Char(S)$, (UTB) holds on $S-Char(S)$.

Since P' is non characteristic, a subarc of γ' starting at P' is contained in $B^s_{Euc}(D_s,r)$, but for the point P'. It suffices to prove the following **claim:** $B^s_{Euc}(D_s,r)$ contains a Heisenberg ball having P' on its boundary. We need an estimate for the Euclidean curvatures of the Heisenberg ball with center at the origin.

Lemma 5.1. *Let $B(0,s)$ be the Heisenberg ball centered in 0 with radius s. For every point $P \in \partial B(0,s)$, $P = P(x,y,t)$, $(x,y) \neq (0,0)$, then there exists a positive number $s_0(P)$ such that the principal Euclidean curvatures in P are respectively, as $s \to 0$:*

$$k_1^E = \frac{1}{s\sqrt{1+\frac{1}{16}\phi^2}}(1+o(1))$$

and

$$k_2^E = \frac{3}{4s^3}(1+o(1)).$$

where $o(1) \to 0$ as $s \to 0$, uniformly w.r.t. $|\phi| \leq C$, for any fixed $C > 0$.

Proof. ¿From (5) and $\alpha = \frac{\phi s}{2}$ we get the following parametrization of the Carnot ball of radius s:

$$\begin{cases} x = s\cos\theta\frac{\sin\alpha}{\alpha} \\ y = s\sin\theta\frac{\sin\alpha}{\alpha} \\ t = \frac{s^2}{2}\frac{2\alpha-\sin(2\alpha)}{\alpha^2}. \end{cases}$$

We can calculate the Euclidean curvatures k_1^E, k_2^E of the Carnot ball obtaining respectively

$$k_1^E = \frac{g'(\alpha)}{f(\alpha)\sqrt{f'(\alpha)^2 + (\frac{s}{2}g'(\alpha))^2}}$$

and

$$k_2^E = \frac{1}{f'(\alpha)}\Big(\frac{g'(\alpha)}{\sqrt{f'(\alpha)^2 + (\frac{s}{2}g'(\alpha))^2}}\Big)',$$

where $f(\alpha) = s\frac{\sin\alpha}{\alpha}$ and $g(\alpha) = s^2\frac{2\alpha-\sin(2\alpha)}{2\alpha^2}$. As a consequence we get

$$k_1^E = \frac{G'(\alpha)}{F(\alpha)\sqrt{F'(\alpha)^2 + (sG'(\alpha))^2}}$$

and

$$k_2^E = \frac{1}{F'(\alpha)}\Big(\frac{G'(\alpha)}{\sqrt{F'(\alpha)^2 + (sG'(\alpha))^2}}\Big)',$$

where $F(\alpha) = \frac{\sin\alpha}{\alpha}$ and $G(\alpha) = \frac{2\alpha-\sin(2\alpha)}{2\alpha^2}$. On the other hand as $\alpha \to 0$ for fixed ϕ,

$$F'(\alpha) = -\frac{\alpha}{3} + \frac{1}{30}\alpha^3 + o(\alpha^4),$$

$$G'(\alpha) = \frac{2}{3} - \frac{2}{5}\alpha^2 + o(\alpha^3),$$

and

$$F''(\alpha) = -\frac{1}{3} + \frac{1}{10}\alpha^2 + o(\alpha^3)$$

$$G''(\alpha) = -\frac{4}{5}\alpha + o(\alpha^2).$$

Hence

$$k_1^E = \frac{1}{s\sqrt{1+\frac{1}{16}\phi^2}}(1+o(1))$$

and

$$k_2^E = \frac{3}{4s^3}(1+o(1)).$$

□

We now complete the proof of Theorem 5.1. If s is small enough, by Lemma 5.1 there exists an open neighborhood $\mathcal{V}$ of P' such that $B(0,s)\cap\mathcal{V}$ is contained in $B_{Euc}(D_s,r)$. Consider now all Heisenberg balls $B(R,\eta)$ with $R\in\gamma'$. By Lemma 3.2, $B(R,\eta)\subseteq B(0,s)$. As $\eta\to 0$, $B(R,\eta)$ shrinks to P', hence $B(R,\eta)$ is contained in $B(0,s)\cap\mathcal{V}\subseteq B_{Euc}(D_s,r)$ for η small enough. Clearly, $B(R,\eta)$ contains P' on its boundary. This proves the claim, hence Theorem 5.1.

□

Following Federer, [8] we introduce the notion of reach.

Definition 5.2. *Let Ω be an open subset of $\mathbb{H}$ with boundary S. We denote by $Unp(S)$ the set of the points P in $\mathbb{H}$ such that there exists a unique point Q in S which is nearest to P,*

$$d(P,S)=d(P,Q).$$

We say that $Q=\xi(P)$ is the **projection** *of P onto S.*

Let U be an open subset of S. S has **locally positive reach** *on U if for all Q in U an open neighborhood of Q in $\mathbb{H}$ is contained in $Unp(S)$.*

The notion of positive reach is related to (UTB) via an exponential-like map.

