BELTRAMI'S MODELS OF NON-EUCLIDEAN GEOMETRY

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Abstract. In two articles published in 1868 and 1869, Eugenio Beltrami provided three models in Euclidean plane (or space) for non-Euclidean geometry. Our main aim here is giving an extensive account of the two articles' content. We will also try to understand how the way Beltrami, especially in the first article, develops his theory depends on a changing attitude with regards to the definition of surface. In the end, an example from contemporary mathematics shows how the boundary at infinity of the non-Euclidean plane, which Beltrami made intuitively and mathematically accessible in his models, made non-Euclidean geometry a natural tool in the study of functions defined on the real line (or on the circle).

Contents

1. Introduction

In two articles published in 1868 and 1869, Eugenio Beltrami, at the time professor at the University of Bologna, produced various models of the hyperbolic version non-Euclidean geometry, the one thought in solitude by Gauss, but developed and written by Lobachevsky and Bolyai. One model is presented in the *Saggio di interpretazione della geometria non-euclidea*[5] [*Essay on the interpretation of non-Euclidean geometry*], and other two models are developed in *Teoria fondamentale degli spazii di curvatura costante* [6] [*Fundamental theory of spaces with constant curvature*]. One of the models in the *Teoria,* the so-called Poincaré disc, had been briefly mentioned by Riemann in his *Habilitationschrift* [26], the text of which was posthumously published in 1868 only, after Beltrami had written his first paper [5][1], and it served as a lead for the second one [6]. Riemann was not so much interested in getting involved in a *querelle* between Euclidean and non-Euclidean geometry, which he had in fact essentially solved in his remark, as he was interested

[1]In a letter to Genocchi in 1868 ([15] p.578-579), Beltrami says that he had the manuscript of the *Saggio* ready in 1867, but that, faced with criticisn from Cremona, he had postponed its publication. After reading Riemann's *Habilitationschrift,* he felt confident in submitting the article

in developing a broad setting for geometry, a "library" of theories of space useful for the contemporary as well as for the future developments of sciences. At the time Beltrami wrote his *Saggio*, however, Riemann's *Habilitationschrift* was not widely available and although its connection to non-Euclidean geometry was rather clear, it was not explicitely stated. It should be mentioned that the second model in the *Teoria* had been previously considered by Liouville, who used it as an example of a surface with constant, negative curvature.

Beltrami's papers were widely read and promptly translated into French by Jules Hoüel. Their impact was manifold. (i) They clearly showed that the postulates of non-Euclidean geometry described the simply connected, complete surfaces of negative curvature (surfaces which, however, only locally could be thought of as surfaces in $\mathbb{R}^3$). (ii) Hence, it was not possible proving the Postulate of the Paralles using the remaining ones, as J. Hoüel explicited in [17][2]. (iii) It was then possible to consider and use non-Euclidean geometry without having an opinion, much less a faith, concerning the "real geometry" (of space, of Pure Reason). (iv) More important, and lasting, the universe of non-Euclidean geometry was not anymore the counter-intuitive world painted by Lobachevsky and Bolyai: any person instructed in Gaussian theory of surfaces could work out all consequences of the non-Euclidean principles directly from Beltrami's models; this legacy is quite evident up to the present day. (v) In all of Beltrami's models, the non-Euclidean plane (or n-space) is confined to a portion of the Euclidean plane (or n-space), whose boundary encodes important geometric features of the non-Euclidean space it encloses.

The presence of an important "boundary at infinity" in non-Euclidean geometry had been realized before. In Beltrami's models, this boundary is (from the Euclidean viewpoint of the "external observer") wholly within reach, easy to visualize, complete with natural coordinate systems (spherical, when the boundary is seen as a the limit of a sphere having fixed finite center and radius tending to infinity; Euclidean-flat when it is seen as the limit of horocycles: distinguished spheres having infinite radius and center at infinity). With this structure in place, it was possible to radically change viewpoint and to see the non-Euclidean space as the "filling" of its boundary, spherical or Euclidean. To wit, some properties of functions defined on the real line or on the unit circle become more transparent when we consider their extensions to the non-Euclidean plane of which the line or the circle are the boundary in Beltrami's models. This line of reasoning is not to be found in Beltrami's articles, but it relies on Beltrami's models, and it is the main reason why non-Euclidean (hyperbolic) geometry has entered the toolbox of such different areas as harmonic and complex analysis, potential theory, electrical engineering and so on. The pioneer of these kind of applications is Poincaré [25]. It is still matter of discussion whether or not Poincaré had had any exposure to Beltrami's work, or if he re-invented one of Beltrami's models ([19], p. 277-278). Even if he had not had first-hand knowledge of Beltrami's work, however, I find

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to the Neapolitan *Giornale di matematiche,* emended of a statement about three dimensional non-Euclidean geometry and with some integration "which I can hazard now, because substantially agreeing with some of Riemann's ideas."

[2]In another letter to Genocchi ([15] p.588), Beltrami writes that it this fact clearly follows from his *Saggio,* and that "in the note of Hoüel I do not find further elements to prove it". Beltrami being generally rather unassuming about his own work, it is likely that he had already thought of this consequence of his model, but that he thought it prudent to leave it to state explicitely to the reader.

it unlikely that he had not heard of the debate about non-Euclidean geometry in which Beltrami's work was so central, and of the possibility of having concrete models of it. It is nowadays very common, for mathematicians from all branches, to start with a class of objects (tipically, but not only, functions) naturally defined on some geometric space, and to look for a "more natural" geometry which might help in understanding some properties of those same objects. The new geometry, this way, has a "model" built on the old one.

Let me end these introductory remarks by reminding the reader that, unfortunately, in the mathematical pop-culture the name of Beltrami is seldom attached to his models. The model of the *Saggio* is generally called the *Klein model* and the two models of the *Teoria* are often credited to Poincaré. There are reasons for this. Klein made more explicit the connections between the model in the *Saggio* and projective geometry, which Beltrami had just mentioned in his article. Poincaré, as I said above, was the first to use the other two models in order to understand phenomena apparently far from the non-Euclidean topic. More informed sources refer to the projective model as the *Beltrami-Klein (projective) disc model;* the other two should perhaps be called *Riemann-Beltrami-Poincaré (conformal) disc model* and *Liouville-Beltrami (conformal) half-plane model.*

The aim of this note is mostly expository. There are excellent accounts of how non-Euclidean geometry developed: from the scholarly and influential monograph of Roberto Bonola [11], which is especially interesting for the treatment of the early history, to the lectures of Federigo Enriques [16], to the recent, flamboyant book by Jeremy Gray [19], in which the development of modern geometry is treated in all detail. To have a taste of what happened after Beltrami, Klein and Poincaré, I reccommend the beautiful article [21] by Milnor, which is also historically accurate, and the less historically concerned, but equally useful article [14] by Cannon, Floyd, Kenyon and Parry. An extensive account of the modern view of hyperbolic spaces (from the metric space perspective) is in Bridson and Haefliger's beautiful monograph [13]. I will just summarize the well known story up to Beltrami for ease of the reader. Then, I will describe in some detail the main mathematical content of the *Saggio* and of the *Teoria.* Not only such content is a masterful piece of mathematics, but it was also in the non-pretentious style of Beltrami to present his work as double faced: his reader could think of it as an investigation on the foundations of geometry, but, if skeptical, he could also give it *a purely analytic meaning* ([6] p.406; Beltrami refers to the geometric terms he uses, but the same distinction applies to the two papers as a whole). We can appreciate the analytic content in itself. With this at hand, we will try to understand what Beltrami claimed to have achieved in geometric (and logical) terms.

To end on a different tune, I will describe how, starting from a reasonable problem about functions defined on the real line (looking at a signal at different scales) one is naturally led to consider non-Euclidean geometry in the upper-half space, the last of Beltrami's models. My aim here is giving a simple example of one of the most important legacies of Beltrami's models, which is point (v) above. For understandandable reasons, this aspect is seldom mentioned in historical accounts, while it is central (and popular) in the research literature and in textbooks.

A disclaimer is due. I am neither trained in geometry, nor in history of mathematics. This surely accounts for the bibliography, which is probably not the one an historian of science would have used, and for other naiveties I can not be aware of. I have tried, however, to be as historically correct as I could and to be honest about anachronisms. I have been using Beltrami's models for many years, as a tool or as an inspiring metaphor, while working in harmonic analysis and complex function theory, and this is my only title to discuss them. I thank Salvatore Coen, who entrusted me with writing this note and for investing so much energy to edit the volume.

Note on bibliography. *Beltrami's papers in the bibliography are given with the coordinates of their publication, except for the page numbers, which refer to the edition of his collected works* [2].

2. Non-Euclidean geometry before Beltrami

Like in many other scientific revolutions, at the roots of the non-Euclidean one we find an orthodox theory and a disturbing asimmetry. The orthodox theory is Euclid's *Elements,* in which the science of measurement and space (ideal space from a Platonic viewpoint, real from an Aristotelian one: it does not matter here) is given a hierarchical structure (axioms, postulates, definitions, theorems), held together by logics. The postulates should encode unquestionable truths about space, from which other truths are deduced. The asimmetry consists in the Fifth Postulate (or Parallel Postulate), concerning properties of parallel lines, which Euclid postpones until after Proposition XXVIII. He has just proved that two straight lines a and b in the plane do not meet, if the internal angles they make on the same side of a third line c meeting both of them sum to a straight angle. The Fifth Postulate states that the converse is true: if the sum is less than a straight angle, then a and b eventually meet on that same side of c. The main disturbig feature of the Fifth Postulate is that, in order to verify the property it states, one has to consider the straight lines in their infinite extension. It was soon realized that the Fifth Postulate is equivalent to the *uniqueness* of the straight line through a given point P, which is parallel to a given straight line a not containing P. Very early, attempts were made to prove it, based on the remaining Postulates and Axioms.

In the effort, several properties were found which, given the other Postulates, were equivalent to the Fifth. Also, the critical thought unleashed in search for a proof of the Fifth Postulate helped in finding a number of hidden assumption (namely, hidden postulates) in Euclid's work: the line divides the plane into two parts, for instance, or the Archimedean property of lengths.