Definition 5.3. *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H}-\overline{\Omega}$ and suppose that S is C^1 in the Euclidean sense. Let*

$$\mathcal{C}=\{(P,s):\ P\in S,\ s\in dom(\mathcal{N}_P^{+}S)\}\subseteq S\times\mathbb{R}$$

where $dom(\mathcal{N}_PS)$ is the domain of $\mathcal{N}_PS$. The **exponential map** *associated with S is the map*

$$\exp_S:\mathcal{C}\to\mathbb{H},\ \exp_S(P,s)=\mathcal{N}_P^{+}S(s).$$

We call this map *exponential* because it associates to a geodesic γ leaving S from P and minimizing the distance from S, which we might identify with P itself, and to a number t, the point $\gamma(t)$, the same way in which the exponential map in Riemannian geometry associates to a geodesic γ leaving a point Q and a number t, the point $\gamma(t)$.

Theorem 5.2. *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H}-\overline{\Omega}$ and suppose that S is C^1 in the Euclidean sense.*

Then, $\exp_S$ is a homeomorphism of $int(\mathcal{C})$ onto an open subset of $int(Unp(S))$ and $S\cap int(Unp(S))=S\cap\exp_S(int\mathcal{C})$.

Proof. Let $\mathcal{U}\subseteq S\times\mathbb{R}$ be the interior of $\mathcal{C}$.

The map $\exp_S$ is $1-1$ on $\mathcal{U}$. Suppose $\exp_S(P,s)=\exp_S(P',s')=Q$. By definition of metric normal, $s=\delta_S(Q)=s'$. Suppose $s>0$, the case $s<0$ being similar. Since $\mathcal{U}$ is open, $\mathcal{N}_PS$ can be extended to an interval $[0,s+\epsilon]$ for some positive ϵ. Let $Q'=\mathcal{N}_PS(s+\epsilon)$. Thus, if $P\neq P'$,

$$d(Q',S)\leq d(Q',P')<d(Q',Q)+d(Q,P')=d(Q',P)=d(Q',S),$$

contradiction . The strict inequality depends on the fact that Q' can not lie on the prolongement of a geodesic between P' and Q. As a consequence, we have that $\exp_S(\mathcal{U})\subseteq Unp(S)$.

The map $\exp_S$ is continuous on $\mathcal{C}$, since S is C^1 in the Euclidean sense and, by (9), η continuously depends on $s,P,\nabla_{\mathbb{H}}g(P)$ and $[X,Y]g(P)$.

The map $\exp_S$ maps $\mathcal{C}$ *onto* $\mathbb{H}$, hence, *a fortiori*, maps $\mathcal{C}$ *onto* $\mathbb{H}$. Let $Q\in\mathbb{H}$, let P be a point on S such that $d(Q,P)=d(Q,S)$ and let $\delta_S(Q)=s$. Then $(P,s)\in\mathcal{C}$ and $\exp_S(P,s)=Q$.

Consider now

$$G:Unp(S)\to S\times\mathbb{R},\ G(Q)=(\xi_S(Q),\delta_S(Q)).$$

By 6.3, G is a continuous function. Since $\exp_S|_{\mathcal{U}}$ is a homeomorphism, $G\circ\exp_S|_{\mathcal{U}}=Id$. Hence, $\mathcal{U}$ and $\exp_S(\mathcal{U})$ are homeomorphic, $\mathcal{U}$ is locally identifiable with a subset of $\mathbb{R}^3$, hence, by Brouwer's theorem, see [20], on the invariance of domain, $\exp_S(\mathcal{U})\subseteq Unp(S)$ is an open subset of $\mathbb{H}$, contained in $Unp(S)$. This shows that $\exp_S$ is a homeomorphism of $\mathcal{U}$ onto an open subset of $int(Unp(S))$.

We still have to show that $S \cap int(Unp(S)) = \exp_S(\mathcal{U})$. Let $\mathcal{U}_1 = \{P \in S : (P, 0) \in \mathcal{U}\}$. $\mathcal{U}_1$ is open in S. Clearly, $\mathcal{U}_1 \subseteq S \cap int(Unp(S))$. We show that $S \cap int(Unp(S)) \subseteq \mathcal{U}_1$. Let $P_0 \in S \cap int(Unp(S))$ and let $\mathcal{V}$ be an open neighborhood of P_0 in $int(Unp(S))$. Then, $G(\mathcal{V}) \subseteq \mathcal{C}$, but G is the inverse function of $\exp_S$, hence $G(\mathcal{V}) = \exp_S^{-1}(\mathcal{V})$, which is open in $\mathcal{C}$. Hence, $G(\mathcal{V}) \subset \mathcal{U}$ and, since $G(P) = (P, 0)$ when $P \in S$, this shows that the open subset $\mathcal{V} \cap S$ of S is contained in $\mathcal{U}_1$.

□

Theorem 5.3. *Let S be a surface in $\mathbb{H}$, $S = \partial\Omega$ and $S = \partial(\mathbb{H} - \overline{\Omega})$, where Ω is open in $\mathbb{H}$, and let S_1 be an open subset of S. If S is C^1 in the Euclidean sense, then S satisfies (UTB) on S_1 if and only if it satisfies (PR) on S_1.*

Proof. One easily verifies that S satisfies (PR) on S_1 if and only if $S_1 \subseteq int(Unp(S))$. On the other hand, S satisfies (UTB) on S_1 if and only if $S_1 \times \{0\} \subset \mathcal{U} = int\mathcal{C}$. The theorem now follows from Theorem 5.2. □

¿From Theorems 5.1 and 5.3 we obtain the following.

Corollary 5.1. *Let S be a surface in $\mathbb{H}$, $S = \partial\Omega$ and $S = \partial(\mathbb{H} - \overline{\Omega})$, where Ω is open in $\mathbb{H}$ and suppose that S is $C^{1,1}$ in the Euclidean sense. Then, Ω satisfies (PR) on $S - Char(S)$.*

6. Regularity of the distance function

The main result of this section is the following theorem.

Theorem 6.1. *Let S be a surface in $\mathbb{H}$ which is the boundary of an open set Ω and of $\mathbb{H} - \overline{\Omega}$.*

(i) If S is C^1, and S satisfies (UTB) on $S - Char(S)$, then $\nabla_{\mathbb{H}}\delta_S$ is a continuous function in an open neighborhood of $S - Char(S)$ in $\mathbb{H}$. In particular, the conclusion holds if S is $C^{1,1}$ in the Euclidean sense.

(ii) If S is C^k in the Euclidean sense, $k \geq 2$, then $\nabla_{\mathbb{H}}\delta_S$ and δ_S are of class C^{k-1}, in the Euclidean sense, in an open neighborhood of $S - Char(S)$ in $\mathbb{H}$.

Theorem 1.1 follows now from Theorems 5.1 and 6.1.

The proof of Theorem 6.1 will come after we prove a number of lemmata.

Lemma 6.1. *Let f be a Lipschitz function in the open set W, g a continuous function in W and V any linear combination of X and Y. If $Vf = g$ where $Vf = g$ exists. Then $Vf = g$ in W hence f is $C^1(W)$.*

Proof. Following Federer, [8] Lemma 4.7, without loss of generality we can assume $V = X$. Let P be any point in W, and $r > 0$ so that $B(P, 2r) \subset W$. By Rademacher's theorem in [18] and Fubini's theorem, for a.e. $Q \in B(P, r)$, whenever $\mid \tau \mid < r$

$$f(Q \cdot (\tau, 0, 0)) - f(Q) = \int_0^\tau Xf(Q \cdot (s, 0, 0))ds = \int_0^\tau g(Q \cdot (s, 0, 0))ds.$$

By the continuity of f and g it follows that

$$f(P \cdot (\tau, 0, 0)) - f(P) = \int_0^\tau g(P \cdot (s, 0, 0))ds.$$

Differentiating with respect to τ we get $Xf(P) = g(P)$. □

Lemma 6.2. *Let P in $Unp(S) \setminus S$, such that δ_S is differentiable at P. Then*

$$\nabla_{\mathbb{H}}\delta_S(P) = \dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P)). \tag{11}$$

Proof. Let $\tilde{\gamma}_P$ be the geodesic connecting P with $\xi(P)$, parametrized to have $\tilde{\gamma}(0) = P$, and $\tilde{\gamma}(1) = \xi(P)$, where $d(P, \xi(P)) = d_S(P)$ is the speed of γ. The geodesic γ is a reparametrization of an arc of $\mathcal{N}^+_{\xi(P)}S$,

$$\mathcal{N}^+_{\xi(P)}S(s) = \tilde{\gamma}\left(1 - \frac{s}{\delta_S(P)}\right).$$

By lemma 3.1,

$$d_S(\tilde{\gamma}(t)) = d(\tilde{\gamma}(t), \xi(P)) = (1-t)d_S(P).$$

Hence

$$-d_S(P) = \frac{d}{dt}|_{t=0} d_S(\tilde{\gamma}(t)) = \langle \dot{\tilde{\gamma}}(0), \nabla_{\mathbb{H}} d_S(P) \rangle.$$

On the other hand, by Cauchy-Schwarz and the Eikonal equation (rather, the easy inequality $|\nabla_{\mathbb{H}} d_S(P)| \le 1$)

$$\langle \dot{\tilde{\gamma}}(0), \nabla_{\mathbb{H}} d_S(P) \rangle \ge -|\dot{\tilde{\gamma}}(0)| \cdot |\nabla_{\mathbb{H}} d_S(P)| \ge -d_S(P).$$

As a consequence inequalities are actual equalities and this implies

$$d_S(P) \nabla_{\mathbb{H}} d_S = -\dot{\tilde{\gamma}}(0),$$

which is equivalent to (11). □

Lemma 6.3. *Let E be a closed subset of $\mathbb{H}$. The map $P \mapsto \xi(P)$ is continuous on $Unp(E)$.*

The proof of Federer, ([8] Theorem 4.8 (4)) extends to the Heisenberg case without changes.