Trying without success to prove the Postulate of Parallels by contradiction, mathematicians went deeper and deeper into a geometric world in which the Postulate did not hold, finding increasingly counterintuitive properties of figures. This kind of research reached maturity with the work if Girolamo Saccheri[3] [1667-1733]. In his *Euclides ab omni naevo vindicatus* (see [11], Chapter II), Saccheri considered a fixed quadrilateral $ABCD$ with right angles in A and B and equal sides $AC = BD$. He considered the three possibilities for the angles $\widehat{C} = \widehat{D}$: (r) the angles are both right (then the Fifth Postulate holds); (o) the angles are both obtuse; (a) the angles are bothe acute. The *obtuse hypothesis* (o) leads to contradiction[4] It remained the

[3] Interestingly, the work of Saccheri, which had an indirect role in the development of non-Euclidean geometry and was then forgotten, was re-discovered by Beltrami [10].

[4] The obtuse hypothesis holds on a sphere, using geodesics (great circles) as straight lines; but on a sphere we do not have uniqueness of the geodesic through two points. This was considered to be a major problem by Beltrami, who was looking for a geometry in which all principles of

acute angle hypothesis, which Saccheri developed at length. I again refer to Bonola's monograph [11] for more on this early history, but let me mention a few among the properties which, in the effort of proving the Fifth Postulate by contradiction, were found to hold in the fictional geometry where the Fifth would be false.

(i) The set of points equidistant from a straight line a (on one side of a) *is not* a straight line (this is already implicit in Posidonius [I cent. a.C.], who *defines* two straight lines to be parallel if they are equidistant).
(ii) For each planar figure F, there is not a similar figure F' (i.e., with the point having reciprocal distances in proportion to F) of arbitrary size (J. Wallis [1616-1703]).
(iii) The sum of the internal angles for one (hence, of all) triangle ABC is less than a straight angle (G. Saccheri).
(iv) More, the quantity $\pi-(\hat{A}+\hat{B}+\hat{C})$ is proportional to the area of ABC (J.E. Lambert [1728-1777]); hence, the proportionality constant is an *absolute quantity.*
(v) There are distinct parallel lines a and b having a common point and a common orthogonal line at infinity (G. Saccheri).
(v) There are points P, Q, R which are not collinear, such that no cricle passes through all of them (W. Bolyai [1775-1856], padre di J. Bolyai).

Lambert went on to observe that (iv) holds, with reversed signs, for geodesic triangles on a sphere of radius r:

$$Area(ABC) = r^2(\hat{A}+\hat{B}+\hat{C}-\pi);$$

hence, that (iv) could be seen as a phenomenon taking place on a sphere of imaginary radius. This viewpoint played a major role in the developement of non-Euclidean geometry.

Finally, independently of each other, N.I Lobachevsky [1793-1856], J. Bolyai [1802-1860] and C.F. Gauss [1777-1855] totally changed perspective. Starting from the assumption that the Parallel Postulate could not be proved based on the remaining ones, they assumed it did not hold and went on developing the corresponding theory, trusting that no contradiction could possibly arise. They all shared the view that geometry was a description of physical space, and Gauss and Lobachevsky even compared their non-Euclidean geometry (as Gauss called it) and the absolute quantity already pointed out by Lambert, with available astronomical evidence. They deduced that, in real space, the value of r must be very large. Lobachevsky and

Euclidean geometry hold true, but the uniqueness of parallels, but not for Riemann. In the *Habilitationschrift,* Riemann offered the sphere as a model for a geometry in which no parallel existed. Some of the principles used by Saccheri had to be abandoned: uniqueness of the line through two points, it seemed; but also the infinite extension of lines (Riemann seems to be the first to point out that the right geometric requirement is not that the straight lines have infinite length -what he calls "infinite extent of the line", translated in "unboudedness" in modern netric space theory-, but that one finds no obstructions while following a straight line -a property he calls "unboundedness", translated nowadays in "metrically complete and without boundary"). Clifford used the two-to-one covering of the real projective plane by the sphere to exhibit a geometry with positive constant curvature in which (i) there was just one line throughtwo points; (ii) space was homogeneous and isotropic; (iii) there are no distinct, parallel straight lines. Kelin realized the role of this model in the discussion of non-Euclidean geometries. Of course, in the projective-space model, often called *Riemann's non-Euclidean geometry,* one not only has to give up the infinite extension of straight lines, but also the fact that a line divides the plane into two parts; or, which is about the same, orientability of the plane.

Bolyai published their foundings, but Gauss did not: because he did not want to be involved in a philosophical-mathematical struggle, but also because he learned of the research of Bolyai and Lobachevsky and was satisfied that others had gone public with the new geometry.

At this stage we have two competing geometries: Euclidean and non-Euclidean, with and without the Fifth Postulate. There also was a substantial body of *absolute geometry,* which was the intersection of the two. Bonola suggests that Kant's doctrine of space, according to which Euclidean Geometry was the main concrete example of *synthetic a priori knowledge,* acted against the acceptance of, or even the debate about non-Euclidean geometry. J. Gray [18] argues that, at the time Bolyai and Lobachevsky published their work, Kantian philosophy was not hegemonic as it had been a generation before. On the mathematical side, Lobachevsky proposed to model non-Euclidean geometry using as analytic model hyperbolic trigonometry, the non-Euclidean version of trigonometry (or, better, the "imaginary" version of spherical trigonometry). He was logically on firm ground, but the mathematical community seem not to have reacted to his proposal.

Another fact has to be taken into account. At the beginning of XIX century, Euclidean geometry was considered the least questionable of all sciences (an ancient state of affairs, of course, that had also directed Kant to exemplify by it the notion of "synthetic a priori"), while the conceptual foundations of calculus were still matter of debate and controversy. It was through the work of Cauchy, Bolzano and, especially, Weierstrass, that mathematical analysis, hence the differential geometry of surfaces, reached a level of philosophical reliability comparable (for the standards of the time) to that of geometry [12]. Weierstrass lectured on the foundations of calculus in 1859-60. The '60's were an excellent decade for developing analytic models for geometry.

Starting with Gauss work on surfaces, a substantial body of knowledge had been accumulating on surfaces of constant curvature. The reason is clear: on these surfaces only could figures be moved freely, at leaast locally, without alteration of their metric properties. Gauss published his *Theorema egregium* in 1827 and it was already clear that, if figures could be moved isometrically, cuvature had to be constant. Minding observed that the converse was true in the 30's, and he found various surfaces of constant negative curvature in Euclidean space, the tractroid among them. Liouville found other examples in 1850, one of which -the Liouville-Beltrami half plane- will be discussed below. Riemann, in 1854, revolutioned the concepts of geometry in his *Habilitationschrift,* in which surfaces of constant curvature played a distinguished and exemplary role. Codazzi, in 1857, found that the trigonometric formulae on the tractroid could be obtained by those on the sphere by considering the radius as imaginary. With the exception of Riemann's work, at that time unpublished. See [19] for an extensive and very readable account of the early history of differential geometry. Beltrami knew all of this litrature, which he brought to sinthesis, and more, in the *Saggio.*

3. The models of Beltrami

Between 1868 and 1869, in two influential articles, Beltrami provided models of the non-Euclidean geometry of Lobachevsky and Bolyai. One of these models is better known as the *Klein model*, another one as the *Poincaré disc model*, the third as the *Poincaré half plane model.* A fourth one, which Beltrami worked out, like the disc model, directly from Riemann's *Habilitationschrift*, has had a much minor impact. In a sense, the first article, *Saggio di interpretazione della Geometria non-euclidea* [5], was written under the influence of Gauss, and the second, *Teoria fondamentale degli spazi di curvatura costante* [6], under the influence of Riemann. In this section we first give an exposition of the mathematical content of the two papers, then we will try to clarify some points of methodology and philosophy.

3.1. **The "projective" model.** In his *Saggio* ([5], p.377), Beltrami introduces a family of Riemannian metrics on the disc of radius $a > 0$ centered at the origin by letting

$$ds^2 = R^2 \frac{(a^2 - v^2)du^2 + 2uvdudv + (a^2 - u^2)dv^2}{(a^2 - u^2 - v^2)^2}. \tag{1}$$

The Gaussian curvature of ds^2 is the constant $-1/R^2$. The advantage of such metric is that its geodesics are straight lines.

Were does the metric ds^2 come from? In 1865 ([3], p.262-280), Beltrami had considered the problem of finding all surfaces which could be parametrized in such a way that all geodesics were given by straight lines. The starting point was Lagrange's rather obvious observation that, by projecting the sphere S onto a plane Π from the center of S, the geodesics of S (great circles) were mapped $2-1$ onto straight lines in Φ, and viceversa. To his suprise, Beltrami found out that the only surfaces which had such distinguished parametrizations where the surfaces of constant curvature. In the positive curvature case, he computed the Riemannian metric

$$ds^2 = R^2 \frac{(a^2 + v^2)du^2 - 2uvdudv + (a^2 + u^2)dv^2}{(a^2 + u^2 + v^2)^2}, \tag{2}$$

which is in fact the spherical metric on a sphere of radius R, written after the sphere has been projected from its center C to a plane Π having distance a from C. He had probably worked out the analogous expression (1) for the negative curvature case, but he did not write it in the article *per abbreviare il discorso,* "for the sake of brevity" ([3], p.276). Beltrami also mentions ([5], Nota I, p.399) that (1) can be obtained from (2) by changing a and R in their imaginary counterparts ia and iR, according to the general euristic principle we have already mentioned, that hyperbolic geometry is spherical geometry on a sphere of imaginary radius. Since the equations of the geodesics are the same in both the real and in the imaginary case, the geodesics with respect to the coordinates u, v are straight lines. Hence, they are chords of the limiting circle.

The angle θ between two "extrisically perpendicular" geodesics $u = const$ and $v = const$ is computed:

$$\cos(\theta) = \frac{uv}{\sqrt{(a^2 - u^2)(a^2 - v^2)}}; \sin(\theta) = \frac{a\sqrt{a^2 - u^2 - v^2}}{\sqrt{(a^2 - u^2)(a^2 - v^2)}}. \tag{3}$$

Form the expression of $sin(\theta)$ we see that, when the two geodesics meet at a point on the circle at infinity, the angle they make vanishes[5]. This is exactly the phenomenon (recalled in Section 2, (v)) that Saccheri had found "repugnant to the nature of the straight line".