Lemma 6.4. *Let S be an Euclidean C^1 surface. The map $P \mapsto \dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P))$ is a continuous section of the horizontal fiber restricted to $Unp(S)$.*

Proof. The key observation is that $\dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P))$ is obtained by $\dot{\mathcal{N}}^+_{\xi(P)}(0) = N^+_P S$ by rotation. Let θ be the angle

$$\theta = -\frac{2\delta_S(P)}{d(\xi(P), C)},$$

where C is the characteristic point of $\Pi_{\xi(P)}S$ ($\theta = 0$ when $d(\xi(P), C) = 0$). Then,

$$\dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P)) = (R_\theta)_{*,P} N^+_{\xi(P)}(0). \tag{12}$$

Here, $(R_\theta)_*$ is rotation by θ in $\mathcal{H}$ and $(R_\theta)_{*,P}V$ is the evaluation at P of the horizontal vector field $(R_\theta)_*V$. In complex notation

$$(R_\theta)_*(aX + ibY) = e^{i\theta}(aX + ibY).$$

In the right hand side of (12), we denote by $N^+_{\xi(P)}(0)$ both the horizontal vector at $\xi(P)$ and its extension to a left invariant (horizontal) vector field. Equation (12) is an immediate consequence of the geodesic equation (2).

In view of Lemma 6.3 and the assumption that S is C^1 in the Euclidean sense, hence that $N^+_{\xi(P)}(0)$ and $d(\xi(P), C)$ are continuous functions of $\xi(P) \in S$, (12) implies that $\dot{\mathcal{N}}^+_{\xi(P)}(\delta_S(P))$ continuously depends on P. □

Theorem 6.2. *Let S be a surface in $\mathbb{H}$, $S = \partial\Omega$ and $S = \partial(\mathbb{H} - \overline{\Omega})$, where Ω is open in $\mathbb{H}$, and suppose that S is C^1 in the Euclidean sense. Then, $\nabla_{\mathbb{H}}\delta_S$ is a continuous function the int(Unp).*

Proof. It follows from Lemmata 6.1, 6.2 and 6.4. □

Proof of Theorem 6.1-(i). From Theorem 6.2 and Theorem 5.3 immediately follows (i). Moreover the last part of the conclusion follows from Corollary 5.1. □

Before giving the proof of Theorem 6.1-(ii) we need some preparation.

Let S be a surface in $\mathbb{H}$, which is the boundary of an open set Ω and of $\mathbb{H}-\overline{\Omega}$, and suppose that S is C^2 in the Euclidean sense. Suppose that in neighborhood $\mathcal{U}\subset S$ of a point P_0 the surface is free of characteristic points and it can be parametrized as

$$\mathcal{U}=\{(u,v,f(u,v)):\ (u,v)\in A\},$$

where $A\subseteq\mathbb{R}^2$ is open and f is C^2. Let $F:A\times\mathbb{R}\to\mathbb{H}$,

$$F(u,v,s)=\exp_S((u,v,f(u,v)),s). \tag{13}$$

The function F is an expression of $\exp_S$ in local coordinates, which easier to work with. It has an intrinsic geometric meaning, since the projection $proj:\mathbb{H}\to\mathbb{R}^2$, $proj(z,t)=z$, pushes the Carnot-Carathéodory metric of $\mathbb{H}$ onto the Euclidean metric of $\mathbb{R}^2$. Without loss of generality, assume that, if $P\in\mathcal{U}$, then $\mathcal{N}_PS$ points upward. In the sequel, we write $Xf=-\partial_uf+2v$ and $Yf=-\partial_vf-2u$. In other words, if $g(u,v,t)=t-f(u,v)$, $(u,v,t)\in\mathbb{H}$, then $Xf=Xg$ and $Yf=Yg$.

Lemma 6.5. *Let S be a C^2 surface in $\mathbb{H}$, $S=\partial\Omega$ and $S=\partial(\mathbb{H}-\overline{\Omega})$, where Ω is open in $\mathbb{H}$. Suppose that, as above, $f:A\to\mathbb{R}$, $A\subseteq\mathbb{R}^2$, gives a C^2 parametrization of a characteristic point free open portion $\mathcal{U}$ of S. Let F be the function defined in (13) and let $P=F(u,v,0)\in S$. Then, the matrix representing $JF(u,v,0)$ with respect to the basis $\{\partial_u,\partial_v,\partial_\sigma\}$ of $\mathbb{R}^2\times\mathbb{R}$, and $\{X,Y,\partial_t\}$ of $\mathbb{H}\equiv\mathbb{R}^3$ is*

$$\begin{pmatrix} 1 & 0 & \frac{Xf}{|\nabla_{\mathbb{H}}f|}\\ 0 & 1 & \frac{Yf}{|\nabla_{\mathbb{H}}f|}\\ -Xf & -Yf & 0\end{pmatrix} \tag{14}$$

Corollary 6.1. *With the assumptions of Lemma 6.5, suppose that C is the charactyeristic point of Π_PS. Then,*

$$\det JF(u,v,0)=2d(P,C)\neq 0. \tag{15}$$

Proof of Lemma 6.5. The map F can be written explicitely, since we know the expression of the normal metric and

$$F(u,v,\tau)=\mathcal{N}_{(u,v,f(u,v))}S(\tau). \tag{16}$$

We are going to use the coordinates

$$(x,y,t)=F(u,v,\tau)=(u,v,f(u,v))\circ(x',y',t')$$

where $(x',y',t')=\gamma_{u,v}(\tau)$ and $\gamma_{u,v}$ is the metric normal's left translate by P^{-1}, whose expression is given in (9):

$$\begin{cases} x'=\frac{1}{4}(Xf\sin\alpha+Yf(1-\cos\alpha))\\ y'=\frac{1}{4}(Yf\sin\alpha-Xf(1-\cos\alpha))\\ t'=\frac{|\nabla_{\mathbb{H}}f|^2}{8}(\alpha-\sin\alpha),\end{cases}$$

where $\alpha=\frac{4\tau}{|\nabla_{\mathbb{H}}f|}$. Since

$$\begin{cases} x=u+x'\\ y=v+y'\\ t=f(u,v)+t'+2(vx'-uy')\end{cases}$$

we have

$$\begin{cases} x_u=1+x'_u\\ y_u=y'_u\\ t_u=f_u(u,v)-2y'+t'_u+2(vx'_u-uy'_u),\end{cases}$$

$$\begin{cases} x_v = x'_v \\ y_v = 1 + y'_v \\ t_v = f_v(u,v) + 2y' + t'_v + 2(vx'_v - uy'_v) \end{cases}$$

and

$$\begin{cases} x_\tau = x'_\tau \\ y_\tau = y'_\tau \\ t_\tau = +t'_\tau + 2(vx'_\tau - uy'_\tau). \end{cases}$$

We compute the derivatives of each coordinate,

$$\begin{aligned} 4x'_u &= (Xf)_u \sin\alpha + (Yf)_u(1-\cos\alpha) + \alpha_u Xf \cos\alpha + \alpha_u Yf \sin\alpha \\ &= (Xf)_u \sin\alpha + (Yf)_u(1-\cos\alpha) \\ &- \cos\alpha \frac{Xf}{2} \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} \big(2Xf(Xf)_u + 2Yf(Yf)_u\big) - \sin\alpha \frac{Xf}{2} \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} \big(2Xf(Xf)_u + 2Yf(Yf)_u\big), \end{aligned}$$

and so $(x'_u)_{|\tau=0} = 0$. Analogously

$$\begin{aligned} 4x'_v &= (Xf)_v \sin\alpha + (Yf)_v(1-\cos\alpha) \\ &- \big(\cos\alpha Xf \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3} + \sin\alpha Yf \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3}\big)\big(Xf(Xf)_u + Yf(Yf)_u\big), \end{aligned}$$

and $(y'_u)_{|\tau=0} = 0$;

$$4x'_\tau = Xf \cos\alpha \frac{4}{\mid \nabla_{\mathbb{H}} f \mid} + Yf \sin\alpha \frac{4}{\mid \nabla_{\mathbb{H}} f \mid},$$

and $(x'_\tau)_{|\tau=0} = \frac{Xf}{|\nabla_{\mathbb{H}} f|}$;

$$\begin{aligned} 4y'_u &= (Yf)_u \sin\alpha - (Xf)_u(1-\cos\alpha) \\ &- \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3}\big(\cos\alpha Yf - \sin\alpha Xf\big)\big(Xf(Xf)_u + Yf(Yf)_u\big), \end{aligned}$$

and $(y'_u)_{|\tau=0} = 0$;

$$\begin{aligned} 4y'_v &= (Yf)_v \sin\alpha - (Xf)_v(1-\cos\alpha) \\ &- \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3}\big(\cos\alpha Yf - \sin\alpha Xf\big)\big(Xf(Xf)_v + Yf(Yf)_v\big), \end{aligned}$$

and $(y'_v)_{|\tau=0} = 0$;

$$4y'_\tau = \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid^3}\big(\cos\alpha Yf - \sin\alpha Xf\big),$$

and $(y'_\tau)_{|\tau=0} = \frac{Yf}{|\nabla_{\mathbb{H}} f|}$.