[5]This is true also for geodesics not in the form $u = const$ and $v = const$. See the formulae at p.381 in the *Saggio*.

Beltrami's first aim is proving two basic properties he singles out as crucial in the introduction to the *Saggio*([5], p.375-76):

(A1) Given two distinct points A and B, there is exactly a line passing through them ([5], p.381).

(A2) Given points A and B and oriented lines $l \ni A$ and $m \ni B$, there is a rigid movement of the plane which maps A to B and l into m, preserving the orientation ([5], Nota II, p.400-405: I summarize in XX century language what Beltrami writes as "principle of superposition" at p.376).

The first property does not hold, he says, in the positive curvature case. He is thinking of the sphere model and the statement, as such, is not wholly accurate. In fact, as Clifford had shown, but not linked to the dibate on the Parallel Postulate, the real projective space has the properties (A1) and (A2). It was known that properties (A1) and (A2) locally hold on surfaces of constant negative curvature: Beltrami's goal is realizing a model in which they hold globally.

Here, Beltrami explains, "point" means "point on the surface of constant curvaturem parametrized by (u, v) in the disc $\{(u, v) : \ u^2 + v^2 < a^2\}$" and "line" is a geodesic for the metric (1). I use the expresion "rigid movement" for any transformation which preserves the metric (Beltrami writes of of the possibility of superposing without restrictions a figure onto another). To have a complete picture of the model, it is useful to know:

(A3) Lines can be indefinitely prolonged ([5], p.380 for geodesics passing through the origin; extend to the general case using (A3)) and they have infinite length in both directions.

(A4) The plane is simply connected ([5], p.378).

(A5) There are more lines passing through a point $A \notin l$, which do not meet l ([5], p.382).

Beltrami calls *psedudospheres* the surfaces bijectively parametrized by the coordinates (u, v) and endowed with the metric (1) (below, we will use idifferently *psedosphere, hyperbolic plane, non-Euclidean plane).* At p.381, he correctly says that the theorems of non-Euclidean geometry apply (translating geometric terms in differential-geometric terms as above) to the pseudospheres. We know that the implication goes in the other direction as well: if a question is posed in terms of non-Euclidean geometry and it is answered working on the pseudosphere model, we can consider that answer as belonging to non-Euclidean geometry; the same way questions in Euclidean geometry can be answered in terms of the Cartesian plane.[6] How aware was Beltrami of this?

It is my opinion that he was personally convinced, but hesitant to state it explicitely. He surely lacked the conceptual frame which was going to be provided, much later, by Peano and Hilbert [27]. He also did not have the philosophical confidence of Riemann, who simply avoided this kind of questions, to go directly to the general science of space. There is little doubt I think, that, for Riemann, non-Euclidean, hyperbolic geometry *was* that of the model he had mentioned in the *Habilitationschrift* and that Beltrami was going to study in depth in his *Teoria fondamentale* [6].

[6]This amounts to say that, for each given value of R there is, but for isometries, exactly one non-Euclidean plane having "radius of curvature" equal to R (where R is connected to the universal geometric constants envisioned by Lobachevsky, Bolyai and Gauss).

Instead of simply saying that his model was, rather than contained, non-Euclidean geometry, Beltrami further proceeds in two directions. On the one hand, he goes on to deduce from his model a number of important theorems in non-Euclidean geometry, as to convince the reader and himself that his model correctly answers all reasonable questions in non-Euclidean geometry. On the other hand, he seems to worry that the surface with the metric (1) might not be considered wholly "real", since it is not clear in which relation it stands with respect to Euclidean three space (the strictest measure of "reality"). Then, he will show that, after cutting pieces of it, the pseudosphere can be isometrically folded onto a "real" constant curvature surface in Euclidean space.[7] If it is true that both directions seem to be suggested by logical and philosophical hesitation, this hesitation induces Beltrami to do some really elegant mathematics, and to unravel some very interesting features of his model.

After one knows that the geodesics for (1) are chords of the unit disc, it is obvious that the geodesics in this metric satisfy the incidence properties of straight lines in non-Euclidean geometry, and Beltrami is very coscious in proving this with all details ([5], p.382). In the *Nota II ([5], Nota II, p.400-405),* as we said in (A2), Beltrami shows that the metric ds^2 is invariant under rotations around the origin and, for each fixed (u_0, v_0), there is a metric preserving transformation of the unit disc, which maps $(0,0)$ to (u_0, v_0): that is, with respect to the metric (1), the disc is -in contemporary terms- homogeneous and isotropic; figures can be moved around with no more restrictions than the ones they are subjected to in Euclidean plane. Being the geodesics straight lines, one is not surprised in learning that the isometries are the projective transformations of the plane fixing the circle at infinity. The isometries of the pseudosphere are compositions of (Euclidean with respect to u, v coordinates) rotations around the origin, reflections in the coordinate axis and maps of the form

$$(u,v) \mapsto \left(\frac{a^2(u - r_0)}{a^2 - r_0 u}, \frac{a\sqrt{a^2 - r_0^2}\,v}{a^2 - r_0 u} \right), \ 0 < r_0 < a.$$

This is one of the two points in which Beltrami makes explicit the connection between his model and projective geometry, which will be later developed in depth by Felix Klein.

The distance ρ between the origin and (u,v) is computed by straightforward integration as

$$\rho = \frac{R}{2} \log \frac{a + \sqrt{u^2 + v^2}}{a - \sqrt{u^2 + v^2}}, \tag{4}$$

which implies (A3). Also, using another integration of (1) on a Euclidean (hence, by (4), non-Euclidean) circle centered at the origin, one deduces that the semiperimeter of the non-Euclidean circle having radius ρ is

$$\pi R \sinh(\rho/R),$$

a formula already known to Gauss ([5], p.380 and 384).

[7] As a matter of fact, later Hilbert showed that no regular surface in Euclidean three space is isometric to the whole pseudosphere. Much later, Nash showed that any surface can be isometrically imbedded in a Euclidean space having dimension large enough.

Beltrami now turns his attention to the *angle of parallelism.* Consider two geodesics α and β meeting at right angles at a point P, a point Q on β at non-Euclidean distance δ from P and let ζ, ξ be the two geodesics passing through Q which are *parallel* to α (that is, which separate the geodesics through Q which meet, and the ones which do not, α). Let Δ be the angle ζ and ξ make with β (by reflection invariance, it is the same). Assume Q is the origin in the model. Using the facts that (1) is conformal to the Euclidean metric at the origin and that geodesics are straight lines, and (4), one easily computes

$$\tan(\Delta) = 1/\sinh(\delta/R),$$

which had been previously found by Battaglini on surfaces of constant negative curvature. After standard manipulation, this becomes Lobachevsky's pivotal formula

$$\tan(\Delta/2) = e^{-\frac{\delta}{R}}. \tag{5}$$

Among the other theorems Beltrami deduces from his models we find the main formulae of non-Euclidean trigonometry and the formula for the area of a geodesic triangle having angles $\hat{A},\ \hat{B},\ \hat{C}$;

$$R^2(\pi - \hat{A} - \hat{B} - \hat{C}).$$

When the angles vanishe, we obtain the single value πR^2 for the area of a geodesic triangle with all vertices at infinity. It is curious that ([5], p.389), when mentioning that the value "is independent of its [the triangle's] shape", Beltrami's fails to observe that all such triangles are in fact isometric and hence, from the pseudospherical point of view, they have in fact the same shape.

At p.387, between the discussion of trigonometry and that about areas, Beltrami writes: "The preceeding results seem to fully show the orrespondence between non-Euclidan (two-dimensional) geometry and geometry of the pseudosphere." Again, this is evidence that he thought of his model as essentially identical to Lobachevsky's geometry, but he lacked a conceptual frame for defending his belief in front of possible criticism; hence, he preferred to keep a reasonably low profile.

We now come to the second appearence of projective geometry in the article ([5], p.391-392). In the Nota II, Beltrami finds that the equation of the pseudospherical circle having radius ρ and center (u_0, v_0) is

$$\frac{a^2 - u_0 u - v_0 v}{\sqrt{(a^2 - u^2 - v^2)(a^2 - u_0^2 - v_0^2)}} = \cosh(\rho/R) \geq 1. \tag{6}$$

From Gauss Lemma, such circles are orthogonal to the geodesics issuing from (u_0, v_0). In Beltrami's model, however, it makes sense to find the curves which are ortogonal to the geodesics passing through (u_0, v_0) even when $u_0^2 + v_0^2 = a^2$ (a *point at infinity*) $u_0^2 + v_0^2 > a^2$ (an *ideal point*). The equation of the orthogonal curve is deduced the same way as (6), and it has a similar form,

$$\frac{a^2 - u_0 u - v_0 v}{\sqrt{a^2 - u^2 - v^2}} = C,\ \ u^2 + v^2 < a^2. \tag{7}$$

We might be called *generalized (metric) circles* the curves γ_C given by (7). The curves γ_C corresponding to the same (u_0, v_0), but to different values of C, are equidistant as it is easy to show. When the center (u_0, v_0) is ideal, the value $C = 0$ is admissible and γ_0 is a geodesic, in pseudospherical terms, and it is the polar of the point (u_0, v_0) with respect to the limiting circle. For $C \neq 0$, then, γ_C is a set of points having a fixed distance from the geodesic γ_0 and γ_C *is not* a geodesic: a fact which reminds us of early attempts to prove Euclid's Fifth Postulate.