$$8t'_u = 2\big(Xf(Xf)_u + Yf(Yf)_u\big)(\alpha - \sin\alpha) - \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid}\big(Xf(Xf)_u + Yf(Yf)_u\big)(1-\cos\alpha),$$

and $(t'_u)_{|\tau=0} = 0$;

$$8t'_v = 2\big(Xf(Xf)_v + Yf(Yf)_v\big)(\alpha - \sin\alpha) + \frac{4\tau}{\mid \nabla_{\mathbb{H}} f \mid}\big(Xf(Xf)_v + Yf(Yf)_v\big)(1-\cos\alpha),$$

and $(t'_v)_{|\tau=0} = 0$, and, eventually

$$8t'_\tau = 4 \mid \nabla_{\mathbb{H}} f \mid (1-\cos\alpha),$$

and

$$t'_\tau = 0.$$

Hence

$$\begin{aligned} x_u &= 1+\frac{1}{4}(Xf)_u\sin\alpha+\frac{1}{4}(Yf)_u(1-\cos\alpha)\\ &- \frac{1}{4}\cos\alpha\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbb{H}}f\mid^3}\big(2Xf(Xf)_u+2Yf(Yf)_u\big)-\frac{1}{4}\sin\alpha\frac{Xf}{2}\frac{4\tau}{\mid\nabla_{\mathbb{H}}f\mid^3}\big(2Xf(Xf)_u+2Yf(Yf)_u\big)\end{aligned}$$

and $(x_u)_{|\tau=0}=1$;

$$x_v = x'_v$$

and $(x_v)_{|\tau=0}=0$;

$$x_\tau = x'_\tau$$

and $(x_\tau)_{|\tau=0}=\frac{Xf}{|\nabla_{\mathbb{H}}f|}$;

$$y_u = y'_u$$

and $(y_u)_{|\tau=0}=0$;

$$\begin{aligned} y_v &= 1+\frac{1}{4}(Yf)_v\sin\alpha-\frac{1}{4}(Xf)_v(1-\cos\alpha)\\ &- \frac{1}{4}\frac{4\tau}{\mid\nabla_{\mathbb{H}}f\mid^3}\big(\cos\alpha Yf-\sin\alpha Xf\big)\big(Xf(Xf)_v+Yf(Yf)_v\big),\end{aligned}$$

and $(y_v)_{|\tau=0}=1$;

$$y_\tau = y'_\tau,$$

and $(y_\tau)_{|\tau=0}\frac{Yf}{|\nabla_{\mathbb{H}}f|}$

$$\begin{aligned} t_u &= f_u+\frac{1}{4}\big(Xf(Xf)_u+Yf(Yf)_u\big)(\alpha-\sin\alpha)-\frac{\tau}{2\mid\nabla_{\mathbb{H}}f\mid}\big(Xf(Xf)_u+Yf(Yf)_u\big)(1-\cos\alpha)\\ &+ 2vx'_u-2uy'_u-2y',\end{aligned}$$

with $(t_u)_{|\tau=0}=f_u$.

$$t_\tau = \frac{1}{2}\mid\nabla_{\mathbb{H}}f\mid(1-\cos\alpha)+2(vx'_\tau-uy'_\tau),$$

and $(t_\tau)_{|\tau=0}=2v\frac{Xf}{|\nabla_{\mathbb{H}}f|}-2u\frac{Yf}{|\nabla_{\mathbb{H}}f|}$.

Moreover

$$\begin{aligned} t_v &= f_v+\frac{1}{4}\big(Xf(Xf)_v+Yf(Yf)_v\big)(\alpha-\sin\alpha)-\frac{1}{2}\frac{4\tau}{\mid\nabla_{\mathbb{H}}f\mid}\big(Xf(Xf)_v+Yf(Yf)_v\big)(1-\cos\alpha)\\ &+ 2vx'_v-2uy'_v+2x'\end{aligned}$$

and $(t_v)_{|\tau=0}=f_v$. As a consequence

$$J\begin{pmatrix} x & y & t\\ u & v & \tau\end{pmatrix}_{\tau=0}=\begin{pmatrix}1 & 0 & \frac{Xf}{|\nabla_{\mathbb{H}}f|}\\ 0 & 1 & \frac{Yf}{|\nabla_{\mathbb{H}}f|}\\ f_u & f_v & \frac{2vXf-2uYf}{|\nabla_{\mathbb{H}}f|}\end{pmatrix} \tag{17}$$

Changing the basis in the target space from $\partial_x,\partial_y,\partial_t$ to X,Y,∂_t, we obtain (14). □

Corollary 6.1 follows immediately,

$$\det JF(u,v,0)=\mid\nabla_{\mathbb{H}}f\mid. \tag{18}$$

Proof of Theorem 6.1-(ii). Suppose that $S = \{(x,y,t) : g(x,y,t) = 0\}$, with $g \in C^k$, $k \geq 2$. Let P_0 be a noncharacteristic point on S such that $\partial_t g(P_0) \neq 0$. Then, in a neighborhood of P_0, we can assume that $g(x,y,t) = t - f(x,y)$, with f as in Lemma 6.5, $f \in C^k$. Observe that F is of class C^{k-1} in the Euclidean sense, since in its definition all the Euclidean derivatives of f appear. Now, if $Q = F(u,v,\tau)$, then $\tau = \delta_S(Q)$. By Lemma 6.5, Corollary 6.1 and the Inverse Function Theorem in $\mathbb{R}^3$, δ_S is a C^{k-1} function in a neighborhood of P_0. Since δ_S satisfies (11), δ_S is C^{k-1}, by Lemma 6.4 we have that $\nabla_{\mathbb{H}}\delta_S$ is C^{k-1} as well.