While the case of the ideal points seems to have been forgotten in contemporary hyperbolic pop-culture, the case of the points at infinity has not. Suppose that $u_1^2+v_1^2=a^2$, rewrite (6) with $\rho'-\rho$ instead of ρ,.

$$\frac{a^2-u_0u-v_0v}{\sqrt{a^2-u^2-v^2}}=\sqrt{a^2-u_0^2-v_0^2}\cosh\left(\frac{\rho'-\rho}{R}\right), \tag{8}$$

and let $\rho'\to\infty$ and $(u_0,v_0)\to(u_1,v_1)$ in such a way that

$$\frac{1}{2}\sqrt{a^2-u_0^2-v_0^2}e^{\rho'/R}\to a.$$

Then, the geodesic circle $S((u_0,v_0),\rho'-\rho)$ having equation (9) stabilizes to the *horocicle* $H((u_1,v_1),\rho)$ having equation

$$\frac{a^2-u_1u-v_1v}{\sqrt{a^2-u^2-v^2}}=ae^{-\frac{\rho}{R}}. \tag{9}$$

The distance between two concentric horocicles $H((u_1,v_1),\rho_0)$ and $H((u_1,v_1),\rho_1)$ is clearly constant and has value $|\rho_1-\rho_0|$. By a limiting argument, it is also easily deduced that the horocycle $H((u_1,v_1),\rho)$ is normal to all geodesics issuing from its center (u_1,v_1); that is, to all geodesics converging to the point at infinity (u_1,v_1).

Beltrami has, this way a unifying interpretation in terms of generalized circles having center in the Euclidean plane for three important different non-Euclidean objects: the (usual) metric circle, the horocycle, the set of the points having a fixed distance from a geodesic. (The reader will figure out without difficuty what the corresponding objects are in Euclidean and in spherical geometry).

Another important feature of the *Saggio,* as said before, is that Beltrami "folds" his pseudosphere onto surfaces of constant curvature in Euclidean space. His clever construction, however, is easier if translated in the "conformal" Liouville-Beltrami and Riemann-Beltrami-Poincaré models; hence, we will postpone it after we have discussed the second article, *Teoria fondamentale.*

It seems evident that the projective nature of the model was perfectly clear to Beltrami: (i) the metric is invariant under projective tansformations; (ii) important geometric objects (the circles with ideal center, for instance) had to do with projective concepts (the polar of a point with respect to a circle, to remain in the example). Probably, Beltrami did not pursue this line of investigation because it was not to his taste. Klein's main contribution was that the Riemannian distance produced by the metric had a projective interpretation (as the logarithm of a biratio; see [11] and [16] for extensive accounts of this), hence that the model could be fully developed within *synthetic* geometry, without resorting to infinitesimal calculus. The preference for synthetic arguments goes in a direction which is opposite to the one taken by Riemann in the footsteps of Gauss, and by Beltrami in the footsteps of both. It has its own merits and it had a great influence in the discussion on the fundation of mathematics, but, as far as geometry is concerned, the major conceptual developments of the past century see synthetic geometry in an ancillary position.

Anisotropy. Let us spend some words for some remarks on the metric (1) itself. We will let $a=1$, since the particular value of a has no importance for us. We can rewrite (1) as

$$ds^2 = R^2 \frac{du^2 + dv^2}{1 - u^2 - v^2} + R^2 \frac{(udu + vdv)^2}{(1 - u^2 - v^2)^2}.$$

In this form, it is easy to see that the infinitesimal discs centered at $(r, 0)$ (this is no loss of generality in view of rotational invariance) and having radius dr are ellipsis having a semiaxis $\frac{(1-r^2)}{R} dr$ in the u direction and $\frac{(1-r^2)^{1/2}}{R} dr$ in the v direction. In fact, the metric at $(r, 0)$ becomes

$$ds^2 = R^2 \frac{du^2 + dv^2}{1 - r^2} + R^2 \frac{(rdu)^2}{(1 - r^2)^2} = R^2 \frac{du^2}{(1 - r^2)^2} + R^2 \frac{dv^2}{1 - r^2}.$$

This important eccentricity says that the metric is very far from being conformal to the Euclidean metric in (u, v) coordinates (a metric ds^2 ic *conformal* to the metric $du^2 + dv^2$ if $ds^2 = f(u, v)(du^2 + dv^2)$ for some strictly positive function $f(u, v)$). It is interesting that, in a foundational article published the year before, *Delle variabili complesse su una superficie curva, 1867* [4] p.318-373, Beltrami himself had considered and solved the problem of finding conformal coordinates on a (Riemannian) surface. It is possible that, at the time of writing the *Saggio,* Beltrami had worked out conformal coordinates for the metric (1), possibly the ones associated with the so-called Poincaré disc model. In fact, Beltrami ([5], p.378) says that "the form of this expression [i.e, ds^2 as in (1)], although *less simple than other equivalent forms which might be obtained by introducing different variabels,* has the peculiar advantage [...] that any linear equation in the u, v variables represents a geodesic line and that, conversely, each geodesic line is represented by a linear equation" [my translation]. Since at the time he sent the article to the journal he had already read Riemann's *Habilitationschrift,* however, Beltrami could also had found there a disc model he had not thought about before.

3.2. **The "conformal" models.** When he learned of Riemann's *Habilitationschrift,* Beltrami was in the position of developing a remark contained in it, that the length element of an n-dimensional manifold having constant (Riemannian) curvature α ($\alpha \in \mathbb{R}$) is

$$ds^2 = \frac{\sum_j dx_j^2}{1 + \frac{\alpha}{4} \sum_j x_j^2}. \tag{10}$$

More important than this, Riemann had given solid philosophical and scientific, as well as geometric, foundations for a theory of manifolds in which Euclidean spaces played no priviledge role (except for the infinitesimal structure of space, about which, by the way, he had interesting questions and observations). To wit: (i) geometry was the study of n-dimensional manifolds endowed with a length structure (an idea which is still actual; see for instance [13]); (ii) the "geometricity" of a space was not, then, less so because the space was not a subspace of Euclidean three-space. Backed by this philosophy, which he however did not fully endorse in the *Teoria,* Beltrami could present his "purely analytic" calculations as (almost) sound geometry.

The starting point of the *Teoria* is a different way to write down the (n-dimensional) length element ds^2 in (1),

$$ds = R \frac{\sqrt{dx^2 + dx_1^2 + \cdots + dx_n^2}}{x}, \tag{11}$$

subjected to the constraint

$$x^2 + x_1^2 + \cdots + x_n^2 = a^2 \tag{12}$$

([6], p.407). A formal calculation, in fact, shows that this is the same as the n-dimensional version of (1), already cited in the *Saggio:*

$$ds^2 = R^2 \left(\frac{|d\underline{x}|^2}{a^2 - |\underline{x}|^2} - \frac{\underline{x} \cdot d\underline{x}}{(a^2 - |\underline{x}|^2)^2} \right), \tag{13}$$

with $\underline{x} = (x_1, \ldots, x_n)$ and obvious vector notation. The interest of the new way to write the metric is that the metric ds in (11) can be thought of as a metric living in the "right" half-space $H_{n+1}^+ = \{(x, x_1, \ldots, x_n) : x > 0\}$, restricted to the half-sphere $S_{n+1}^+(a)$ given by the constraint. The space H_{n+1}^+ with the metric (11) is itself a model for the $(n+1)$-dimensional non-Euclidean geometry, as we shall see shortly.

We will use the notation $B_n(a) = \{(x_1, \ldots, x_n) : \ x_1^2 + \cdots + x_n^2 < a^2\}$. With Beltrami, we take x to be a variable dependent on $(x_1, \ldots, x_n)$ in $B_n(a)$. A variational argument shows that the geodesics for the metric (11) are chords of the ball $B_n(a)$, as in the 2-dimensional case. This also garantees that the Riemannian manifold here introduced is simply connected and that it has the property that there is exactly one geodesic passing through any two given points. Next, Beltrami shows that his metric is invariant under a (Euclidean) rotation around the center of $B_n(a)$. This would immediately follow from the expression (13) for (11), but Beltrami prefers to point out that each of the following is invariant under such rotations: (i) the metric (11) when thought of as a metric on H_{n+1}^+ (here, the rotation acts on the coordinates $x_1, \ldots, x_n$ and leaves x fixed); (ii) both sets H_{n+1}^+ and $S_{n+1}(a)$. He then shows by a lengthy argument that metric isometries act transitively on $B_n(a)$. We skip here the argument and only give its conclusion ([6], p.416): the isometries for (11) are those projective transformations of Euclidean n-space which map $B_n(a)$ onto itself.

Next, the metric (11) is written in polar coordinates $\underline{x} = r\Lambda = r(\lambda_1, \ldots, \lambda_n)$, with $0 \leq r < a$ and $|\Lambda| = 1$:

$$ds^2 = R^2 \frac{a^2 dr^2}{a^2 - r^2} + R^2 \frac{!d\Lambda!^2}{(a^2 - r^2)^2}.$$

Changing from r to ρ, the non-Euclidean distance from the origin, $d\rho = Radr/(a^2 - r^2)$, one obtains

$$ds^2 = d\rho^2 + R^2 \sinh^2(\rho/R)|d\Lambda|^2. \tag{14}$$

Introducing new coordinates by radially stretching the metric,

$$\xi_j = 2R \tanh(\rho/2R)\lambda_j, \ j = 1, \ldots, n,$$

and doing elementary calculations with hyperbolic functions, one finally obtains that

$$ds^2 = \frac{\sum_j d\xi_j^2}{1 - \frac{\sum_j \xi_j^2}{4R^2}}, \tag{15}$$

which is Riemann's choice for the coordinates of the n-dimensional space of constant curvature. From here, closely following Riemann's exposition in [26], Beltrami computes that the curvature of the space metrized by (15) (hence, the metric (11) itself) is $-1/R^2$. In the two-dimensional case, (15) was independently rediscovered by Poincaré in [25].

The third model is obtained by a fractional transformation of the coordinates $(x, x_1, \dots, x_{n-1})$ in (11) (while keeping the constraint (12)):

$$(\eta, \eta_1, \dots, \eta_{n-1}) = \left(\frac{Rx}{a - x_n}, \frac{Rx_1}{a - x_n}, \dots \frac{Rx_{n-1}}{a - x_n}\right).$$

In the new coordinates,

$$ds^2 = R^2 \frac{d\eta^2 + d\eta_1^2 + \dots + \eta_{n-1}^2}{\eta}. \tag{16}$$

In dimension two, as Beltrami notes, the metric had been computed by Liouville in his *Note IV* to Monge's *Application de l'Analyse à la Géométrie* ([22], p.600), at the end of a discussion on Gauss' *Theorema Egregium.* What is interesting, Beltrami notes, is that the metric (16) is no other than the metric (11) in the ambient space (without the constraint (12)), with one less dimension.

The original metric (11) (hence, the two-dimensional metric (1) of the *Saggio*) might then so interpreted. Start with the (conformal) metric (11) on the right half-space H^+_{n+1}. The metric (11) restricted to the half n-dimensional sphere $S^+_{n+1}(a)$, makes $S^+_{n+1}(a)$ in a model of n-dimensional pseudo-sphere. Beltrami proves so, but we know that $S^+_{n+1}(a)$ contains the whole geodesic for (11) connecting any two of its points (this is clear, once we know that the geodesics for (11) are arcs of circles or straight lines, which are perpendicular to the boundary of H^+_{n+1}). The projective metric (1), and its n-dimensional generalizations, is the projection of the the conformal metric (11) on the disc cut by $S_{n+1}(a)$ on the boundary of H^+_{n+1}.

A special feature of the metric (16) is that the horocycles Γ_k having center at the point at infinity of H^+_{n+1} have equation $\eta = k$, for a positive constant k. It is immediate, then, that the restriction of the pseudospherical metric to Γ_k is (a rescaled version of) the Euclidean metric; a fact already noted by Bolyai in the three-dimensional case.

Imbedding the psudosphere in Euclidean space. In his *Saggio,* Beltrami found three surfaces in Euclidean space which carried his metric (1), by cleverly folding the pseudosphere.[8] One at least of these surfaces was previously known. What Beltrami was interested in was finding "real“ models for non-Euclidean geometry; i.e. surfaces of constant curvature in Euclidean space. This he did in the *Saggio,* written before reading Riemann; at a time when Beltrami kept a rather conservatve profile about "reality" of geometric objects.

The construction of the three surfaces is easier to see, I believe, if carried out in the conformal models of the later *Teoria*; I will then translate them in that context. I will also let $R = 1$, leaving to the reader to figure out the elementary modifications

[8] At ths point only I disagree with Gray's account of Beltrami's work. He writes, a p.208 in [19]: "...but all Beltrami did was hint that the pseudosphere [the tractroid, in this paper's terminology] must be cut open before there is any chance of a map between it and the disc [Beltrami's projective model]“. In fact, Beltrami's hint at p.390 follows his formula (12), which can be used to find (after a cut) a surface of revolution in Euclidean space, what we call here S_3. Formula (14) at p. 393 leads to our S_2 and, finally, formula (17) at p.394 leads to the tractroid, our S_1 below. The tractrix, called by Beltrami *linea dalle tangenti costanti* (*constant tangent curve*) is explicitely mentioned at p.395. I find it indicative of Beltrami virtuosism, that he managed to exhibit three different surfaces of revolution by using the projective model, which is not especially amenable to this sort of calculations.

for different values of the curvature. The general frame adopted by Beltrami is the following. The metric

$$ds^2 = d\rho^2 + G(\rho)^2 d\theta^2$$

is the length element on a surface of revolution S, θ being the angular coordinate and $d\rho$ being the length element on the generatrix of S, provided $|G'(\rho)| \leq 1$, because dG is the projection of $d\rho$ in the direction orthogonal to the axis of revolution.

We start from the *horocycle construction,* which also is the best known. Consider the Liouville-Beltrami metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},\ y > 0. \tag{17}$$

Let $d\rho = dy/y$ (then, ρ is hyperbolic length on any geodesic $x = const$) and consider $0 \leq x \leq 2\pi$. We set $\rho = \log y$ (we will recover the other primitives using an isometry). The metric ds^2 becomes

$$ds^2 = d\rho^2 + e^{-2\rho} dx^2. \tag{18}$$

The condition $|dG| \leq d\rho$ becomes $\rho \geq 0$; i.e. $y \geq 1$. The surface of revolution S_1 so obtained is the one often finds in articles and web pages concerning Beltrami's models. It is rather confusing the use of calling such surface "pseudosphere", the term Beltrami uses for the (simply connected) model he has for the non-Euclidean plane. Being the generating curve of S_1 a *tractrix,* the term *tractroid,* which is sometimes used, is better (but ugly). The part of the pseudosphere which can be mapped onto the surface S_1 in a one-to-one way (but for a curve) is $\{(x, y) :\ y \geq 1,\ 0 \leq x \leq 2\pi\}$. The vertical sides $x = 0,\ y \geq 1$ and $x = 2\pi,\ y \geq 1$ are half-geodesics which have to be glued together. The resulting surface in Euclidean space looks like an exponentially thin thorn as $y \to +\infty$ and as a trumpet near $y = 1$. At the aperture of the trumpet, the metric of S_1 can not be carried by Euclidean space anymore.

The Euclidean dilation $(x, y) \mapsto (\lambda x, \lambda y)$, $\lambda > 0$, is clearly an isometry for ds^2 in (17). By composing with these dilations, we find other regions in the pseudosphere which can be mapped onto the surface S_1; regions which might be otherwise be found by choosing the primitives $\rho = \log(y/y_0)$ with $y_0 > 0$, as Beltrami did in the *Saggio.* By isometry, however, the tractroid does not change with y_0.

The contemporary way to think of S_1 woud be to take a quotient of the pseudosphere with the parameters x, y with respect to the action of the group of isometries generated by $(x, y) \mapsto (x + 2\pi, y)$. The resulting Riemann surface, then, can be embedded in Euclidean space for the part where $y \geq 1$ as a tractroid, as we have just seen.

We now give the correspondng construction with two *geodesic circles having ideal center* playing the role that was above played by a horocycle. Again in the upper plane model, consider the geodesic γ_∞ having equation $x = 0$. The lines γ_m with equation $y = mx$, $m \in \mathbb{R}$, are then equidistant from γ_0 (proof: the lines $y = \pm mx$ are the envelope of a family of circles having constant hyperbolic radius and centers on γ_0). Writing $x = r\cos(\theta),\ y = r\sin(\theta)$ in polar coordinates, the metric becomes

$$ds^2 = \frac{d\theta^2}{\sin^2(\theta)} + \frac{dr^2}{r^2\sin^2(\theta)}.$$

Let $d\theta/\sin(\theta) = d\rho$, $\rho = \log(\tan(\theta/2))$, and $dr/r = dt$, $r = e^t$. Then,

$$ds^2 = d\rho^2 + z(\rho)^2 dt^2, \tag{19}$$

where $z(\rho) = 1/\sin(\theta)$. It follows that $|dz/d\rho| = |1/\tan(\theta)| \leq 1$ if and only if $|\theta - \pi/2| \leq \pi/4$. Keeping in mind that we want the "angle" coordinate t in $[A, A+2\pi]$, we have that any Euclidean cicular sector

$$\Omega = \{(x,y) : \ y \geq |x|, \ r_0 \leq \sqrt{x^2+y^2} \leq r_0 e^{2\pi}\},$$

with $r_0 > 0$, is isometric to a surface of revolution S_2 in Euclidean space. The two arcs of circles (which are geodesics for the metric) are glued together (identifying points lying on the same line $y = mx$). The segments of the lines $y = \pm x$ lying on the boundary of Ω are apertures corresponding to the aperture of the trumpet in the horocycle construction. While in S_1 the aperture is a piece of horocycle, in this case it is made by two pieces of circel having ideal center. Contrary to the case of the surface S_1, in this case we can construct a one-parameter family of non-isometric surfaces $S_2(k)$ by integrating $dr/r = kdt$ $(k > 0)$, having then $z(\rho) = 1/[k\sin(\theta)]$: as k grows larger, the interval of the allowable r's becomes thinner, while the circles with ideal center $y = mx$ can range over a greater range of m's. In this case, reverting to $k = 1$, group of isometries generated by $(x,y) \mapsto (e^{2\pi}x, e^{2\pi}y)$ (Euclidean dilations) is the one giving (an extension of) S_2 as quotient.

Lastly, we consider the conformal disc model instead,

$$ds^2 = \frac{dx^2+dy^2}{\left(1-\frac{x^2+y^2}{4}\right)^2} = \frac{dr^2+r^2d\theta^2}{\left(1-\frac{r^2}{4}\right)^2},$$

in usual polar coordinates (r,θ). We set $d\rho = dr/(1-r^2/4)$ (ρ is then non-Euclidean distance from the origin), so that $\tanh(\rho/2) = r/2$ and $r/(1-r^2/4) = \sinh(\rho)$. The metric then becomes

$$ds^2 = d\rho^2 + \sinh^2(\rho)d\theta^2. \tag{20}$$

Since $(d/d\rho)\sinh(\rho) = \cosh(\theta) \geq 1$, (20) can not be the length element of a surface of revolution S_3 whose angular coordinate is θ. In fact, Beltrami points out that, if such were the case, then S_3 would have, by symmetry, equal principal curvatures at the point corresponding to $(x,y) = (0,0)$; it could not then have negative curvature there. The remedy is considering $0 \leq \theta \leq 2\pi\epsilon$ only, with $\epsilon < 1$, and letting $t = \theta/\epsilon$ be the angular coordinate. This way, the metric becomes $ds^2 = d\rho^2 + \epsilon^2\sinh^2(\rho)dt^2$, which defines the metric for a surface of revolution S_3^ϵ provided that $\cosh(\rho) \leq 1/\epsilon$. The surface S_3^ϵ has a cusp at the point corresponding to the origin, where the total angle is only $2\pi\epsilon$. The aperture at its other hand is an arc of the geodesic circle centered at the origin and having radius $\rho_\epsilon = \cosh^{-1}(1/\epsilon)$. It is intuitive (and can be proved) that S_3^ϵ converges to S_1 as ϵ tends to zero. The isometry group producing S_3^ϵ as a quotient, in the case $\epsilon = 1/n$ ($n \geq 2$ integer), is the one generated by a rotation of an angle $2\pi/n$.