We now consider the set S_0 of those points (x_0,y_0,t_0) in S where $\partial_t g(x_0,y_0,t_0) = 0$. We consider two cases. Suppose that (x_0,y_0,t_0) has a neighborhood $\mathcal{U}$ in S such that the metric normal at any point of $\mathcal{U}$ is a Euclidean straight line, which is normal in the Euclidean sense to S. Hence, in an open neighborhood of (x_0,y_0,t_0), δ_S is the Euclidean distance, and the required smoothness of δ_S and $\nabla_{\mathbb{H}}$ follows.

The second case is that where (x_0,y_0,t_0) is the limit of points (x,y,t) of S where $\partial_t g(x,y,t) \neq 0$. Since $\nabla g(x_0,y_0,t_0) \neq 0$, we can assume that, for (x,y,t) in a neighborhhod $\mathcal{U}$ of (x_0,y_0,t_0), $\partial_y g(x,y,t) \neq 0$. By the Implicit Function Theorem, restricting $\mathcal{U}$ if necessary, we can assume that $g(x,y,t) = 0$ if and only if $y = h(x,t)$, where h is defined in an open neighborhood of (x_0,t_0) in $\mathbb{R}^2$. We consider a function $H : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{H}$, defined in an open neighborhood of $(x_0,t_0,0)$,

$$H(x,t,\tau) = \nu_{(x,h(x,t),t)}S(\tau) \tag{19}$$

Observe that H is a smooth map, in the Euclidean sense, at points where $\partial_t g = 0$ as well, by (9) and (10). Now,

$$|\det JH| = |h_t \cdot JF|$$

and the latter, when $\tau = 0$, is equal to

$$2|h_t|d(P,C) = |h_t|\frac{|\nabla_{\mathbb{H}}g|}{|\partial_t g|}$$

Since

$$0 = g_y h_t + g_t$$

we have that

$$\frac{|\nabla_{\mathbb{H}}g|}{|\partial_y g|} = |\nabla_{\mathbb{H}}h|$$

hence we have an expression analogous to (18),

$$\det JH(x,h(x,t),t) =\mid \nabla_{\mathbb{H}}h(x,t) \mid . \tag{20}$$

where $Xh = X(h(x,t) - y) = h_x + 2hh_t$ and $Yh = Y(h(x,t) - y) = -1 - 2xh_t$. On vertical points P, in particular,

$$|\det JH(P)| = \sqrt{h_x^2 + 1} > 0.$$

Now, the proof that δ_S and $\nabla_{\mathbb{H}}\delta_S$ have the required smoothness proceeds as in the previous cases. □

As a byproduct, we have the proof of a special case of a theorem of Monti and Serra-Cassano [15].

Corollary 6.2. *Let $S = \partial\Omega$ be the boundary of an open C^2 subset of $\mathbb{H}$ and let d_S be the distance function for S. Then, d_S satisfies the eikonal equation*

$$|\nabla_{\mathbb{H}}d_S| = 1, \tag{21}$$

on the surface S. Moreover if $\Omega = \{g < 0\}$, where g is a smooth function, then

$$\nabla_{\mathbb{H}}\delta_S = \frac{\nabla_{\mathbb{H}}g}{|\nabla_{\mathbb{H}}g|}.$$

¿From Lemma 6.5 and the analiticity of the geodesics' equations (2) it follows that δ_S is analytic in a neighborhood of $S - Char(S)$ if S itself is analytic.

7. Cutlocus

Next, we give an elementary description of the set of the metric normals' endpoints.

Let $S = \partial\Omega$ be the boundary of an open, connected set in $\mathbb{H}$ and assume that S is at least C^2. For $P \in S$, let Q be endpoint in $\overline{\Omega}$ other than P of the metric normal $\mathcal{N}_P S$ (when $\mathcal{N}_P S$ reduces to the point P, we set $Q = P$). The *cut-locus* in $\overline{\Omega}$ of S (the *skeleton* of Ω) is the set K_S of such points Q as P varies over S.

Below, $\mathcal{N}_P S$ refers to the portion of the metric normal at P which lies inside $\overline{\Omega}$.

Here are some properties of K_S.