In all the three types of surfaces we have seen, the gluing is done along geodesic segments. The apertures due to the partial imbeddability in Euclidean space are always arcs of generalized circles: a horocycle for S_1; two circles with ideal center for S_2 and a metric circle for S_3. In this case as well, we see the tri-partition of the "projective" generalized circles emerging very elegantly from Beltrami's *Saggio*.

Anachronism (for the complex reader). If we replace the unit disc in the real plane $\mathbb{R}^2$ by the unit ball $\mathbb{B}_{\mathbb{C}^2} = \{(z,w) : |z|^2 + |w|^2 < 1\}$ in the complex 2-space $\mathbb{C}^2$ and we replace real by complex variables, we might consider (1) as the restriction to the real disc $\mathbb{B}_{\mathbb{R}^2} = \{(x,y) : \ x^2+u^2<1\} \subset \mathbb{B}_{\mathbb{C}^2}$ of the metric

$$d\sigma^2 = R^2 \frac{|dz|^2+|dw|^2}{1-|z|^2-|w|^2} + R^2 \frac{|\overline{z}dz+\overline{w}dw|^2}{(1-|z|^2-|w|^2)^2}. \tag{21}$$

People working several complex variables will immediately recognize $d\sigma^2$ as the *Bergman metric* on $\mathbb{B}_{\mathbb{C}^2}$; i.e. of the (essentially) unique bi-holomorphically invariant Riemannian metric $\mathbb{B}_{\mathbb{C}^2}$. Beltrami's first model, then, can be interpreted as a totally geodesic surface in Bergman's complex ball. Easy considerations show that the automorphism group of $\mathbb{B}_{\mathbb{C}^2}$ acts transitively on the tangent space of the real disc $\mathbb{B}_{\mathbb{R}^2} = \{(z,w,) \in \mathbb{B}_{\mathbb{C}^2} : \ z,w \in \mathbb{R}\}$, hence that -by a theorem of Gauss, as noted by Beltrami- the metric ds^2 has constant curvature. It is easy to see that another totally geodesic surface in $B_{\mathbb{C}^2}$ can be obtained by letting $w=0$ in. This gives the metric

$$d\tau^2 = R^2 \frac{|dz|^2}{(1-|z|^2)^2} \tag{22}$$

on the complex disc $\mathbb{B}_{\mathbb{C}^1}$. The metric $d\tau^2$ makes the unit complex disc $\mathbb{B}_{\mathbb{C}^1} = \{z \in \mathbb{C} : \ |z| < 1\}$ into the Riemann-Beltrami-Poincaré disc model. Note that the projective model $\mathbb{B}_{\mathbb{R}^2}$ has here curvature $K=-1$, while the conformal model $\mathbb{B}_{\mathbb{C}^1}$ has curvature $K=-4$: the two surfaces in $\mathbb{B}_{\mathbb{C}^2}$ *are not isometric.* It is also interesting to note that, if we pass to higher (complex) dimension, we still find this way higher dimensional (real) projective models of non-Euclidean geometry, but we always find two-dimensional only conformal disc models of it.

The bi-holomorphisms of the unit disc are exactly the projective tranformations of the complex two-space, which map the unit ball into itself: in the light of Beltrami's discussion of his projective model, this is almost obvious. Among these transformations, the subgroup formed by the ones having real coefficients also map the unit real disc $\mathbb{B}_{\mathbb{R}^2}$ onto itself, and exhaust the projective self maps of $\mathbb{B}_{\mathbb{R}^2}$ which preserve orientation. This are the (sense preserving) isometries of the Beltrami-Klein (projective) model. On the other hand, the projective self maps of $\mathbb{B}_{\mathbb{C}^2}$ fixing the second coordinate w, which we can identify with projective transformations of the projective line $\mathbb{C} \cup \{\infty\}$ (i.e. fractional linear tranformations) fixing the unit disc in $\mathbb{C}$, are the isometries of the Riemann-Beltrami-Poincaré (conformal) model. They have the form

$$z \mapsto e^{i\lambda} \frac{a-z}{1-\overline{a}z}$$

for fixed a in $\mathbb{C}$, $|a|<1$, and λ in $\mathbb{R}$. It interesting that the two-dimensional disc model and the (eventually) higher dimensional projective model studied by Beltrami find a unification in the context of complex projective geometry.

The metric (21) has not constant curvature in the sense of Riemann (if it had, we could not have found in it geodesic surfaces having different curvature), it is not, then, by itself a model for non-Euclidean geometry. It is interesting to see in one of its features how much it departs from having constant curvature. We saw before that the projective metric (1) exhibits an extrinsic (hence, illusory), but important anisotropy. The metric (21), which is defined similarly, exhibits a similar anisotropy, which is not, this time, illusory at all. A way to see this is the following. Consider the non-Euclidean plane in the Riemann-Beltrami-Poincaré three-dimensional model

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{(1 - x^2 - y^2 - z^2)^2}$$

considered before and the family of circles $\Gamma_r = \{(x, y, z) : \ x^2 + y^2 + z^2 = r^2\}$, $0 < r < 1$. The metric ds^2 restricted to each Γ_r is a spherical metric and all the metrics so obtained are just rescalings of each other. (Keeping track of the rescaling factors, we can this way give spherical coordinates to the boundary at infinity of the non-Euclidean space). The situation is much different with the metric (21). Its restrictions to spheres centered at the origin (which are spheres for the Bergman metric itself!) are not rescaled versions of each other: there is not a map Λ from one of the spheres (S_1) to another one (S_2) such that the $d_2(\Lambda(A), \Lambda(B))/d_1(A, B)$ is constant (here, d_j is the distance on S_j which comes from the restriction of the Bergman metric). Moreover, if we normalize the metrics on the spheres of (Bergman) radius ρ in order to that they stabilize to a metric on the boundary at infinity as $\rho \to \infty$, such metric turns out *not to be a Riemannian metric.* In fact, it becomes what is called a *sub-riemannian metric,* in which, for instance, the uniqueness of the geodesic through two points fails on any (arbitrary small) small open set.[9]

Beltrami's projective model coupled with complex variable opens the way for a world which is outside (but in the limit of) Riemannian geometry.

3.3. **What was Beltrami's interpretation of his own work?** I will try to argue here that Beltrami had a model of non-Euclidean geometry, that this model was geometrically rather "real" and that he thought so. What is uncertain is whether, at the time he wrote the *Saggio,* he claimed (or believed) that this model was a faithful representation of non-Euclidean geometry; that is, if he thought that a property which could be proved in his model was necessarily true in the non-Euclidean (synthetic) theory of Bolyai and Lobachevsky. In his [27], M.J. Scanlan argues otherwise. In particular, he mantains that Beltrami's pseudosphere *is* the imbedded tractroid S_1 we have met before; rather, two of such surfaces glued together, which would not be a model for non-Euclidean plane (try to see what such union becomes on the Liouville-Beltrami model).

Beltrami begins, we have seen, by giving the metric (1), where (u, v) are coordinates satisfing $u^2 + v^2 < a^2$. The coordinates u, v are meant to bijecively parametrize the points of a surface S. A tthis point, after all properties of the metric (1) are proved, we surely have an *analytic model:* a class of objects verifying the assumptions of non-Euclidean (hyperbolic geometry).

What is this surface and where does it live? Beltrami does not say, but it is clear that the question has for him the greatest relevance. If the model has to *geometric,* then the surface S has to be "real". The strictest standard for "reality" is being a surface in Euclidean three-space.

That Beltrami considers S a surface, not merely an anlytic fiction, becomes evident at p.379, in a passage which might be confusing for the contemporary reader,

[9]The main reason for this is that, at the point $(r, 0)$ in $\mathbb{B}_{\mathbb{C}^2}$ $(0 < r < 1)$, infinitesimal balls have two (real) large directions and two (real) small directions. Let $z = x + iy$: the x (normal) direction is small exactly as in the Beltrami-Klein model. Due to the holomorphic coupling of the variables in the metric (21), the (tangential) direction y is small as well. We have then two large and one small tangential directions, causing an interesting degeneracy of the metric in the limit.

who is accustomed to the abstract definition of manifolds (i.e.: set theory coupled with analytic considerations). Beltrami considers the *geometric* disc $x^2+y^2 < a^2$ in a Euclidean plane endowed with Cartesian coordinates x, y. Setting $x = u$ and $y = v$ we have a map from the psudosphere S (parametrized by u and v) and a region of the Euclidean plane. We have then two geometric objects (the pseudosphere, a Euclidean disc), endowed with coordinate systems, and we use the equality of the coordinates' values to establishe a map between these two geometric objects. The confusion for us is that we are not used anymore to consider Euclidean discs geometrically, and equating the coordinates makes for us litte sense.

The surface S exists as an unquestionably "real object" if it lives in Euclidean three space (otherwise, someone could object that it is more like an "analytic object", at least before Riemann's paper). It lives there, however, not as we are used to think of it nowadays, as an imbedded submanifold; but, rather, as a physical surface, which remains the same if moved without alteration of the mutual metric relations within the surface; without taking into account self intersections. This idea of surface in space is shortly explained in the introductory remarks to [4] (p.318): *Each couple of values for u, v determine a point of the surface, which remains essentially distinct from that corresponding to a different couple of values. The fact that, in same place in space, two points having different curvilinear coordinates might coincide, does not matter unless we consider a given configuration of the surface.* That is to say: the contemporary imbedded surface is for Beltrami a *configuration* of his own surface, and it is possible that parts of the surface overlap. What Beltrami calls (real) *surface,* then, is the (equivalence) class of all "configurations" which are isometric; i.e. for which a parametrization u, v such that the length element on the surface can be expresse by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

exists (the couple u, v ranging on a fixed set and E, F, G being given functions of u and v). This definition of Riemannian surface is perfectly consistent with Gauss theory and its motivations, and with physical intuition. Of course, in our highly formalized mathematics this definition is hard to work with. Beltrami, however, felt perfectly at ease with it.

Being such the definition, Beltrami does not find any problem when he has to fold his pseudosphere. Rather, he might have problems when the imbedding can not be locally done, due to the obstruction that $|G'(\rho)| > 1$ on his surfaces of revolution. The tractroid in Euclidean space, for instance, can only model the part of the pseudosphere which is internal to the horocycle bounding it (i.e., between the horocycle and its point at infinity).

However, having freed the surface from its particular "configuration" in space, it is possible to identify it with the range of the parameters u, v and with the length element ds^2: what we would call a patch of (abstract) surface with a Riemannian metric. Namely, further extending the family of the possible "configurations" to those in which the imbedding is just locally possible, we come to a definition of Riemannian surface which is almost equivalent to the contemporary one. Beltrami appears to be open to this possibility in the first few sentences of [4]: *It is not useless to remind that, when considering a surface as defined solely by its linear element [i.e., ds^2], one has to abstract from any concept or image which implies a concrete determination of its shape which is related to external objects; e.g. [its position with respect to] a system of orthogonal axis [in three space].*A surface of this kind, which is intermediate between "real" and "purely analytic" is the pseudosphere in his *Saggio.* In the *Teoria,* relying -at least as two-dimensional manifolds are concerned- on Riemann's authority, Beltrami does not even mention the problem of finding a Euclidean spece where he can imbed his pseudospheres.

Let now get to the second problem: the relation between pseudospherical and non-Euclidean geometry. Beltrami shows that, properly interpreted, terms and postulates of non-Euclidean geometry hold on the pseudosphere. Hence, all theorems of non-Euclidean geometry hold on the pseudosphere[10].

As a consequence, as Hoüel made explicit in [17], the Postulate of Parallels can not be proved from the remaining ones (on this point, I desagree with Scanlan). In fact, any proposition which can be proved by the remaining postulates would also hold on the pseudosphere (no matter if it is real-geometric or purely-analytic), but on the pseudosphere we have not uniqueness of parallels, so we have a contradiction. Scanlan says that Beltrami and Hoüel did not have enough matematical logics behind to fully justify this. In my opinion, on this point Scanlan is close to the anachronisms he is castigating.

Let me use a metaphor. Suppose a child has realized that Odd and Even satisfy, with respect to sum, the same relation that Minus and Plus signs satisfy with respect to product. Lacking the idea of group isomorphism, she might be hesitant in describing this discovery. Challenged to prove that an expression with one hundred sums of Odds and Evens ends correspondingly to the corresponding expression of one hundred products of Minuses and Pluses, she will probably compute some shorter expresions and then she will use (as we sometimes do when teaching) the non-rigorous expression "and so on". Now, it is true that the higher level of formalism provided by isomorphisms would give her a tool to extend this observation and to be more fully aware of its meaning; but, in my opinion, the clever observation of the child remains solid even in its naïf form.

Another point of interest is the following: was Beltrami convinced that the pseudosphere had to the non-Euclidean plane the same relation that Cartesian coordinates have to the Euclidean one? That is, as I was saying above: if a question in non-Euclidean geometry is answered doing analytic calculations on the pseudosphere, was Beltrami certain that the answer was the correct one in non-Euclidean geometry as well? In the *Saggio,* there seems to be some hesitation on this point. In fact, Beltrami proves a number of knowm results in non-Euclidean geometry, to conclude (p.387): *The preceeding results seem to us to fully show the correspondence between non-Euclidean planimetry and pseudospherical geometry.* But he is not yet satisfied, and goes on: *To verify the same thing from a differen viewpoint, we also want to directly establish, by our analysis, the theorem about the sum of the three angles of a triangle.* These passages, hesitan as they are, seem to show that Beltrami was personally convinced, but that, like the child of the metaphor, he did not have the language to write it down convincingly. Beltrami certainly knew that the pseudosphere and the non-Euclidean plane are isometric (he introduced the coordinates u, v in non-Euclidean plane in place of Cartesian coordinates, and using known metric properties of non-Euclidean triangles the isometry follows rather easily) Then, all geometric quantities (angles, areas, lengths) which can be expressed in terms of distances correspond. For carrying out this program in all details, however, the mathematical technology available at the time was probaby

[10] A similar remark is in *Teoria,* p. 427.

not sufficient (defining the area of a generic figure through distance alone, for instance, requires concepts like Hausdorff measure, which were developed only much later). Notwithstanding, Beltrami seemed rather confident that he could answer one by one all possible problems in non-Euclidean geometry by his model, even if he did not have a general *logical* recipe to do so, but rather a *method.*

4. From the boundary to the interior: an example from signal processing

In this section, I would like to give the reader a hint of what it means that, in Beltrami's models, the non-Eucean space can be usefully thought of as the "filling" of the Euclidean space. I make no attempt to trace the history of this idea, which is probably rather old. In the context of complex analysis, an idea with this flavor it was first formalized by Georg Pick [24] (I have learned the details of this story from a nice article by Robert Osserman [23]) by noting that the classical Schwarz Lemma could be interpreted as saying that holomorphic functions from the unit disc D in the complex plane to itself, decrease hyperbolic distance. Let me remind the reader that Schwarz Lemma states that if $f : D \to D$ is holomorphic and $f(0) = 0$, then $|f(0)| \leq |z|$ in D; equality occurring for some $z \neq 0$ if and only if for some λ with $|\lambda| = 1$ one has $f(z) = \lambda z$ holds for all z in D. Changing variables by means of a fractional tranformation of the unit disc, Pick showed that, if $f : D \to D$ is holomorphic, then $\frac{2|df(z)|}{(1-|f(z)|^2)} \leq \frac{2|dz|}{(1-|z|^2)} = ds$, but ds is the length element in the Riemann-Beltrami-Poincaré model (corresponding to curvature $K = -1$), proving the remark. Lars Valerian Ahlfors [1] extended this observation to holomorphic maps from D to Riemann surfaces S having curvature $K \leq -1$. Bounded holomorphic functions can be reconstructed from their boundary values by means of an integral reproducing formula, which is obviously invariant under rotations of D: it was not a priori obvious that the underlying geometry was much richer. (tthe first step in this direction had been taken by Poincaré in [25]).

It would be interesting, I think, knowing more about the history of how non-Euclidean (hyperbolic) geometry came to be recognized as the geometry underlying classes of objects arising in a Euclidean or spherical setting. Here, instead, I offer a small sample of this relation, showing how non-Euclidean geometry naturally arises from the problem of "looking at a signal at a given scale". What I am going to say is not at all original, perhaps except for the order in which the argument is presented.

Suppose we want to analyze *signals* which can be modeled as functions in $L^\infty(\mathbb{R}) \cup L^1(\mathbb{R})$ (Lebesgue spaces with respect to the Lebesgue measure in $\mathbb{R}$; where $\mathbb{R}$ could be thought of as time). Suppose that we have a device $\mathcal{T} = \{T_t \ \ t > 0\}$ which looks at the signal *at the scale* t for each $t > 0$; i.e., dropping in an orderly fashion the information at smaller scales, while keeping information at larger scales. Here a short list of reasonable (not all independent) *desiderata* for the device.

(L) Linearity. $T_t(af + bg) = aT_tf + bT_tg$.

(T) Translation invariance. If

$$\tau_a f(x) = f(x - a),$$

is (forward) translation in time, then

$$\tau_a \circ T_t = T_t \circ \tau_a. \tag{23}$$

The obvious meaning of the requirement is that the device acts uniformly in time.

(D) Dilations. Let
$$\delta_\lambda f(x) = \lambda^{-1} f(\lambda^{-1}x)$$
be a *dilation* in time by a factor $\lambda > 0$ (normalized to keep invariant the $L^1(\mathbb{R})$-norm of the signal). Then,
$$T_t \circ \delta_\lambda = \delta_\lambda \circ T_{t/\lambda}. \tag{24}$$
The meaning is: dilating a signal by λ an then looking it at the scale $t > 0$ is the same as looking at it at the scale t/λ, then dilating it by λ: in both cases we loose the detail .

(S) Stability. We want the device $\mathcal{T}$ to give an output which can be put in the same device. For instance,
$$\|T_t f\|_{L^p} \leq C(t)\|f\|_{L^p},$$
for all $p \in [1, \infty]$.

(C) Continuity. As t goes 0, $T_t f$ should be "close" to f. For instance,
$$\lim_{t \to 0^+} T_t f = f \text{ in } L^p(\mathbb{R}),\ 1 \leq p < \infty, \text{ and in } C_0(\mathbb{R}),$$
where $C_0(\mathbb{R})$ is the space of the continuous functions vanising at infinity.

(SG) Semigroup. Looking at the signal at the scale s, then at the scale t, we loose all detail at scale t, then s. It is reasonable that, overall, we have lost details at scale $t + s$:
$$T_t \circ T_s = T_{s+t}$$

(P) Positivity. $f \geq 0 \implies T_t f \geq 0$.

None of the requirements is arbitray, but we can indeed think of useful devices which do not satisfy some of them.

It is easy to see that (L), (T), (D), (S), (C), (SG), (P) are satisfied by the operators $T_t = P_t$, $f \mapsto P_t f$, where
$$P_t f(x) = \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{t}{y^2 + t^2} f(x - y) dy, \tag{25}$$
which is the *Poisson integral* of f. Using elementary arguments involving Fourier tranforms, it can also be shown that such operators are essentially the only ones satisfying all the requirements. More precisely, either $T_t f = f$ for all $t > 0$ (but this way we would not loose any detail!), or there is some $k > 0$ such that $T_t = P_{kt}$.

We now look at all of the scaled versions of a signal together:
$$u(x, t) = P_t f(x),$$
where $u : H_2^+ = \mathbb{R} \times [0, +\infty) \to \mathbb{R}$. It is easy to see that u is *harmonic* on H_2^+, $\partial_{xx} u + \partial_{tt} u = 0$; but we ll not use this fact. We ask, now, how much can the (scaled) signal change with respect to x or t. We consider a positive signal $f \geq 0$. The answer, we will see, can be interpreted in terms of non-Euclidean geometry.

Theorem 1. *Harnack's inequality. Let $u(x, t) = P_t f(x)$, with $f \geq 0$, $f \in L^1$. If $(x, t),\ (x', t') \in \mathbb{R}^2_+$ and*
$$1/2 \leq t'/t \leq 2,\ |x - x'| \leq \max(t, t\prime),$$
then
$$u(x, t) \leq Cu(x', t'),$$

where C is a universal constant.