Lemma 7.1. *Let $P \in S$ be non characteristic. If $\mathcal{N}_P S$ is not a straight line, then it is a proper subarc of a maximal length minimizing geodesic starting at P.*

Proof. Without loss of generality, assume that $P = O$. Let $\mathcal{N}_P S = \gamma|_{[0,b]}$, where γ is a maximal length-minimizing geodesic starting at P and having length $a \geq b$. If $b = a$, then $\gamma(b) = Q$ belongs to the t-axis, hence $R_\theta\gamma$ satisfies $l_{\mathbb{H}}(R_\theta\gamma) = l_{\mathbb{H}}(\gamma) = d(P,Q) = d(Q,S)$, i.e., for all θ, $R_\theta\gamma$ lies on $\mathcal{N}_P S$. This implies that $R_\theta\dot{\gamma}(0)$ is perpendicular to $T_O S$, for all θ, which is absurd. □

Lemma 7.2. *Let C be a characteristic point of S. Then, $\mathcal{N}_C S = \{C\}$.*

Proof. Suppose that $Q \in \mathcal{N}_C S = \{C\}$ is different from C. Then, S does not intersect $B(Q, d(Q,C))$ and meets its closure in C. Hence, S is not smooth at C. □

Proposition 7.1. *The cut locus K_S of S has the following properties.*

(i) *K_S has empty interior.*
(ii) *K_S contains the characteristic points of S and each characteristic point of S is an accumulation point of $K_S - S$.*

Proof. The assertion in (i) is a consequence of the following.

Lemma 7.3. *Let $Q \in K_S$ and let γ be a geodesic from Q to S such that $d(Q,S) = l_{\mathbb{H}}(\gamma)$. Then $\gamma - \{Q\}$ is free of points of K_S.*

Proof. of the lemma. Let P be the endpoint of γ in S. Let R be point of K_S on γ, other than Q. By definition of K_S, there exists a maximal geodesic η through R, having length $a > d(R,S)$, such that $\eta(0) = P' \in S$, $R = \eta(b)$, where $b = d(R,S) > a$, and for $b < c \leq a$, $d(\eta(c),S) < c$ (otherwise R would not be the endpoint of $\mathcal{N}_{P'} S$). Clearly, $d(R,P') = d(R,P)$.

By the triangle inequality, then

$$d(Q,S) \leq d(Q,P') \leq d(Q,R) + d(R,P') = d(Q,R) + d(R,P) = d(Q,P) = d(Q,S)$$

but this is possible only if R lies on the geodesic between Q and P', and so $Q \in \eta$, contradicting the fact that η's length does not realize the distance of its points from S past R. This proves the lemma. □

The first assertion in (ii) was proved in Lemma 7.2. Let now C be a characteristic point of S and let P_n be a sequence of noncharacteristic points of S tending to C. (Such sequence exists because the horizontal distribution is not closed under Lie brackets and by Frobenius Theorem). The formula for the geodesic normal in Theorem 4.1 implies that the length of $\mathcal{N}_{P_n} S$ is no more that $d(P_n, C_n)$, where C_n is the characteristic point of $T_{P_n} S$. By continuity, $d(P_n, C_n) \to d(C,C) = 0$ as $n \to \infty$. Let $R_n \in K_S$ be the endpoint other than P_n of $\mathcal{N}_{P_n} S$. Then,

$$d(R_n, C) \leq d(R_n, P_n) + d(P_n, C) \leq d(R_n, C_n) + d(P_n, C) \to 0$$

and this proves (ii). □

Proposition 7.2. *Let $R \in \mathbb{H} - K_S$. Then there is a unique geodesic γ from R to S such that $l_{\mathbb{H}}\gamma = d(R,S)$. i.e., there exists a unique $P \in S$ such that $R \in \mathcal{N}_P S$.*

Proof. Suppose there are two such geodesics, γ and γ', having the other endpoint, resp. P and P', on S. We can not have $P = P'$, otherwise, with a reasoning similar to that of Lemma 7.1, one shows that P is characteristic, hence γ reduces to a point and $R \in S$.

Suppose first that γ is not a straight line. Let $Q \in K_S$ be the endpoint of $\mathcal{N}_P \subseteq \gamma$ other than P. If γ' dos not lie in the (possibly non length-minimizing) prolongment of γ, then, as in the proof of Proposition 7.1, we have that $d(P',Q) < d(P,Q) = d(Q,S)$, a contradiction. The other possibility is that γ' and γ have in common the arc between Q and R. This leads to a contradiction, too, since it would imply

$$d(Q,P') = d(P',R) - d(Q,R) < d(P,R) + d(Q,R) = d(P,Q)$$

, hence Q could not belong to $\mathcal{N}_P$.

If $\gamma = [P,R]$ and $\gamma' = [P',R]$ are straight lines, either they lie on the prolongment of each other (but this is not possible, otherwise $R \in K_S$), or they have different directions. In the second case, there exists a point Q such that $R \in [Q,P]$ and $d(Q,S) = l_{\mathbb{H}}([P,Q])$ (otherwise R would be the endpoint of $\mathcal{N}_P$, then $R \in K_S$) and, as above, we would have $d(Q,P') < d(P,Q)$, contradicting the fact that $Q \in \mathcal{N}_P$. $\square$

A detailed study of the cut-locus K_S and of its properties will be the object of further research. For the case when S is analytic, [1] contains a detailed study of the cut locus, which is defined, however, in different way.

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