The geometric meaning of Harnack's inequality is this: u is essentially constant (in multiplicative terms) on squares having the form

$$Q(x_0, t_0) = \{(x,t) : \ |t - t_0| \le t_0/2, \ |x - x_0| \le t_0/2\}.$$

It is immediate from (17), the expression of the non-Euclidean distance in the Liouville-Beltrami model,

$$ds^2 = \frac{dx^2 + dt^2}{t^2}$$

(with $R = 1$) that $Q(t_0, x_0)$ is comparable with a disc having fixed hyperbolic radius; i.e. that there are positive constants $c_1 < c_2$ such that $D_{hyp}((x_0,t_0), c_1) \subseteq Q(x_0,t_0) \subseteq D_{hyp}((x_0,t_0), c_2)$. As a consequence, if (x_1,t_1) and (X_2,t_2) lie a hyperbolic distance d apart, then $u(x_2,t_2) \le C^d u(x_1,t_1)$. The usual proof of Harnack's inequality makes use of the harmonicity of u, which might be thought of as an infinitesimal version of the mean value property on circles, which implies a Poisson integral formula like (25), with discs contained in H_2^+ instead of H_2^+ itself. We have preferred a different approach to show how the inequality directly follows for the geometric requirements we have made.

Proof.

$$\begin{aligned} u(0,t) &= \frac{1}{\pi}\int_{\mathbb{R}} f(y)\frac{t}{t^2+y^2}dy \\ &\approx \frac{1}{t}\left[\int_{|y/t|\le 1} f(y)dy + \sum_{n=1}^{\infty}\frac{1}{1+2^{2n}}\int_{2^{n-1}\le|y/t|\le 2^n} f(y)dy\right] \\ &\approx u(0,2t). \end{aligned}$$

For the next calculation, consider that, if $|x/t| \le 1$ e $2^{n-1} \le |(x-y)/t| \le 2^n$, then

$$(2^n - 1)t \le |x-y| - |x| \le |y| \le |x-y| + |x| \le (2^n+1)t.$$

$$\begin{aligned} u(x,t) &\approx \frac{1}{t}\left[\int_{|(x-y)/t|\le 1} f(y)dy + \sum_{n=1}^{\infty}\frac{1}{1+2^{2n}}\int_{2^{n-1}\le|(x-y)/t|\le 2^n} f(y)dy\right] \\ &\lesssim \frac{1}{t}\left[\int_{|y/t|\le 1} f(y)dy + \sum_{n=1}^{\infty}\frac{1}{1+2^{2n}}\int_{2^{n-1}\le|y/t|\le 2^n} f(y)dy\right] \\ &\approx u(0,t). \end{aligned}$$

□

In order to better understand Harnack's inequality and its proof, divide H_2^+ in dyadic squares:

$$Q_{n,j} = \{(x,t) : \ \frac{1}{2^{n+1}} \le t < \frac{1}{2^n}, \ \frac{j-1}{2^n} \le x \le \frac{1}{2^{n+1}}\}.$$

The inequality implies that u changes by at most a multiplicative constant when we move from a square $Q = Q_{n,j}$ to a square $Q' = Q_{n',j'}$ such that $Q \cap Q' \neq \emptyset$. Let $\mathcal{Q}$ be the set of the dyadic squares and define a *graph structure* G on $\mathcal{Q}$ by saying that there is an edge of the graph between the squares Q_1 and Q_2 if $Q \cap Q' \neq \emptyset$. We can define a distance d_G on the graph G by saying that $d_G(Q_1, Q_2)$ is the minimum number of edges of the graph G one has to cross while going from Q_1 to Q_2. Let

d be non-Euclidean distance (in the Liouville-Beltrami model). It is clear that, if (x_j, t_j) lis in Q_j, $j = 1, 2$, then

$$d((x_1, t_1), (Q_1, Q_2)) + 1 \approx d_G(Q_1, Q_2) + 1 : \tag{26}$$

the graph and the hyperbolic distance are, *in the large,* comparable. This is the main reason why Cannon *et al* ([14]) call the dyadic decomposition of the upper half space *the Fifth model* of non-Euclidean (hyperbolic) geometry, after the three models by Beltrami and the "Fourth", commonly used model living in Minkovsky space. In the language of the metric d_G, Harnack's inequality reads as

$$|\log u(x_1, t_1) - \log u(x_2, t_2)| \le C' d_G(Q_1, Q_2) + K'. \tag{27}$$

for some constants C' and K'. More precisely, it could be proved that there is a universal constant $K > 0$ such that

$$|\log u(x_1, t_1) - \log u(x_2, t_2)| \le K d((x_1, t_1), (x_2, t_2)). \tag{28}$$

and that, for given $(x_1, t_1),\ (x_2, t_2)$, the constant on the right of (28) is best possible for Harnak's inequality. The proof is in the books and xploits the harmonicity of u and a version of the Poisson integral on circles. Also, Harnack's inequality hold as well for positive harmonic functions u, which do not arise as Poisson integrals of positive, integrable functions on the real line. Both (28) and (27) can be read as Lipschitz conditions for $|\log u|$ in terms of the discretized, or of the continuous hyperbolic distance.

The example above reflects the modern view that "geometry" has not to be related to "real space". Such view was endorsed by Riemann, but it found a coherent and rather complete logical and mathematical framework only much later, with the work of Peano and Hilbert on the axiomatization of mathematical theories. Although Beltrami seems to have felt safer on the old ground of reality, his models are nonetheless an essential step in the direction of the contemporary, freer way to think and use geometry.

References

[1] Lars Valerian Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), no. 3, 359-364.

[2] Eugenio Beltrami, *Opere matematiche di Eugenio Beltrami, pubblicate per cura della Facoltà di Scienze dalla Regia Università di Roma*, vol. I (1902) vii+437 p.; vol. II (1904) 468 p., Hoepli, Milano.

[3] ______, *Risoluzione del problema: "Riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette",* Ann. Mat. Pura App. **I** (1865), no. 7, 185-204.

[4] ______, *Delle variabili complesse sopra una superificie qualunque,* Ann. Mat. Pura App. **Ser. II vol. I** (1867), 329-366.

[5] ______, *Saggio di interpretazione della geometria non-euclidea*, Giornale di Matematiche **VI** (1868), 284-312; English transl., *Essai d'interpretation de la géométrie noneuclidéenne*, Annales scientifiques de l'É.N.S. I^{re} série **6** (1869), 251-288.

[6] ______, *Teoria fondamentale degli spazii di curvatura costante*, Ann. Mat. Pura App. **Serie II, vol. II** (Unknown Month 1868), 232-255; English transl., *Théorie fondamentale des espace de courbure constante*, Annales scientifiques de l'É.N.S. I^{re} série **6** (1869), 347-375-288.

[7] ______, *Osservazione sulla nota del prof. L. Schlaefli alla memoria del sig. Beltrami "Sugli spazi di curvatura costante".*, Ann. Mat. Pura Appl. **V** (Unknown Month 1871), 194-198.
[8] ______, *Teorema di geometria pseudosferica.*, Giorn. Mat. **X** (1872), 53.
[9] ______, *Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche.*, Giorn. Mat. **X** (1872), 147-159.
[10] ______, *Un precursore italiano di Legendre e di Lobatschewsky.*, Rendiconti della R. Accad. dei Lincei **V** (1889), no. I semestre, 441-448.
[11] Roberto Bonola, *La geometria non-euclidea*, Zanichelli, Bologna, 1906.
[12] Carl B. Boyer, *The History of the Calculus and Its Conceptual Development,* Dover Publications, Inc., New York, 1959.
[13] Martin R. Bridson and André Häfliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften, Vol. 319, Springer, Berlin, 1999.
[14] James W. Cannon, William J. Floyd, Richard Kenyon, and Walter R., *Hyperbolic Geometry*, Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 1997.
[15] Salvatore Cicenia, *Le lettere di E. Beltrami ad A. Genocchi sulle geometrie non euclidee.*, Nuncius (1989), 567-593.
[16] Federigo Enriques, *Conferenze sulla geometria non-euclidea*, Zanichelli, Bologna, 1918.
[17] Jules Hoüel, *Note sur l'impossibilité de démotrer par une construction plane le principe des parallèles, dit Postulatum d'Euclide*, Nouvelles annales de mathématiques 2^{e} série **tome 9** (1870), 93-96.
[18] Jeremy Gray, *Non-Euclidean geometry: a re-interpretation.*, Historia Mathematica **6** (1979), 236-258.
[19] ______, *Worlds out of nothing: a course in the history of geometry in the 19th Century.*, Springer Undergraduate Mathematics Series, Springer, Berlin, 2007.
[20] Teri Merrick, *What Frege Meant When He Said: Kant is Right about Geometry*, Philophia Math. (III) (2006), 1-32.
[21] John Milnor, *Hyperbolic Geometry: the first 150 years*, Bull. Am. Math. SOc. **6** (1982), no. 1, 9-24.
[22] Monge, *Application de l'Analyse à la Géométrie, 5. é?d., rev., cor. et annoté?e par m. Liouville*, Bachelier, Paris, 1850.
[23] Robert Osserman, *From Schwarz to Pick to Ahlfors and beyond*, Notices Amer. Math. Soc. **46** (1999), no. 8, 868-873.
[24] Georg Pick, *Über eine Eigenschaft der konformen Abbildung kreisfrmiger Bereiche*, Math. Ann. **77** (1915), no. 1, 1-6.
[25] Henri Poincaré, *Théorie des groupes Fuchsiens*, Acta Math. **1** (1882), no. 1, 1-76.
[26] Bernhard Riemann, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, Abhandlungen der Kniglichen Gesellschaft der Wissenschaften zu Göttingen **13** (1867); English transl., *On the Hypotheses which lie at the Bases of Geometry*, Nature **VIII** (1873), no. 183, 14-17.
[27] Michael J. Scanlan, *Beltrami's Model and the Independance of the Parallel Postulate*, Hist. Philos. Logic **9** (1988), no. 1, 13-34.
[28] R. M. Winger, *Gauss and non-Euclidean geometry*, Bull. Amer. Math. Soc. **31** (1925), no. 7, 356-358.

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